Supplemental material

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In this supplemental material we provide for the interested reader additional details on the derivation of the Heisenberg-Langevin equations of motion, the Martin-Siggia-Rose construction, the nature of the observed phase transition and the optimal path approximation.

HEISENBERG-LANGEVIN EQUATIONS OF MOTION

In order to derive the Heisenberg-Langevin equations of motion of an observable O we employ the conjugate Master equation

$$\dot{O} = i[H, O] + \sum_{a,k} \mathbf{D}'[L_{a,k}]O + \hat{\xi}^O, \qquad \text{(S1)}$$

where $\mathbf{D}'[X]O = X^\dagger O X - (X^\dagger X O + O X^\dagger X)/2$ and $\hat{\xi}^O$ is the quantum noise term for the operator O. The noise-less equation of motion is only formally correct on the level of single-operator expectation values, while the noise contributes by preserving the (anti-)commutation relations of the operators during the evolution. [1]. For a linear coupling of the system to the bath, the noise is typically Gaussian, with zero mean, but non-vanishing time- and space-local correlations. This prescription leads to Eqs. (2)-(4) of the main text; below we provide the main conceptual steps.

There exist several canonical (and equivalent) strategies to determine the properties of the noise operators $\hat{\xi}^{n,x,y}$, as for instance outlined in Ref. [1]. Here, we follow a path relying on the unitary Heisenberg equations of motion for system operators in the presence of a bath. As a simplifying assumption, we imagine the spatial correlations of this bath to be shorter than the typical interparticle distance in the system. This allows us to describe every spin as coupled to its own bath. We can therefore focus on a single spin as a representative and we model the spontaneous emission dynamics via the simple Hamiltonian

$$\widetilde{H} = H_{\text{s-b}} + H_{\text{b}} = \sum_q \lambda_q(\sigma^+ b_q + b_q^\dagger \sigma^-) + \sum_q \omega_q b_q^\dagger b_q, \text{(S2)}$$

where the b_ls represent a set of bosonic bath operators. We further assume that this bosonic reservoir is kept at zero temperature and that the number of modes is sufficiently large to allow a continuum description with a density of states $D(\omega) = \sum_q \delta(\omega - \omega_q)$. Taking the von Neumann equation for the global (spin plus bath) density matrix and eliminating the bath degrees of freedom leads then to the jump operator $L_d = \sqrt{\gamma} \sigma^-$ with $\gamma = 2\pi[\lambda(0)]^2 D(0)$ being proportional to the bath density of states $D(0)$ and the couplings $\lambda(0)$ evaluated at zero frequency (see e.g. Chapter 8 of [1]). The Heisenberg equations of motion for the operators are therefore

$$\dot{\sigma}^+ = i[\widetilde{H}, \sigma^+] = -i\sum_q \lambda_q b_q^\dagger \sigma^z, \qquad \text{(S3)}$$

$$\dot{n} = i\sum_q \lambda_q(b_q^\dagger \sigma^- - \sigma^+ b_q), \qquad \text{(S4)}$$

$$\dot{b}_q^\dagger = i\lambda_q \sigma^+ + i\omega_q b_q^\dagger. \qquad \text{(S5)}$$

Formally solving Eq. (S5) yields

$$b_q^\dagger(t) = b_q^\dagger(0)e^{i\omega_q t} + i\lambda_q \int_0^t dt' \sigma^+(t')e^{i\omega_q(t-t')}. \text{(S6)}$$

Inserting this solution into Eqs. (S3), (S4) and performing the Born-Markov approximation leads to

$$\dot{\sigma}^+ = -\frac{\gamma}{2}\sigma^+ + i\underbrace{\sum_q \lambda_q b_q^\dagger(0)\sigma^z e^{i\omega_q t}}_{\tilde{\xi}^+(t)}, \qquad \text{(S7)}$$

$$\dot{n} = -\gamma n + i\underbrace{\sum_q \lambda_q(b_q^\dagger(0)\sigma^- e^{i\omega_q t} - \sigma^+ b_q(0)e^{-i\omega_q t})}_{\tilde{\xi}^n(t)}. $$

$$\text{(S8)}$$

Defining $\tilde{\xi}^-(t) = (\tilde{\xi}^+(t))^\dagger$ and taking the bath to be in the vacuum state (corresponding to spontaneous emission) $\rho_B = |0\rangle\langle 0|$. This entails $b_q \rho_B = \rho_B b_q^\dagger = 0 \,\forall q$. In the Born-Markov approximation, the back-action of the system onto the bath is ignored, alongside the resulting memory effects; therefore, the noise average corresponds to averaging the bath variables onto ρ_B at all times. For example, this implies that any average of the form

