Nonassociative geometry in quasi-Hopf representation categories II: Connections and curvature

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Abstract
We continue our systematic development of noncommutative and nonassociative differential geometry internal to the representation category of a quasitriangular quasi-Hopf algebra. We describe derivations, differential operators, differential calculi and connections using universal categorical constructions to capture algebraic properties such as Leibniz rules. Our main result is the construction of morphisms which provide prescriptions for lifting connections to tensor products and to internal homomorphisms. We describe the curvatures of connections within our formalism, and also the formulation of Einstein-Cartan geometry as a putative framework for a nonassociative theory of gravity.

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Contents

1 Introduction and summary

This paper is the second part in a series of articles whose goal is to systematically develop a formalism for differential geometry on noncommutative and nonassociative spaces. The main physical inspiration behind this work is sparked by the recent observations from closed string theory that certain non-geometric flux compactifications experience a nonassociative deformation of the spacetime geometry [BHM06, BP11, Lus10, BDLPR11, MSS12, BFHLS14] (see [Lus11, MSS13, Blu14] for reviews and further references), together with the constructions of [MSS14, ASz15] which show that the corresponding nonassociative algebras and their basic geometric structures can be obtained by cochain twist quantization, and hence are commutative and associative quantities when regarded as objects in a suitable braided monoidal category. See the first paper in this series [BSS15], hereafter referred to as Part I, for further motivation and a more complete list of relevant references.

Earlier categorical approaches to nonassociative geometry along these lines were pursued in [BHM11, BM10]. In the present paper we develop important notions of differential geometry internal to the representation category ${}^{H}\mathscr{M}$ of a quasitriangular quasi-Hopf algebra H. In particular, we develop the notions of derivations, differential operators, differential calculi and connections by using universal categorical constructions such as categorical limits. In contrast to the approach of [BM10], our geometric structures are described by internal homomorphisms instead of morphisms in the category ${}^{H}\mathscr{M}$. This leads to a much richer framework, because the conditions for being a morphism in ${}^{H}\mathscr{M}$ (i.e. H-equivariance) are very restrictive and hence the framework in [BM10] allows for only very special geometric structures. Our internal homomorphism approach is inspired by the formalism of [AS14] (see [Sch11, Asc12] for overviews), and it clarifies these ideas and constructions in categorical terms.

We begin in Section 2 with a brief review of the categorical framework which was developed in Part I. In contrast to that paper, in the present paper we consider the case where all modules are $\mathbb{Z}$-graded; this allows us later on to regard graded objects such as differential calculi naturally as objects in our categories.

In Section 3 we introduce derivations $\mathrm{der}(A)$ on braided commutative algebras A in ${}^{H}\mathscr{M}$ by formalizing the Leibniz rule in terms of an equalizer in ${}^{H}\mathscr{M}$. We analyse structural properties of $\mathrm{der}(A)$ and in particular prove that, in the case where H is triangular, $\mathrm{der}(A)$ together with an internal commutator $[\,\cdot\,,\,\cdot\,]$ is a Lie algebra in ${}^{H}\mathscr{M}$. We then introduce differential operators $\mathrm{diff}(V)$ on symmetric A-bimodules V in ${}^{H}\mathscr{M}$ by again using a suitable equalizer in ${}^{H}\mathscr{M}$ to capture the relevant algebraic properties. We show that $\mathrm{diff}(V)$ is an algebra in ${}^{H}\mathscr{M}$ and we also prove that the zeroth order differential operators are the internal endomorphisms $\mathrm{end}_A(V)$ in the category of symmetric A-bimodules ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$. Using the product structure on differential operators to formalize nilpotency of a differential, we can then give a definition of a differential calculus in ${}^{H}\mathscr{M}$.

In Section 4 we develop an appropriate notion of connections $\mathrm{con}(V)$ on objects V in

${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. The idea is to formalize a generalization of the usual Leibniz rule with respect to a differential calculus in terms of an equalizer in ${}^{H}\mathscr{M}$. The resulting object $\mathrm{con}(V)$ is analysed in detail and it is shown that the usual affine space of ordinary connections arises as a certain proper subset of $\mathrm{con}(V)$. Our more flexible definition of connections has the advantage that $\mathrm{con}(V)$ also forms an object in ${}^{H}\mathscr{M}$ in addition to being an affine space. We then develop a lifting prescription for connections to tensor products $V \otimes_A W$ of objects V, W in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. It is important to notice that our notion of tensor product connections differs from the standard one: Although our techniques are only applicable to braided commutative algebras and their bimodules in ${}^{H}\mathscr{M}$, they are more flexible in the sense that *any* two connections can be lifted to a tensor product connection, not only those which satisfy the very restrictive 'bimodule connection' property proposed in [Mou95, D-VM96, BM-HDS96, D-V01]. We also develop a lifting prescription for connections to internal homomorphisms $\mathrm{hom}_A(V, W)$ of objects V, W in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. These lifts are all important ingredients in (noncommutative and nonassociative) Riemannian geometry for extending e.g. tangent bundle connections to all tensor fields, and they play an instrumental role in physical applications of our formalism to noncommutative and nonassociative gravity theories such as those anticipated to arise in non-geometric string theory. All of these constructions moreover generalize and clarify the corresponding constructions of [AS14] in categorical terms.

Finally, in Section 5 we assign curvatures to connections and show that they are internal endomorphisms in the category ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$, provided that H is triangular. We also obtain a Bianchi tensor, which in classical differential geometry would identically vanish; in general it is not necessarily equal to 0, and hence in this sense it characterizes the noncommutativity and nonassociativity of our geometries. We further observe that the curvature of any tensor product connection is the sum of the two individual curvatures, which means that curvatures behave additively in an appropriate sense. We conclude with a brief outline of how our formalism could be used to describe a noncommutative and nonassociative theory of gravity coupled to Dirac fields; our considerations are based on Einstein-Cartan geometry and its noncommutative generalization which was developed in [AC09].

2 Categorical preliminaries

Let k be an associative and commutative ring with unit $1 \in k$. In contrast to Part I, in this paper we shall work with $\mathbb{Z}$-graded k-modules. This will have the advantage later on that naturally graded objects such as differential calculi can be described as objects in the categories we define below, and also that minus signs will be absorbed into the formalism. The goal of this section is to adapt the material developed in [BSS15] to the graded setting and to thereby also fix our notation for the present paper.

2.1 $\mathbb{Z}$-graded k-modules

The category $\mathscr{M}$ of bounded $\mathbb{Z}$-graded k-modules is defined as follows: The objects in $\mathscr{M}$ are the bounded $\mathbb{Z}$-graded k-modules

$$\underline{V} = \bigoplus_{n\in\mathbb{Z}} \underline{V}_n \ , \tag{2.1}$$

where the k-modules $\underline{V}_n = 0$ for all but finitely many n. The morphisms in $\mathscr{M}$ are the degree preserving k-linear maps $f : \underline{V} \to \underline{W}$, i.e. $f(\underline{V}_n) \subseteq \underline{W}_n$ for all $n \in \mathbb{Z}$. For any object $\underline{V}$ in $\mathscr{M}$ there is a map

$$|\cdot| : \bigsqcup_{n\in\mathbb{Z}} \underline{V}_n \longrightarrow \mathbb{Z} \ , \tag{2.2}$$

which assigns to elements $v \in \underline{V}_n$ their degree $|v| = n$. Elements of $\underline{V}_n$ are said to be homogeneous of degree n.

The category $\mathscr{M}$ is monoidal with monoidal functor $\otimes : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ given by the $\mathbb{Z}$-graded tensor product: For any two objects $\underline{V}, \underline{W}$ in $\mathscr{M}$ we define

$$\underline{V} \otimes \underline{W} := \bigoplus_{n\in\mathbb{Z}} \big(\underline{V} \otimes \underline{W}\big)_n := \bigoplus_{n\in\mathbb{Z}} \bigoplus_{m+l=n} \underline{V}_m \otimes \underline{W}_l \ , \tag{2.3}$$

where $\underline{V}_m \otimes \underline{W}_l$ is the usual tensor product of k-modules. To any $\mathscr{M} \times \mathscr{M}$-morphism $(f : \underline{V} \to \underline{V}',\ g : \underline{W} \to \underline{W}')$ the monoidal functor assigns the $\mathscr{M}$-morphism

$$f \otimes g : \underline{V} \otimes \underline{W} \longrightarrow \underline{V}' \otimes \underline{W}' \ , \qquad v \otimes w \longmapsto f(v) \otimes g(w) \ . \tag{2.4}$$

The unit object in $\mathscr{M}$ is given by the ring k itself, but regarded as a $\mathbb{Z}$-graded k-module with $k_n = 0$, for all $n \neq 0$, and $k_0 = k$. The associator in $\mathscr{M}$ is the natural isomorphism

$$\underline{\Phi} : \otimes \circ \big(\otimes \times \mathrm{id}_{\mathscr{M}}\big) \Longrightarrow \otimes \circ \big(\mathrm{id}_{\mathscr{M}} \times \otimes\big) \ , \tag{2.5}$$

whose components are identity maps. The unitors in $\mathscr{M}$ are the natural isomorphisms

$$\underline{\lambda} : k \otimes - \Longrightarrow \mathrm{id}_{\mathscr{M}} \qquad \text{and} \qquad \underline{\rho} : - \otimes k \Longrightarrow \mathrm{id}_{\mathscr{M}} \ , \tag{2.6}$$

whose components are $\underline{\lambda} : k \otimes \underline{V} \to \underline{V},\ c \otimes v \mapsto c\,v$ and $\underline{\rho} : \underline{V} \otimes k \to \underline{V},\ v \otimes c \mapsto c\,v$. Here and in the following we shall refrain from writing indices on the components of natural transformations.

The monoidal category $\mathscr{M}$ is also braided. Denoting by $\otimes^{\mathrm{op}} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ the functor taking the opposite $\mathbb{Z}$-graded tensor product, i.e. $\underline{V} \otimes^{\mathrm{op}} \underline{W} := \underline{W} \otimes \underline{V}$ and similarly for morphisms, the braiding is the natural isomorphism

$$\underline{\tau} : \otimes \Longrightarrow \otimes^{\mathrm{op}} \ , \tag{2.7}$$

whose components are given by

$$\underline{\tau} : \underline{V} \otimes \underline{W} \longrightarrow \underline{W} \otimes \underline{V} \ , \qquad v \otimes w \longmapsto (-1)^{|v|\,|w|}\ w \otimes v \ , \tag{2.8}$$

for all homogeneous $v \in \underline{V}$ and $w \in \underline{W}$.

Finally, $\mathscr{M}$ is a braided closed monoidal category with internal hom-functor

$$\underline{\mathrm{hom}} : \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathscr{M} \ . \tag{2.9}$$

For any object $(\underline{V}, \underline{W})$ in $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$ we define

$$\underline{\mathrm{hom}}\big(\underline{V}, \underline{W}\big) := \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{hom}}\big(\underline{V}, \underline{W}\big)_n := \bigoplus_{n\in\mathbb{Z}} \bigoplus_{l-m=n} \mathrm{Hom}_k\big(\underline{V}_m, \underline{W}_l\big) \ , \tag{2.10}$$

where $\mathrm{Hom}_k(\underline{V}_m, \underline{W}_l)$ denotes the k-module of k-linear maps between the homogeneous components $\underline{V}_m$ and $\underline{W}_l$. To any $\mathscr{M}^{\mathrm{op}} \times \mathscr{M}$-morphism $(f^{\mathrm{op}} : \underline{V} \to \underline{V}', g : \underline{W} \to \underline{W}')$ the internal hom-functor assigns the $\mathscr{M}$-morphism

$$\underline{\mathrm{hom}}(f^{\mathrm{op}}, g) : \underline{\mathrm{hom}}\big(\underline{V}, \underline{W}\big) \longrightarrow \underline{\mathrm{hom}}\big(\underline{V}', \underline{W}'\big) \ , \qquad L \longmapsto g \circ L \circ f \ . \tag{2.11}$$

The natural currying isomorphism

$$\underline{\zeta} : \mathrm{Hom}_{\mathscr{M}}(- \otimes -, -) \Longrightarrow \mathrm{Hom}_{\mathscr{M}}(-, \underline{\mathrm{hom}}(-,-)) \tag{2.12}$$

has components given by

$$\underline{\zeta}(f) : \underline{V} \longrightarrow \underline{\mathrm{hom}}(\underline{W}, \underline{X}) \ , \qquad v \longmapsto f\big(v \otimes (\,\cdot\,)\big) \ , \tag{2.13}$$

for all $\mathscr{M}$-morphisms $f : \underline{V} \otimes \underline{W} \to \underline{X}$. The natural inverse currying isomorphism has components given by

$$\underline{\zeta}^{-1}(g) : \underline{V} \otimes \underline{W} \longrightarrow \underline{X} \ , \qquad v \otimes w \longmapsto g(v)(w) \ , \tag{2.14}$$

for all $\mathscr{M}$-morphisms $g : \underline{V} \to \underline{\mathrm{hom}}(\underline{W}, \underline{X})$.

2.2 $\mathbb{Z}$-graded quasi-Hopf representation categories

Let H be a quasitriangular quasi-Hopf algebra (see e.g. [Dri90, BSS15] for definitions), which we regard as being $\mathbb{Z}$-graded and sitting in degree 0. A left H-action on an object $\underline{V}$ in $\mathscr{M}$ is an $\mathscr{M}$-morphism

$$\triangleright : H \otimes \underline{V} \longrightarrow \underline{V} \ , \tag{2.15}$$

such that

$$1 \triangleright v = v \qquad \text{and} \qquad (h\,h'\,) \triangleright v = h \triangleright (h' \triangleright v) \ , \tag{2.16}$$

for all $v \in \underline{V}$ and $h, h' \in H$. The bounded $\mathbb{Z}$-graded representation category ${}^{H}\mathscr{M}$ of H is defined as follows: The objects in ${}^{H}\mathscr{M}$ are the bounded $\mathbb{Z}$-graded left H-modules, i.e. the pairs $V = (\underline{V}, \triangleright)$ consisting of an object $\underline{V}$ in $\mathscr{M}$ and a left H-action $\triangleright : H \otimes \underline{V} \to \underline{V}$ on $\underline{V}$. The morphisms in ${}^{H}\mathscr{M}$ are the H-equivariant $\mathscr{M}$-morphisms $f : V \to W$, i.e.

$$f(h \triangleright v) = h \triangleright f(v) \ , \tag{2.17}$$

for all $h \in H$ and $v \in \underline{V}$.

