Layer methods for stochastic Navier-Stokes equations using simplest characteristics

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Abstract

We propose and study a layer method for stochastic Navier-Stokes equations (SNSE) with spatial periodic boundary conditions and additive noise. The method is constructed using conditional probabilistic representations of solutions to SNSE and exploiting ideas of the weak sense numerical integration of stochastic differential equations. We prove some convergence results for the proposed method including its first mean-square order. Results of numerical experiments on two model problems are presented.

Keywords Navier-Stokes equations, Oseen-Stokes equations, Helmholtz-Hodge-Leray decomposition, stochastic partial differential equations, conditional Feynman-Kac formula, weak approximation of stochastic differential equations and layer methods.

AMS 2000 subject classification. 65C30, 60H15, 60H35

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(w(t), \mathcal{F}_t^w) = ((w_1(t), \dots, w_q(t))^\top, \mathcal{F}_t^w)$ be a q-dimensional standard Wiener process, where $\mathcal{F}_t^w,\ 0 \le t \le T$, is an increasing family of σ-subalgebras of $\mathcal{F}$ induced by $w(t)$. We consider the system of stochastic Navier-Stokes equations (SNSE) with additive noise for velocity v and pressure p in a viscous incompressible flow:

$$dv(t) = \left[\frac{\sigma^2}{2}\Delta v - (v, \nabla)v - \nabla p + f(t,x)\right] dt + \sum_{r=1}^{q} \gamma_r(t,x) dw_r(t), \tag{1.1}$$
$$0 \le t \le T,\ x \in \mathbf{R}^n,$$
$$\operatorname{div}\ v = 0, \tag{1.2}$$

with spatial periodic conditions

$$v(t, x + Le_i) = v(t,x),\ p(t, x + Le_i) = p(t,x), \tag{1.3}$$
$$0 \le t \le T, \quad i = 1, \dots, n,$$

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and the initial condition

$$v(0,x) = \varphi(x). \tag{1.4}$$

In (1.1)-(1.2), v, f, and γ_r are n-dimensional functions; p is a scalar; $\{e_i\}$ is the canonical basis in $\mathbf{R}^n$ and $L > 0$ is the period (for simplicity in writing, the periods in all the directions are taken the same). The functions $f = f(t,x)$ and $\gamma_r(t,x)$ are supposed to be spatial periodic as well. Further, we require that $\gamma_r(t,x)$ are divergence free:

$$\operatorname{div} \gamma_r(t,x) = 0, \ \ r = 1, \ldots, q. \tag{1.5}$$

We note that the noise term in (1.1) contains a finite number of Wiener processes and it can be considered as an approximation of an infinite dimensional Wiener process [12, 38]. From this point of view, it is possible to view (1.1)-(1.2) as an approximation of SNSE driven by the infinite dimensional Wiener process. Consequently, the numerical methods proposed in this paper for (1.1)-(1.2) can, in principle, be used for approximating SNSE with infinite dimensional Wiener process but this aspect is not studied here.

SNSE can be useful for explaining the turbulence phenomenon (see [8, 15, 26] and references therein). They have complicated dynamics and some interesting properties (e.g., ergodicity of solutions [19, 10, 11, 24]). At the same time, rather little has been done in numerics for SNSE. In [20] algorithms based on Wiener Chaos expansion for SNSE are considered, these algorithms can work on relatively short time intervals only. In [6, 4] implicit and semi-discrete Euler time and finite element based space-time discretizations are studied, convergence is proved in the mean-square (strong) sense. The work [14] deals with a time-splitting scheme combined with a Galerkin approximation in the space variable for SNSE exploiting the semi-group and cubature techniques, a weak convergence is proved for the proposed method. In [3] the authors consider a method based on splitting SNSE in a deterministic NSE and stochastic Stokes equation, they prove convergence in the mean-square sense and in probability of the method. They used splitting ideas similar to the ones considered in [17, 18] for linear parabolic stochastic partial differential equations (SPDE).

Here we suggest to exploit probabilistic representations of solutions to SNSE (i.e., the method of characteristics) for constructing numerical methods of the layer type. In this paper we concentrate on a layer method which derivation is based on the simplest characteristics. The proposed scheme is remarkable in its simplicity for implementation and it is promising to be effective and reliable for studying SNSE. We provide theoretical analysis of this method and, in particular, prove its first-order mean-square convergence, which is the optimal order for schemes for equations with additive noise that use Wiener increments at each time step as the only information about the Wiener process.

Layer methods for deterministic semilinear and quasilinear partial differential equations of parabolic type were proposed in [28, 30] (see also [29, 23, 13]), making use of the well-known probabilistic representations of solutions to linear parabolic equations and the ideas of weak sense numerical integration of stochastic differential equations. The probabilistic approach takes into account coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. For deterministic NSEs, layer methods were first considered in [1] and further developed in [34, 33]. Layer methods for linear and semilinear SPDE were constructed and analyzed in [32], which does not cover the case of SNSE. Here we combine and extend the layer method approach from its use for deterministic NSE [1, 34] and parabolic SPDE [32] to SNSE. Layer methods can be viewed as an alternative to finite difference schemes. This paper is an important step in development of layer methods and their application to various problems. At the same time, their more comprehensive numerical testing and continuation of their theoretical analysis require further efforts.

While the layer method approach is applied here to SNSE with periodic boundary conditions, it can also be exploited in the case of other types of boundary conditions imposed on SNSE, in particular using ideas of [33], where layer methods for deterministic NSEs with no-slip boundary conditions were proposed.

The rest of the paper is organized as follows. In Section 2 we introduce additional notation and write down probabilistic representations for linearized SNSE (i.e., stochastic Oseen-Stokes equations) and for the SNSE (1.1)-(1.4) which we use in Section 3 for constructing layer methods for the SNSE. Numerical error analysis is done in Section 4, it includes local error analysis and global convergence in the a.s. and mean-square senses. Results of numerical experiments on two test models are presented in Section 5, where we also prove a remarkable property of the layer method based on the simplest characteristics that for the test models the method contains only those modes which are present in the exact solution.

2 Preliminaries

In this section we recall the required function spaces [7, 40, 41, 26, 27] and write probabilistic representations of solutions to linearized SNSE and to SNSE resting on results from [21, 22, 36, 38].

2.1 Function spaces, the Helmholtz-Hodge-Leray decomposition, and notation

Let $\{e_i\}$ be the canonical basis in $\mathbf{R}^n$. We shall consider spatial periodic n-vector functions $u(x) = (u^1(x), \ldots, u^n(x))^\top$ in $\mathbf{R}^n : u(x + Le_i) = u(x),\ i = 1, \ldots, n,$ where $L > 0$ is the period in ith direction. Denote by $Q = (0, L)^n$ the cube of the period (of course, one may consider different periods $L_1, \ldots, L_n$ in the different directions). We denote by $\mathbf{L}^2(Q)$ the Hilbert space of functions on Q with the scalar product and the norm

$$(u, v) = \int_Q \sum_{i=1}^n u^i(x)v^i(x)dx,\ \|u\| = (u, u)^{1/2}.$$

We keep the notation $|\cdot|$ for the absolute value of numbers and for the length of n-dimensional vectors, for example,

$$|u(x)| = [(u^1(x))^2 + \cdots + (u^n(x))^2]^{1/2}.$$

We denote by $\mathbf{H}_p^m(Q),\ m = 0, 1, \ldots,$ the Sobolev space of functions which are in $\mathbf{L}^2(Q)$, together with all their derivatives of order less than or equal to m, and which are periodic functions with the period Q. The space $\mathbf{H}_p^m(Q)$ is a Hilbert space with the scalar product and the norm

$$(u, v)_m = \int_Q \sum_{i=1}^n \sum_{[\alpha^i]\leq m} D^{\alpha^i} u^i(x) D^{\alpha^i} v^i(x)dx,\ \|u\|_m = [(u, u)_m]^{1/2},$$

where $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_n),\ \alpha^i_j \in \{0, \ldots, m\},\ [\alpha^i] = \alpha^i_1 + \cdots + \alpha^i_n$, and

$$D^{\alpha^i} = D_1^{\alpha^i_1} \cdots D_n^{\alpha^i_n} = \frac{\partial^{[\alpha^i]}}{(\partial x^1)^{\alpha^i_1} \cdots (\partial x^n)^{\alpha^i_n}}\ ,\ i = 1, \ldots, n.$$

Note that $\mathbf{H}^0_p(Q) = \mathbf{L}^2(Q)$.

Introduce the Hilbert subspaces of $\mathbf{H}^m_p(Q)$:

$$\mathbf{V}^m_p = \{v :\ v \in \mathbf{H}^m_p(Q),\ \operatorname{div} v = 0\},\ m > 0,$$
$$\mathbf{V}^0_p = \text{the closure of } \mathbf{V}^m_p,\ m > 0 \text{ in } \mathbf{L}^2(Q).$$

Clearly,

$$\mathbf{V}^{m_1}_p = \text{the closure of } \mathbf{V}^{m_2}_p \text{ in } \mathbf{H}^{m_1}_p(Q) \text{ for any } m_2 \geq m_1.$$

Denote by P the orthogonal projection in $\mathbf{H}^m_p(Q)$ onto $\mathbf{V}^m_p$ (we omit m in the notation P here). The operator P is often called the Leray projection. Due to the Helmholtz-Hodge-Leray decomposition, any function $u \in \mathbf{H}^m_p(Q)$ can be represented as

$$u = Pu + \nabla g,\ \operatorname{div} Pu = 0,$$

where $g = g(x)$ is a scalar Q-periodic function such that $\nabla g \in \mathbf{H}^m_p(Q)$. It is natural to introduce the notation $P^\perp u := \nabla g$ and hence write

$$u = Pu + P^\perp u$$

with

$$P^\perp u \in (\mathbf{V}^m_p)^\perp = \{v :\ v \in \mathbf{H}^m_p(Q),\ v = \nabla g\}.$$

Let

$$u(x) = \sum_{\mathbf{n} \in \mathbf{Z}^n} u_{\mathbf{n}} e^{i(2\pi/L)(\mathbf{n},x)},\ g(x) = \sum_{\mathbf{n} \in \mathbf{Z}^n} g_{\mathbf{n}} e^{i(2\pi/L)(\mathbf{n},x)},\ g_{\mathbf{0}} = 0, \tag{2.1}$$
$$Pu(x) = \sum_{\mathbf{n} \in \mathbf{Z}^n} (Pu)_{\mathbf{n}} e^{i(2\pi/L)(\mathbf{n},x)},\ P^\perp u(x) = \nabla g(x) = \sum_{\mathbf{n} \in \mathbf{Z}^n} (P^\perp u)_{\mathbf{n}} e^{i(2\pi/L)(\mathbf{n},x)}$$

be the Fourier expansions of u, g, Pu, and $P^\perp u = \nabla g$. Here $u_{\mathbf{n}}$, $(Pu)_{\mathbf{n}}$, and $(P^\perp u)_{\mathbf{n}} = (\nabla g)_{\mathbf{n}}$ are n-dimensional vectors and $g_{\mathbf{n}}$ are scalars. We note that $g_{\mathbf{0}}$ can be any real number but for definiteness we set $g_{\mathbf{0}} = 0$. The coefficients $(Pu)_{\mathbf{n}}$, $(P^\perp u)_{\mathbf{n}}$, and $g_{\mathbf{n}}$ can be easily expressed in terms of $u_{\mathbf{n}}$:

$$(Pu)_{\mathbf{n}} = u_{\mathbf{n}} - \frac{u_{\mathbf{n}}^\top \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n},\ (P^\perp u)_{\mathbf{n}} = i\frac{2\pi}{L} g_{\mathbf{n}} \mathbf{n} = \frac{u_{\mathbf{n}}^\top \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}, \tag{2.2}$$
$$g_{\mathbf{n}} = -i\frac{L}{2\pi}\frac{u_{\mathbf{n}}^\top \mathbf{n}}{|\mathbf{n}|^2},\ \mathbf{n} \neq \mathbf{0},\ g_{\mathbf{0}} = 0.$$

We have

$$\nabla e^{i(2\pi/L)(\mathbf{n},x)} = \mathbf{n} e^{i(2\pi/L)(\mathbf{n},x)} \cdot i\frac{2\pi}{L},$$

hence $u_{\mathbf{n}} e^{i(2\pi/L)(\mathbf{n},x)} \in \mathbf{V}^m_p$ if and only if $(u_{\mathbf{n}}, \mathbf{n}) = 0$. We obtain from here that the orthogonal basis of the subspace $(\mathbf{V}^m_p)^\perp$ consists of $\mathbf{n} e^{i(2\pi/L)(\mathbf{n},x)},\ \mathbf{n} \in \mathbf{Z}^n,\ \mathbf{n} \neq \mathbf{0}$; and an orthogonal basis of $\mathbf{V}_p^m$ consists of ${}_ku_{\mathbf{n}}e^{i(2\pi/L)(\mathbf{n},x)}$, $k=1,\ldots,n-1$, $\mathbf{n}\in\mathbf{Z}^n$, where under $\mathbf{n}\neq\mathbf{0}$ the vectors ${}_ku_{\mathbf{n}}$ are orthogonal to $\mathbf{n}$: $({}_ku_{\mathbf{n}},\mathbf{n})=0$, $k=1,\ldots,n-1$, and they are orthogonal among themselves: $({}_ku_{\mathbf{n}},\ {}_mu_{\mathbf{n}})=0$, $k,m=1,\ldots,n-1$, $m\neq k$, and finally, for $\mathbf{n}=\mathbf{0}$, the vectors ${}_ku_{\mathbf{0}}$, $k=1,\ldots,n$, are orthogonal.