$$\langle \xi^+(t) \dots \rangle \propto \sum_q \lambda_q \text{tr}_B \left\{ \rho_B(0) b_q^\dagger(0) \dots \right\} \qquad \text{(S9)}$$

identically vanishes. Analogously, $\langle \dots \xi^- \rangle \equiv 0$. Noticing that $\tilde{\xi}^n \equiv -\tilde{\xi}^+ \sigma^- - \sigma^+ \tilde{\xi}^-$, one therefore finds

$$\langle \tilde{\xi}^+(t)\tilde{\xi}^+(t') \rangle = \langle \tilde{\xi}^+(t)\tilde{\xi}^-(t') \rangle = \langle \tilde{\xi}^+(t) \rangle = \langle \tilde{\xi}^n(t) \rangle = 0. $$
$$\text{(S10)}$$

while the non-vanishing correlations are

$$\langle \tilde{\xi}^-(t)\tilde{\xi}^+(t') \rangle = \gamma \delta(t - t'), \qquad \text{(S11)}$$

$$\langle \tilde{\xi}^n(t)\tilde{\xi}^n(t') \rangle = \gamma n \delta(t - t'), \qquad \text{(S12)}$$

$$\langle \tilde{\xi}^n(t)\tilde{\xi}^+(t') \rangle = -\gamma \sigma^+ \delta(t - t'), \qquad \text{(S13)}$$

$$\langle \tilde{\xi}^n(t)\tilde{\xi}^+(t') \rangle = -\gamma \sigma^- \delta(t - t'). \qquad \text{(S14)}$$

The first one, for example, can be calculated via

$$\left\langle \tilde{\xi}^-(t)\tilde{\xi}^+(t') \right\rangle = \sum_{q,q'} \lambda_q \lambda_{q'} e^{i(\omega_q t - \omega_{q'} t')} \times$$
$$\times \left\langle \sigma^z(t) b_q(0) b_{q'}^\dagger(0) \sigma^z(t') \right\rangle \tag{S15}$$

Using the fact that $b_q b_{q'}^\dagger = \delta_{qq'} + b_{q'}^\dagger b_q$ we identify two contributions. The second one is by construction equivalent to

$$\left\langle \tilde{\xi}^-(t)\tilde{\xi}^+(t') \right\rangle = \sum_{q,q'} \lambda_q \lambda_{q'} e^{i(\omega_q t - \omega_{q'} t')} \times$$
$$\times \left\langle \left[\sigma^z(t), b_{q'}^\dagger(0)\right] \left[b_q(0), \sigma^z(t')\right] \right\rangle, \tag{S16}$$

where the commutators are subleading corrections in the system-bath coupling. The first one can, for sufficiently high number of bosonic mode, can be re-expressed as the integral

$$\int d\omega D(\omega)\lambda(\omega)^2 e^{i\omega(t-t')} \left\langle \sigma^z(t)\sigma^z(t') \right\rangle. \tag{S17}$$

Considering slowly-varying density of states D and coupling λ around $\omega = 0$, we then find

$$\langle \tilde{\xi}^-(t)\tilde{\xi}^+(t') \rangle \approx 2\pi D(0)\lambda(0)^2 \delta(t-t') = \gamma\delta(t-t'). \tag{S18}$$

Rotating into the (x,y,n) basis and introducing $\tilde{\xi}^x = \tilde{\xi}^+ + \tilde{\xi}^-$ and $\tilde{\xi}^y = -i\tilde{\xi}^+ + i\tilde{\xi}^-$ (and analogously σ^x and σ^y) one finds $\left\langle \tilde{\xi}^i(t)\tilde{\xi}^j(t') \right\rangle = \gamma\delta(t-t')M^{ij}$ with

$$M = \begin{pmatrix} 1 & -i & -\frac{\sigma^x - i\sigma^y}{2} \\ i & 1 & -\frac{i\sigma^x + \sigma^y}{2} \\ -\frac{\sigma^x + i\sigma^y}{2} & i\frac{\sigma^x + i\sigma^y}{2} & n \end{pmatrix}. \tag{S19}$$

The dependence of the ξ^n noise on the density keeps track of the fact that the absorbing configuration $n = 0$ represents a fluctuationless state in the entire parameter regime, which forbids a density independent contribution to ξ^n in Eq. (S12). The Markovian noise level introduced by the decay terms $L_d = \gamma\sigma^-$ only appears as an additive noise in the $\sigma^{x,y}$ variables.