The category ${}^{H}\mathscr{M}$ is monoidal with monoidal functor $\otimes : {}^{H}\mathscr{M} \times {}^{H}\mathscr{M} \to {}^{H}\mathscr{M}$ (denoted by the same symbol as that of $\mathscr{M}$): For any two objects V,W in ${}^{H}\mathscr{M}$ we define $V \otimes W$ as the $\mathbb{Z}$-graded tensor product of the underlying $\mathbb{Z}$-graded k-modules (2.3) together with the left H-action

$$\triangleright : H \otimes \underline{V} \otimes \underline{W} \longrightarrow \underline{V} \otimes \underline{W} \ , \qquad h \otimes v \otimes w \longmapsto (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w) \ , \tag{2.18}$$

where we have used the Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ (with summation understood) for the coproduct of H. To any ${}^{H}\mathscr{M} \times {}^{H}\mathscr{M}$-morphism $(f : V \to V',\ g : W \to W'\,)$ the monoidal functor assigns the ${}^{H}\mathscr{M}$-morphism induced by (2.4). The unit object in ${}^{H}\mathscr{M}$ is $I := (k, \triangleright)$ with trivial left H-action $\triangleright : H \otimes k \to k\ ,\ h \otimes c \mapsto \epsilon(h)\, c$ given by the counit of H. The associator in ${}^{H}\mathscr{M}$ is the natural isomorphism

$$\Phi : \otimes \circ \big(\otimes \times \mathrm{id}_{{}^{H}\mathscr{M}} \big) \Longrightarrow \otimes \circ \big(\mathrm{id}_{{}^{H}\mathscr{M}} \times \otimes \big) \ , \tag{2.19}$$

whose components are given in terms of the associator $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)} \in H \otimes H \otimes H$ of H by

$$\begin{aligned} \Phi : (V \otimes W) \otimes X &\longrightarrow V \otimes (W \otimes X) \ , \\ (v \otimes w) \otimes x &\longmapsto (\phi^{(1)} \triangleright v) \otimes \big((\phi^{(2)} \triangleright w) \otimes (\phi^{(3)} \triangleright x) \big) \ . \end{aligned} \tag{2.20}$$

The unitors in ${}^{H}\mathscr{M}$ are the natural isomorphisms

$$\lambda : I \otimes - \Longrightarrow \mathrm{id}_{{}^{H}\mathscr{M}} \qquad \text{and} \qquad \rho : - \otimes I \Longrightarrow \mathrm{id}_{{}^{H}\mathscr{M}} \ , \tag{2.21}$$

whose components are $\lambda : I \otimes V \to V\ ,\ c \otimes v \mapsto c\,v$ and $\rho : V \otimes I \to V\ ,\ v \otimes c \mapsto c\,v$.

The monoidal category ${}^{H}\mathscr{M}$ is also braided. The braiding is the natural isomorphism $\tau : \otimes \Rightarrow \otimes^{\mathrm{op}}$ with components given in terms of the universal R-matrix $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ of H by

$$\tau : V \otimes W \longrightarrow W \otimes V \ , \qquad v \otimes w \longmapsto (-1)^{|v|\,|w|} \, \big(R^{(2)} \triangleright w \big) \otimes \big(R^{(1)} \triangleright v \big) \ , \tag{2.22}$$

for all homogeneous $v \in V$ and $w \in W$.

Finally, ${}^H\mathscr{M}$ is a braided closed monoidal category with internal hom-functor

$$\hom : \big({}^H\mathscr{M}\big)^{\mathrm{op}} \times {}^H\mathscr{M} \longrightarrow {}^H\mathscr{M} \ . \tag{2.23}$$

For any object (V,W) in $({}^H\mathscr{M})^{\mathrm{op}} \times {}^H\mathscr{M}$ the internal hom-object $\hom(V,W) := (\underline{\hom}(\underline{V},\underline{W}), \triangleright)$ in ${}^H\mathscr{M}$ is given by the $\mathbb{Z}$-graded k-module (2.10) equipped with the left adjoint H-action

$$\begin{aligned} \triangleright : H \otimes \underline{\hom}\big(\underline{V},\underline{W}\big) &\longrightarrow \underline{\hom}\big(\underline{V},\underline{W}\big) \ , \\ h \otimes L &\longmapsto \big(h_{(1)} \triangleright \cdot\big) \circ L \circ \big(S(h_{(2)}) \triangleright \cdot\big) \ , \end{aligned} \tag{2.24}$$

where S is part of the quasi-antipode (S,α,β) of H. To any $({}^H\mathscr{M})^{\mathrm{op}} \times {}^H\mathscr{M}$-morphism $(f^{\mathrm{op}} : V \to V', g : W \to W')$ the internal hom-functor assigns the ${}^H\mathscr{M}$-morphism induced by (2.11). The natural currying isomorphism

$$\zeta : \mathrm{Hom}_{{}^H\mathscr{M}}(-\otimes -,-) \Longrightarrow \mathrm{Hom}_{{}^H\mathscr{M}}(-,\hom(-,-)) \tag{2.25}$$

has components given by

$$\begin{aligned} \zeta(f) : V &\longrightarrow \hom(W,X) \ , \\ v &\longmapsto f\Big(\big(\phi^{(-1)} \triangleright v\big) \otimes \big(\big(\phi^{(-2)}\,\beta\, S(\phi^{(-3)})\big) \triangleright (\,\cdot\,)\big)\Big) \ , \end{aligned} \tag{2.26}$$

for all ${}^H\mathscr{M}$-morphisms $f : V \otimes W \to X$, where $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)} \in H \otimes H \otimes H$ denotes the inverse associator of H. The natural inverse currying isomorphism has components given by

$$\begin{aligned} \zeta^{-1}(g) : V \otimes W &\longrightarrow X \ , \\ v \otimes w &\longmapsto \phi^{(1)} \triangleright \Big(g(v)\big(\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\big) \triangleright w\big)\Big) \ , \end{aligned} \tag{2.27}$$

for all ${}^H\mathscr{M}$-morphisms $g : V \to \hom(W,X)$.

2.3 Internal evaluation, composition and tensor product

In any closed monoidal category there are evaluation and composition morphisms for its internal homomorphisms. If the category is in addition braided then there are also tensor product morphisms for its internal homomorphisms. These morphisms are derived using the currying bijection, see e.g. [Maj95, Proposition 9.3.13] and [BSS15], and they induce important structures on the internal homomorphisms that make them look like morphisms.

We shall denote the internal evaluation ${}^H\mathscr{M}$-morphisms by

$$\mathrm{ev} : \hom\big(V,W\big) \otimes V \longrightarrow W \ , \tag{2.28}$$

the internal composition ${}^H\mathscr{M}$-morphisms by

$$\bullet : \hom\big(W,X\big) \otimes \hom\big(V,W\big) \longrightarrow \hom\big(V,X\big) \ , \tag{2.29}$$

and the internal tensor product ${}^H\mathscr{M}$-morphisms by

$$\circledast : \hom\big(V,W\big) \otimes \hom\big(X,Y\big) \longrightarrow \hom\big(V \otimes X, W \otimes Y\big) \ . \tag{2.30}$$

For the explicit forms of these morphisms in terms of the currying map see e.g. [BSS15, Propositions 2.11 and 5.5]. The next three lemmas summarize important properties of these morphisms, which we shall frequently use throughout this paper.

Lemma 2.1. *Let V, W, X, Y be any four objects in ${}^H\mathscr{M}$. Then*

$$\mathrm{ev}\big((L' \bullet L) \otimes v\big) = \mathrm{ev}\Big((\phi^{(1)} \triangleright L') \otimes \mathrm{ev}\Big((\phi^{(2)} \triangleright L) \otimes (\phi^{(3)} \triangleright v)\Big)\Big) \ , \tag{2.31a}$$

$$\big(L'' \bullet L'\big) \bullet L = (\phi^{(1)} \triangleright L'') \bullet \big((\phi^{(2)} \triangleright L') \bullet (\phi^{(3)} \triangleright L)\big) \ , \tag{2.31b}$$

for all $L \in \mathrm{hom}(V, W)$, $L' \in \mathrm{hom}(W, X)$, $L'' \in \mathrm{hom}(X, Y)$ and $v \in V$.

Proof. The proof follows the same steps as in [BSS15, Proposition 2.12]. □

Lemma 2.2. *Let V, W, X, Y be any four objects in ${}^H\mathscr{M}$. Then*

$$L \circledast L' = \big(L \circledast 1_Y\big) \bullet \big(1_V \circledast L'\big) \ , \tag{2.32a}$$

$$\big(K \bullet L\big) \circledast 1_Y = \big(K \circledast 1_Y\big) \bullet \big(L \circledast 1_Y\big) \ , \tag{2.32b}$$

$$1_Y \circledast \big(K \bullet L\big) = \big(1_Y \circledast K\big) \bullet \big(1_Y \circledast L\big) \ , \tag{2.32c}$$

$$(-1)^{|L|\,|L'|} \, \big(R^{(2)} \triangleright L\big) \circledast \big(R^{(1)} \triangleright L'\big) = \big(1_W \circledast L'\big) \bullet \big(L \circledast 1_X\big) \ , \tag{2.32d}$$

for all $L \in \mathrm{hom}(V, W)$, $K \in \mathrm{hom}(W, X)$ and $L' \in \mathrm{hom}(X, Y)$, where $1_V := (\beta \triangleright \cdot\,) \in \mathrm{hom}(V, V)$, for all objects V in ${}^H\mathscr{M}$, are the unit internal homomorphisms.

Proof. The proof follows the same steps as in [BSS15, Lemmas 5.6 and 5.7]. □

Lemma 2.3. *Let U, V, W, X, Y, Z be any six objects in ${}^H\mathscr{M}$. Then*

$$\Phi \circ \big((L \circledast 1_W) \circledast 1_Y\big) \circ \Phi^{-1} = L \circledast \big(1_W \circledast 1_Y\big) \ , \tag{2.33a}$$

$$\Phi \circ \big((1_U \circledast L') \circledast 1_Y\big) \circ \Phi^{-1} = 1_U \circledast \big(L' \circledast 1_Y\big) \ , \tag{2.33b}$$

$$\Phi \circ \big((1_U \circledast 1_W) \circledast L''\big) \circ \Phi^{-1} = 1_U \circledast \big(1_W \circledast L''\big) \ , \tag{2.33c}$$

for all $L \in \mathrm{hom}(U, V)$, $L' \in \mathrm{hom}(W, X)$ and $L'' \in \mathrm{hom}(Y, Z)$.

Proof. The proof follows the same steps as in [BSS15, Proposition 5.9]. □

Remark 2.4. To simplify notation, in what follows we shall drop the labels on the unit internal homomorphisms and simply write $1 := (\beta \triangleright \cdot\,) \in \mathrm{hom}(V, V)$, for any object V in ${}^H\mathscr{M}$.

Finally, we prove a technical lemma that will be useful for our analysis throughout this paper.

Lemma 2.5. *Let V, W, X be any three objects in ${}^H\mathscr{M}$.*

(i) For any ${}^H\mathscr{M}$-morphism $g : V \to \mathrm{hom}(W, X)$ there is an identity

$$\zeta^{-1}(g) = \mathrm{ev} \circ \big(g \otimes \mathrm{id}\big) : V \otimes W \longrightarrow X \ . \tag{2.34}$$

(ii) Let $f : V \otimes W \to X$ be any ${}^H\mathscr{M}$-morphism. Then $\zeta(f) \circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}) = 0$, for all ${}^H\mathscr{M}$-morphisms $j : U \to V$.

Proof. The proof of item (i) is exactly as in [BSS15, Proposition 2.12 (i)]. To prove item (ii), let us first suppose that $\zeta(f) \circ j = 0$. Then

$$0 = \mathrm{ev} \circ \big((\zeta(f) \circ j) \otimes \mathrm{id}\big) = \mathrm{ev} \circ \big(\zeta(f) \otimes \mathrm{id}\big) \circ (j \otimes \mathrm{id}) = f \circ (j \otimes \mathrm{id}) \ , \tag{2.35}$$

where in the last equality we have used item (i). Let us now assume that $f \circ (j \otimes \mathrm{id}) = 0$. Then

$$\begin{aligned} 0 &= \zeta\big(f \circ (j \otimes \mathrm{id})\big) \\ &= \zeta\big(\mathrm{Hom}_{{}^H\mathscr{M}}(j^{\mathrm{op}} \otimes \mathrm{id}^{\mathrm{op}}, \mathrm{id})\big(f\big)\big) \\ &= \mathrm{Hom}_{{}^H\mathscr{M}}(j^{\mathrm{op}}, \mathrm{hom}(\mathrm{id}^{\mathrm{op}}, \mathrm{id}))\big(\zeta(f)\big) = \zeta(f) \circ j \ , \end{aligned} \tag{2.36}$$

where in the third equality we have used naturality of the currying bijection, see also the proof of [BSS15, Theorem 2.10]. □

2.4 Algebras and bimodules

An algebra in the braided closed monoidal category ${}^H\mathscr{M}$ is an object A in ${}^H\mathscr{M}$ together with two ${}^H\mathscr{M}$-morphisms $\mu : A \otimes A \to A$ (product) and $\eta : I \to A$ (unit), such that (denoting the product by juxtaposition and the unit element by $1 := \eta(1) \in A$)

$$(a\,a'\,)\,a'' = (\phi^{(1)} \triangleright a)\,\big((\phi^{(2)} \triangleright a'\,)\,(\phi^{(3)} \triangleright a''\,)\big) \ , \tag{2.37}$$

for all $a, a', a'' \in A$, and

$$1\,a = a = a\,1 \ , \tag{2.38}$$

for all $a \in A$. To simplify notation, we shall denote an algebra in ${}^H\mathscr{M}$ simply by its underlying bounded $\mathbb{Z}$-graded left H-module A, suppressing the product and unit from the notation. We denote by ${}^H\mathscr{A}$ the category of all algebras in ${}^H\mathscr{M}$; the morphisms in ${}^H\mathscr{A}$ are given by all ${}^H\mathscr{M}$-morphisms $f : A \to B$ that preserve the products and units, i.e. $f(a\,a'\,) = f(a)\,f(a'\,)$, for all $a, a' \in A$, and $f(1) = 1$.

Example 2.6. Let V be any object in ${}^H\mathscr{M}$. Then the internal endomorphism object $\mathrm{end}(V) := \hom(V, V)$, together with the ${}^H\mathscr{M}$-morphisms $\bullet : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$ and $\eta : I \to \mathrm{end}(V)\ ,\ c \mapsto c\,(\beta \triangleright \,\cdot\,)$, is an object in ${}^H\mathscr{A}$. We call this object the algebra of internal endomorphisms of V.

We say that an object A in ${}^H\mathscr{A}$ is braided commutative if its product is compatible with the braiding in ${}^H\mathscr{M}$, i.e. $\mu \circ \tau = \mu$ or

$$a\,a' = (-1)^{|a|\,|a'|}\,\big(R^{(2)} \triangleright a'\,\big)\,\big(R^{(1)} \triangleright a\big) \ , \tag{2.39}$$

for all homogeneous $a, a' \in A$. The full subcategory of braided commutative algebras in ${}^H\mathscr{M}$ is denoted by ${}^H\mathscr{A}^{\mathrm{com}}$.

Example 2.7. Classical examples of braided commutative algebras are given by function algebras $C^\infty(M)$ on G-manifolds M, where G is a Lie group with Lie algebra $\mathfrak{g}$, see [BSS15, Section 6]. The relevant quasitriangular quasi-Hopf algebra in this case is the universal enveloping Hopf algebra $U\mathfrak{g}$ (with trivial R-matrix and associator). These examples are braided commutative algebras concentrated in $\mathbb{Z}$-degree 0. Similarly to [BSS15, Section 6], one can show that the exterior algebras of differential forms $\Omega^\sharp(M)$ on G-manifolds are braided commutative algebras according to our definition. These algebras are now nontrivially $\mathbb{Z}$-graded and they satisfy our assumption of boundedness of $\mathbb{Z}$-graded k-modules for all finite-dimensional manifolds. These examples and cochain twist deformations thereof (cf. [BSS15, Section 6]) are our main examples of interest.