In what follows we suppose that the below assumptions hold.

Assumptions 2.1. *We assume that the coefficients $f(t,x)$ and $\gamma_r(s,x)$, $r=1,\ldots,q$, are sufficiently smooth and the problem* (1.1)-(1.4) *has a unique classical solution $v(t,x)$, $p(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}^n$, which has continuous derivatives in the space variable x up to some order l, and the solution and the derivatives have uniformly in (t,x) bounded moments of a sufficiently high order m, where $m\geq 2$ is a positive number or $m=\infty$ if the moments of any order are finite.*

The solution $v(t,x)$, $p(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}^n$, to (1.1)-(1.4) is $\mathcal{F}_t^w$-adaptive, $v(t,\cdot)\in\mathbf{V}_p^m$ and $\nabla p(t,\cdot)\in(\mathbf{V}_p^m)^{\perp}$ for every $t\in[0,T]$ and $\omega\in\Omega$.

Assumptions of this kind are rather usual for works dedicated to numerics. They are rested on results concerning regularity of solutions (see, e.g., the corresponding theory for deterministic NSE in [40, 41]). Unfortunately, we could not find explicit results on the classical solution for SNSE in literature. At the same time, the question about existence of the unique sufficiently regular (with respect to x) solution of the SNSE (1.1)-(1.4) on a time interval $[0,T]$ is analogous to the one in the deterministic case (see also Remark 2.2 below). Indeed, the following remark reduces this problem of regularity for the SNSE to regularity of solutions to NSE with random coefficients which is close to the theory of deterministic NSE treated in [2, 40, 41].

Remark 2.1 *Let $\Gamma(t,x)=\sum_{r=1}^q\int_0^t\gamma_r(s,x)dw_r(s)$. Then $V(t,x)=v(t,x)+\Gamma(t,x)$ together with $p(t,x)$ solves the following 'usual' NSE with random coefficients:*

$$\frac{\partial}{\partial t}V=\frac{\sigma^2}{2}\Delta V-(V-\Gamma(t,x),\nabla)(V-\Gamma(t,x))-\nabla p+f(t,x)-\frac{\sigma^2}{2}\Delta\Gamma(t,x),$$
$$0\leq t\leq T,\ x\in\mathbf{R}^n,$$
$$\text{div }\ V=0,$$

with spatial periodic conditions

$$V(t,x+Le_i)=V(t,x),\ p(t,x+Le_i)=p(t,x),$$
$$0\leq t\leq T,\quad i=1,\ldots,n,$$

and the initial condition

$$V(0,x)=\varphi(x).$$

Remark 2.2 *In the case $n=2$, there are established existence and uniqueness results for deterministic NSE [2, 40, 41], which can be used for justifying Assumption 2.1. For $n>2$ such results are not available. But we consider approximations of (1.1)-(1.4) with arbitrary n here since if Assumption 2.1 holds then construction of the presented layer method and the convergence proofs are essentially the same for all n.*

2.2 Probabilistic representations of solutions to linearized SNSE

We start with considering a linearized version of the SNSE (1.1)-(1.4), i.e., the stochastic Oseen-Stokes equations (see [25]):

$$dv_a(t) = \left[\frac{\sigma^2}{2}\Delta v_a - (a,\nabla)v_a - \nabla p_a + f(t,x)\right]dt + \sum_{r=1}^{q}\gamma_r(t,x)dw_r(t), \tag{2.3}$$

$$0 \le t \le T,\ x \in \mathbf{R}^n,$$

$$\text{div } v_a = 0, \tag{2.4}$$

with spatial periodic conditions

$$v_a(t, x + Le_i) = v_a(t,x),\ p_a(t, x+Le_i) = p_a(t,x), \tag{2.5}$$

$$0 \le t \le T,\ i = 1,\dots,n,$$

and the initial condition

$$v_a(0,x) = \varphi(x), \tag{2.6}$$

where $a = a(t,x)$ is an n-dimensional vector $a = (a^1,\dots,a^n)^\intercal$ with a^i being Q-periodic deterministic functions which have continuous derivatives with respect to x up to some order; and the rest of the notation is the same as in (1.1)-(1.4).

We re-write the problem (2.3)-(2.6) with positive direction of time into the problem with negative direction of time which is more convenient for making use of probabilistic representations. To this end, introduce the new time variable $s = T - t$ and the functions $u_a(s,x) := v_a(T-s,x)$, $\tilde{a}(s,x) := a(T-s,x)$, $\tilde{f}(s,x) := f(T-s,x)$, $\tilde{\gamma}_r(s,x) := \gamma_r(T-s,x)$, and $\tilde{p}_a(s,x) := p_a(T-s,x)$.

Further, we recall the definition of a backward Ito integral [38]. Introduce the "backward" Wiener processes

$$\tilde{w}_r(t) := w_r(T) - w_r(T-t),\quad r = 1,\dots,q,\quad 0 \le t \le T, \tag{2.7}$$

and a decreasing family of σ-subalgebras $\mathcal{F}^w_{t,T}$, $0 \le t \le T$, induced by the increments $w_r(T) - w_r(t')$, $r = 1,\dots,q$, $t' \ge t$. The increasing family of σ-subalgebras $\mathcal{F}^{\tilde{w}}_t$ induced by $\tilde{w}_r(s')$, $s' \le t$, coincides with $\mathcal{F}^w_{T-t,T}$, while $\mathcal{F}^{\tilde{w}}_{t,T}$ is induced by the increments $\tilde{w}_r(T) - \tilde{w}_r(t')$, $r = 1,\dots,q$, $t' \ge t$, and coincides with $\mathcal{F}^w_{T-t}$. The backward Ito integral with respect to $\tilde{w}_r(s)$ is defined as the Ito integral with respect to $w_r(s)$:

$$\int_t^{t'} \psi(t'') * d\tilde{w}_r(t'') := \int_{T-t'}^{T-t} \psi(T-t'')dw_r(t''),\quad 0 \le t \le t' \le T, \tag{2.8}$$

where $\psi(T-t)$, $t \le T$, is an $\mathcal{F}^w_t$-adapted square-integrable function and $\psi(t)$ is $\mathcal{F}^{\tilde{w}}_t$-adapted. Note that $w_r(t) = \tilde{w}_r(T) - \tilde{w}_r(T-t)$, $r = 1,\dots,q$, $0 \le t \le T$.

The backward stochastic Oseen-Stokes equations can be written as

$$-du_a(s) = \left[\frac{\sigma^2}{2}\Delta u_a - (\tilde{a},\nabla)u_a - \nabla\tilde{p}_a + \tilde{f}(s,x)\right]ds + \sum_{r=1}^{q}\tilde{\gamma}_r(s,x) * d\tilde{w}_r(s), \tag{2.9}$$

$$0 \le s \le T,\ x \in \mathbf{R}^n,$$

$$\text{div } u_a = 0, \tag{2.10}$$

with spatial periodic conditions

$$u_a(s, x + Le_i) = u_a(s, x),\ \tilde{p}_a(s, x + Le_i) = \tilde{p}_a(s, x), \quad (2.11)$$
$$0 \le s \le T,\ i = 1, \dots, n,$$

and the terminal condition

$$u_a(T, x) = \varphi(x). \quad (2.12)$$

We note that (2.8) implies

$$\int_s^T \tilde{\gamma}_r(s', x) * d\tilde{w}_r(s') = \int_0^{T-s} \gamma_r(s', x) dw_r(s').$$

The processes $u_a(s, x),\ \tilde{p}_a(s, x)$ are $\mathcal{F}^{\tilde{w}}_{s,T}$-adapted (and $\mathcal{F}^{w}_{T-s}$-adapted), they depend on $\tilde{w}_r(T) - \tilde{w}_r(s') = w_r(T - s'),\ s \le s' \le T$.

Let $u_a(s, x), \tilde{p}_a(s, x)$ be a solution of the problem (2.9)-(2.12). For the function $u_a(s, x)$, one can use the following probabilistic representation of solutions to the Cauchy problem for linear SPDE of parabolic type (usually called the conditional Feynman-Kac formula or the averaging over characteristics formula, see, e.g., [38] and [32]):

$$u_a(s, x) = E^{\tilde{w}} \left[\varphi(X_{s,x}(T)) Y_{s,x,1}(T) + Z_{s,x,1,0}(T)\right],\ 0 \le s \le T, \quad (2.13)$$

where $X_{s,x}(s'),\ Y_{s,x,y}(s'),\ Z_{s,x,y,z}(s'),\ s' \ge s$, solves the system of Ito stochastic differential equations:

$$dX = (-\tilde{a}(s', X) - \sigma\mu(s', X))ds' + \sigma dW(s'),\ X(s) = x, \quad (2.14)$$
$$dY = \mu^{\intercal}(s', X) Y dW(s'),\ Y(s) = y, \quad (2.15)$$
$$dZ = (-\nabla\tilde{p}_a(s', X) + \tilde{f}(s', X))Y ds' + F(s', X) Y dW(s') \quad (2.16)$$
$$+ \sum_{r=1}^{q} \tilde{\gamma}_r(s', X) Y d\tilde{w}_r(s'),\ Z(s) = z.$$

In (2.13)-(2.16), $W(s)$ is a standard n-dimensional Wiener process independent of $\tilde{w}_r(s)$ on the probability space $(\Omega, \mathcal{F}, P)$; Y is a scalar, and Z is an n-dimensional column-vector; $\mu(s, x)$ is an arbitrary n-dimensional spatial periodic vector function and $F(s, x)$ is an arbitrary $n \times n$-dimensional spatial periodic matrix function, which are sufficiently smooth in s, x; the expectation $E^{\tilde{w}}$ in (2.13) is taken over the realizations of $W(s)$, $t \le s \le T$, for a fixed $\tilde{w}_r(s')$, $r = 1, \dots, q$, $s \le s' \le T$, in other words, $E^{\tilde{w}}(\cdot)$ means the conditional expectation:

$$E\left(\cdot | \tilde{w}_r(s') - \tilde{w}_r(s),\ r = 1, \dots, q,\ s \le s' \le T\right).$$

The probabilistic representation like (2.13)-(2.16) for the Cauchy problem (2.9), (2.12) is obtained (see, e.g., [38]) for linear SPDEs with deterministic coefficients. However here $\tilde{p}_a(s, x)$ is a part of the solution of the problem (2.9)-(2.12) and it is random (more precisely it is $\mathcal{F}^{\tilde{w}}_{s,T}$-adapted). In this case the representation (2.13)-(2.16) can be rigorously justified in the following way. The solution u_a of (2.9), (2.12) can be represented in the form of the sum

$$u_a = u_a^{(0)} + u_a^{(1)},$$

where $u_a^{(0)}$ satisfies the Cauchy problem for the backward deterministic linear parabolic PDE with random parameters:

$$-\frac{\partial u_a^{(0)}}{\partial s} = \frac{\sigma^2}{2}\Delta u_a^{(0)} - (\tilde{a},\nabla)u_a^{(0)} - \nabla\tilde{p}_a, \tag{2.17}$$
$$u_a^{(0)}(T,x) = 0,$$

and $u_a^{(1)}$ satisfies the Cauchy problem for the backward stochastic linear parabolic PDE with deterministic parameters:

$$-du_a^{(1)}(s) = \left[\frac{\sigma^2}{2}\Delta u_a^{(1)} - (\tilde{a},\nabla)u_a^{(1)} + \tilde{f}(s,x)\right]ds + \sum_{r=1}^{q}\tilde{\gamma}_r(s,x) * d\tilde{w}_r(s), \tag{2.18}$$
$$u_a^{(1)}(T,x) = \varphi(x).$$

Clearly,

$$u_a^{(0)}(s,x) = E^{\tilde{w}}\left[Z_{s,x,1,0}^{(0)}(T)\right] = -E^{\tilde{w}}\int_s^T \nabla\tilde{p}_a(s',X_{s,x}(s'))\,Y_{s,x,1}(s')ds'.$$

The Feynman-Kac formula for $u_a^{(1)}$ coincides with (2.13)-(2.16) under $\nabla\tilde{p}_a(s,x) = 0$.

Let $\mathcal{F}_{s,t}^W$ be a σ-algebra induced by $W_r(s') - W_r(s)$, $r = 1,\dots,n,\ s \le s' \le t$. We note that $\nabla\tilde{p}_a(s',X_{s,x}(s'))$ in (2.16) is $\mathcal{F}_{s,s'}^W \vee \mathcal{F}_{s',T}^{\tilde{w}}$-adapted, where the family of σ-algebras $\mathcal{F}_{s,s'}^W \vee \mathcal{F}_{s',T}^{\tilde{w}}$ is neither increasing nor decreasing in s'. Consequently, $Z_{s,x,y,z}(s')$ is measurable with respect to $\mathcal{F}_{s,s'}^W \vee \mathcal{F}_{s',T}^{\tilde{w}}$ for every $s' \in [s,T]$. Since $\tilde{\gamma}_r(s',X_{s,x}(s'))Y(s')$ are independent of $\tilde{w}_r$, the Ito integral in (2.16) is well defined.

On the basis of the probabilistic representation (2.13)-(2.16) we, first, construct layer methods for the stochastic Oseen-Stokes equations and, second, using the obtained methods as a guidance, we construct the corresponding methods for the SNSE (this way of deriving numerical methods for nonlinear SPDEs was proposed in [32]). We remark that within the non-anticipating stochastic calculus the probabilistic representation (2.13)-(2.16) for the linear problem (2.9)-(2.12) cannot be carried over to the backward SNSE problem by changing the coefficient $\tilde{a}(s,x)$ to $u(s,x)$ since then the integrand $\tilde{\gamma}_r(s',X_{s,x}(s'))Y(s')$ would be $\mathcal{F}_{s,s'}^W \vee \mathcal{F}_{s',T}^{\tilde{w}}$-measurable. That is why we prefer to use the mimicry approach.

In our preliminary study [35], we also wrote down two direct probabilistic representations for solutions of the SNSE and used them for constructing a layer method. The first of these direct representations follows from a specific probabilistic representation for a linear SPDE which differs from (2.13)-(2.16), and the second one uses backward doubly stochastic differential equations [37].

Each choice of $\mu(s,x)$ and $F(s,x)$ in (2.13)-(2.16) gives us a particular probabilistic representation for the solution of the stochastic Oseen-Stokes equations (2.9)-(2.12) which can be used for deriving the corresponding layer method. In this paper we concentrate on the layer method based on the probabilistic representations (2.13)-(2.16) with $F(s,x) = 0$ and $\mu(s,x)$ turning the equation (2.14) for $X(s)$ into pure diffusion – the simplest characteristics. From the algorithmic point of view, this method is substantially better than the standard one, i.e., the one based on (2.13)-(2.16) with $F(s,x) = 0$ and $\mu(s,x) = 0$ (see the preliminary study [35]).

3 Layer algorithms based on the probabilistic representation with simplest characteristics

Let us introduce a uniform partition of the time interval $[0,T]: 0 = t_0 < t_1 < \cdots < t_N = T$ and the time step $h = T/N$ (we restrict ourselves to the uniform partition for simplicity only).

If we put $\mu(s,x) = -\tilde{a}(s,x)/\sigma$ and $F(s,x) = 0$ in (2.13)-(2.16), we can obtain the following local probabilistic representation for the solution to the backward stochastic Oseen-Stokes equation (2.9)-(2.12):

$$\begin{aligned} u_a(t_k,x) &= E^{\tilde{w}}[u_a(t_{k+1}, X_{t_k,x}(t_{k+1}))Y_{t_k,x,1}(t_{k+1})] \\ &+E^{\tilde{w}}\Bigg[-\int_{t_k}^{t_{k+1}} \nabla \tilde{p}_a(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)ds + \int_{t_k}^{t_{k+1}} \tilde{f}(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)ds \\ &+\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}} \tilde{\gamma}_r(s, X_{t_k,x}(s))Y_{t_k,x,1}(s)d\tilde{w}_r(s)\Bigg], \end{aligned} \tag{3.1}$$

where $X_{t,x}(s)$, $Y_{t,x,1}(s)$, $s \geq t$, solve the system of stochastic differential equations

$$dX = \sigma dW(s), \ X(t) = x, \tag{3.2}$$

$$dY = -\frac{1}{\sigma} Y \tilde{a}^\top(s,X)dW(s), \ Y(t) = 1. \tag{3.3}$$

We apply a slightly modified explicit Euler scheme with the simplest noise simulation to (3.2)-(3.3):

$$\bar{X}_{t_k,x}(t_{k+1}) = x + \sigma\sqrt{h}\xi, \ \bar{Y}_{t_k,x,1}(t_{k+1}) = 1 - \frac{1}{\sigma}\tilde{a}^\top(t_{k+1},x)\sqrt{h}\xi, \tag{3.4}$$

where $\xi = (\xi^1, \ldots, \xi^n)^\top$ and $\xi^1, \ldots, \xi^n$ are i.i.d. random variables with the law $P(\xi^i = \pm 1) = 1/2$. Approximating $X_{t_k,x}(t_{k+1})$ and $Y_{t_k,x,1}(t_{k+1})$ in (3.1) by $\bar{X}_{t_k,x}(t_{k+1})$ and $\bar{Y}_{t_k,x,1}(t_{k+1})$ from (3.4), we obtain

$$\begin{aligned} u_a(t_k,x) &= E^{\tilde{w}}[u_a(t_{k+1}, x+\sigma\sqrt{h}\xi)(1 - \frac{1}{\sigma}\tilde{a}^\top(t_{k+1},x)\sqrt{h}\xi)] - \nabla\tilde{p}_a(t_{k+1},x)h \\ &+\tilde{f}(t_{k+1},x)h + \sum_{r=1}^{q}\tilde{\gamma}_r(t_{k+1},x)\Delta_k\tilde{w}_r + \rho \\ &= 2^{-n}\sum_{j=1}^{2^n} u_a(t_{k+1}, x+\sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma}\breve{u}_a(t_{k+1},x) - \nabla\tilde{p}_a(t_{k+1},x)h \\ &+\tilde{f}(t_{k+1},x)h + \sum_{r=1}^{q}\tilde{\gamma}_r(t_{k+1},x)\Delta_k\tilde{w}_r + \rho, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \breve{u}_a(t_{k+1},x) &= E^{\tilde{w}}[u_a(t_{k+1}, x+\sigma\sqrt{h}\xi)\xi^\top]\tilde{a}(t_{k+1},x) \\ &= 2^{-n}\sum_{j=1}^{2^n} u_a(t_{k+1}, x+\sigma\sqrt{h}\xi_j)\xi_j^\top\tilde{a}(t_{k+1},x) \end{aligned} \tag{3.6}$$

$\rho = \rho(t_k, x)$ is a remainder and ξ_j, $j = 1, \ldots, 2^n$, are all possible realizations of the random vector ξ, i.e., $\xi_1 = (1, 1, \ldots, 1)^\top$, $\ldots$, $\xi_{2^n} = (-1, -1, \ldots, -1)^\top$.

Using the Helmholtz-Hodge-Leray decomposition and taking into account that

$$\operatorname{div} u_a(t_{k+1}, x + \sigma\sqrt{h}\xi_j) = 0, \quad \operatorname{div} \gamma_r = 0,$$

we get from (3.5)-(3.6):

$$u_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} u_a(t_{k+1}, x + \sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma} P\breve{u}_a(t_{k+1}, x) + P\tilde{f}(t_{k+1}, x)h$$
$$-\frac{\sqrt{h}}{\sigma} P^\perp \breve{u}_a(t_{k+1}, x) + P^\perp \tilde{f}(t_{k+1}, x)h - \nabla \tilde{p}_a(t_{k+1}, x)h$$
$$+\sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x)\Delta_k \tilde{w}_r + \rho,$$

whence we obtain after applying the operator P :

$$u_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} u_a(t_{k+1}, x + \sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma} P\breve{u}_a(t_{k+1}, x) + P\tilde{f}(t_{k+1}, x)h \tag{3.7}$$
$$+\sum_{r=1}^{q} \tilde{\gamma}_r(t_{k+1}, x)\Delta_k \tilde{w}_r + P\rho.$$

Dropping the remainder in (3.7) and re-writing the obtained approximation in the one with positive direction of time, we obtain the one-step approximation for the forward-time stochastic Oseen-Stokes equation (2.3)-(2.6):

$$\hat{v}_a(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} v_a(t_k, x + \sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma} P\breve{v}_a(t_k, x) + Pf(t_k, x)h \tag{3.8}$$
$$+\sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_k w_r,$$

where

$$\breve{v}_a(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} v_a(t_k, x + \sigma\sqrt{h}\xi_j)\xi_j^\top a(t_k, x). \tag{3.9}$$

Using (3.8)-(3.9) as a guidance, we arrive at the one-step approximation for the SNSE (1.1)-(1.4):

$$\hat{v}(t_{k+1}, x) = 2^{-n} \sum_{j=1}^{2^n} v(t_k, x + \sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma} P\breve{v}(t_k, x) \tag{3.10}$$
$$+Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_k w_r,$$

where

$$\breve{v}(t_k, x) = 2^{-n} \sum_{j=1}^{2^n} v(t_k, x + \sigma\sqrt{h}\xi_j)\xi_j^\top v(t_k, x). \tag{3.11}$$

It is easy to see that under Assumptions 2.1 $\operatorname{div}\hat{v}(t_{k+1},x)=0$. The corresponding layer method for the SNSE (1.1)-(1.4) has the form

$$\bar{v}(0,x)=\varphi(x),\ \bar{v}(t_{k+1},x)=2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)-\frac{\sqrt{h}}{\sigma}P\breve{v}(t_k,x) \tag{3.12}$$
$$+Pf(t_k,x)h+\sum_{r=1}^{q}\gamma_r(t_k,x)\Delta_k w_r,\quad k=0,\ldots,N-1,$$

where

$$\breve{v}(t_k,x)=2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top\bar{v}(t_k,x). \tag{3.13}$$

3.1 Practical implementation of the layer method

Practical implementation of the layer method (3.12)-(3.13) is straightforward and efficient. Let us write the corresponding numerical algorithm for simplicity in the two-dimensional ($n=2$) case. We choose a positive integer M as a cut-off frequency and write the approximate velocity at the time t_{k+1} as the partial sum:

$$\bar{v}(t_{k+1},x)=\sum_{n_1=-M}^{M-1}\sum_{n_2=-M}^{M-1}\bar{v}_{\mathbf{n}}(t_{k+1})e^{i(2\pi/L)(\mathbf{n},x)}, \tag{3.14}$$

where $\mathbf{n}=(n_1,n_2)^\top$.

We note that we use the same notation $\bar{v}(t_{k+1},x)$ for the partial sum in (3.14) instead of writing $\bar{v}_M(t_{k+1},x)$ while in (3.12) $\bar{v}(t_{k+1},x)$ denotes the approximate velocity containing all frequencies but this should not lead to any confusion.

Further, we have

$$\frac{1}{4}\sum_{j=1}^{4}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)=\sum_{n_1=-M}^{M-1}\sum_{n_2=-M}^{M-1}\bar{v}_{\mathbf{n}}(t_k)e^{i(2\pi/L)(\mathbf{n},x)}\frac{1}{4}\sum_{j=1}^{4}e^{i(2\pi\sigma\sqrt{h}/L)(\mathbf{n},\xi_j)}. \tag{3.15}$$

Then

$$\begin{aligned}\breve{v}(t_k,x)&=\frac{1}{4}\sum_{j=1}^{4}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top\bar{v}(t_k,x)\\&=\sum_{n_1=-N}^{M-1}\sum_{n_2=-N}^{M-1}\bar{v}_{\mathbf{n}}(t_k)e^{i(2\pi/L)(\mathbf{n},x)}\frac{1}{4}\sum_{j=1}^{4}e^{i(2\pi\sigma\sqrt{h}/L)(\mathbf{n},\xi_j)}\xi_j^\top\bar{v}(t_k,x)\\&=\sum_{n_1=-M}^{M-1}\sum_{n_2=-M}^{M-1}V_{\mathbf{n}}(t_k)e^{i(2\pi/L)(\mathbf{n},x)}\bar{v}(t_k,x),\end{aligned}$$

where

$$V_{\mathbf{n}}(t_k)=\bar{v}_{\mathbf{n}}(t_k)\cdot\frac{1}{4}\sum_{j=1}^{4}e^{i(2\pi\sigma\sqrt{h}/L)(\mathbf{n},\xi_j)}\xi_j^\top.$$

Note that $V_{\mathbf{n}}(t_k)$ is a 2×2-matrix. Let

$$V(t_k, x) := \sum_{n_1=-M}^{M-1} \sum_{n_2=-M}^{M-1} V_{\mathbf{n}}(t_k) e^{i(2\pi/L)(\mathbf{n},x)} \tag{3.16}$$

then

$$\breve{v}(t_k, x) = V(t_k, x)\bar{v}(t_k, x).$$

We obtain the algorithm:

$$\bar{v}_{\mathbf{n}}(0) = \varphi_{\mathbf{n}}, \tag{3.17}$$
$$\bar{v}_{\mathbf{n}}(t_{k+1}) = \bar{v}_{\mathbf{n}}(t_k)\frac{1}{4}\sum_{j=1}^{4} e^{i(2\pi\sigma\sqrt{h}/L)(\mathbf{n},\xi_j)} - \frac{\sqrt{h}}{\sigma}\left(\breve{v}_{\mathbf{n}}(t_k) - \frac{\breve{v}_{\mathbf{n}}^{\top}(t_k)\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n}\right) + f_{\mathbf{n}}(t_k)h$$
$$- h\frac{f_{\mathbf{n}}^{\top}(t_k)\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} + \sum_{r=1}^{q} \gamma_{r,\mathbf{n}}(t_k)\ \Delta_k w_r,$$

where

$$\breve{v}_{\mathbf{n}}(t_k) = (\breve{v}(t_k, x))_{\mathbf{n}} = (V(t_k, x)\bar{v}(t_k, x))_{\mathbf{n}}. \tag{3.18}$$

To find $\breve{v}_{\mathbf{n}}(t_k)$ one can either multiply two partial sums of the form (3.14) and (3.16) or exploit fast Fourier transform in the usual fashion (see, e.g. [5]) to speed up the algorithm. The algorithm (3.17) can be viewed as analogous to spectral methods. It is interesting that the layer method (3.12)-(3.13) is, on the one hand, related to a finite difference scheme (see below) and on the other hand, to spectral methods.