Let A be any object in ${}^H\mathscr{A}$. An A-bimodule in ${}^H\mathscr{M}$ is an object V in ${}^H\mathscr{M}$ together with two ${}^H\mathscr{M}$-morphisms $l : A \otimes V \to V$ (left A-action) and $r : V \otimes A \to V$ (right A-action), such that (denoting also the A-actions by juxtaposition)

$$(v\,a)\,a' = (\phi^{(1)} \triangleright v)\,\big((\phi^{(2)} \triangleright a)\,(\phi^{(3)} \triangleright a'\,)\big) \ , \tag{2.40a}$$

$$a\,(a'\,v) = \big((\phi^{(-1)} \triangleright a)\,(\phi^{(-2)} \triangleright a'\,)\big)\,(\phi^{(-3)} \triangleright v) \ , \tag{2.40b}$$

$$a\,(v\,a'\,) = \big((\phi^{(-1)} \triangleright a)\,(\phi^{(-2)} \triangleright v)\big)\,(\phi^{(-3)} \triangleright a'\,) \ , \tag{2.40c}$$

for all $a, a' \in A$ and $v \in V$, and

$$1\,v = v = v\,1 \ , \tag{2.41}$$

for all $v \in V$. To simplify notation, we shall denote an A-bimodule in ${}^H\mathscr{M}$ simply by its underlying bounded $\mathbb{Z}$-graded left H-module V, suppressing the A-actions from the notation. We denote by ${}^H{}_A\mathscr{M}_A$ the category of all A-bimodules in ${}^H\mathscr{M}$; the morphisms in ${}^H{}_A\mathscr{M}_A$ are given by all ${}^H\mathscr{M}$-morphisms $f : V \to W$ that preserve the A-actions, i.e. $f(v\,a) = f(v)\,a$ and $f(a\,v) = a\,f(v)$, for all $v \in V$ and $a \in A$. If A is an object in ${}^H\mathscr{A}^{\mathrm{com}}$, we may demand that the left and right A-actions in an A-bimodule V are compatible with the braiding in ${}^H\mathscr{M}$, i.e. $r \circ \tau = l$ and $l \circ \tau = r$, or

$$a\,v = (-1)^{|a|\,|v|}\,\big(R^{(2)} \triangleright v\big)\,\big(R^{(1)} \triangleright a\big)\ , \tag{2.42a}$$

$$v\,a = (-1)^{|a|\,|v|}\,\big(R^{(2)} \triangleright a\big)\,\big(R^{(1)} \triangleright v\big)\ , \tag{2.42b}$$

for all homogeneous $a \in A$ and $v \in V$. We shall call such A-bimodules symmetric and denote the full subcategory of all symmetric A-bimodules in ${}^H\mathscr{M}$ by ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$.

Example 2.8. Following [BSS15, Section 6] and Example 2.7, the $C^\infty(M)$-bimodules of sections $\Gamma^\infty(E)$ of G-equivariant vector bundles $E \to M$ over G-manifolds M are symmetric $C^\infty(M)$-bimodules concentrated in $\mathbb{Z}$-degree 0. Similarly, the $\Omega^\sharp(M)$-bimodules of E-valued differential forms $\Omega^\sharp(M,E)$ are symmetric $\Omega^\sharp(M)$-bimodules that are now nontrivially $\mathbb{Z}$-graded. These examples and cochain twist deformations thereof (cf. [BSS15, Section 6]) are our main examples of interest.

The category ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ is a braided monoidal category. The monoidal functor is denoted $\otimes_A : {}^H{}_A\mathscr{M}_A^{\mathrm{sym}} \times {}^H{}_A\mathscr{M}_A^{\mathrm{sym}} \to {}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ and it assigns to any two objects V, W in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ the object

$$V \otimes_A W = \frac{V \otimes W}{\mathrm{Im}\big(r \otimes \mathrm{id} - (\mathrm{id} \otimes l) \circ \Phi\big)} \tag{2.43}$$

in ${}^H\mathscr{M}$, together with left and right A-actions given by the ${}^H\mathscr{M}$-morphisms

$$\begin{aligned} l : A \otimes \big(V \otimes_A W\big) &\longrightarrow V \otimes_A W\ ,\\ a \otimes \big(v \otimes_A w\big) &\longmapsto a\,(v \otimes_A w) := \big((\phi^{(-1)} \triangleright a)\,(\phi^{(-2)} \triangleright v)\big) \otimes_A (\phi^{(-3)} \triangleright w)\ , \end{aligned} \tag{2.44a}$$

and

$$\begin{aligned} r : \big(V \otimes_A W\big) \otimes A &\longrightarrow V \otimes_A W\ ,\\ \big(v \otimes_A w\big) \otimes a &\longmapsto (v \otimes_A w)\,a := (\phi^{(1)} \triangleright v) \otimes_A \big((\phi^{(2)} \triangleright w)\,(\phi^{(3)} \triangleright a)\big)\ . \end{aligned} \tag{2.44b}$$

To any ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}} \times {}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$-morphism $\big(f : V \to X, g : W \to Y\big)$ the monoidal functor assigns the ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$-morphism

$$f \otimes_A g : V \otimes_A W \longrightarrow X \otimes_A Y\ , \qquad v \otimes_A w \longmapsto f(v) \otimes_A g(w)\ . \tag{2.45}$$

The unit object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ is A itself with left and right A-actions given by the product in A. The unitors in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ are the natural isomorphisms $\rho^A : - \otimes_A A \Rightarrow \mathrm{id}_{{}^H{}_A\mathscr{M}_A^{\mathrm{sym}}}$ and $\lambda^A : A \otimes_A - \Rightarrow \mathrm{id}_{{}^H{}_A\mathscr{M}_A^{\mathrm{sym}}}$ with components given by $\lambda^A : A \otimes_A V \to V\,,\ a \otimes_A v \mapsto a\,v$ and $\rho^A : V \otimes_A A \to V\,,\ v \otimes_A a \mapsto v\,a$. The associator is the natural isomorphism $\Phi^A : \otimes_A \circ \big(\otimes_A \times \mathrm{id}_{{}^H{}_A\mathscr{M}_A^{\mathrm{sym}}}\big) \Rightarrow \otimes_A \circ \big(\mathrm{id}_{{}^H{}_A\mathscr{M}_A^{\mathrm{sym}}} \times \otimes_A\big)$ whose components are given by

$$\begin{aligned} \Phi^A : (V \otimes_A W) \otimes_A X &\longrightarrow V \otimes_A (W \otimes_A X)\ ,\\ (v \otimes_A w) \otimes_A x &\longmapsto (\phi^{(1)} \triangleright v) \otimes_A \big((\phi^{(2)} \triangleright w) \otimes_A (\phi^{(3)} \triangleright x)\big)\ . \end{aligned} \tag{2.46}$$

Finally, the braiding in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ is the natural isomorphism $\tau^A : \otimes_A \Rightarrow \otimes_A^{\mathrm{op}}$ with components given by

$$\tau^A : V \otimes_A W \longrightarrow W \otimes_A V\ , \qquad v \otimes_A w \longmapsto (-1)^{|v|\,|w|}\,\big(R^{(2)} \triangleright w\big) \otimes_A \big(R^{(1)} \triangleright v\big)\ , \tag{2.47}$$

for all homogeneous $v \in V$ and $w \in W$. The braided monoidal category ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ is also closed; we shall give an explicit description of the internal hom-functor in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ in Section 3.4.

3 Derivations and differential operators

In the remainder of this paper we shall systematically build up notions of differential geometry internal to the bounded $\mathbb{Z}$-graded representation category ${}^H\mathscr{M}$ of a quasitriangular quasi-Hopf algebra H. In this section we shall address the notions of derivations, differential operators and differential calculi. We describe derivations and differential operators as subobjects of the internal endomorphisms in ${}^H\mathscr{M}$ by expressing the algebraic properties which characterize them in terms of universal categorical constructions. See [Mac98] for an introduction to the notions of limits and colimits in a category that we use below.

3.1 Internal commutators

Recalling Example 2.6, for any object V in ${}^H\mathscr{M}$ there exists an algebra in ${}^H\mathscr{M}$ given by the internal endomorphisms $\mathrm{end}(V)$ with product the internal composition $\bullet$ and unit element $1 := (\beta \triangleright \cdot\,)$.

Definition 3.1. The internal commutator in the algebra of internal endomorphisms $\mathrm{end}(V)$ is the ${}^H\mathscr{M}$-morphism

$$[\,\cdot\,,\,\cdot\,] : \mathrm{end}(V) \otimes \mathrm{end}(V) \longrightarrow \mathrm{end}(V)\ , \qquad L \otimes L' \longmapsto (\bullet - \bullet \circ \tau)(L \otimes L'\,)\ . \tag{3.1}$$

Proposition 3.2. *The internal commutator in* $\mathrm{end}(V)$ *satisfies the following properties:*

(i) If H is triangular, i.e. its R-matrix satisfies $R = R_{21}^{-1}$, then $[\,\cdot\,,\,\cdot\,]$ is braided antisymmetric, i.e.

$$[\,\cdot\,,\,\cdot\,] = -[\,\cdot\,,\,\cdot\,] \circ \tau\ , \tag{3.2}$$

or

$$\big[L, L'\,\big] = -(-1)^{|L|\,|L'|}\, \big[R^{(2)} \triangleright L', R^{(1)} \triangleright L\big]\ , \tag{3.3}$$

for all homogeneous $L, L' \in \mathrm{end}(V)$.

(ii) If H is triangular, then $[\,\cdot\,,\,\cdot\,]$ satisfies the braided Jacobi identity $\mathrm{Jac} = 0$, with Jacobiator given by the ${}^H\mathscr{M}$-morphism $\mathrm{Jac} : (\mathrm{end}(V) \otimes \mathrm{end}(V)) \otimes \mathrm{end}(V) \longrightarrow \mathrm{end}(V)$ defined as

$$\mathrm{Jac} := [\,\cdot\,,\,\cdot\,] \circ \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \big(((\mathrm{id} \otimes \mathrm{id}) \otimes \mathrm{id}) + (\tau \circ \Phi) + (\Phi^{-1} \circ \tau)\big)\ , \tag{3.4}$$

or

$$\begin{aligned} 0 = \big[\big[L, L'\,\big], L''\,\big] & \\ + (-1)^{|L|\,(|L'|+|L''|)}\, & \big[\big[R^{(2)}_{(1)}\, \phi^{(2)} \triangleright L', R^{(2)}_{(2)}\, \phi^{(3)} \triangleright L''\,\big], R^{(1)}\, \phi^{(1)} \triangleright L\big] \\ + (-1)^{|L''|\,(|L|+|L'|)}\, & \big[\big[\phi^{(-1)}\, R^{(2)} \triangleright L'', \phi^{(-2)}\, R^{(1)}_{(1)} \triangleright L\big], \phi^{(-3)}\, R^{(1)}_{(2)} \triangleright L'\,\big]\ , \end{aligned} \tag{3.5}$$

for all homogeneous $L, L', L'' \in \mathrm{end}(V)$.

(iii) For generic quasitriangular H, $[\,\cdot\,,\,\cdot\,]$ satisfies the braided derivation property

$$[\,\cdot\,,\,\cdot\,] \circ (\bullet \otimes \mathrm{id}) = \bullet \circ \Big(\big(\mathrm{id} \otimes [\,\cdot\,,\,\cdot\,]\big) + \big([\,\cdot\,,\,\cdot\,] \otimes \mathrm{id}\big) \circ \Phi^{-1} \circ \big(\mathrm{id} \otimes \tau\big)\Big) \circ \Phi\ , \tag{3.6}$$

or

$$\begin{aligned} \big[L \bullet L', L''\,\big] = {} & \big(\phi^{(1)} \triangleright L\big) \bullet \big[\phi^{(2)} \triangleright L', \phi^{(3)} \triangleright L''\,\big] \\ & + (-1)^{|L'|\,|L''|}\, \big[\,\widetilde{\phi}^{(-1)}\, \phi^{(1)} \triangleright L, \widetilde{\phi}^{(-2)}\, R^{(2)}\, \phi^{(3)} \triangleright L''\big] \bullet \big(\widetilde{\phi}^{(-3)}\, R^{(1)}\, \phi^{(2)} \triangleright L'\,\big)\ , \end{aligned} \tag{3.7}$$

for all homogeneous $L, L', L'' \in \mathrm{end}(V)$.

Proof. Item (i) follows from a short calculation

$$[\,\cdot\,,\,\cdot\,] = \bullet - \bullet\circ\tau = -(\bullet\circ\tau - \bullet) = -(\bullet - \bullet\circ\tau^{-1})\circ\tau = -(\bullet - \bullet\circ\tau)\circ\tau = -[\,\cdot\,,\,\cdot\,]\circ\tau\ , \quad (3.8)$$

where in the fourth equality we have used triangularity of the R-matrix which implies $\tau^{-1} = \tau$. The proofs of items (ii) and (iii) involve standard manipulations using the weak associativity of the internal composition (2.31b) and standard properties of the R-matrix (see e.g. [BSS15, Section 5.1]). □

Corollary 3.3. *Let H be a triangular quasi-Hopf algebra and V any object in ${}^H\mathscr{M}$. Then the ${}^H\mathscr{M}$-object given by the internal endomorphisms* $\mathrm{end}(V)$*, together with the internal commutator $[\,\cdot\,,\,\cdot\,]$ given in (3.1), is a Lie algebra in ${}^H\mathscr{M}$.*

3.2 Derivations

We give a description of the derivations on an object A in ${}^H\mathscr{A}^{\mathrm{com}}$ by using universal constructions in the braided closed monoidal category ${}^H\mathscr{M}$ to formalize a suitable version of the Leibniz rule, that is compatible with the structures in ${}^H\mathscr{M}$, in terms of an equalizer. Let us start by noticing that for any object V in ${}^H_{\ A}\mathscr{M}^{\mathrm{sym}}_A$ there is an ${}^H\mathscr{M}$-morphism

$$\widehat{l} := \zeta(l) : A \longrightarrow \mathrm{end}(V)\ , \qquad (3.9)$$

which is obtained by currying the left A-action $l : A\otimes V \to V$. Similarly to [BSS15, Lemma 4.1], one can show that (3.9) is moreover an ${}^H\mathscr{A}$-morphism to the algebra of internal endomorphisms, cf. Example 2.6. In particular, this implies that for any object A in ${}^H\mathscr{A}^{\mathrm{com}}$ there exist two parallel ${}^H\mathscr{M}$-morphisms

$$\mathrm{end}(A)\otimes A \ \underset{\widehat{l}\circ\mathrm{ev}}{\overset{[\,\cdot\,,\,\cdot\,]}{\rightrightarrows}}\ \mathrm{end}(A)\ , \qquad (3.10)$$

where for brevity we denote by $[\,\cdot\,,\,\cdot\,]$ the composition $[\,\cdot\,,\,\cdot\,]\circ\big(\mathrm{id}\otimes\widehat{l}\,\big)$ with $l : A\otimes A \to A$ the left A-action induced by the product in A.

Definition 3.4. Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$. The derivations of A is the object $\mathrm{der}(A)$ in ${}^H\mathscr{M}$ which is defined by the equalizer

$$\mathrm{der}(A) \longrightarrow \mathrm{end}(A) \ \underset{\zeta(\widehat{l}\circ\mathrm{ev})}{\overset{\zeta([\,\cdot\,,\,\cdot\,])}{\rightrightarrows}}\ \mathrm{hom}(A,\mathrm{end}(A)) \qquad (3.11)$$

in ${}^H\mathscr{M}$.

In the category ${}^H\mathscr{M}$ equalizers may be computed by taking the kernel of the difference of the two parallel morphisms. In particular, $\mathrm{der}(A)$ can be represented explicitly as the kernel

$$\mathrm{der}(A) = \mathrm{Ker}\Big(\zeta\big([\,\cdot\,,\,\cdot\,] - \widehat{l}\circ\mathrm{ev}\big)\Big)\ . \qquad (3.12)$$

The following lemma will allow us to establish a relation between our definition of derivations and the standard definition in terms of a Leibniz rule.