3.2 Relationship between the layer method and finite difference methods

Let us discuss a relationship between the layer method (3.12)-(3.13) and finite difference methods. For simplicity in writing, we give this illustration in the two-dimensional case. It is not difficult to notice that the two-dimensional analog of the layer approximation (3.12) can be re-written as the following finite difference scheme for the SNSE (1.1)-(1.4):

$$\frac{\bar{v}(t_{k+1}, x) - \bar{v}(t_k, x)}{h} \tag{3.19}$$
$$= \frac{\bar{v}(t_k, x^1 + \sigma\sqrt{h}, x^2 + \sigma\sqrt{h}) + \bar{v}(t_k, x^1 - \sigma\sqrt{h}, x^2 + \sigma\sqrt{h}) - 4\bar{v}(t_k, x^1, x^2)}{4h}$$
$$+ \frac{\bar{v}(t_k, x^1 + \sigma\sqrt{h}, x^2 - \sigma\sqrt{h}) + \bar{v}(t_k, x^1 - \sigma\sqrt{h}, x^2 - \sigma\sqrt{h})}{4h}$$
$$- \frac{1}{\sigma\sqrt{h}}P\breve{v}(t_k, x) + Pf(t_k, x) + \sum_{r=1}^{q} \gamma_r(t_k, x)\frac{\Delta w_r(t_{k+1})}{h}$$

with

$$\frac{\breve{v}(t_k,x)}{\sigma\sqrt{h}} = \bar{v}^1(t_k,x)\frac{\bar{v}(t_k,x^1+\sigma\sqrt{h},x^2+\sigma\sqrt{h}) - \bar{v}(t_k,x^1-\sigma\sqrt{h},x^2+\sigma\sqrt{h})}{4\sigma\sqrt{h}} \tag{3.20}$$
$$+ \bar{v}^1(t_k,x)\frac{\bar{v}(t_k,x^1+\sigma\sqrt{h},x^2-\sigma\sqrt{h}) - \bar{v}(t_k,x^1-\sigma\sqrt{h},x^2-\sigma\sqrt{h})}{4\sigma\sqrt{h}}$$
$$+ \bar{v}^2(t_k,x)\frac{\bar{v}(t_k,x^1+\sigma\sqrt{h},x^2+\sigma\sqrt{h}) - \bar{v}(t_k,x^1+\sigma\sqrt{h},x^2-\sigma\sqrt{h})}{4\sigma\sqrt{h}}$$
$$+ \bar{v}^2(t_k,x)\frac{\bar{v}(t_k,x^1-\sigma\sqrt{h},x^2+\sigma\sqrt{h}) - \bar{v}(t_k,x^1-\sigma\sqrt{h},x^2-\sigma\sqrt{h})}{4\sigma\sqrt{h}} \quad .$$

As one can see, $\bar{v}(t_k,\cdot)$ in the right-hand side of (3.19) is evaluated at the nodes (x^1,x^2), $(x^1\pm\sigma\sqrt{h},x^2\pm\sigma\sqrt{h})$, which is typical for a standard explicit finite difference scheme with the space discretization step h_x taken equal to $\sigma\sqrt{h}$ and h being the time-discretization step. We also note that if in the approximation (3.4) we choose a different random vector ξ then we can obtain another layer method for the SNSE which can be again re-written as a finite difference scheme (see such a discussion in the case of the deterministic NSE in [34]).

We recall [28, 29, 32] that convergence theorems for layer methods (in comparison with the theory of finite difference methods) do not contain any conditions on stability of their approximations. In layer methods we do not need to a priori prescribe space nodes: they are obtained automatically depending on choice of a probabilistic representation and a numerical scheme. We note that our error analysis for the layer methods (see Section 4) immediately implies the same error estimates for the corresponding finite difference scheme (3.19).

3.3 Other layer methods based on simplest characteristics

In this paper we study the explicit method (3.12)-(3.13) but one can also derive other layer methods based on the simplest characteristics. For instance, we can obtain an implicit version of (3.12)-(3.13) replacing $\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)$ in $\breve{v}(t_k,x)$ by $\bar{v}(t_{k+1},x+\sigma\sqrt{h}\xi_j)$. This implicit method can be efficiently realized similarly to the algorithm (3.17)-(3.18) (resolving the implicitness at each step consists in solving a system of linear equations). Further, in our preliminary work [35] we derived a method based on a direct probabilistic representation which, in comparison with (3.12)-(3.13), has $hP(\bar{v}(t_k,x),\nabla)\bar{v}(t_k,x)$ instead of $\frac{\sqrt{h}}{\sigma}P\breve{v}(t_k,x)$. Below we write its implicit version:

$$\bar{v}(0,x) = \varphi(x), \tag{3.21}$$
$$\bar{v}(t_{k+1},x) = 2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j) - P\left[(\bar{v}(t_k,x),\nabla)\bar{v}(t_{k+1},x)\right]h$$
$$+Pf(t_k,x)h + \sum_{r=1}^{q}\gamma_r(t_k,x)\Delta_k w_r, \quad k=0,\ldots,N-1.$$

This method can be realized in an efficient fashion similar to the algorithm (3.17)-(3.18).

Let us also note that it is not difficult to see from (3.20) that

$$(\bar{v}(t_k,x),\nabla)\bar{v}(t_k,x)\approx\frac{\breve{v}(t_k,x)}{\sigma\sqrt{h}}\ . \tag{3.22}$$

If we put the exact $v(t_k,x)$ in (3.22) (both in its left and right-hand sides) instead of the approximate $\bar{v}(t_k,x)$ then the accuracy of the approximation in (3.22) is of order $O(h)$.

3.4 Approximation of pressure

We have constructed numerical methods for velocity $v(t,x)$, here we propose approximations for pressure $p(t,x)$.

Applying the projection operator $P^{\perp}$ to SNSE (1.1)-(1.4), we get (see also (1.5)):

$$\nabla p(t,x)=-P^{\perp}\left[(v(t,x),\nabla)v(t,x)\right]+P^{\perp}f(t,x). \tag{3.23}$$

Based on (3.23), we can complement the layer method (3.12) for the velocity by the approximation of pressure as follows:

$$\nabla\tilde{p}(t_{k+1},x)=-P^{\perp}\left[(\bar{v}(t_{k+1},x),\nabla)\bar{v}(t_{k+1},x)\right]+P^{\perp}f(t_{k+1},x). \tag{3.24}$$

But in order to avoid computing derivatives of $\bar{v}(t_{k+1},x)$, we approximate (see (3.22)) the term $(\bar{v}(t_{k+1},x),\nabla)\bar{v}(t_{k+1},x)$ in (3.24) by $\breve{v}(t_{k+1},x)/\sigma\sqrt{h}$ with $\breve{v}(t_{k+1},x)$ from (3.13) (with t_{k+1} instead of t_k). We obtain

$$\nabla\bar{p}(t_{k+1},x)=-\frac{1}{\sigma\sqrt{h}}P^{\perp}\breve{v}(t_{k+1},x)+P^{\perp}f(t_{k+1},x), \tag{3.25}$$

where $\breve{v}(t_{k+1},x)$ is from (3.13). Note that in the velocity approximation (3.12) we use $\breve{v}(t_k,x)$ while in the pressure approximation (3.25) we use $\breve{v}(t_{k+1},x)$.

As a result, we obtain *the layer method* (3.12)-(3.13), (3.25) for the solution of SNSE (1.1)-(1.4).

To provide an algorithm involving an approximation of pressure, let us return to the algorithm (3.17) for velocity. Based on (3.25) (see also (2.2)), we obtain

$$\bar{p}_{\mathbf{n}}(t_{k+1})=i\frac{L}{2\pi}\left(\frac{\breve{v}_{\mathbf{n}}^{\top}(t_{k+1})\mathbf{n}}{\sigma\sqrt{h}|\mathbf{n}|^2}-\frac{f_{\mathbf{n}}^{\top}(t_{k+1})\mathbf{n}}{|\mathbf{n}|^2}\right),\quad \mathbf{n}\neq\mathbf{0},\ \bar{p}_{\mathbf{0}}(t_{k+1})=0, \tag{3.26}$$

where $\breve{v}_{\mathbf{n}}^{\top}(t_{k+1})$ are as in (3.18) with t_{k+1} instead of t_k.

As a result, we obtain *the algorithm* (3.17)-(3.18), (3.26) for the solution of SNSE (1.1)-(1.4) which corresponds to the layer method (3.12)-(3.13), (3.25).

4 Error analysis

In this section we provide theoretical support for the layer method (3.12)-(3.13).

As before, $||u(\cdot)||=||u(x)||$ denotes the $\mathbf{L}^2$-norm of a function $u(x)$, $x\in Q$. In this section we use the same letter K for various deterministic constants and $C=C(\omega)$, $EC^2(\omega)<\infty$, for various positive random variables.

We start with analysis of the local error (Section 4.1). Then we consider the global error in the almost sure sense (Section 4.2) and in the mean-square sense (Section 4.3).

4.1 Local error

The following theorem gives estimates for the local error of the layer method (3.12)-(3.13).

Theorem 4.1 *Let Assumptions 2.1 hold with $l \geq 4$ and sufficiently high m. The one-step error*

$$\rho(t_{k+1}, x) = \hat{v}(t_{k+1}, x) - v(t_{k+1}, x) \tag{4.1}$$

of the one-step approximation (3.10)-(3.11) *for the SNSE* (1.1)-(1.4) *is estimated as*

$$||E(\rho(t_{k+1}, x)|\mathcal{F}_{t_k}^w)|| \leq C(\omega)h^2; \tag{4.2}$$

when $m \geq 3$ and for $m \geq 6$ and $1 \leq p \leq m/6$:

$$\left(E||\rho(t_{k+1}, \cdot)||^{2p}\right)^{1/2p} \leq Kh^{3/2}, \tag{4.3}$$

where a random constant $C(\omega) > 0$ with $EC^2 < \infty$ does not depend on h and k, a deterministic constant $K > 0$ does not depend on h and k but depends on p.

Proof. Taking into account Assumptions 2.1, the equality (3.11), and the relations

$$\sum_{j=1}^{2^n} \xi_j^i = 0,\ i = 1, \dots, n;\ 2^{-n} \sum_{j=1}^{2^n} \xi_j^{i_1} \xi_j^{i_2} = \begin{cases} 1, & i_1 = i_2 \\ 0, & i_1 \neq i_2 \end{cases},\ i_1, i_2 = 1, \dots, n; \tag{4.4}$$
$$\sum_{j=1}^{2^n} \xi_j^{i_1} \xi_j^{i_2} \xi_j^{i_3} = 0,\ i_1, i_2, i_3 = 1, \dots, n,$$

we expand the right-hand side of (3.10) at (t_k, x) and obtain

$$\hat{v}(t_{k+1}, x) = v(t_k, x) - hP\left[(v(t_k, x), \nabla)v(t_k, x)\right] + \frac{\sigma^2}{2} h \Delta v(t_k, x) \tag{4.5}$$
$$+ Pf(t_k, x)h + \sum_{r=1}^{q} \gamma_r(t_k, x)\Delta_k w_r + r_1(t_k, x),$$

where the remainder $r_1(t_k, x)$ satisfies the inequality

$$\left(E||r_1(t_k, \cdot)||^{2p}\right)^{1/2p} \leq Kh^2. \tag{4.6}$$

It follows from (4.6) that

$$||E\left(r_1(t_k, x)|\mathcal{F}_{t_k}^w\right)|| \leq C(\omega)h^2. \tag{4.7}$$

We write the solution $v(s, x),\ s \geq t_k$, of (1.1)-(1.4) as

$$v(s, x) = v(t_k, x) + \int_{t_k}^{s} \left[\frac{\sigma^2}{2} \Delta v(s', x) - (v(s', x), \nabla)v(s', x) + f(s', x)\right] ds' \tag{4.8}$$
$$- \int_{t_k}^{s} \nabla p(s', x) ds' + \sum_{r=1}^{q} \int_{t_k}^{s} \gamma_r(s', x) dw_r(s')$$

and, in particular,

$$v(t_{k+1},x)=v(t_k,x)+\int_{t_k}^{t_{k+1}}\left[\frac{\sigma^2}{2}\Delta v(s,x)-(v(s,x),\nabla)v(s,x)+f(s,x)\right]ds \qquad (4.9)$$
$$-\int_{t_k}^{t_{k+1}}\nabla p(s,x)ds+\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\gamma_r(s,x)dw_r(s).$$

Substituting $v(s,x)$ from (4.8) in the integrand of the first integral in (4.9) and expanding $\gamma_r(s,x)$ at (t_k,x), we obtain

$$v(t_{k+1},x)=v(t_k,x)+h\frac{\sigma^2}{2}\Delta v(t_k,x)-h(v(t_k,x),\nabla)v(t_k,x)+hf(t_k,x) \qquad (4.10)$$
$$-\int_{t_k}^{t_{k+1}}\nabla p(s,x)ds+\sum_{r=1}^{q}\gamma_r(t_k,x)\Delta_k w_r+r_2(t_k,x),$$

where

$$r_2(t_k,x)=r_2^{(1)}(t_k,x)+r_2^{(2)}(t_k,x)$$

and

$$r_2^{(1)}(t_k,x)=\frac{\sigma^2}{2}\int_{t_k}^{t_{k+1}}\left[\int_{t_k}^{s}\Delta\left(\frac{\sigma^2}{2}\Delta v(s',x)-(v(s',x),\nabla)v(s',x)\right.\right.$$
$$\left.\left.+f(s',x)\right)ds'\right]ds-\frac{\sigma^2}{2}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\Delta\nabla p(s',x)ds'ds$$
$$-\int_{t_k}^{t_{k+1}}(v(s,x),\nabla)\left[\int_{t_k}^{s}\left(\frac{\sigma^2}{2}\Delta v(s',x)-(v(s',x),\nabla)v(s',x)\right.\right.$$
$$\left.\left.+f(s',x)\right)ds'\right]ds$$
$$+\int_{t_k}^{t_{k+1}}(v(s,x),\nabla)\int_{t_k}^{s}\nabla p(s',x)ds'ds$$
$$-\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{s}\left(\frac{\sigma^2}{2}\Delta v(s',x)-(v(s',x),\nabla)v(s',x)\right.\right.$$
$$\left.\left.+f(s',x)\right)ds',\nabla\right)v(s,x)ds$$
$$+\int_{t_k}^{t_{k+1}}\left(\int_{t_k}^{s}\nabla p(s',x)ds',\nabla\right)v(s,x)ds$$
$$+\int_{t_k}^{t_{k+1}}(t_{k+1}-s)\frac{\partial}{\partial s}f(s,x)ds,$$

$$r_2^{(2)}(t_k,x)=\frac{\sigma^2}{2}\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\Delta\gamma_r(s',x)dw_r(s')ds$$
$$-\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\left[(v(s,x),\nabla)\int_{t_k}^{s}\gamma_r(s',x)dw_r(s')\right]ds$$

$$- \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{s} \gamma_r(s', x) dw_r(s'), \nabla \right) v(s, x) ds$$
$$+ \sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} (w_r(t_{k+1}) - w_r(s)) \frac{\partial}{\partial s} \gamma_r(s, x) ds.$$

We see that the remainder $r_2(t_k, x)$ consists of 1) $r_2^{(1)}(t_k, x)$ with terms of mean-square order h^2 and 2) $r_2^{(2)}(t_k, x)$ with terms containing $\mathcal{F}^w_{t_{k+1}}$-measurable Ito integrals of mean-square order $h^{3/2}$ which expectations with respect to $\mathcal{F}^w_{t_k}$ equal zero. Further, using Assumptions 2.1, one can show that

$$|E \left(r_2(t_k, x) | \mathcal{F}^w_{t_k} \right) | \leq C(\omega) h^2, \quad \left(E \left| r_2(t_k, x) \right|^{2p} \right)^{1/2p} \leq K h^{3/2}, \tag{4.11}$$

where $C(\omega) > 0$ and $K > 0$ do not depend on k, x, and h. Based on the second inequality in (4.11), we obtain

$$E \left|\left| r_2(t_k, \cdot) \right|\right|^{2p} = E \left(\int_Q \left| r_2(t_k, x) \right|^2 dx \right)^p \leq K E \int_Q \left| r_2(t_k, x) \right|^{2p} dx \leq K h^{2p \times 3/2}. \tag{4.12}$$

Applying the projector operator P to the left- and right-hand sides of (4.10), we arrive at

$$v(t_{k+1}, x) = v(t_k, x) + h \frac{\sigma^2}{2} \Delta v(t_k, x) - h P[(v(t_k, x), \nabla) v(t_k, x)] + h P f(t_k, x) \tag{4.13}$$
$$+ \sum_{r=1}^{q} \gamma_r(t_k, x) \Delta_k w_r + r_3(t_k, x),$$

where the new remainder $r_3(t_k, x) = P r_2(t_k, x)$. Using (4.12), we get

$$E||r_3(t_k, \cdot)||^{2p} = E||P r_2(t_k, \cdot)||^{2p} \leq E||r_2(t_k, \cdot)||^{2p} \leq K h^{2p \times 3/2}. \tag{4.14}$$

From (4.1), (4.5), and (4.13), we have $\rho = r_1 - r_3$. Hence, from (4.6) and (4.14) we obtain (4.3).

Observing that expectation of projection P of Ito integrals remains equal to zero, we get $E \left(P r_2^{(2)}(t_k, x) | \mathcal{F}^w_{t_k} \right) = 0$. Since $r_2^{(1)}(t_k, x)$ consists of terms of mean-square order h^2, we obtain

$$\begin{aligned} ||E \left(r_3(t_k, x) | \mathcal{F}^w_{t_k} \right) ||^2 &= ||E \left(P r_2^{(1)}(t_k, x) | \mathcal{F}^w_{t_k} \right) ||^2 \\ &= \int_Q \left[E \left(P r_2^{(1)}(t_k, x) | \mathcal{F}^w_{t_k} \right) \right]^2 dx \\ &\leq \int_Q E \left(\left| P r_2^{(1)}(t_k, x) \right|^2 | \mathcal{F}^w_{t_k} \right) dx \\ &\leq E \left(\int_Q \left| r_2^{(1)}(t_k, x) \right|^2 dx | \mathcal{F}^w_{t_k} \right) \leq C(\omega) h^4 \end{aligned}$$

whence

$$||E \left(r_3(t_k, x) | \mathcal{F}^w_{t_k} \right) || \leq C(\omega) h^2 \ . \tag{4.15}$$

Then the estimate (4.2) follows from (4.6), (4.15) and (4.5), (4.13). □

Remark 4.1 *We recall that in Assumptions 2.1 we require existence of moments of order m of the solution and its spatial derivatives. The higher the m, the higher p, $1 \leq p \leq m/6$, can be taken in* (4.3). *In particular, to guarantee* (4.3) *with $p = 1$, we need existence of the moments of order $m = 6$, while if the moments of any order m are finite then* (4.3) *is valid for any p. We also note that the smoothness conditions on the SNSE solution (see Assumptions 2.1) required for proving Theorem 4.1 are so that $v(t, x)$ should have continuous spatial derivatives up to order four and $p(t, x)$ – up to order three.*

Theorem 4.2 *Let Assumptions 2.1 hold with the bounded moments of any order $m \geq 6$. Then for almost every trajectory $w(\cdot)$ and any $0 < \varepsilon < 3/2$ there exists a constant $C(\omega) > 0$ such that the one-step error from* (4.1) *is estimated as*

$$||\rho(t_{k+1}, \cdot)|| \leq C(\omega) h^{3/2-\varepsilon}, \tag{4.16}$$

i.e., the layer method (3.12)-(3.13) *has the one-step error of order $3/2 - \varepsilon$ a.s. .*

Proof. Here we follow the recipe used in [16] (see also [31, 32]). The Markov inequality together with (4.3) implies

$$P(||\rho(t_{k+1}, \cdot)|| > h^\gamma) \leq \frac{E||\rho(t_{k+1}, \cdot)||^{2p}}{h^{2p\gamma}} \leq K h^{2p(3/2-\gamma)}.$$

Then for any $\gamma = 3/2 - \varepsilon$ there is a sufficiently large $p \geq 1$ such that (recall that $h = T/N$)

$$\sum_{N=1}^{\infty} P\left(||\rho(t_{k+1}, \cdot)|| > \frac{T^\gamma}{N^\gamma}\right) \leq K T^{2p(3/2-\gamma)} \sum_{N=1}^{\infty} \frac{1}{N^{2p(3/2-\gamma)}} < \infty.$$

Hence, due to the Borel-Cantelli lemma, the random variable

$$\varsigma := \sup_{h>0} h^{-\gamma} ||\rho(t_{k+1}, \cdot)||$$

is a.s. finite which implies (4.16). □

4.2 Global error in the almost sure sense

In this section we prove an almost sure (a.s.) convergence of the method (3.12)-(3.13) with order of $1/2 - \varepsilon$ for arbitrary $\varepsilon > 0$. In the next section, under slightly stronger assumptions, we prove the expected first-order convergence in the mean-square sense.

Since we assumed in Assumptions 2.1 that the problem (1.1)-(1.4) has a unique classical solution $v(t, x)$, $p(t, x)$ which has continuous derivatives in the space variable x up to some order and since we are considering the periodic case, then $v(t, x)$, $p(t, x)$ and their derivatives are a.s. finite on $[0, T] \times Q$.

We note that thanks to the assumed smoothness of the coefficients of the SNSE (1.1)-(1.4) and the initial condition (see Assumptions 2.1), properties of the projection operator P and the construction of the layer method (3.12)-(3.13), the functions $\bar{v}(t_k, x)$ are sufficiently smooth in x. To prove the below a.s. convergence Theorem 4.3, we make the following assumptions on uniform boundedness of the approximate solution $\bar{v}(t_k, x)$ and its derivatives.

Assumptions 4.1. *Assume that $\bar{v}(t_k,x)$, $k=0,\dots,N$, and its derivatives satisfy the following inequalities*

$$|\bar{v}(t_k,x)| \le C(\omega), \quad |\partial\bar{v}(t_k,x)/\partial x^i| \le C(\omega), \ \ i=1,\dots,n, \qquad (4.17)$$
$$|\partial^2\bar{v}(t_k,x)/\partial x^i\partial x^j| \le C(\omega), \ \ i,j=1,\dots,n,$$

where $C(\omega)>0$ is an a.s. finite constant independent of x, h, k and with bounded moments of a sufficiently high order.

The first inequality in (4.17) is necessary for a.s. convergence of the layer method (3.12)-(3.13). The second and third inequalities are also necessary if one expects convergence of spatial derivatives of $\bar{v}(t,x)$. We note that we have succeeded in proving convergence of the layer method (3.21) in the deterministic case (i.e., when $\gamma_r(t,x)=0$) imposing only conditions on the solution $v(t,x)$ of the deterministic NSE, without using assumptions like (4.17); and also, in the case of deterministic Oseen-Stokes equations we derived estimates like (4.17) for approximate solutions (these results are not presented here).

Theorem 4.3 *Let Assumptions 2.1 hold with the bounded moments of any order $m\ge 6$ and Assumptions 4.1 also hold. Then for almost every trajectory $w(\cdot)$ and any $0<\varepsilon<1/2$ there exists a constant $C(\omega)>0$ with $EC^2<\infty$ such that*

$$||\bar{v}(t_k,\cdot)-v(t_k,\cdot)|| \le C(\omega)h^{1/2-\varepsilon}, \qquad (4.18)$$

i.e., the layer method (3.12)-(3.13) *for the SNSE* (1.1)-(1.4) *converges with order* $1/2-\varepsilon$ *a.s..*

Proof. Denote the error of the method (3.12)-(3.13) on the kth layer by

$$\varepsilon(t_k,x)=\bar{v}(t_k,x)-v(t_k,x).$$

Due to (3.12) and (3.13) and due to the fact, that

$$\operatorname{div} v(t_k,x)=\operatorname{div}\bar{v}(t_k,x)=0, \qquad (4.19)$$

we get

$$\begin{aligned}
\varepsilon(t_{k+1},x)+v(t_{k+1},x) &= \bar{v}(t_{k+1},x) \\
&= 2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)-\frac{\sqrt{h}}{\sigma}P\breve{v}(t_k,x) \\
&+ Pf(t_k,x)h+\sum_{r=1}^{q}\gamma_r(t_k,x)\Delta_k w_r \\
&= 2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j) \\
&-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n}P[\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top\bar{v}(t_k,x)]
\end{aligned}$$

$$+ Pf(t_k,x)h + \sum_{r=1}^{q} \gamma_r(t_k,x)\Delta_k w_r.$$

Substituting $\bar{v} = v + \varepsilon$ in the right-hand side of the previous formula, we obtain

$$\begin{aligned} \varepsilon(t_{k+1},x) + v(t_{k+1},x) &= 2^{-n}\sum_{j=1}^{2^n} v(t_k, x+\sigma\sqrt{h}\xi_j) + 2^{-n}\sum_{j=1}^{2^n} \varepsilon(t_k, x+\sigma\sqrt{h}\xi_j) \\ &-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[v(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top v(t_k,x)] \\ &-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top \varepsilon(t_k,x)] \\ &-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[\varepsilon(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^T v(t_k,x)] \\ &+Pf(t_k,x)h + \sum_{r=1}^{q} \gamma_r(t_k,x)\Delta_k w_r. \end{aligned} \tag{4.20}$$

Due to (3.10) and (4.1), we have

$$\begin{aligned} &2^{-n}\sum_{j=1}^{2^n} v(t_k, x+\sigma\sqrt{h}\xi_j) - \frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[v(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top v(t_k,x)] \\ &+Pf(t_k,x)h + \sum_{r=1}^{q} \gamma_r(t_k,x)\Delta_k w_r = \hat{v}(t_{k+1},x) = v(t_{k+1},x) + \rho(t_{k+1},x), \end{aligned}$$

where ρ satisfies (4.16).