Lemma 3.5. *Let A be any object in ${}^H\mathscr{A}^{\mathrm{com}}$. An ${}^H\mathscr{M}$-subobject $U \subseteq \mathrm{end}(A)$ is an ${}^H\mathscr{M}$-subobject of* $\mathrm{der}(A)$ *if and only if*

$$[L,a] = \widehat{l}(\mathrm{ev}(L\otimes a))\ , \qquad (3.13)$$

for all $L\in U$ and $a\in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,] - \widehat{l}\circ \mathrm{ev} : \mathrm{end}(A)\otimes A \to \mathrm{end}(A)$ and $j : U \to \mathrm{end}(A)$ the inclusion ${}^H\mathscr{M}$-morphism, we have to show that $\zeta(f)\circ j = 0$ if and only if $f\circ(j\otimes \mathrm{id}) = 0$. This is a consequence of item (ii) of Lemma 2.5. □

Remark 3.6. We explain how our definition of derivations is related to the standard definition in terms of a Leibniz rule: Let $L \in \mathrm{der}(A)$ be any derivation. Then Lemma 3.5 implies that $[L,a] = \widehat{l}(\mathrm{ev}(L\otimes a))$. Evaluating this equation on some $a' \in A$, we obtain

$$\mathrm{ev}\big([L,a]\otimes a'\,\big) = \mathrm{ev}\big(\,\widehat{l}(\mathrm{ev}(L\otimes a))\otimes a'\,\big)\ . \tag{3.14}$$

Using now the evaluation identity (2.31a) and also item (i) of Lemma 2.5, we can simplify this equation and obtain

$$\begin{aligned}\mathrm{ev}\Big(&(\phi^{(1)}\triangleright L)\otimes\big((\phi^{(2)}\triangleright a)\,(\phi^{(3)}\triangleright a'\,)\big)\Big)\\ &-(-1)^{|L|\,|a|}\,(\phi^{(1)}\,R^{(2)}\triangleright a)\,\mathrm{ev}\Big((\phi^{(2)}\,R^{(1)}\triangleright L)\otimes(\phi^{(3)}\triangleright a')\Big) = \mathrm{ev}\big(L\otimes a\big)\,a'\ ,\end{aligned} \tag{3.15}$$

for all homogeneous $a,a' \in A$ and $L\in \mathrm{der}(A)$. For the special case of trivial R-matrix $R = 1\otimes 1$ and associator $\phi = 1\otimes 1\otimes 1$ the last equation reduces to $L(a\,a'\,) = L(a)\,a' + (-1)^{|L|\,|a|}\,a\,L(a'\,)$, which is exactly the Leibniz rule for a graded derivation. Hence, the equalizer (3.11) provides us with a suitable generalization of the graded Leibniz rule that is consistent with the structures in the braided closed monoidal category ${}^H\mathscr{M}$.

Finally, we prove a structural result for our derivations.

Proposition 3.7. *Let H be a triangular quasi-Hopf algebra and A any object in ${}^H\mathscr{A}^{\mathrm{com}}$. Then the ${}^H\mathscr{M}$-object given by the derivations $\mathrm{der}(A)$, together with the internal commutator $[\,\cdot\,,\,\cdot\,]$ given in (3.1), is a Lie algebra in ${}^H\mathscr{M}$.*

Proof. We already know from Corollary 3.3 that, under our hypotheses, $\mathrm{end}(A)$ together the internal commutator $[\,\cdot\,,\,\cdot\,]$ is a Lie algebra in ${}^H\mathscr{M}$. Moreover, $\mathrm{der}(A)$ is by construction an ${}^H\mathscr{M}$-subobject of $\mathrm{end}(A)$, so it remains to prove that the image of the restricted internal commutator

$$[\,\cdot\,,\,\cdot\,] : \mathrm{der}(A)\otimes\mathrm{der}(A) \longrightarrow \mathrm{end}(A) \tag{3.16}$$

is an ${}^H\mathscr{M}$-subobject of $\mathrm{der}(A)$. Using Lemma 3.5 this is the case if and only if

$$[[L,L'\,],a] = \widehat{l}\big(\mathrm{ev}\big([L,L'\,]\otimes a\big)\big)\ , \tag{3.17}$$

for all $L,L' \in \mathrm{der}(A)$ and $a\in A$. One can now easily show that this equality holds true by using the braided Jacobi identity and antisymmetry (cf. items (ii) and (i) of Proposition 3.2), the derivation property of Lemma 3.5 and finally the evaluation identity (2.31a). For simplifying the resulting expressions one also needs standard R-matrix properties, which are listed in e.g. [BSS15, Section 5.1]. □

3.3 Cochain twisting of derivations

We shall briefly study the deformation of derivations under cochain twisting. For a more complete introduction to these deformation techniques we refer to Part I. Let H be a quasitriangular quasi-Hopf algebra and F a cochain twisting element, i.e. $F\in H\otimes H$ is an invertible element with the normalization $(\epsilon\otimes\mathrm{id})(F) = 1 = (\mathrm{id}\otimes\epsilon)(F)$. Any cochain twisting element defines a braided closed monoidal functor $\mathcal{F} : {}^H\mathscr{M} \to {}^{H_F}\mathscr{M}$, where H_F is the twisted quasitriangular quasi-Hopf algebra of H by F, see e.g. [BSS15, Theorem 5.11]. The functor $\mathcal{F} : {}^H\mathscr{M} \to {}^{H_F}\mathscr{M}$

acts on objects and morphisms as the identity, and the coherence maps for the braided monoidal structures are the ${}^{H_F}\mathscr{M}$-isomorphisms

$$\begin{aligned} \varphi : \mathcal{F}(V) \otimes_F \mathcal{F}(W) &\longrightarrow \mathcal{F}(V \otimes W) \ , \\ v \otimes_F w &\longmapsto \big(F^{(-1)} \triangleright v\big) \otimes \big(F^{(-2)} \triangleright w\big) \ , \end{aligned} \tag{3.18a}$$

where $F^{-1} = F^{(-1)} \otimes F^{(-2)}$ denotes the inverse cochain twisting element, and

$$\psi : I_F \longrightarrow \mathcal{F}(I) \ , \qquad c \longmapsto c \ . \tag{3.18b}$$

The coherence maps for the internal hom-structures are the ${}^{H_F}\mathscr{M}$-isomorphisms

$$\begin{aligned} \gamma : \hom_F\big(\mathcal{F}(V), \mathcal{F}(W)\big) &\longrightarrow \mathcal{F}\big(\hom(V,W)\big) \ , \\ L &\longmapsto \big(F^{(-1)} \triangleright \cdot\,\big) \circ L \circ \big(S\big(F^{(-2)}\big) \triangleright \cdot\,\big) \ . \end{aligned} \tag{3.19}$$

The braided closed monoidal functor $\mathcal{F} : {}^H\mathscr{M} \to {}^{H_F}\mathscr{M}$ induces functors (denoted with abuse of notation by the same symbols) $\mathcal{F} : {}^H\mathscr{A}^{\mathrm{com}} \to {}^{H_F}\mathscr{A}^{\mathrm{com}}$ and $\mathcal{F} : {}^H{}_A\mathscr{M}_A^{\mathrm{sym}} \to {}^{H_F}{}_{\mathcal{F}(A)}\mathscr{M}_{\mathcal{F}(A)}^{\mathrm{sym}}$, which allow us to twist quantize algebras and bimodules in ${}^H\mathscr{M}$ to algebras and bimodules in ${}^{H_F}\mathscr{M}$. Details can be found in [BSS15, Proposition 5.16].

Proposition 3.8. *Let A be any object in ${}^H\mathscr{A}^{\mathrm{com}}$ and let F be any cochain twisting element based on H. Then the coherence map $\gamma : \mathrm{end}_F(\mathcal{F}(A)) \to \mathcal{F}(\mathrm{end}(A))$ restricts to an ${}^{H_F}\mathscr{M}$-isomorphism*

$$\gamma : \mathrm{der}_F(\mathcal{F}(A)) \longrightarrow \mathcal{F}(\mathrm{der}(A)) \ . \tag{3.20}$$

Proof. The braided closed monoidal functor $\mathcal{F} : {}^H\mathscr{M} \to {}^{H_F}\mathscr{M}$ is an equivalence of categories, hence it preserves all limits and colimits. It then follows that $\mathcal{F}(\mathrm{der}(A))$ is the equalizer of the ${}^{H_F}\mathscr{M}$-diagram

$$\xymatrix{ \mathcal{F}(\mathrm{end}(A)) \ar@<0.5ex>[r]^-{\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]))} \ar@<-0.5ex>[r]_-{\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}))} & \mathcal{F}(\hom(A,\mathrm{end}(A))) } \ . \tag{3.21}$$

On the other hand, the object $\mathrm{der}_F(\mathcal{F}(A))$ in ${}^{H_F}\mathscr{M}$ is defined according to Definition 3.4 as the equalizer of the ${}^{H_F}\mathscr{M}$-diagram

$$\xymatrix{ \mathrm{end}_F(\mathcal{F}(A)) \ar@<0.5ex>[r]^-{\zeta_F([\,\cdot\,,\cdot\,]_F)} \ar@<-0.5ex>[r]_-{\zeta_F(\widehat{l_F}\circ\mathrm{ev}_F)} & \hom_F(\mathcal{F}(A),\mathrm{end}_F(\mathcal{F}(A))) } \ . \tag{3.22}$$

A straightforward but slightly lengthy calculation shows that the ${}^{H_F}\mathscr{M}$-diagrams (3.21) and (3.22) are isomorphic: The ${}^{H_F}\mathscr{M}$-diagram

$$\xymatrix{ \mathrm{end}_F(\mathcal{F}(A)) \ar@<0.5ex>[r]^-{\zeta_F([\,\cdot\,,\cdot\,]_F)} \ar@<-0.5ex>[r]_-{\zeta_F(\widehat{l_F}\circ\mathrm{ev}_F)} \ar[dd]_-{\gamma} & \hom_F(\mathcal{F}(A),\mathrm{end}_F(\mathcal{F}(A))) \ar[d]^-{\gamma\circ(\,\cdot\,)} \\ & \hom_F(\mathcal{F}(A),\mathcal{F}(\mathrm{end}(A))) \ar[d]^-{\gamma} \\ \mathcal{F}(\mathrm{end}(A)) \ar@<0.5ex>[r]^-{\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]))} \ar@<-0.5ex>[r]_-{\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}))} & \mathcal{F}(\hom(A,\mathrm{end}(A))) } \tag{3.23}$$

commutes (i.e. the diagram obtained by taking either both upper or lower horizontal arrows commutes) and the vertical arrows are all ${}^{H_F}\mathscr{M}$-isomorphisms. Due to the universality of limits there must be a unique isomorphism between $\mathrm{der}_F(\mathcal{F}(A))$ and $\mathcal{F}(\mathrm{der}(A))$. The assertion now follows from the fact that we describe our derivations as a subobject of the internal endomorphisms (cf. (3.12)) and hence the unique isomorphism between $\mathrm{der}_F(\mathcal{F}(A))$ and $\mathcal{F}(\mathrm{der}(A))$ is the one induced by the isomorphism between $\mathrm{end}_F(\mathcal{F}(A))$ and $\mathcal{F}(\mathrm{end}(A))$, which is precisely γ. □

3.4 Internal homomorphisms

In [BSS15, Section 4] we gave an explicit description of the internal hom-functor $\mathrm{hom}_A : \big({}^H{}_A\mathscr{M}_A\big)^{\mathrm{op}} \times {}^H{}_A\mathscr{M}_A \to {}^H{}_A\mathscr{M}_A$ by imposing a suitable weak right A-linearity condition on the internal hom-functor hom in ${}^H\mathscr{M}$. We shall now give an easier but equivalent construction of hom_A for the case where A is an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and we restrict ourselves to the full subcategory ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ of symmetric A-bimodules in ${}^H\mathscr{M}$. This construction involves a generalization of the internal commutator $[\,\cdot\,,\,\cdot\,]$ from Definition 3.1, and it will allow us later on to interpret the internal hom-objects $\mathrm{hom}_A(V,W)$ as zeroth order differential operators.

Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and let V,W be two objects in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. We define an ${}^H\mathscr{M}$-morphism (denoted with abuse of notation by the same symbol as the internal commutator)

$$[\,\cdot\,,\,\cdot\,] := \bullet\circ\big(\mathrm{id}\otimes\widehat{l}\,\big) - \bullet\circ\big(\widehat{l}\otimes\mathrm{id}\big)\circ\tau : \mathrm{hom}(V,W)\otimes A \longrightarrow \mathrm{hom}(V,W)~, \tag{3.24}$$

where $\widehat{l}$ was defined in (3.9). Then

$$[L,a] = L\bullet\widehat{l}(a) - (-1)^{|L|\,|a|}\;\widehat{l}\big(R^{(2)}\triangleright a\big)\bullet\big(R^{(1)}\triangleright L\big)~, \tag{3.25}$$

for all homogeneous $L\in\mathrm{hom}(V,W)$ and $a\in A$.

Definition 3.9. The object $\mathrm{hom}_A(V,W)$ in ${}^H\mathscr{M}$ is defined by the equalizer

$$\xymatrix{\mathrm{hom}_A(V,W) \ar[r] & \mathrm{hom}(V,W) \ar@<0.5ex>[r]^-{\zeta([\,\cdot\,,\,\cdot\,])} \ar@<-0.5ex>[r]_-{0} & \mathrm{hom}(A,\mathrm{hom}(V,W))} \tag{3.26}$$

in ${}^H\mathscr{M}$. This equalizer can be realized explicitly in terms of the ${}^H\mathscr{M}$-subobject

$$\mathrm{hom}_A(V,W) = \mathrm{Ker}\big(\zeta([\,\cdot\,,\,\cdot\,])\big) \subseteq \mathrm{hom}(V,W) \tag{3.27}$$

of the internal hom-object $\mathrm{hom}(V,W)$ in ${}^H\mathscr{M}$.

Lemma 3.10. *Let A be any object in ${}^H\mathscr{A}^{\mathrm{com}}$ and let V,W be any two objects in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. An ${}^H\mathscr{M}$-subobject $U\subseteq\mathrm{hom}(V,W)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{hom}_A(V,W)$ if and only if*

$$[L,a] = 0~, \tag{3.28}$$

for all $L\in U$ and $a\in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,] : \mathrm{hom}(V,W)\otimes A\to\mathrm{hom}(V,W)$ and $j : U\to\mathrm{hom}(V,W)$ the inclusion ${}^H\mathscr{M}$-morphism, we have to show that $\zeta(f)\circ j = 0$ if and only if $f\circ(j\otimes\mathrm{id}) = 0$. This is a consequence of item (ii) of Lemma 2.5. □

The object $\mathrm{hom}_A(V,W)$ in ${}^H\mathscr{M}$ given by (3.27) carries a natural left and right A-action given by the ${}^H\mathscr{M}$-morphisms

$$l := \bullet\circ\big(\widehat{l}\otimes\mathrm{id}\big) : A\otimes\mathrm{hom}_A(V,W) \longrightarrow \mathrm{hom}_A(V,W)~, \tag{3.29a}$$

$$r := \bullet\circ\big(\mathrm{id}\otimes\widehat{l}\,\big) : \mathrm{hom}_A(V,W)\otimes A \longrightarrow \mathrm{hom}_A(V,W)~. \tag{3.29b}$$

It is moreover an object in ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$ because the result of Lemma 3.10 is precisely the symmetry condition for the left and right A-action given in (3.29) (see also (3.24)). The assignment of these objects $\hom_A(V,W)$ in ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$ is functorial and we denote the corresponding functor by

$$\hom_A : \big({}^H_A\mathscr{M}_A^{\mathrm{sym}}\big)^{\mathrm{op}} \times {}^H_A\mathscr{M}_A^{\mathrm{sym}} \longrightarrow {}^H_A\mathscr{M}_A^{\mathrm{sym}} ~. \tag{3.30}$$

To any $\big({}^H_A\mathscr{M}_A^{\mathrm{sym}}\big)^{\mathrm{op}} \times {}^H_A\mathscr{M}_A^{\mathrm{sym}}$-morphism $(f^{\mathrm{op}} : V \to V',~ g : W \to W'\,)$ this functor assigns

$$\hom_A(f^{\mathrm{op}},g) : \hom_A(V,W) \longrightarrow \hom_A(V',W'\,) ~,\qquad L \longmapsto g\circ L\circ f ~. \tag{3.31}$$

Finally, we show that (3.30) is an internal hom-functor in ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$.