Therefore, we get from (4.20):

$$\begin{aligned} \varepsilon(t_{k+1},x) &= 2^{-n}\sum_{j=1}^{2^n} \varepsilon(t_k, x+\sigma\sqrt{h}\xi_j) \\ &-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top \varepsilon(t_k,x)] \\ &-\frac{\sqrt{h}}{\sigma}2^{-n}\sum_{j=1}^{2^n} P[\varepsilon(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^\top v(t_k,x)] + \rho(t_{k+1},x). \end{aligned} \tag{4.21}$$

For

$$P^{(1)} := 2^{-n}\sum_{j=1}^{2^n} \varepsilon(t_k, x+\sigma\sqrt{h}\xi_j), \tag{4.22}$$

we have

$$||P^{(1)}|| \le 2^{-n}\sum_{j=1}^{2^n} ||\varepsilon(t_k, \cdot+\sigma\sqrt{h}\xi_j)|| = ||\varepsilon(t_k,\cdot)||. \tag{4.23}$$

Due to the first equality from (4.4), we have

$$\begin{aligned} P^{(2)} \quad : \quad &= -\frac{\sqrt{h}}{\sigma} 2^{-n} \sum_{j=1}^{2^n} P[\bar{v}(t_k, x + \sigma\sqrt{h}\xi_j)\xi_j^\top \varepsilon(t_k, x)] \\ &= \quad -\frac{\sqrt{h}}{\sigma} 2^{-n} \sum_{j=1}^{2^n} P[(\bar{v}(t_k, x + \sigma\sqrt{h}\xi_j) - \bar{v}(t_k, x))\xi_j^\top \varepsilon(t_k, x)], \end{aligned} \tag{4.24}$$

whence, using boundedness of the spatial derivatives, we get

$$||P^{(2)}|| \le \frac{\sqrt{h}}{\sigma} 2^{-n} \sum_{j=1}^{2^n} ||[(\bar{v}(t_k, \cdot + \sigma\sqrt{h}\xi_j) - \bar{v}(t_k, \cdot))\xi_j^T \varepsilon(t_k, \cdot)]|| \le C(\omega)h||\varepsilon(t_k, \cdot)||. \tag{4.25}$$

Further

$$\begin{aligned} &-\frac{\sqrt{h}}{\sigma} 2^{-n} \sum_{j=1}^{2^n} P[\varepsilon(t_k, x + \sigma\sqrt{h}\xi_j)\xi_j^T v(t_k, x)] \\ = \quad &-\frac{\sqrt{h}}{\sigma} 2^{-n} \sum_{j=1}^{2^n} P[(\varepsilon(t_k, x + \sigma\sqrt{h}\xi_j) - \varepsilon(t_k, x))\xi_j^T v(t_k, x)] \\ = \quad &-h 2^{-n} \sum_{j=1}^{2^n} P[\sum_{m=1}^{n} \frac{\partial \varepsilon}{\partial x^m}(t_k, x)\xi_j^m \sum_{l=1}^{n} \xi_j^l v^l(t_k, x) + R := P^{(3)} + R, \end{aligned} \tag{4.26}$$

where the remainder R is estimated by

$$||R(t_k, \cdot)|| \le C(\omega)h^{3/2}, \ EC^2(\omega) < \infty, \tag{4.27}$$

because of boundedness of the second derivatives of $\varepsilon = \bar{v} - v$.

Due to the second equality in (4.4), we get

$$\begin{aligned} P^{(3)} \quad : \quad &= -h 2^{-n} \sum_{j=1}^{2^n} P[\sum_{m=1}^{n} \frac{\partial \varepsilon}{\partial x^m}(t_k, x)\xi_j^m \sum_{l=1}^{n} \xi_j^l v^l(t_k, x)] \\ &= \quad -h \sum_{m=1}^{n} P[\frac{\partial \varepsilon}{\partial x^m}(t_k, x) v^m(t_k, x)]. \end{aligned} \tag{4.28}$$

Hence

$$||P^{(3)}|| = h||P \sum_{m=1}^{n} [\frac{\partial \varepsilon}{\partial x^m}(t_k, \cdot) v^m(t_k, \cdot)]||$$

$$\le h|| \sum_{m=1}^{n} [\frac{\partial \varepsilon}{\partial x^m}(t_k, x) v^m(t_k, \cdot)]|| \le C(\omega)h \sum_{m=1}^{n} ||\frac{\partial \varepsilon}{\partial x^m}(t_k, \cdot)||$$

$$= C(\omega)h \sum_{m=1}^{n} [\int_Q \sum_{i=1}^{n} (\frac{\partial \varepsilon^i}{\partial x^m}(t_k, x))^2 dx]^{1/2} \le nC(\omega)h[\int_Q \sum_{i,m=1}^{n} (\frac{\partial \varepsilon^i}{\partial x^m}(t_k, x))^2 dx]^{1/2}$$

or

$$||P^{(3)}||^2 \le C(\omega)h^2 \sum_{i,m=1}^{n} \int_Q (\frac{\partial \varepsilon^i}{\partial x^m}(t_k,x))^2 dx. \tag{4.29}$$

Integrating by parts, we obtain

$$\int_Q (\frac{\partial \varepsilon^i}{\partial x^m}(t_k,x))^2 dx = -\int_Q \varepsilon^i(t_k,x)\frac{\partial^2 \varepsilon^i}{\partial (x^m)^2}(t_k,x)dx,$$

hence

$$\sum_{i,m=1}^{n} \int_Q (\frac{\partial \varepsilon^i}{\partial x^m}(t_k,x))^2 dx = -(\varepsilon, \Delta\varepsilon). \tag{4.30}$$

Taking into account boundedness of $\Delta\varepsilon$, we get from (4.29) and (4.30)

$$||P^{(3)}||^2 \le C(\omega)h^2||\varepsilon(t_k,\cdot)|| \le C(\omega)h^3 + h||\varepsilon(t_k,\cdot)||^2. \tag{4.31}$$

Let us also note the following inequality

$$||P^{(3)}|| \le C(\omega)h\sqrt{||\varepsilon(t_k,\cdot)||}. \tag{4.32}$$

Thus, we have

$$\varepsilon(t_{k+1},x) = P^{(1)} + P^{(2)} + P^{(3)} + R + \rho,$$

$$\begin{aligned} ||\varepsilon(t_{k+1},\cdot)||^2 &= ||P^{(1)}||^2 + ||P^{(2)}||^2 + ||P^{(3)}||^2 + 2(P^{(1)},P^{(2)}) + 2(P^{(1)},P^{(3)}) \\ &+2(P^{(2)},P^{(3)}) + 2(P^{(1)}+P^{(2)}+P^{(3)},R+\rho) + (R+\rho,R+\rho). \end{aligned} \tag{4.33}$$

According to Lemma 4.1 which follows this proof, we have

$$|(P^{(1)},P^{(3)})| \le C(\omega)h^2. \tag{4.34}$$

Let us now write down the bounds for the other terms in (4.33) (below we also use Young's inequality in addition to the mentioned relations):

$$\begin{gathered} (4.23) \Longrightarrow ||P^{(1)}||^2 \le ||\varepsilon(t_k,\cdot)||^2, \\ (4.25) \Longrightarrow ||P^{(2)}||^2 \le C(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \le C(\omega)h^3 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.31) \Longrightarrow ||P^{(3)}||^2 \le C(\omega)h^2||\varepsilon(t_k,\cdot)|| \le C(\omega)h^3 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.23) \text{ and } (4.25) \Longrightarrow |(P^{(1)},P^{(2)})| \le C(\omega)h||\varepsilon(t_k,\cdot)||^2, \\ (4.25) \text{ and } (4.32) \Longrightarrow |(P^{(2)},P^{(3)})| \le C(\omega)h^2||\varepsilon(t_k,\cdot)||^{3/2} \le C(\omega)h^5 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.23) \text{ and } (4.27) \Longrightarrow |(P^{(1)},R)| \le C(\omega)h^{3/2}||\varepsilon(t_k,\cdot)|| \le C(\omega)h^2 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.25) \text{ and } (4.27) \Longrightarrow |(P^{(2)},R)| \le C(\omega)h^{5/2}||\varepsilon(t_k,\cdot)|| \le C(\omega)h^4 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.32) \text{ and } (4.27) \Longrightarrow |(P^{(3)},R)| \le C(\omega)h^{5/2}||\varepsilon(t_k,\cdot)||^{1/2} \le C(\omega)h^3 + h||\varepsilon(t_k,\cdot)||^2, \\ (4.23) \text{ and } (4.16) \Longrightarrow |(P^{(1)},\rho)| \le C(\omega)h^{3/2-\varepsilon}||\varepsilon(t_k,\cdot)|| \le C(\omega)h^{2-2\varepsilon} + h||\varepsilon(t_k,\cdot)||^2, \\ (4.25) \text{ and } (4.16) \Longrightarrow |(P^{(2)},\rho)| \le C(\omega)h^{5/2-\varepsilon}||\varepsilon(t_k,\cdot)|| \le C(\omega)h^{4-2\varepsilon} + h||\varepsilon(t_k,\cdot)||^2, \end{gathered} \tag{4.35}$$

$$(4.32) \text{ and } (4.16) \Longrightarrow |(P^{(3)},\rho)| \le C(\omega)h^{5/2-\varepsilon}||\varepsilon(t_k,\cdot)||^{1/2} \le C(\omega)h^{3-4\varepsilon/3} + h||\varepsilon(t_k,\cdot)||^2,$$

$$(4.27) \Longrightarrow |(R,R)| \le C(\omega)h^3, \qquad (4.27) \text{ and } (4.16) \Longrightarrow |(R,\rho)| \le C(\omega)h^{3-\varepsilon},$$

$$(4.16) \Longrightarrow |(\rho,\rho)| \le C(\omega)h^{3-2\varepsilon}.$$

The above bounds together with (4.33) and (4.34) imply

$$||\varepsilon(t_{k+1},\cdot)||^2 \le ||\varepsilon(t_k,\cdot)||^2 + C_1(\omega)h||\varepsilon(t_k,\cdot)||^2 + C_2(\omega)h^{2-2\varepsilon}, \tag{4.36}$$

from which it follows that

$$||\varepsilon(t_k,\cdot)||^2 \le C(\omega)h^{1-2\varepsilon}. \tag{4.37}$$

This proves Theorem 4.3. $\square$

Now let us prove (4.34).

Lemma 4.1 *Under the assumptions of Theorem 4.3, the following estimate holds:*

$$|(P^{(1)},P^{(3)})| \le C(\omega)h^2, \tag{4.38}$$

where $P^{(1)}$ *and* $P^{(3)}$ *are defined in* (4.22) *and* (4.28), *respectively.*

Proof. We have

$$\begin{aligned} P^{(1)} &= 2^{-n}\sum_{j=1}^{2^n}\varepsilon(t_k, x+\sigma\sqrt{h}\xi_j) \\ &= \varepsilon(t_k,x) + 2^{-n}\sum_{j=1}^{2^n}\sum_{l=1}^{n}\frac{\partial\varepsilon}{\partial x^l}(t_k,x)\sigma\sqrt{h}\xi_j^l + r \\ &= \varepsilon(t_k,x)+r, \end{aligned} \tag{4.39}$$

where

$$||r|| \le C(\omega)h. \tag{4.40}$$

As the sum in (4.39) is equal to zero, we get

$$P^{(1)} = \varepsilon(t_k,x)+r. \tag{4.41}$$

Further, as $\operatorname{div}\varepsilon = 0$, we get

$$\begin{aligned} (\varepsilon,P^{(3)}) &= -h\sum_{m=1}^{n}(\varepsilon, P[\frac{\partial\varepsilon}{\partial x^m}(t_k,x)v^m(t_k,x)]) \\ &= -h\sum_{m=1}^{n}(\varepsilon, \frac{\partial\varepsilon}{\partial x^m}(t_k,x)v^m(t_k,x)). \end{aligned} \tag{4.42}$$

Integrating by parts, we find

$$\int_Q \varepsilon^i(t_k,x)\frac{\partial\varepsilon^i}{\partial x^m}(t_k,x)v^m dx = -\int_Q \varepsilon^i(t_k,x)\frac{\partial(\varepsilon^i v^m)}{\partial x^m}dx$$

$$= -\int_Q \varepsilon^i(t_k,x)\frac{\partial\varepsilon^i}{\partial x^m}(t_k,x)v^m dx - \int_Q[\varepsilon^i(t_k,x)]^2\frac{\partial v^m}{\partial x^m}dx,$$

hence

$$\int_Q \varepsilon^i(t_k,x)\frac{\partial \varepsilon^i}{\partial x^m}(t_k,x)v^m dx = -\frac{1}{2}\int_Q [\varepsilon^i(t_k,x)]^2 \frac{\partial v^m}{\partial x^m}dx,\ i,m=1,\dots,n. \tag{4.43}$$

It follows from (4.42), (4.43) and $\operatorname{div} v = 0$ that

$$(\varepsilon, P^{(3)}) = 0 \tag{4.44}$$

while according to (4.40), (4.32):

$$|(r,P^{(3)})| \le C(\omega)h^2\sqrt{||\varepsilon(t_k,\cdot)||} \le C(\omega)h^2. \tag{4.45}$$

Finally, (4.41), (4.44), and (4.45) give (4.38). □

Let us now consider the error of the approximation of pressure considered in Section 3.4.