Proposition 3.11. *The braided monoidal category ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$ is closed: There is a natural bijection $\zeta^A : \mathrm{Hom}_{{}^H_A\mathscr{M}_A^{\mathrm{sym}}}(-\otimes_A -,-) \Rightarrow \mathrm{Hom}_{{}^H_A\mathscr{M}_A^{\mathrm{sym}}}(-,\hom_A(-,-))$ with components given by*

$$\begin{aligned}\zeta^A(f) : V &\longrightarrow \hom_A(W,X) ~,\\ v &\longmapsto f\Big(\big(\phi^{(-1)}\triangleright v\big)\otimes_A \big(\big(\phi^{(-2)}\,\beta\, S(\phi^{(-3)})\big)\triangleright (\,\cdot\,)\big)\Big) ~,\end{aligned} \tag{3.32}$$

for all ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$-morphisms $f : V\otimes_A W \to X$. The components of its inverse are

$$\begin{aligned}(\zeta^A)^{-1}(g) : V\otimes_A W &\longrightarrow X ~,\\ v\otimes_A w &\longmapsto \phi^{(1)}\triangleright\Big(g(v)\big(\big(S(\phi^{(2)})\,\alpha\,\phi^{(3)}\big)\triangleright w\big)\Big) ~,\end{aligned} \tag{3.33}$$

for all ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$-morphisms $g : V\to \hom_A(W,X)$.

Proof. With a proof analogous to [BSS15, Lemma 4.2] one shows that (3.32) and (3.33) are the components of a natural bijection between the functors $\mathrm{Hom}_{{}^H_A\mathscr{M}}(-\otimes_A -,-)$ and $\mathrm{Hom}_{{}^H_A\mathscr{M}_A}(-,\hom(-,-))$. It thus remains to prove that (1) the image of $\zeta^A(f)$ is contained in $\hom_A(W,X)$ for all ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$-morphisms $f : V\otimes_A W\to X$, and that (2) $(\zeta^A)^{-1}(g)$ is a right A-linear map for all ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$-morphisms $g : V\to \hom_A(X,Y)$.

Due to Lemma 3.10, point (1) is shown by the calculation

$$\begin{aligned}\big(\zeta^A(f)(v)\big)\,a &= \zeta^A(f)(v\,a)\\ &= (-1)^{|a|\,|v|}~\zeta^A(f)\big((R^{(2)}\triangleright a)\,(R^{(1)}\triangleright v)\big)\\ &= (-1)^{|a|\,|v|}~\big(R^{(2)}\triangleright a\big)\,\big(R^{(1)}\triangleright \zeta^A(f)(v)\big) ~,\end{aligned} \tag{3.34}$$

for all homogeneous $a\in A$ and $v\in V$. In the first equality we have used the right A-linearity of $\zeta^A(f)$, in the second equality the symmetry of the A-bimodule V, and in the last equality the left A-linearity and H-equivariance of $\zeta^A(f)$.

Point (2) is likewise shown by a short calculation

$$\begin{aligned}(\zeta^A)^{-1}(g)\big((v\otimes_A w)\,a\big) &= (\zeta^A)^{-1}(g)\big((\phi^{(1)}\triangleright v)\otimes_A \big((\phi^{(2)}\triangleright w)\,(\phi^{(3)}\triangleright a)\big)\big)\\ &= \mathrm{ev}\big(g(\phi^{(1)}\triangleright v)\otimes_A \big((\phi^{(2)}\triangleright w)\,(\phi^{(3)}\triangleright a)\big)\big)\\ &= \mathrm{ev}\big((g(v)\otimes_A w)\,a\big)\\ &= (-1)^{|a|\,(|v|+|w|)}~\big(R^{(2)}\triangleright a\big)\,\big(R^{(1)}\triangleright \mathrm{ev}\big(g(v)\otimes_A w\big)\big)\\ &= \big((\zeta^A)^{-1}(g)(v\otimes_A w)\big)\,a ~,\end{aligned} \tag{3.35}$$

for all homogeneous $a\in A$, $v\in V$ and $w\in W$. The second equality holds by direct inspection (see also Lemma 2.5 (i) for a similar statement) and in the fourth equality we have used the symmetry of the A-bimodules W and $\hom_A(V,W)$ as well as the H-equivariance of ev. The last equality uses the symmetry of the A-bimodule X. □

3.5 Differential operators and calculi

Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and V any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. We define the internal multi-commutator of order $n \in \mathbb{Z}_{>0}$ to be the ${}^H\mathscr{M}$-morphism

$$[\,\cdot\,,\,\cdot\,]^{(n)} : \big(\cdots\big((\mathrm{end}(V)\otimes A)\otimes A\big)\cdots\big)\otimes A \longrightarrow \mathrm{end}(V) \ , \tag{3.36a}$$

where the source contains n factors of A, given by the composition

$$[\,\cdot\,,\,\cdot\,]^{(n)} := [\,\cdot\,,\,\cdot\,]\circ\big([\,\cdot\,,\,\cdot\,]\otimes\mathrm{id}\big)\circ\cdots\circ\big((\cdots(([\,\cdot\,,\,\cdot\,]\otimes\mathrm{id})\otimes\mathrm{id})\cdots)\otimes\mathrm{id}\big) \ . \tag{3.36b}$$

We have suppressed as before the precomposition of the internal multi-commutator with $(\cdots((\mathrm{id}\otimes\widehat{l}\,)\otimes\widehat{l}\,)\cdots)\otimes\widehat{l}$, where $\widehat{l}$ is the ${}^H\mathscr{A}$-morphism given in (3.9). We further denote by $\Phi^{(-n)}$ the combination of associators required to re-bracket the expressions

$$\mathrm{end}(V)\otimes\big(A\otimes(A\otimes(\cdots(A\otimes A)\cdots))\big)\xrightarrow{\ \Phi^{(-n)}\ }\big(\cdots\big((\mathrm{end}(V)\otimes A)\otimes A\big)\cdots\big)\otimes A \ , \tag{3.37}$$

where again the source and target contain n factors of A. We shall denote the source of this ${}^H\mathscr{M}$-isomorphism also by $\mathrm{end}(V)\otimes A^{\otimes n}$.

Definition 3.12. Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and V any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. The differential operators of order $n\in\mathbb{Z}_{\geq 0}$ of V is the object $\mathrm{diff}^n(V)$ in ${}^H\mathscr{M}$ which is defined by the equalizer

$$\mathrm{diff}^n(V)\longrightarrow\mathrm{end}(V)\ \underset{0}{\overset{\zeta([\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\Phi^{(-(n+1))})}{\rightrightarrows}}\ \mathrm{hom}(A^{\otimes n},\mathrm{end}(V)) \tag{3.38}$$

in ${}^H\mathscr{M}$. This equalizer can be realized explicitly in terms of the ${}^H\mathscr{M}$-subobject

$$\mathrm{diff}^n(V) = \mathrm{Ker}\big(\zeta\big([\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\Phi^{(-(n+1))}\big)\big) \tag{3.39}$$

of the internal endomorphism object $\mathrm{end}(V)$ in ${}^H\mathscr{M}$.

Remark 3.13. Comparing Definitions 3.12 and 3.9 we observe that the order 0 differential operators $\mathrm{diff}^0(V)$ are the internal endomorphisms $\mathrm{end}_A(V)$ in the category ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$.

Lemma 3.14. *Let A be any object in ${}^H\mathscr{A}^{\mathrm{com}}$ and let V be any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. An ${}^H\mathscr{M}$-subobject $U\subseteq\mathrm{end}(V)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{diff}^n(V)$ if and only if*

$$\big[\big[\cdots\big[[L,a_1],a_2\big],\cdots\big],a_{n+1}\big] = 0 \ , \tag{3.40}$$

for all $L\in U$ and $a_1,a_2,\dots,a_{n+1}\in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\Phi^{(-(n+1))} : \mathrm{end}(V)\otimes A^{\otimes n}\to\mathrm{end}(V)$ and $j : U\to\mathrm{end}(V)$ the inclusion ${}^H\mathscr{M}$-morphism, it follows from Lemma 2.5 (ii) that $\zeta(f)\circ j = 0$ if and only if $f\circ(j\otimes\mathrm{id}) = 0$. The latter condition is equivalent to $[\,\cdot\,,\,\cdot\,]^{(n+1)}\circ\big((\cdots((j\otimes\mathrm{id})\otimes\mathrm{id})\cdots)\otimes\mathrm{id}\big)\circ\Phi^{(-(n+1))} = 0$, and the assertion now follows because $\Phi^{(-(n+1))}$ is an isomorphism. □

There is an ${}^H\mathscr{M}$-subobject relation $\mathrm{diff}^n(V)\subseteq\mathrm{diff}^m(V)$ for all $n\leq m$, which immediately follows from Lemma 3.14 and (3.39). These subobject relations give rise to the sequence of ${}^H\mathscr{M}$-monomorphisms

$$\mathrm{diff}^0(V)\longrightarrow\mathrm{diff}^1(V)\longrightarrow\mathrm{diff}^2(V)\longrightarrow\ \cdots\ \longrightarrow\mathrm{diff}^n(V)\longrightarrow\ \cdots\ . \tag{3.41}$$

We shall now show that differential operators can be composed with respect to the internal composition.

Proposition 3.15. *The internal composition* $\bullet : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$ *restricts to an* ${}^H\mathscr{M}$*-morphism*

$$\bullet : \mathrm{diff}^n(V) \otimes \mathrm{diff}^m(V) \longrightarrow \mathrm{diff}^{n+m}(V) \ , \tag{3.42}$$

for all $n, m \in \mathbb{Z}_{\geq 0}$.

Proof. Restricting $\bullet : \mathrm{end}(V) \otimes \mathrm{end}(V) \to \mathrm{end}(V)$ to the corresponding ${}^H\mathscr{M}$-subobjects of differential operators yields an ${}^H\mathscr{M}$-morphism $\bullet : \mathrm{diff}^n(V) \otimes \mathrm{diff}^m(V) \to \mathrm{end}(V)$ and we have to prove that its image lies in $\mathrm{diff}^{n+m}(V)$. As the image of this ${}^H\mathscr{M}$-morphism is an ${}^H\mathscr{M}$-subobject of $\mathrm{end}(V)$, by Lemma 3.14 it is enough to show that

$$\big[\big[\cdots\big[[L \bullet L', a_1], a_2\big], \cdots\big], a_{n+m+1}\big] = 0 \ , \tag{3.43}$$

for all $L \in \mathrm{diff}^n(V)$, $L' \in \mathrm{diff}^m(V)$ and $a_1, a_2, \dots, a_{n+m+1} \in A$. This equality follows by iteratively using the derivation property of the internal commutator, cf. item (iii) of Proposition 3.2, and applying Lemma 3.14 to L and L'. □

Forming the colimit in ${}^H\mathscr{M}$ of the diagram given in (3.41) we can define the object $\mathrm{diff}(V)$ of differential operators on V. This colimit can be represented explicitly as the union of differential operators of all orders $n \in \mathbb{Z}_{\geq 0}$, i.e.

$$\mathrm{diff}(V) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathrm{diff}^n(V) \subseteq \mathrm{end}(V) \ . \tag{3.44}$$

Corollary 3.16. *The differential operators* $\mathrm{diff}(V)$ *is an* ${}^H\mathscr{A}$*-subobject of the algebra of internal endomorphisms* $\mathrm{end}(V)$ *(cf. Example 2.6).*

Proof. By Proposition 3.15 the internal composition closes on $\mathrm{diff}(V)$, i.e. there is an ${}^H\mathscr{M}$-morphism

$$\bullet : \mathrm{diff}(V) \otimes \mathrm{diff}(V) \longrightarrow \mathrm{diff}(V) \ . \tag{3.45}$$

The unit $\eta : I \to \mathrm{end}(V)$ has its image in the degree 0 differential operators because of the calculation

$$[c\,1, a] = c\,1 \bullet \widehat{l}(a) - c\,\widehat{l}\big(R^{(2)} \triangleright a\big) \bullet \big(R^{(1)} \triangleright 1\big) = c\,\widehat{l}(a) - c\,\widehat{l}\big(R^{(2)}\, \epsilon(R^{(1)}) \triangleright a\big) = 0 \tag{3.46}$$

and Lemma 3.14; here we used the normalization $(\epsilon \otimes \mathrm{id})(R) = 1$ of the R-matrix. □

Remark 3.17. Combining Lemmas 3.5 and 3.14 we see that for any object A in ${}^H\mathscr{A}^{\mathrm{com}}$, $\mathrm{der}(A) \subseteq \mathrm{diff}^1(A)$ is an ${}^H\mathscr{M}$-subobject, i.e. the derivations of A are differential operators of order 1.

With the techniques developed above we can now introduce the notion of a differential calculus in ${}^H\mathscr{M}$. In the following we shall denote by $I[1]$ the object in ${}^H\mathscr{M}$ which is obtained by shifting the unit object $I = (k, \triangleright)$ in $\mathbb{Z}$-degree by 1: $I[1]_1 = k$ and $I[1]_n = 0$, for all $n \neq 1$.

Definition 3.18. Let H be a quasitriangular quasi-Hopf algebra. A differential calculus (A, d) in ${}^H\mathscr{M}$ is an object A in ${}^H\mathscr{A}^{\mathrm{com}}$ together with an ${}^H\mathscr{M}$-morphism $\mathrm{d} : I[1] \to \mathrm{der}(A)$ which is nilpotent in the sense that the composition of ${}^H\mathscr{M}$-morphisms

$$I[1] \otimes I[1] \xrightarrow{\ \mathrm{d}\otimes\mathrm{d}\ } \mathrm{der}(A) \otimes \mathrm{der}(A) \longrightarrow \mathrm{diff}(A) \otimes \mathrm{diff}(A) \xrightarrow{\ \bullet\ } \mathrm{diff}(A) \tag{3.47}$$

is 0; here the second arrow is defined using Remark 3.17.

Remark 3.19. Given a differential calculus (A, d) in ${}^H\mathscr{M}$ there is a distinguished H-invariant derivation of $\mathbb{Z}$-degree 1, which is given by $\mathrm{d}(1) \in \mathrm{der}(A)$ and is called the differential.

Example 3.20. Building upon Example 2.7, examples of differential calculi are provided by the exterior algebras of differential forms $\Omega^\sharp(M)$ on G-manifolds M, equipped with the de Rham differential, and cochain twist quantizations thereof. See Proposition 3.22 for details on the twist deformation quantization of differential calculi.

3.6 Cochain twisting of differential operators and calculi

The cochain twist deformation quantization functor preserves differential operators and differential calculi.

Proposition 3.21. *Let A be any object in ${}^{H}\mathscr{A}^{\mathrm{com}}$, V any object in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ and F any cochain twisting element based on H. Then the coherence map $\gamma : \mathrm{end}_F(\mathcal{F}(V)) \to \mathcal{F}(\mathrm{end}(V))$ restricts to an ${}^{H_F}\mathscr{M}$-isomorphism*

$$\gamma : \mathrm{diff}_F^n(\mathcal{F}(V)) \longrightarrow \mathcal{F}(\mathrm{diff}^n(V)) \ , \tag{3.48}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. The proof is analogous to the proof of Proposition 3.8. □

Proposition 3.22. *Let $\big(A, \mathrm{d} : I[1] \to \mathrm{der}(A)\big)$ be any differential calculus in ${}^{H}\mathscr{M}$ and let F be any cochain twisting element based on H. Then $\mathcal{F}(A)$ together with the ${}^{H_F}\mathscr{M}$-morphism*

$$\mathrm{d}_F := \gamma^{-1} \circ \mathcal{F}(\mathrm{d}) \circ \psi : I_F[1] \longrightarrow \mathrm{der}_F(\mathcal{F}(A)) \tag{3.49}$$

is a differential calculus in ${}^{H_F}\mathscr{M}$, where ψ is the coherence morphism in (3.18b).

Proof. By Proposition 3.8, the target of d_F is as claimed in (3.49). Moreover, d_F is nilpotent (in $\mathrm{diff}_F(\mathcal{F}(A))$) because of the short calculation

$$\begin{aligned} \mathrm{d}_F(c) \bullet_F \mathrm{d}_F(c'\,) &= \gamma^{-1}\Big(\big(F^{(-1)} \triangleright \gamma(\mathrm{d}_F(c))\big) \bullet \big(F^{(-2)} \triangleright \gamma(\mathrm{d}_F(c'\,))\big)\Big) \\ &= \gamma^{-1}\Big(\mathrm{d}\big(F^{(-1)} \triangleright c\big) \bullet \mathrm{d}\big(F^{(-2)} \triangleright c'\,\big)\Big) \\ &= \gamma^{-1}\big(\mathrm{d}(c) \bullet \mathrm{d}(c'\,)\big) = 0 \ , \end{aligned} \tag{3.50}$$

for all $c, c' \in I[1]$. In the first equality we have used [BSS15, Proposition 2.16], in the second equality the definition of d_F and the H-equivariance of d, in the third equality the normalization of the cochain twist, and in the last equality the nilpotency of d. □

4 Connections

For a given differential calculus (A, d) in ${}^{H}\mathscr{M}$, we shall develop the notion of connections on objects in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ by again using universal constructions in the category ${}^{H}\mathscr{M}$. We will show that connections of objects V, W in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ can be canonically lifted to connections on the tensor product object $V \otimes_A W$ and on the internal hom-object $\hom_A(V, W)$. Throughout this section H is an arbitrary quasitriangular quasi-Hopf algebra.

4.1 Connections on symmetric bimodules

Let $\big(A, \mathrm{d} : I[1] \to \mathrm{der}(A)\big)$ be a differential calculus in ${}^{H}\mathscr{M}$. Connections on an object V in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ are distinguished differential operators of order 1 which satisfy a Leibniz rule with respect to the ${}^{H}\mathscr{M}$-morphism d. We shall again formalize this algebraic property in terms of an equalizer in ${}^{H}\mathscr{M}$. Denoting by $\times$ the categorical product in ${}^{H}\mathscr{M}$ and recalling that $I[1]$ denotes the shifted unit object in ${}^{H}\mathscr{M}$, there are two parallel ${}^{H}\mathscr{M}$-morphisms

$$\big(\mathrm{end}(V) \times I[1]\big) \otimes A \ \underset{\widehat{l} \circ \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}) \circ (\mathrm{pr}_2 \otimes \mathrm{id})}{\overset{[\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \otimes \mathrm{id})}{\rightrightarrows}} \ \mathrm{end}(V) \ \ , \tag{4.1}$$

where $\mathrm{pr}_1 : \mathrm{end}(V) \times I[1] \to \mathrm{end}(V)$ and $\mathrm{pr}_2 : \mathrm{end}(V) \times I[1] \to I[1]$ are the projection ${}^H\mathscr{M}$-morphisms. The upper arrow in (4.1) is the mapping

$$(L,c) \otimes a \longmapsto [L,a] \tag{4.2a}$$

and the lower arrow is the mapping

$$(L,c) \otimes a \longmapsto \widehat{l}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \ . \tag{4.2b}$$

Definition 4.1. Let (A,d) be a differential calculus in ${}^H\mathscr{M}$ and V any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. The connections of V is the object $\mathrm{con}(V)$ in ${}^H\mathscr{M}$ which is defined by the equalizer

$$\mathrm{con}(V) \longrightarrow \mathrm{end}(V) \times I[1] \underset{\zeta(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id}))}{\overset{\zeta([\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id}))}{\rightrightarrows}} \mathrm{hom}(A,\mathrm{end}(V)) \tag{4.3}$$

in ${}^H\mathscr{M}$. This equalizer can be realized explicitly in terms of the ${}^H\mathscr{M}$-subobject

$$\mathrm{con}(V) = \mathrm{Ker}\Big(\zeta\big([\,\cdot\,,\,\cdot\,] \circ (\mathrm{pr}_1 \otimes \mathrm{id}) - \widehat{l} \circ \mathrm{ev} \circ (\mathrm{d} \otimes \mathrm{id}) \circ (\mathrm{pr}_2 \otimes \mathrm{id})\big)\Big) \tag{4.4}$$

of the object $\mathrm{end}(V) \times I[1]$ in ${}^H\mathscr{M}$.

Lemma 4.2. *Let (A,d) be any differential calculus in ${}^H\mathscr{M}$ and let V be any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. An ${}^H\mathscr{M}$-subobject $U \subseteq \mathrm{end}(V) \times I[1]$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{con}(V)$ if and only if*

$$[L,a] = \widehat{l}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \ , \tag{4.5}$$

for all $(L,c) \in U$ and $a \in A$.

Proof. Denoting by $f := [\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id}) - \widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id}) : (\mathrm{end}(V)\times I[1])\otimes A \to \mathrm{end}(V)$ and $j : U \to \mathrm{end}(V) \times I[1]$ the inclusion ${}^H\mathscr{M}$-morphism, we have to show that $\zeta(f) \circ j = 0$ if and only if $f \circ (j \otimes \mathrm{id}) = 0$. This is a consequence of item (ii) of Lemma 2.5. □

Remark 4.3. By Lemma 4.2, any element $(L,c) \in \mathrm{con}(V)$ satisfies the condition (4.5) for all $a \in A$. In particular, the $\mathbb{Z}$-degree 1 elements $\nabla = (L,1) \in \mathrm{con}(V)$ satisfy the Leibniz rule with respect to the differential $\mathrm{d}(1)$. Hence, our notion of connections contains the standard notion of connections as distinguished points. It is important to notice that our definition has the advantage that $\mathrm{con}(V)$ is by construction an object in ${}^H\mathscr{M}$ while the subset of all ordinary connections $\nabla = (L,1) \in \mathrm{con}(V)$ is just an affine space over the k-module of all $\mathbb{Z}$-degree 1 elements $(L,0) \in \mathrm{con}(V)$, hence it is not an object in ${}^H\mathscr{M}$.

Finally, we prove an important structural result for connections.

Proposition 4.4. *Let (A,d) be any differential calculus in ${}^H\mathscr{M}$ and let V be any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. Then $\mathrm{con}(V)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{diff}^1(V) \times I[1]$.*

Proof. The object $\mathrm{con}(V)$ is by construction an ${}^H\mathscr{M}$-subobject of $\mathrm{end}(V) \times I[1]$ and hence the image of $\mathrm{pr}_1 : \mathrm{con}(V) \to \mathrm{end}(V)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{end}(V)$. We have

$$[[L,a],a'\,] = \big[\,\widehat{l}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big), a'\,\big] = 0 \ , \tag{4.6}$$

for all $(L,c) \in \mathrm{con}(V)$, which by using Lemma 3.14 shows that the image of $\mathrm{pr}_1 : \mathrm{con}(V) \to \mathrm{end}(V)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{diff}^1(V)$ and hence that $\mathrm{con}(V)$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{diff}^1(V) \times I[1]$. □

4.2 Connections on tensor products

We shall now develop a lifting prescription for connections to tensor products of objects in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$. Let us first notice that for any two objects V, W in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ there are two ${}^{H}\mathscr{M}$-morphisms given by the compositions

$$\mathrm{end}(V) \xrightarrow{\rho^{-1}} \mathrm{end}(V) \otimes I \xrightarrow{\mathrm{id}\otimes\eta} \mathrm{end}(V) \otimes \mathrm{end}(W) \xrightarrow{\circledbullet} \mathrm{end}(V \otimes W) \tag{4.7a}$$

and

$$\mathrm{end}(W) \xrightarrow{\lambda^{-1}} I \otimes \mathrm{end}(W) \xrightarrow{\eta\otimes\mathrm{id}} \mathrm{end}(V) \otimes \mathrm{end}(W) \xrightarrow{\circledbullet} \mathrm{end}(V \otimes W) \ . \tag{4.7b}$$

These ${}^{H}\mathscr{M}$-morphisms are given explicitly by the mappings

$$L \longmapsto L \circledbullet 1 \qquad \text{and} \qquad L' \longmapsto 1 \circledbullet L' \ , \tag{4.8}$$

respectively.

Definition 4.5. For any two objects V, W in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$ we define the ${}^{H}\mathscr{M}$-morphism

$$\begin{aligned} \boxdot : \big(\mathrm{end}(V) \times I[1]\big) \times \big(\mathrm{end}(W) \times I[1]\big) &\longrightarrow \mathrm{end}(V \otimes W) \times I[1] \ , \\ \big((L,c),(L',c')\big) &\longmapsto \big(L \circledbullet 1 + 1 \circledbullet L', c\big) \ . \end{aligned} \tag{4.9}$$

In order to prove that $\boxdot$ restricts to connections, i.e. to an ${}^{H}\mathscr{M}$-morphism $\boxdot : \mathrm{con}(V) \times \mathrm{con}(W) \to \mathrm{con}(V \otimes W)$, we require the following technical lemma.

Lemma 4.6. *Let A be an object in ${}^{H}\mathscr{A}^{\mathrm{com}}$ and let V, W be any two objects in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$.*

(i) Recalling the ${}^{H}\mathscr{M}$-morphisms $\widehat{l}_V : A \to \mathrm{end}(V)$ and $\widehat{l}_{V\otimes W} : A \to \mathrm{end}(V \otimes W)$ given in (3.9), one has

$$\widehat{l}_{V\otimes W}(a) = \widehat{l}_V(a) \circledbullet 1 \ , \tag{4.10}$$

for all $a \in A$.

(ii) For any $L, K \in \mathrm{end}(V)$ and $L' \in \mathrm{end}(W)$, one has

$$[K \circledbullet 1, L \circledbullet 1] = [K, L] \circledbullet 1 \qquad \text{and} \qquad [1 \circledbullet L', L \circledbullet 1] = 0 \ . \tag{4.11}$$

(iii) For any $L \in \mathrm{end}(V)$, $L' \in \mathrm{end}(W)$ and $a \in A$, one has

$$\big[L \circledbullet 1 + 1 \circledbullet L', a\big] = [L, a] \circledbullet 1 \ . \tag{4.12}$$

Proof. Let us first prove item (i). By definition of $\widehat{l}_{V\otimes W}$ we have

$$\mathrm{ev}\big(\widehat{l}_{V\otimes W}(a) \otimes (v \otimes w)\big) = a\,(v \otimes w) \ , \tag{4.13}$$

for all $v \in V, w \in W$ and $a \in A$. On the other hand, using [BSS15, Equation (5.7a)] we have

$$\mathrm{ev}\big((\widehat{l}_V(a) \circledbullet 1) \otimes (v \otimes w)\big) = \big((\phi^{(-1)} \triangleright a)\,(\phi^{(-2)} \triangleright v)\big) \otimes (\phi^{(-3)} \triangleright w) = a\,(v \otimes w) \ , \tag{4.14}$$

for all $v \in V, w \in W$ and $a \in A$. The assertion then follows by using Lemma 2.5 (i) and invertibility of ζ. Item (ii) follows immediately from Lemma 2.2. Item (iii) is a consequence of item (i) and (ii), as

$$\begin{aligned} \big[L \circledbullet 1 + 1 \circledbullet L', a\big] &= \big[L \circledbullet 1 + 1 \circledbullet L', \widehat{l}_{V\otimes W}(a)\big] \\ &= \big[L \circledbullet 1 + 1 \circledbullet L', \widehat{l}_V(a) \circledbullet 1\big] = [L, a] \circledbullet 1 \ , \end{aligned} \tag{4.15}$$

and the assertion follows. □

Proposition 4.7. *Let (A, d) be a differential calculus in ${}^{H}\mathscr{M}$ and let V, W be two objects in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$. Then $\boxdot$ restricts to an ${}^{H}\mathscr{M}$-morphism*

$$\boxdot : \mathrm{con}(V) \times \mathrm{con}(W) \longrightarrow \mathrm{con}(V \otimes W) \ . \tag{4.16}$$

Proof. We have to show that the image of $\boxdot : \mathrm{con}(V) \times \mathrm{con}(W) \to \mathrm{end}(V \otimes W) \times I[1]$ is an ${}^{H}\mathscr{M}$-subobject of $\mathrm{con}(V \otimes W)$. Using Lemma 4.2 this can be shown by the computation

$$\big[L \odot 1 + 1 \odot L', a\big] = [L, a] \odot 1 = \widehat{l}_V\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \odot 1 = \widehat{l}_{V \otimes W}\big(\mathrm{ev}(\mathrm{d}(c) \otimes a)\big) \ , \tag{4.17}$$

for all $(L, c) \in \mathrm{con}(V)$, $(L', c') \in \mathrm{con}(W)$ and $a \in A$. In the first equality we used item (iii) and in the last equality item (i) of Lemma 4.6. $\square$

The ${}^{H}\mathscr{M}$-morphism (4.16) describes the construction of connections on the object $V \otimes W$ but not on the object $V \otimes_A W$, which is obtained by using the correct monoidal functor $\otimes_A$ in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$. As $V \otimes_A W$ can be obtained by taking a quotient of $V \otimes W$ (cf. (2.43)), we may ask if (4.16) induces an ${}^{H}\mathscr{M}$-morphism with target given by $\mathrm{con}(V \otimes_A W)$. For this to hold true, we have to restrict the source of (4.16) to the fibred product $\mathrm{con}(V) \times_{I[1]} \mathrm{con}(W)$ given by the pullback

$$\begin{array}{ccc} \mathrm{con}(V) \times_{I[1]} \mathrm{con}(W) & \longrightarrow & \mathrm{con}(W) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathrm{con}(V) & \xrightarrow[\mathrm{pr}_2]{} & I[1] \end{array} \tag{4.18}$$

in the category ${}^{H}\mathscr{M}$. Then $\mathrm{con}(V) \times_{I[1]} \mathrm{con}(W)$ is the ${}^{H}\mathscr{M}$-subobject of $\mathrm{con}(V) \times \mathrm{con}(W)$ with elements given by pairs $((L, c), (L', c'))$ such that $c = c'$. We can now state one of the main results of this section.