Proposition 4.1 *Let assumptions of Theorem 4.3 hold. Then for almost every trajectory* $w(\cdot)$ *and any* $0 < \varepsilon < 1/3$ *there exists a constant* $C(\omega) > 0$ *such that the approximate pressure* $\bar{p}(t_k, x)$ *from* (3.25), (3.12)-(3.13) *satisfies the following inequality*

$$\|\bar{p}(t_k,\cdot) - p(t_k,\cdot)\| \le C(\omega)h^{1/3-\epsilon}. \tag{4.46}$$

Proof. We have

$$\frac{\partial v^i}{\partial x^j}(t_k,x) = \frac{v^i(t_k,x+\delta e_j) - v^i(t_k,x-\delta e_j)}{2\delta} + O(\delta^2), \tag{4.47}$$
$$\frac{\partial \bar{v}^i}{\partial x^j}(t_k,x) = \frac{\bar{v}^i(t_k,x+\delta e_j) - \bar{v}^i(t_k,x-\delta e_j)}{2\delta} + O(\delta^2),$$

where δ is a positive sufficiently small number and $|O(\delta^2)| \le C(\omega)\delta^2$. Due to Theorem 4.3,

$$\left\| \frac{v(t_k,x+\delta e_j) - v(t_k,x-\delta e_j)}{2\delta} - \frac{\bar{v}(t_k,x+\delta e_j) - \bar{v}(t_k,x-\delta e_j)}{2\delta} \right\| \le C(\omega)\frac{h^{1/2-\epsilon/2}}{\delta} \text{ a.s.} \tag{4.48}$$

Choosing $\delta = ch^{1/6+\epsilon/2}$ with some $c > 0$, we obtain from (4.47) and (4.48) that

$$\left\| \frac{\partial v}{\partial x^j}(t_k,\cdot) - \frac{\partial \bar{v}}{\partial x^j}(t_k,\cdot) \right\| \le C(\omega)h^{1/3-\epsilon} \quad \text{a.s. .} \tag{4.49}$$

Subtracting (3.23) with $t = t_k$ from (3.24) with t_k instead of t_{k+1}, we get

$$\begin{aligned}\|\nabla\bar{p}(t_k,\cdot) - \nabla p(t_k,\cdot)\| &= \left\|P^\perp\left[(v(t_k,\cdot),\nabla)v(t_k,\cdot)\right] - P^\perp\left[(\bar{v}(t_k,\cdot),\nabla)\bar{v}(t_k,\cdot)\right]\right\| \\ &\le \left\|P^\perp\left[(v(t_k,\cdot),\nabla)(v(t_k,\cdot) - \bar{v}(t_k,\cdot))\right]\right\| + \left\|P^\perp\left[(v(t_k,\cdot) - \bar{v}(t_k,\cdot),\nabla)\bar{v}(t_k,\cdot)\right]\right\| \\ &\le \|(v(t_k,\cdot),\nabla)(v(t_k,\cdot) - \bar{v}(t_k,\cdot))\| + \|(v(t_k,\cdot) - \bar{v}(t_k,\cdot),\nabla)\bar{v}(t_k,\cdot)\| \ .\end{aligned} \tag{4.50}$$

Due to Assumptions 2.1 and (4.49),

$$\|(v(t_k,\cdot),\nabla)(v(t_k,\cdot) - \bar{v}(t_k,\cdot))\| \le C(\omega)h^{1/3-\epsilon} \ \text{ a.s. .} \tag{4.51}$$

Due to Assumptions 4.1 and Theorem 4.3,

$$\|(v(t_k,\cdot)-\bar{v}(t_k,\cdot),\nabla)\bar{v}(t_k,\cdot)\| \leq C(\omega)h^{1/2-\epsilon} \quad \text{a.s. .} \tag{4.52}$$

The estimates (4.50)-(4.52) imply

$$\|\nabla\tilde{p}(t_k,\cdot)-\nabla p(t_k,\cdot)\| \leq C(\omega)h^{1/3-\epsilon} \quad \text{a.s. .} \tag{4.53}$$

Using (3.24), (3.25), (3.13) and boundedness of $\bar{v}(t_k,x)$ and its second derivatives assumed in Assumptions 4.1, we obtain

$$\begin{aligned} & \|\nabla\tilde{p}(t_k,\cdot)-\nabla\bar{p}(t_k,\cdot)\| \\ = & \left\| P^{\perp}\left[\sum_{m=1}^{n}\bar{v}^m(t_k,x)\left(\frac{\partial\bar{v}(t_k,x)}{\partial x^m}-\frac{1}{\sigma\sqrt{h}}2^{-n}\sum_{j=1}^{2^n}\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j)\xi_j^m\right)\right]\right\| \\ \leq & \ C(\omega)h^{1/2-\epsilon} \quad \text{a.s.} \end{aligned} \tag{4.54}$$

The estimate (4.46) follows from (4.53) and (4.54). □

Remark 4.2 *The intuition built on numerics for ordinary stochastic differential equations (see, e.g. [29]) and on layer methods for SPDEs [31, 32] suggests that the one-step error properties of Theorem 4.1 should lead to first mean-square convergence order (proved in Theorem 4.4 below) and to* $1-\varepsilon$ *a.s. convergence order for the velocity approximation instead of* $1/2-\varepsilon$ *a.s. order proved in Theorem 4.3. Analogously, we expect that spatial derivatives of the approximate velocity converge with a.s. order* $1-\varepsilon$ *instead of* $1/3-\varepsilon$ *shown in* (4.49)*. It is not difficult to see from the proof of Proposition 4.1 that a.s. convergence of both velocity and its first-order spatial derivatives with order* $1-\varepsilon$ *implies a.s. convergence of pressure with order* $1-\varepsilon$*. In our numerical experiments (see Section 5) we observed convergence (both mean-square and a.s.) of velocity and pressure with order one.*

4.3 Mean-square global error

To prove the mean-square convergence of the layer method (3.12)-(3.13), we need stronger assumptions that Assumptions 4.1 used for proving the almost sure convergence in the previous section.

Assumptions 4.2. *Assume that* $\bar{v}(t_k,x)$, $k=0,\ldots,N$, *and its derivatives satisfy the following inequalities*

$$\begin{gathered} |\bar{v}(t_k,x)| \leq C(\omega), \quad |\partial\bar{v}(t_k,x)/\partial x^i| \leq C(\omega), \quad i=1,\ldots,n, \\ |\partial^2\bar{v}(t_k,x)/\partial x^i\partial x^j| \leq C(\omega), \quad i,j=1,\ldots,n, \\ |\partial^3\bar{v}(t_k,x)/\partial x^i\partial x^j\partial x^l| \leq C(\omega), \quad i,j,l=1,\ldots,n, \end{gathered} \tag{4.55}$$

where $C(\omega)>0$ *is an a.s. finite constant independent of* x, h, k, *which has finite moments up to a sufficiently high order.*

The following result takes place.

Theorem 4.4 *Let Assumptions* 2.1 *and* 4.2 *hold. Then*

$$(E||\bar{v}(t_k,\cdot) - v(t_k,\cdot)||^2)^{1/2} \le Kh, \tag{4.56}$$

i.e., the layer method (3.12)-(3.13) *for the SNSE* (1.1)-(1.4) *converges with mean-square order* 1.

Proof. In this proof the notation $C_k(\omega) > 0$ denote various $\mathcal{F}^w_{t_k}$-measurable random variables with bounded moments of a sufficiently high order. Note that even in the two sides of equalities $C_k(\omega)$ can correspond to different random variables.

Recall (4.33):

$$||\varepsilon(t_{k+1},\cdot)||^2 = ||P^{(1)}||^2 + ||P^{(2)}||^2 + ||P^{(3)}||^2 + 2(P^{(1)},P^{(2)}) + 2(P^{(1)},P^{(3)}) \tag{4.57}$$
$$+2(P^{(2)},P^{(3)}) + 2(P^{(1)} + P^{(2)} + P^{(3)}, R+\rho) + (R+\rho, R+\rho).$$

According to the error estimates obtained in Theorem 4.3 (see (4.35)), we have

$$||P^{(1)}||^2 + ||P^{(2)}||^2 + ||P^{(3)}||^2 + 2(P^{(2)},P^{(3)}) + 2(P^{(2)} + P^{(3)}, R) + (R,R) \tag{4.58}$$
$$\le ||\varepsilon(t_k,\cdot)||^2 + Kh||\varepsilon(t_k,\cdot)||^2 + C_k(\omega)h^3.$$

By (4.3) from Theorem 4.1, we get

$$E(\rho,\rho) \le K_0 h^3. \tag{4.59}$$

Using (4.27) and (4.2), we obtain

$$|E(R,\rho)| = |E(R, E(\rho|\mathcal{F}^w_{t_k}))| \le K_0 h^{7/2}. \tag{4.60}$$

Let us analyze the term $(P^{(1)},P^{(2)})$ in (4.57). We have (see (4.22), (4.24), (4.4) and the proof of Lemma 4.1):

$$\begin{aligned}
&|(P^{(1)},P^{(2)})| \qquad\qquad (4.61)\\
= \;& \frac{\sqrt{h}}{\sigma}\left|\left(2^{-n}\sum_{j=1}^{2^n}\varepsilon(t_k, x+\sigma\sqrt{h}\xi_j), 2^{-n}\sum_{j=1}^{2^n} P[(\bar{v}(t_k,x+\sigma\sqrt{h}\xi_j) - \bar{v}(t_k,x))\xi_j^\top\varepsilon(t_k,x)]\right)\right| \\
\le \;& h\left|\left(2^{-n}\sum_{j=1}^{2^n}\varepsilon(t_k,x+\sigma\sqrt{h}\xi_j), P\sum_{m=1}^{n}\frac{\partial\bar{v}(t_k,x)}{\partial x^m}\varepsilon^m(t_k,x)\right)\right| + C_k(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \\
= \;& h2^{-n}\left|\sum_{j=1}^{2^n}\left(\varepsilon(t_k,x+\sigma\sqrt{h}\xi_j), \sum_{m=1}^{n}\frac{\partial\bar{v}(t_k,x)}{\partial x^m}\varepsilon^m(t_k,x)\right)\right| + C_k(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \\
= \;& h\left|\sum_{m=1}^{n}\left(\varepsilon(t_k,x) + r, \frac{\partial\bar{v}(t_k,x)}{\partial x^m}\varepsilon^m(t_k,x)\right)\right| + C_k(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \\
= \;& h\left|\sum_{m=1}^{n}\left(r, \frac{\partial\bar{v}(t_k,x)}{\partial x^m}\varepsilon^m(t_k,x)\right)\right| + C_k(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \\
\le \;& C_k(\omega)h^2||\varepsilon(t_k,\cdot)|| + C_k(\omega)h^2||\varepsilon(t_k,\cdot)||^2 \le C_k(\omega)h^3 + h||\varepsilon(t_k,\cdot)||^2.
\end{aligned}$$

Consider the term $(P^{(1)}+P^{(2)}+P^{(3)},\rho)$ in (4.57). By (4.2) and (4.23), we obtain

$$\begin{aligned} |E(P^{(1)},\rho)| &= |E(P^{(1)},E(\rho|\mathcal{F}^w_{t_k}))| \leq E(||P^{(1)}||\,||E(\rho|\mathcal{F}^w_{t_k})||) \\ &\leq E(||\varepsilon(t_k,\cdot)||C_k(\omega)h^2) \leq hE||\varepsilon(t_k,\cdot)||^2+K_0h^3. \end{aligned} \tag{4.62}$$

We have (see (4.23) and (4.25)):

$$\begin{aligned} |E(P^{(2)},\rho)| &\leq |E(P^{(2)},E(\rho|\mathcal{F}^w_{t_k}))| \leq E(||P^{(2)}||\,||E(\rho|\mathcal{F}^w_{t_k})||) \\ &\leq E(||\varepsilon(t_k,\cdot)||C_k(\omega)h^3) \leq hE||\varepsilon(t_k,\cdot)||^2+K_0h^5. \end{aligned} \tag{4.63}$$

We get from (4.23) and (4.32):

$$\begin{aligned} |E(P^{(3)},\rho)| &\leq |E(P^{(3)},E(\rho|\mathcal{F}^w_{t_k}))| \leq E(||P^{(3)}||\,||E(\rho|\mathcal{F}^w_{t_k})||) \\ &\leq E\left[||\varepsilon(t_k,\cdot)||^{1/2}C_k(\omega)h^3\right] \leq hE||\varepsilon(t_k,\cdot)||^2+K_0h^{11/3}. \end{aligned} \tag{4.64}$$

Using the assumption that third spatial derivatives of $\varepsilon(t_k,x)$ are bounded and also the relationships (4.4), it is not difficult to see that instead of (4.27):

$$||R(t_k,\cdot)|| \leq C_k(\omega)h^2. \tag{4.65}$$

It follows from (4.65) and the estimate for $P^{(1)}$ (see (4.23)) that

$$\left|(P^{(1)},R)\right| \leq C_k(\omega)h^3+h||\varepsilon(t_k,\cdot)||^2. \tag{4.66}$$

Due to boundedness of third spatial derivatives of $\varepsilon(t_k,x)$, we have for r from (4.39):

$$r=\frac{\sigma^2}{2}h\sum_{i,l=1}^{n}\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x)+r',\quad ||r'||\leq C_k(\omega)h^{3/2},$$

and (see (4.28))

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x),P^{(3)}\right) &= -h\sum_{m=1}^{n}\left(\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x),P\frac{\partial}{\partial x^m}\varepsilon(t_k,x)v^m(t_k,x)\right) \\ &= -h\sum_{m=1}^{n}\left(\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x),\frac{\partial}{\partial x^m}\varepsilon(t_k,x)v^m(t_k,x)\right) \\ &= h\sum_{m=1}^{n}\left(\frac{\partial}{\partial x^m}\left(v^m(t_k,x)\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x)\right),\varepsilon(t_k,x)\right), \end{aligned}$$

where we used that $\operatorname{div}\varepsilon=0$ (and hence $\operatorname{div}\frac{\partial^2}{\partial x^l\partial x^i}\varepsilon(t_k,x)=0$) and integration by parts. Therefore, using also (4.32), we get

$$|(r,P^{(3)})| \leq C_k(\omega)h^2||\varepsilon(t_k,\cdot)||+C_k(\omega)h^{5/2}||\varepsilon(t_k,\cdot)||^{1/2} \tag{4.67}$$

which together with (4.44) implies (cf. (4.38)):

$$|(P^{(1)},P^{(3)})| \leq C_k(\omega)h^3+h||\varepsilon(t_k,\cdot)||^2. \tag{4.68}$$

It follows from (4.57), (4.58), (4.59), (4.60), (4.61), (4.62), (4.63), (4.64), (4.66), and (4.68) that

$$E\left[||\varepsilon(t_{k+1},\cdot)||^2\right] \leq E||\varepsilon(t_k,\cdot)||^2+KhE||\varepsilon(t_k,\cdot)||^2+K_0h^3,$$

which implies (4.56). $\square$

5 Numerical examples

In this section we test the numerical algorithm (3.17), (3.26) from Section 3 on two model problems. The experiments indicate that the algorithm has the first order mean-square convergence.