Theorem 4.8. *Let (A, d) be a differential calculus in ${}^{H}\mathscr{M}$ and let V, W be two objects in ${}^{H}{}_{A}\mathscr{M}_{A}^{\mathrm{sym}}$. Then $\boxdot$ induces an ${}^{H}\mathscr{M}$-morphism*

$$\boxdot : \mathrm{con}(V) \times_{I[1]} \mathrm{con}(W) \longrightarrow \mathrm{con}(V \otimes_A W) \ . \tag{4.19}$$

Proof. Let $((L, c), (L', c)) \in \mathrm{con}(V) \times_{I[1]} \mathrm{con}(W)$ be an arbitrary element. Applying $\boxdot$ gives the element

$$\big(L \odot 1 + 1 \odot L', c\big) \in \mathrm{con}(V \otimes W) \subseteq \mathrm{end}(V \otimes W) \times I[1] \ , \tag{4.20}$$

where we regard $K := L \odot 1 + 1 \odot L'$ simply as a k-linear map $K : V \otimes W \to V \otimes W$. We have to prove that K descends to a well-defined k-linear map $K : V \otimes_A W \to V \otimes_A W$ on the quotient (2.43). Denoting by $\pi : V \otimes W \to V \otimes_A W$ the quotient map, this amounts to showing that

$$\pi \circ K\Big(\big(v\, a\big) \otimes w - (\phi^{(1)} \triangleright v) \otimes \big((\phi^{(2)} \triangleright a)\, (\phi^{(3)} \triangleright w)\big)\Big) = 0 \ , \tag{4.21}$$

for all $v \in V$, $w \in W$ and $a \in A$. Let us for the moment consider the case of trivial associator $\phi = 1 \otimes 1 \otimes 1$. Then the equality (4.21) can be easily verified on homogeneous elements by using

$$\begin{aligned} \pi \circ (L \odot 1)\big((v\, a) \otimes w\big) &= L(v\, a) \otimes_A w \\ &= (-1)^{|v|\,|a|}\ L\big((R^{(2)} \triangleright a)\, (R^{(1)} \triangleright v)\big) \otimes_A w \\ &= (-1)^{(|v|+|L|)\,|a|}\ \big(\widetilde{R}^{(2)}\, R^{(2)} \triangleright a\big)\ \big(\widetilde{R}^{(1)} \triangleright L\big)\big(R^{(1)} \triangleright v\big) \otimes_A w \\ &\qquad + (-1)^{|v|\,|a|}\ \big(\mathrm{d}(c)\big)\big(R^{(2)} \triangleright a\big)\ \big(R^{(1)} \triangleright v\big) \otimes_A w \\ &= L(v) \otimes_A \big(a\, w\big) + (-1)^{|v|\,|L|}\ v \otimes_A \big(\big(\mathrm{d}(c)\big)(a)\, w\big) \\ &= \pi \circ (L \odot 1)\big(v \otimes (a\, w)\big) + (-1)^{|v|\,|L|}\ v \otimes_A \big(\big(\mathrm{d}(c)\big)(a)\, w\big) \end{aligned} \tag{4.22a}$$

and

$$\begin{aligned}
\pi \circ (1 \circledast L')\big((v\,a) \otimes w\big) &= (-1)^{(|v|+|a|)\,|L'|}\ \big((R^{(2)} \triangleright v)\,(\widetilde{R}^{(2)} \triangleright a)\big) \otimes_A \big(\widetilde{R}^{(1)}\, R^{(1)} \triangleright L'\big)(w) \\
&= (-1)^{|v|\,|L'|}\ \Big((R^{(2)} \triangleright v) \otimes_A \big(R^{(1)} \triangleright L'\big)(a\,w) - v \otimes_A \big(\big(\mathrm{d}(c)\big)(a)\,w\big)\Big) \\
&= \pi \circ (1 \circledast L')\big(v \otimes (a\,w)\big) - (-1)^{|v|\,|L|}\ v \otimes_A \big(\big(\mathrm{d}(c)\big)(a)\,w\big)\ , \qquad (4.22b)
\end{aligned}$$

where in the last equality we used $|L| = |L'|$. The equality (4.21) also holds for the case of nontrivial associators, however the corresponding calculation is much more lengthy and involved, and hence we will not write it out in detail. □

The following result allows us to consistently lift connections to tensor products of an arbitrary (finite) number of objects in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$.

Theorem 4.9. *Let (A,d) be a differential calculus in ${}^{H}\mathscr{M}$ and let V,W,X be three objects in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. Then the ${}^{H}\mathscr{M}$-diagram*

$$\begin{array}{ccc}
\mathrm{con}(V) \times_{I[1]} \mathrm{con}(W) \times_{I[1]} \mathrm{con}(X) & \xrightarrow{\ \boxdot\circ(\boxdot\times\mathrm{id})\ } & \mathrm{con}\big((V \otimes_A W) \otimes_A X\big) \\
{\scriptstyle \boxdot\circ(\mathrm{id}\times\boxdot)}\big\downarrow & \swarrow{\scriptstyle \Phi\circ(\,\cdot\,)\circ\Phi^{-1}} & \\
\mathrm{con}\big(V \otimes_A (W \otimes_A X)\big) & &
\end{array} \qquad (4.23)$$

commutes.

Proof. Let $\big((L,c),(L',c),(L'',c)\big) \in \mathrm{con}(V) \times_{I[1]} \mathrm{con}(W) \times_{I[1]} \mathrm{con}(X)$ be an arbitrary element. Applying $\boxdot \circ (\boxdot \times \mathrm{id})$ yields

$$\boxdot \circ (\boxdot \times \mathrm{id})\,\Big(\big((L,c),(L',c),(L'',c)\big)\Big) = \big((L \circledast 1) \circledast 1 + (1 \circledast L') \circledast 1 + (1 \circledast 1) \circledast L'', c\big) \qquad (4.24a)$$

while applying $\boxdot \circ (\mathrm{id} \times \boxdot)$ yields

$$\boxdot \circ (\mathrm{id} \times \boxdot)\,\Big(\big((L,c),(L',c),(L'',c)\big)\Big) = \big(L \circledast (1 \circledast 1) + 1 \circledast (L' \circledast 1) + 1 \circledast (1 \circledast L''), c\big)\ . \qquad (4.24b)$$

The assertion then follows by using Lemma 2.3. □

4.3 Connections on internal homomorphisms

We shall now develop a lifting prescription for connections to the internal hom-objects in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. Let (A,d) be a differential calculus in ${}^{H}\mathscr{M}$ and V,W two objects in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$. Then there are two ${}^{H}\mathscr{M}$-morphisms

$$\mathscr{L} := \zeta(\bullet) : \mathrm{end}(W) \longrightarrow \mathrm{end}(\mathrm{hom}(V,W))\ , \qquad (4.25a)$$

$$\mathscr{R} := \zeta(\bullet \circ \tau) : \mathrm{end}(V) \longrightarrow \mathrm{end}(\mathrm{hom}(V,W))\ . \qquad (4.25b)$$

Definition 4.10. For any two objects V,W in ${}^{H}_{A}\mathscr{M}^{\mathrm{sym}}_{A}$ we define the ${}^{H}\mathscr{M}$-morphism

$$\begin{aligned}
\mathrm{ad}_\bullet : \big(\mathrm{end}(W) \times I[1]\big) \times \big(\mathrm{end}(V) \times I[1]\big) &\longrightarrow \mathrm{end}(\mathrm{hom}(V,W)) \times I[1]\ , \\
\big((L',c'),(L,c)\big) &\longmapsto \big(\mathscr{L}(L') - \mathscr{R}(L), c'\big)\ . \qquad (4.26)
\end{aligned}$$

We shall require the following two technical lemmas.

Lemma 4.11. *Let A be an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and let V,W be any two objects in ${}^H_{\ A}\mathscr{M}^{\mathrm{sym}}_A$. Recalling the ${}^H\mathscr{M}$-morphisms $\widehat{l}_W : A \to \mathrm{end}(W)$ and $\widehat{l}_{\mathrm{hom}(V,W)} : A \to \mathrm{end}(\mathrm{hom}(V,W))$ given in (3.9), one has*

$$\widehat{l}_{\mathrm{hom}(V,W)}(a) = \mathscr{L}\big(\widehat{l}_W(a)\big) \ , \tag{4.27}$$

for all $a \in A$.

Proof. Recalling (3.29) and using naturality of the currying bijection yields

$$\begin{aligned}
\widehat{l}_{\mathrm{hom}(V,W)}(a) &= \zeta\big(\,\bullet\circ(\widehat{l}_W\otimes\mathrm{id})\big)(a)\\
&= \zeta\big(\mathrm{Hom}_{{}^H\mathscr{M}}(\widehat{l}_W^{\,\mathrm{op}}\otimes\mathrm{id}^{\mathrm{op}},\mathrm{id})(\bullet)\big)(a)\\
&= \mathrm{Hom}_{{}^H\mathscr{M}}\big(\widehat{l}_W^{\,\mathrm{op}},\mathrm{hom}(\mathrm{id}^{\mathrm{op}},\mathrm{id})\big)\big(\zeta(\bullet)\big)(a)\\
&= \zeta(\bullet)\big(\widehat{l}_W(a)\big) = \mathscr{L}\big(\widehat{l}_W(a)\big) \ ,
\end{aligned} \tag{4.28}$$

for all $a \in A$. □

Lemma 4.12. *Let V,W be two objects in ${}^H_{\ A}\mathscr{M}^{\mathrm{sym}}_A$. Then*

$$\mathscr{L}(L)\bullet\mathscr{L}(L') = \mathscr{L}(L\bullet L') \ , \tag{4.29a}$$

$$\mathscr{R}(K)\bullet\mathscr{L}(L) = (-1)^{|K|\,|L|}\,\mathscr{L}\big(R^{(2)}\triangleright L\big)\bullet\mathscr{R}\big(R^{(1)}\triangleright K\big) \ , \tag{4.29b}$$

for all homogeneous $L, L' \in \mathrm{end}(W)$ and $K \in \mathrm{end}(V)$.

Proof. First, let us notice that both sides of (4.29a) can be regarded as ${}^H\mathscr{M}$-morphisms $\mathrm{end}(W)\otimes\mathrm{end}(W) \to \mathrm{end}(\mathrm{hom}(V,W))$: The morphism on the left-hand side is given by $\bullet\circ(\mathscr{L}\otimes\mathscr{L})$ and on the right-hand side by $\mathscr{L}\circ\bullet$. By invertibility of the natural currying bijections, these two morphisms agree if and only $\zeta^{-1}(\bullet\circ(\mathscr{L}\otimes\mathscr{L})) = \zeta^{-1}(\mathscr{L}\circ\bullet)$ as morphisms $(\mathrm{end}(W)\otimes\mathrm{end}(W))\otimes\mathrm{hom}(V,W) \to \mathrm{hom}(V,W)$. This can be shown by using Lemma 2.5 (i) and the calculation

$$\begin{aligned}
\zeta^{-1}(\bullet\circ(\mathscr{L}\otimes\mathscr{L}))\big((L\otimes L')\otimes M\big) &= \mathrm{ev}\big((\mathscr{L}(L)\bullet\mathscr{L}(L'))\otimes M\big)\\
&= \mathrm{ev}\,\Big(\mathscr{L}(\phi^{(1)}\triangleright L)\otimes\mathrm{ev}\big(\mathscr{L}(\phi^{(2)}\triangleright L')\otimes(\phi^{(3)}\triangleright M)\big)\Big)\\
&= (\phi^{(1)}\triangleright L)\bullet\big((\phi^{(2)}\triangleright L')\bullet(\phi^{(3)}\triangleright M)\big)\\
&= (L\bullet L')\bullet M\\
&= \mathrm{ev}\big(\mathscr{L}(L\bullet L')\otimes M\big)\\
&= \zeta^{-1}(\mathscr{L}\circ\bullet)\big((L\otimes L')\otimes M\big) \ ,
\end{aligned} \tag{4.30}$$

for all $L, L' \in \mathrm{end}(W)$ and $M \in \mathrm{hom}(V,W)$, where we have also used Lemma 2.1. The equality (4.29b) can be shown similarly. □

Proposition 4.13. *Let (A,d) be a differential calculus in ${}^H\mathscr{M}$ and let V,W be two objects in ${}^H_{\ A}\mathscr{M}^{\mathrm{sym}}_A$. Then $\mathrm{ad}_\bullet$ restricts to an ${}^H\mathscr{M}$-morphism*

$$\mathrm{ad}_\bullet : \mathrm{con}(W)\times\mathrm{con}(V) \longrightarrow \mathrm{con}(\mathrm{hom}(V,W)) \ . \tag{4.31}$$

Proof. We have to show that the image of $\mathrm{ad}_\bullet : \mathrm{con}(W) \times \mathrm{con}(V) \to \mathrm{end}(\mathrm{hom}(V,W)) \times I[1]$ is an ${}^H\mathscr{M}$-subobject of $\mathrm{con}(\mathrm{hom}(V,W))$. Using Lemma 4.2 this can be shown by the computation

$$\begin{aligned}\big[\mathscr{L}(L'\,) - \mathscr{R}(L), a\big] &= \big[\mathscr{L}(L'\,) - \mathscr{R}(L), \widehat{l}_{\mathrm{hom}(V,W)}(a)\big]\\ &= \big[\mathscr{L}(L'\,) - \mathscr{R}(L), \mathscr{L}(\widehat{l}_W(a))\big]\\ &= \mathscr{L}\big([L', a]\big)\\ &= \mathscr{L}\big(\widehat{l}_W(\mathrm{ev}(\mathrm{d}(c'\,) \otimes a))\big)\\ &= \widehat{l}_{\mathrm{hom}(V,W)}\big(\mathrm{ev}(\mathrm{d}(c'\,) \otimes a)\big) ~,\end{aligned} \tag{4.32}$$

for all $(L', c'\,) \in \mathrm{con}(W)$, $(L, c) \in \mathrm{con}(V)$ and $a \in A$. In the second and last equality we have used Lemma 4.11 and in the third equality we have used Lemma 4.12. □

Restricting the source of $\mathrm{ad}_\bullet$ to the fibred product $\mathrm{con}(W) \times_{I[1]} \mathrm{con}(V)$ we obtain a lifting prescription of connections to the internal hom-objects $\mathrm{hom}_A(V,W)$ in the category ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$.

Theorem 4.14. *Let (A, d) be a differential calculus in ${}^H\mathscr{M}$ and let V, W be two objects in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. Then $\mathrm{ad}_\bullet$ induces an ${}^H\mathscr{M}$-morphism*

$$\mathrm{ad}_\bullet : \mathrm{con}(W) \times_{I[1]} \mathrm{con}(V) \longrightarrow \mathrm{con}(\mathrm{hom}_A(V,W)) ~. \tag{4.33}$$

Proof. Let $((L', c), (L, c)) \in \mathrm{con}(W) \times_{I[1]} \mathrm{con}(V)$ be an arbitrary element. Applying $\mathrm{ad}_\bullet$ gives the element

$$\big(\mathscr{L}(L'\,) - \mathscr{R}(L), c\big) \in \mathrm{con}(\mathrm{hom}(V,W)) \subseteq \mathrm{end}(\mathrm{hom}(V,W)) \times I[1] ~, \tag{4.34}$$

where we regard $K := \mathscr{L}(L'\,) - \mathscr{R}(L)$ as a k-linear map $K : \mathrm{hom}(V,W) \to \mathrm{hom}(V,W)$. We have to prove that K restricts to a k-linear map $K : \mathrm{hom}_A(V,W) \to \mathrm{hom}_A(V,W)$ on the k-submodules $\mathrm{hom}_A(V,W) \subseteq \mathrm{hom}(V,W)$ given in (3.27). This amounts to showing that

$$\zeta([\,\cdot\,,\,\cdot\,])\big(K(M)\big) = 0 \in \mathrm{hom}(A, \mathrm{hom}(V,W)) ~, \tag{4.35}$$

for all $M \in \mathrm{hom}_A(V,W)$. Let us for the moment consider the case of trivial associator $\phi = 1 \otimes 1 \otimes 1$. Then the equality (4.35) can be easily verified by acting on generic homogeneous elements $a \in A$, which yields the equation

$$\begin{aligned}\big(\zeta([\,\cdot\,,\,\cdot\,])\big(K(M)\big)\big)\,(a) &= \big[K(M), a\big]\\ &= \big[L' \bullet M - (-1)^{|M|\,|L|}\,(R^{(2)} \triangleright M) \bullet (R^{(1)} \triangleright L), a\big] = 0 ~.\end{aligned} \tag{4.36}$$

This equation follows from

$$\begin{aligned}\big[L' \bullet M, a\big] &= L' \bullet \big[M, a\big] + (-1)^{|M|\,|a|}\,\big[L', R^{(2)} \triangleright a\big] \bullet \big(R^{(1)} \triangleright M\big)\\ &= 0 + (-1)^{|M|\,|a|}\,\big(\mathrm{d}(c)\big)\big(R^{(2)} \triangleright a\big)\,\big(R^{(1)} \triangleright M\big)\\ &= (-1)^{|L|\,|M|}\,M\,\big(\mathrm{d}(c)\big)(a) ~,\end{aligned} \tag{4.37a}$$

where in the last equality we used $|L| = |L'\,|$, and

$$\begin{aligned}\big[(R^{(2)} \triangleright M) \bullet (R^{(1)} \triangleright L), a\big] &= (R^{(2)} \triangleright M) \bullet \big[R^{(1)} \triangleright L, a\big]\\ &\quad + (-1)^{|L|\,|a|}\,\big[R^{(2)} \triangleright M, \widetilde{R}^{(2)} \triangleright a\big] \bullet \big((\widetilde{R}^{(1)}\,R^{(1)}) \triangleright L\big)\\ &= M\big(\mathrm{d}(c)\big)(a) + 0 ~.\end{aligned} \tag{4.37b}$$

The equality (4.35) is also true for the case of nontrivial associators, however the corresponding calculation is much more lengthy and involved, and hence we will not write it out in detail. □

4.4 Cochain twisting of connections

The cochain twist deformation quantization functor preserves connections.

Proposition 4.15. *Let (A, d) be any differential calculus in ${}^H\mathscr{M}$, V any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ and F any cochain twisting element based on H. Then the coherence map $\gamma\times\psi : \mathrm{end}_F(\mathcal{F}(V))\times I_F[1] \to \mathcal{F}(\mathrm{end}(V)) \times \mathcal{F}(I[1])$ restricts to an ${}^{H_F}\mathscr{M}$-isomorphism*

$$\gamma \times \psi : \mathrm{con}_F(\mathcal{F}(V)) \longrightarrow \mathcal{F}(\mathrm{con}(V)) \ . \tag{4.38}$$

Proof. The proof follows that of Proposition 3.8 and it requires showing commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{end}_F(\mathcal{F}(V)) \times I_F[1] & \overset{\zeta_F([\,\cdot\,,\cdot\,]_F\circ(\mathrm{pr}_1\otimes_F\mathrm{id}))}{\underset{\zeta_F(\widehat{l}_F\circ\mathrm{ev}_F\circ(\mathrm{d}_F\otimes_F\mathrm{id})\circ(\mathrm{pr}_2\otimes_F\mathrm{id}))}{\rightrightarrows}} & \mathrm{hom}_F(\mathcal{F}(A),\mathrm{end}_F(\mathcal{F}(V))) \\
 & & \big\downarrow{\scriptstyle \gamma\circ(\,\cdot\,)} \\
\big\downarrow{\scriptstyle \gamma\times\psi} & & \mathrm{hom}_F(\mathcal{F}(A),\mathcal{F}(\mathrm{end}(V))) \\
 & & \big\downarrow{\scriptstyle \gamma} \\
\mathcal{F}(\mathrm{end}(V)) \times \mathcal{F}(I[1]) & \overset{\mathcal{F}(\zeta([\,\cdot\,,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id})))}{\underset{\mathcal{F}(\zeta(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id})))}{\rightrightarrows}} & \mathcal{F}\big(\mathrm{hom}(A,\mathrm{end}(V))\big)
\end{array} \tag{4.39}$$

in ${}^{H_F}\mathscr{M}$, which is a straightforward but slightly lengthy calculation. □

Remark 4.16. Applying this result to Example 2.8, we find in particular that on any cochain twist deformation of any G-equivariant vector bundle there exists at least one connection, because there exist connections on classical G-equivariant vector bundles (in the smooth category).

5 Curvature

We shall develop the notion of curvature of connections on objects in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ and compute explicitly the curvatures of tensor product connections given by our construction in Theorem 4.8. We conclude by giving a brief sketch of how our formalism can be used to describe a noncommutative and nonassociative theory of gravity that is based on Einstein-Cartan geometry. Throughout this section we have to make the assumption that H is a triangular quasi-Hopf algebra, which is in particular satisfied for our main examples of interest, see Examples 2.7, 2.8 and 3.20.

5.1 Definition and properties

For any object V in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$, we define the ${}^H\mathscr{M}$-morphism

$$\begin{aligned}
[\![\,\cdot\,,\,\cdot\,]\!] : (\mathrm{end}(V)\times I[1]) \otimes (\mathrm{end}(V)\times I[1]) &\longrightarrow \mathrm{end}(V) \ , \\
(L,c)\otimes(L',c') &\longmapsto [L,L'] \ .
\end{aligned} \tag{5.1}$$

Lemma 5.1. *Let H be a triangular quasi-Hopf algebra. Let (A,d) be a differential calculus in ${}^H\mathscr{M}$ and V any object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. Then (5.1) restricts to an ${}^H\mathscr{M}$-morphism*

$$[\![\,\cdot\,,\,\cdot\,]\!] : \mathrm{con}(V)\otimes\mathrm{con}(V) \longrightarrow \mathrm{end}_A(V) \ . \tag{5.2}$$

Proof. By Lemma 3.10 it is sufficient to show that

$$\big[[\![(L,c),(L',c')]\!]\,,\,a\big] = \big[[L,L'],a\big] = 0\ , \tag{5.3}$$

for all homogeneous $(L,c),(L',c') \in \mathrm{con}(V)$ and $a \in A$. Using the braided Jacobi identity and braided antisymmetry of Proposition 3.2 (this is where we need triangularity) we obtain

$$\begin{aligned}
\big[[L,L'],a\big] &= -(-1)^{|L|\,(|L'|+|a|)}\,\big[\big[R^{(2)}_{(1)}\,\phi^{(2)} \triangleright L', R^{(2)}_{(2)}\,\phi^{(3)} \triangleright a\big]\,,\,R^{(1)}\,\phi^{(1)} \triangleright L\big]\\
&\qquad + (-1)^{|a|\,|L'|}\,\big[\big[\widetilde{R}^{(2)}\,\phi^{(-2)}\,R^{(1)}_{(1)} \triangleright L, \widetilde{R}^{(1)}\,\phi^{(-1)}\,R^{(2)} \triangleright a\big]\,,\,\phi^{(-3)}\,R^{(1)}_{(2)} \triangleright L'\,\big]\\
&= -(-1)^{|L|\,(|L'|+|a|)}\,\big[\mathrm{ev}\big(\mathrm{d}(c') \otimes (R^{(2)} \triangleright a)\big)\,,\,R^{(1)} \triangleright L\big]\\
&\qquad + (-1)^{|a|\,|L'|}\,\big[\mathrm{ev}\big(\mathrm{d}(c) \otimes (R^{(2)} \triangleright a)\big)\,,\,R^{(1)} \triangleright L'\,\big]\\
&= \big[L, \mathrm{ev}\big(\mathrm{d}(c') \otimes a\big)\big] - (-1)^{|L|\,|L'|}\,\big[L', \mathrm{ev}\big(\mathrm{d}(c) \otimes a\big)\big]\\
&= \mathrm{ev}\big(\mathrm{d}(c) \otimes \mathrm{ev}\big(\mathrm{d}(c') \otimes a\big)\big) - (-1)^{|L|\,|L'|}\,\mathrm{ev}\big(\mathrm{d}(c') \otimes \mathrm{ev}\big(\mathrm{d}(c) \otimes a\big)\big)\\
&= \mathrm{ev}\big(\big(\mathrm{d}(c) \bullet \mathrm{d}(c')\big) \otimes a\big) - (-1)^{|L|\,|L'|}\,\mathrm{ev}\big(\big(\mathrm{d}(c') \bullet \mathrm{d}(c)\big) \otimes a\big) = 0\ ,
\end{aligned} \tag{5.4}$$

where we have also used Lemma 4.2, Lemma 2.1 together with the normalization $(\epsilon\otimes\epsilon\otimes\mathrm{id})(\phi) = 1$ of the associator, and nilpotency of $\mathrm{d} : I[1] \to \mathrm{der}(A)$ from Definition 3.18. □

With these techniques we can now define the curvature of a connection. Since the curvature is supposed to be quadratic in the connections, we cannot realize the assignment of curvatures as an ${}^H\mathscr{M}$-morphism. We shall employ the following element-wise definition.

Definition 5.2. Let (A,d) be a differential calculus in ${}^H\mathscr{M}$ and let V be an object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. The curvature of a connection $\nabla := (L,1) \in \mathrm{con}(V)$ is the element

$$\mathrm{Curv}(\nabla) := [\![\nabla,\nabla]\!] \in \mathrm{end}_A(V)\ . \tag{5.5}$$

Remark 5.3. Given any connection $\nabla := (L,1) \in \mathrm{con}(V)$, we can define the Bianchi tensor corresponding to ∇ as

$$\mathrm{Bianchi}(\nabla) := \mathrm{ev}\big(\mathrm{ad}_\bullet(\nabla,\nabla) \otimes \mathrm{Curv}(\nabla)\big) \in \mathrm{end}_A(V)\ . \tag{5.6}$$

In contrast to the situation in classical differential geometry, here the Bianchi tensor in general does not vanish. Hence, it may be interpreted as a measure of the noncommutativity and nonassociativity of A, V and ∇.

Finally, we observe an additive property of the curvature of the tensor product connections constructed in Theorem 4.8.

Proposition 5.4. *Let H be a triangular quasi-Hopf algebra, (A,d) a differential calculus in ${}^H\mathscr{M}$ and V,W two objects in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. Given any two connections $\nabla_V := (L,1) \in \mathrm{con}(V)$ and $\nabla_W := (L',1) \in \mathrm{con}(W)$, the curvature of their sum satisfies*

$$\mathrm{Curv}(\nabla_V \boxplus \nabla_W) = \mathrm{Curv}(\nabla_V) \circledast 1 + 1 \circledast \mathrm{Curv}(\nabla_W)\ . \tag{5.7}$$

Proof. The proof follows from a simple calculation

$$\begin{aligned}
\mathrm{Curv}(\nabla_V \boxplus \nabla_W) &= [L \circledast 1 + 1 \circledast L', L \circledast 1 + 1 \circledast L'\,]\\
&= [L,L] \circledast 1 + 1 \circledast [L',L'\,]\\
&= \mathrm{Curv}(\nabla_V) \circledast 1 + 1 \circledast \mathrm{Curv}(\nabla_W)\ ,
\end{aligned} \tag{5.8}$$

where we have used the properties in Lemma 2.2 and the braided antisymmetry of the internal commutator from Proposition 3.2 (i). □

5.2 Einstein-Cartan geometry

We conclude with a brief sketch of how our formalism can be used to describe a noncommutative and nonassociative theory of gravity coupled to Dirac fields. Our considerations are based on Einstein-Cartan geometry and its generalization to noncommutative geometry which was developed in [AC09]. Our strategy is to formulate classical Einstein-Cartan geometry in our abstract language and then to give an outline of its cochain twist deformation quantization, which will lead to a noncommutative and nonassociative gravity theory.

Let M be any parallelizable manifold of dimension m. Associated to M is the Hopf algebra $H = U\,\mathrm{Vec}(M)$ (with trivial associator $\phi = 1 \otimes 1 \otimes 1$) given by the universal enveloping algebra of the Lie algebra of vector fields on M, which is triangular with trivial R-matrix $R = 1 \otimes 1$. We take $A := \Omega^\sharp(M)$ to be the exterior algebra of differential forms on M and $V := \Omega^\sharp(M, S)$ to be the A-bimodule of spinor-valued differential forms. Then A is an object in ${}^H\mathscr{A}^{\mathrm{com}}$ and V is an object in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$ for our choice of triangular Hopf algebra H. A vielbein is an invertible element $E \in \mathrm{end}_A(V) \simeq \Omega^\sharp(M, \mathrm{end}(S))$ of the form (here and in the following summations over repeated pairs of indices are understood)

$$E = E^a\,\gamma_a \ , \tag{5.9}$$

where $E^a \in \Omega^1(M)$ are the components of the vielbein, γ_a are the gamma-matrices associated with the spin representation S and $a = 1, \dots, m$. A spin connection is a connection $\nabla := (L, 1) \in \mathrm{con}(V)$ of the form

$$\nabla = \big(\mathrm{d} - \tfrac{1}{2}\,\omega^{ab}\,[\gamma_a, \gamma_b]\,,\,1\big) \ , \tag{5.10}$$

where $\omega^{ab} \in \Omega^1(M)$ are the components of the spin connection and d is the exterior derivative. A Dirac field is an element $\psi \in V_0$ of $\mathbb{Z}$-degree 0 and a conjugate Dirac field is an element $\overline{\psi} \in V^\vee{}_0 := \mathrm{hom}_A(V, A)_0$ in the dual module of $\mathbb{Z}$-degree 0. Using Theorem 4.14 we can induce a connection on conjugate Dirac fields by taking

$$\nabla^\vee := \mathrm{ad}_\bullet\big(\nabla\,,\,(\mathrm{d}, 1)\big) \in \mathrm{con}(V^\vee) \ . \tag{5.11}$$

Since V is a free A-module there is an isomorphism $V \otimes_A V^\vee \simeq \mathrm{end}_A(V)$, which we shall always suppress from our notation.

We can now write down the Lagrangian of Einstein-Cartan gravity coupled to a Dirac field in m dimensions within our formalism as the top form

$$L^{(m)} := \mathrm{Tr}\big(\mathcal{L}^{(m)}\big) \in A_m = \Omega^m(M) \ , \tag{5.12}$$

where

$$\begin{aligned}\mathcal{L}^{(m)} := \sqrt{-1}\ \mathrm{Curv}(\nabla) \bullet \underbrace{E \bullet E \bullet \cdots \bullet E}_{m-2 \text{ times}} \bullet \gamma_5 \\ - \Big(\mathrm{ev}(\nabla \otimes \psi) \otimes_A \overline{\psi} - \psi \otimes_A \mathrm{ev}\big(\nabla^\vee \otimes \overline{\psi}\big)\Big) \bullet \underbrace{E \bullet E \bullet \cdots \bullet E}_{m-1 \text{ times}} \bullet \gamma_5\end{aligned} \tag{5.13}$$

and $\gamma_5 \in \mathrm{end}_A(V)$ (of $\mathbb{Z}$-degree 0) is absent when m is odd. The trace in (5.12) is the ${}^H\mathscr{M}$-morphism

$$\mathrm{Tr} : \mathrm{end}_A(V) \longrightarrow A \tag{5.14}$$

given by the pointwise trace in $\mathrm{end}_A(V) \simeq \Omega^\sharp(M, \mathrm{end}(S))$.

To obtain a noncommutative and nonassociative theory of gravity one can quantize A and V by a suitable cochain twist F, see e.g. [MSS14, ASz15] and [BSS15, Example 6.5] for the explicit examples of relevance to the string theory applications mentioned in Section 1. The construction of the Lagrangian (5.12) then proceeds in the same way as above, but now with all internal constructions made in the braided closed monoidal category ${}^{H_F}\mathscr{M}$ rather than in ${}^{H}\mathscr{M}$. As cochain twisting may induce a nontrivial associator in ${}^{H_F}\mathscr{M}$, our definition of the Lagrangian (5.12) has to be supplemented with a bracketing convention for the internal compositions in (5.13). The study of which choice of bracketing leads to a physically reasonable model for noncommutative and nonassociative gravity is beyond the scope of the present paper and will be addressed in future work.

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