5.1 Model problems

We introduce two model examples of SNSE (1.1)-(1.4) which solutions can be written in an analytic form. Both examples are generalizations of the deterministic model of laminar flow from [39] to the stochastic case.

First model problem. Let

$$f(t,x) = 0, \quad \varphi(x) = 0, \tag{5.1}$$

$$\begin{aligned} q &= 1, \\ \gamma_1^1(t,x) &= A \sin \frac{2\pi\kappa\ x^1}{L} \cos \frac{2\pi\kappa\ x^2}{L} \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right), \\ \gamma_1^2(t,x) &= -A \cos \frac{2\pi\kappa\ x^1}{L} \sin \frac{2\pi\kappa\ x^2}{L} \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right), \quad \kappa \in \mathbf{Z}, \quad A \in \mathbf{R}, \end{aligned} \tag{5.2}$$

then it is easy to check that the problem (1.1)-(1.4), (5.1)-(5.2) has the following solution

$$\begin{aligned} v^1(t,x) =& A \sin \frac{2\pi\kappa\ x^1}{L} \cos \frac{2\pi\kappa\ x^2}{L} \\ & \times \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right) w(t), \\ v^2(t,x) =& -A \cos \frac{2\pi\kappa\ x^1}{L} \sin \frac{2\pi\kappa\ x^2}{L} \\ & \times \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right) w(t), \\ p(t,x) =& \frac{A^2}{4} \left(\cos \frac{4\pi\kappa\ x^1}{L} + \cos \frac{4\pi\kappa\ x^2}{L}\right) \\ & \times \exp\left(-2\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right) (w(t))^2 . \end{aligned} \tag{5.3}$$

Second model problem. To construct this example, we recall the following proposition from [20].

Proposition 5.1 *Let* $V(t,x)$, $P(t,x)$ *be a solution of the deterministic NSE with zero forcing (i.e., of* (1.1)-(1.4) *with all* $\gamma_r = 0$ *and* $f(t,x) = 0$*) then the solution* $v(t,x)$,

$p(t,x)$ *of* (1.1)-(1.4) *with constant* $\gamma_r(t,x) = \gamma_r$ *and* $f(t,x) = 0$ *is equal to*

$$v(t,x) = V\left(t, x - \int_0^t \sum_{r=1}^{q} \gamma_r w_r(s)ds\right) + \sum_{r=1}^{q} \gamma_r w_r(t), \tag{5.4}$$

$$p(t,x) = P\left(t, x - \int_0^t \sum_{r=1}^{q} \gamma_r w_r(s)ds\right). \tag{5.5}$$

Combining this proposition with the deterministic model of laminar flow from [39], we obtain that if

$$f(t,x) = 0, \quad \varphi(x) = \left(A \sin \frac{2\pi\kappa\ x^1}{L} \cos \frac{2\pi\kappa\ x^2}{L}, -A \cos \frac{2\pi\kappa\ x^1}{L} \sin \frac{2\pi\kappa\ x^2}{L}\right)^\top, \tag{5.6}$$
$$\kappa \in \mathbf{Z}, \quad A \in \mathbf{R},$$

and

$$q = 1,\ \gamma_1^1(t,x) = \gamma^1, \quad \gamma_1^2(t,x) = \gamma^2\ . \tag{5.7}$$

then the problem (1.1)-(1.4), (5.6)-(5.7) has the following solution

$$\begin{aligned} v^1(t,x) &= A \sin \frac{2\pi\kappa\ (x^1 - \gamma^1 I(t))}{L} \cos \frac{2\pi\kappa\ (x^2 - \gamma^2 I(t))}{L} \\ &\quad \times \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right) + \gamma^1 w(t), \\ v^2(t,x) &= -A \cos \frac{2\pi\kappa\ (x^1 - \gamma^1 I(t))}{L} \sin \frac{2\pi\kappa\ (x^2 - \gamma^2 I(t))}{L} \\ &\quad \times \exp\left(-\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right) + \gamma^2 w(t), \\ p(t,x) &= \frac{A^2}{4}\left(\cos \frac{4\pi\kappa\ (x^1 - \gamma^1 I(t))}{L} + \cos \frac{4\pi\kappa\ (x^2 - \gamma^2 I(t))}{L}\right) \\ &\quad \times \exp\left(-2\sigma^2 \left(\frac{2\pi\kappa}{L}\right)^2 t\right), \end{aligned} \tag{5.8}$$

where

$$I(t) = \int_0^t w(s)ds,\ w(s) = w_1(s).$$

It is clear that both model problems satisfy Assumptions 2.1.

5.2 Results of numerical experiments

In our numerical experiments we test the algorithm (3.17)-(3.18), (3.26) which is a realization of the layer method (3.12)-(3.13), (3.25). This algorithm possesses the following properties.

Proposition 5.2 *1. The approximate solution of the problem* (1.1)-(1.4), (5.1)-(5.2) *obtained by the algorithm* (3.17)-(3.18), (3.26) *contains only those modes which are present in the coefficient* $\gamma_1(t,x)$ *from* (5.2), *i.e., which are present in the exact solution* (5.3).

2. The approximate solution of the problem (1.1)-(1.4), (5.6)-(5.7) *obtained by the algorithm* (3.17)-(3.18), (3.26) *contains only those modes which are present in the initial condition* $\varphi(x)$ *from* (5.6) *and the zero mode, i.e., which are present in the exact solution* (5.8).

The proof of this proposition is analogous to the proof of a similar result in the deterministic case [34] and it is omitted here. One can deduce from the proof of Proposition 5.2 that in the case of the considered model problems the algorithm (3.17)-(3.18) satisfies Assumptions 4.1 and 4.2.

We measure the numerical error in the experiments as follows. First, we consider the relative mean-square error defined as

$$err^{v}_{msq} = \frac{\sqrt{E\sum_{\mathbf{n}} |\bar{v}_{\mathbf{n}}(T) - v_{\mathbf{n}}(T)|^2}}{\sqrt{E\sum_{\mathbf{n}} |v_{\mathbf{n}}(T)|^2}} \ , \quad err^{p}_{msq} = \frac{\sqrt{E\sum_{\mathbf{n}} |\bar{p}_{\mathbf{n}}(T) - p_{\mathbf{n}}(T)|^2}}{\sqrt{E\sum_{\mathbf{n}} |p_{\mathbf{n}}(T)|^2}} \ . \tag{5.9}$$

Analysis of this error provides us with information about mean-square convergence of the numerical algorithm considered. To evaluate this error in the experiments, we use the Monte Carlo technique for finding the expectations in (5.9) by running K independent (with respect to realizations of the Wiener process $w(t)$) realizations of $\bar{v}_{\mathbf{n}}(T)$, $v_{\mathbf{n}}(T)$, $\bar{p}_{\mathbf{n}}(T)$, $p_{\mathbf{n}}(T)$. Second, we consider the relative L_2-error for a fixed trajectory of $w(t)$:

$$err^{v} = \frac{\sqrt{\sum_{\mathbf{n}} |\bar{v}_{\mathbf{n}}(T) - v_{\mathbf{n}}(T)|^2}}{\sqrt{\sum_{\mathbf{n}} |v_{\mathbf{n}}(T)|^2}} \ , \quad err^{p} = \frac{\sqrt{\sum_{\mathbf{n}} |\bar{p}_{\mathbf{n}}(T) - p_{\mathbf{n}}(T)|^2}}{\sqrt{\sum_{\mathbf{n}} |p_{\mathbf{n}}(T)|^2}} \ . \tag{5.10}$$

Analysis of this error provides us with information about a.s. convergence of the numerical algorithm. To evaluate this error in the tests, we fix a trajectory $w(t)$, $0 \le t \le T$, which is obtained with a small time step.

We note that in the case of the considered examples and the tested algorithm (see Proposition 5.2) $v_{\mathbf{n}}(T)$ are nonzero only for $|\mathbf{n}^1| = |\mathbf{n}^2| = |\kappa|$ and $p_{\mathbf{n}}(T)$ are nonzero only for $|\mathbf{n}^1| = 2|\kappa|$, $\mathbf{n}^2 = 0$ and $\mathbf{n}^1 = 0$, $|\mathbf{n}^2| = 2|\kappa|$. Hence, the sums in (5.9) and (5.10) are finite here. This also implies that it is sufficient here to take the cut-off parameter M in the algorithm (3.17)-(3.18), (3.26) to be equal to $2|\kappa|$.

The test results for the algorithm (3.17)-(3.18), (3.26) applied to the first model problem (1.1)-(1.4), (5.1)-(5.2) are presented in Tables 5.1 and 5.2. In Table 5.1 the "$\pm$" reflects the Monte Carlo errors in evaluating of err^{v}_{msq} and err^{p}_{msq}, they give the confidence intervals for the corresponding values with probability 0.95.

We can conclude from Table 5.1 that both velocity and pressure found due to the algorithm (3.17)-(3.18), (3.26) demonstrate the mean-square convergence with order 1. We also see from Table 5.2 that both velocity and pressure converge with order 1 for a particular, fixed trajectory of $w(t)$. We note that we repeated the experiment for other realizations of $w(t)$ and observed the same behavior. The observed first order convergence of the algorithm is consistent with our predictions.

The test results for the algorithm (3.17)-(3.18), (3.26) applied to the second model problem (1.1)-(1.4), (5.6)-(5.7) are presented in Table 5.3. In these tests we limit ourselves to simulation for a particular, fixed trajectory of $w(t)$ and observation of a.s. convergence. We note that evaluation of the exact solution (5.8) requires simulation of the integral $I(t)$. This was done in the following way. At each time step $k+1$, $k = 0, \ldots, N-1$, we

Table 5.1: Mean-square relative errors err^v_{msq} and err^p_{msq} from (5.9) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.1)-(5.2) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$ by the algorithm (3.17)-(3.18), (3.26) with $M = 2$ and various time steps h. The "±"reflects the Monte Carlo error in evaluating err^v_{msq} and err^p_{msq} via the Monte Carlo technique with $K = 4000$ independent runs. The exact values (up to 5 d.p.) of the denominators in (5.9) are 0.37470 and 0.12159, respectively.

h	velocity	pressure
0.2	0.0537 ± 0.0012	0.0710 ± 0.0038
0.1	0.0263 ± 0.0006	0.0337 ± 0.0016
0.05	0.0130 ± 0.0003	0.0170 ± 0.0009
0.02	0.0052 ± 0.0001	0.0066 ± 0.0003
0.01	0.0025 ± 0.00006	0.0031 ± 0.0001

Table 5.2: Relative errors err^v and err^p from (5.10) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.1)-(5.2) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$ for a fixed trajectory of the Wiener process $w(t)$ by the algorithm (3.17)-(3.18), (3.26) with $M = 2$ and various time steps h. The exact values (up to 5 d.p.) of the denominators in (5.10) are 0.43950 and 0.09658, respectively.

h	velocity	pressure
0.2	0.0485	0.0585
0.1	0.0237	0.0284
0.05	0.0117	0.0141
0.02	0.0047	0.0056
0.01	0.0023	0.0028

simulate a Wiener increment $\Delta_k w$ as i.i.d. Gaussian $\mathcal{N}(0, h)$ random variables (and we find $w(t_{k+1}) = w(t_k) + \Delta_k w$) and i.i.d. Gaussian $\mathcal{N}(0, 1)$ random variables η_k. Then (see [29, Chapter 1]):

$$I(t_{k+1}) = I(t_k) + hw(t_k) + \frac{h}{2}\Delta_k w + \frac{h^{3/2}}{\sqrt{12}}\eta_k \ .$$

Again, the observed first order convergence of the algorithm in Table 5.3 is consistent with our prediction. We note that the remarkable property of the layer method proved in Proposition 5.2 and the numerical tests done on the two model problems indicate that the method has good computational features and could be a useful numerical tool for studying SNSE. At the same time, additional numerical testing as well as continuation of the theoretical analysis deserve further attention.

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Table 5.3: Relative errors err^v and err^p from (5.10) at $T = 3$ in simulation of the problem (1.1)-(1.4), (5.6)-(5.7) with $\sigma = 0.1$, $A = 1$, $\kappa = 1$, $L = 1$, $\gamma^1 = 0.5$, $\gamma^2 = 0.2$ for a fixed trajectory of the Wiener process $w(t)$ by the algorithm (3.17)-(3.18), (3.26) with $M = 2$ and various time steps h. The exact values (up to 6 d.p.) of the denominators in (5.10) are 0.505620 and 0.000548, respectively.

h	velocity	pressure
0.01	0.166	0.973
0.005	0.068	0.384
0.002	0.024	0.134
0.001	0.0118	0.0645
0.0005	0.0058	0.0313

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