Including classical coagulation and branching processes yields additional noise terms. However, due to the presence of the term $\sum_j n_j$ these contributions will always be higher-order in the density and are therefore subleading with respect to the ones derived above in the absorbing phase. Extending the system from a single spin to a lattice of individual spins, an equivalent computation shows

$$\langle \hat{\xi}_k^n(t)\hat{\xi}_{k'}^n(t') \rangle = n_k\gamma\delta_{k,k'}\delta(t-t') + O(n^2), \tag{S20}$$
$$\langle \hat{\xi}_k^x\hat{\xi}_{k'}^x \rangle = \gamma\delta_{k,k'}\delta(t-t') + O(n) = \langle \hat{\xi}_k^y\hat{\xi}_{k'}^y \rangle. \tag{S21}$$

To leading order in the density, this yields the same noise terms reported in the main text. Since the coherent branching and coagulation does not produce an additional noise, this concludes the derivation of the Heisenberg-Langevin equations.

MARTIN-SIGGIA-ROSE CONSTRUCTION

In this section, we provide the derivation of the Martin-Siggia-Rose (MSR) path integral for the present quantum contact process, which results in the effective long wavelength action for the density, Eq. (5) in the main text. As a first step, we take the continuum limit of the equations of motion for n_k, σ_k^x and σ_k^y, such that

$$\sum_{j \text{ nn } x} n_j \to (r^2\nabla^2 + z)n_x, \tag{S22}$$

where z is the coordination number, r is the lattice spacing, ∇ is the the common d-dimensional gradient and $x = rk$ the position. We then re-interpret the operators as stochastic fields subject to the continuum noise sources ξ_X^x, ξ_X^y, ξ_X^n — where $X = (t,x)$ is shorthand for the spatio-temporal argument — which have vanishing mean and correlations $\left\langle \xi_X^i\xi_Y^j \right\rangle = \gamma\delta(X - Y)M^{ij}$, where

$$M = \begin{pmatrix} 1 & 0 & -\frac{\sigma^x}{2} \\ 0 & 1 & -\frac{\sigma^y}{2} \\ -\frac{\sigma^x}{2} & -\frac{\sigma^y}{2} & n \end{pmatrix}. \tag{S23}$$

From this point onwards, as done in the main text, we assume $\gamma \neq 0$ and measure all times and energies in its units, i.e., we effectively set $\gamma = 1$. The equations of motion can thus be expressed as

$$\dot{n}_X = \mathcal{F}_n(n_X, \sigma_X^x, \sigma_X^y) + \xi_X^n, \tag{S24}$$
$$\dot{\sigma}_X^x = \mathcal{F}_{\sigma^x}(n_X, \sigma_X^x, \sigma_X^y) + \xi_X^x, \tag{S25}$$
$$\dot{\sigma}_X^y = \mathcal{F}_{\sigma^y}(n_X, \sigma_X^x, \sigma_X^y) + \xi_X^y, \tag{S26}$$

where

$$\mathcal{F}_n = -n_X + [\Omega\sigma_X^y - \kappa(2n_X - 1)](r^2\nabla^2 + z)n_X, \tag{S27}$$

$$\mathcal{F}_{\sigma^x} = -\frac{z\kappa+1}{2}\sigma_X^x - \kappa\sigma_X^x(r^2\nabla^2 + z)n_X +$$
$$+ \Omega\sigma_X^y(r^2\nabla^2 + z)\sigma_X^x, \tag{S28}$$

$$\mathcal{F}_{\sigma^y} = -\frac{z\kappa+1}{2}\sigma_X^y - [\Omega(4n_X - 2) + \kappa\sigma_X^y](r^2\nabla^2 + z)n_X +$$
$$+ \Omega\sigma_X^x(r^2\nabla^2 + z)\sigma_X^x. \tag{S29}$$

As shown in Refs. [2, 3], the MSR construction defines a path integral in the variables σ_X^y, n_X for the equations of motion (S24)-(S26). The MSR partition function represents the sum over all allowed field configurations, i.e.

$$Z = \int \mathcal{D}[n_X, \sigma_X^x, \sigma_X^y] \int \mathcal{D}[\xi_X^n, \xi_X^x, \xi_X^y]P(\xi_X^n, \xi_X^x, \xi_X^y) \times$$
$$\times J[n_X, \sigma_X^x, \sigma_X^y]\,\delta\left(\dot{n}_X - \mathcal{F}_n(n_X, \sigma_X^x, \sigma_X^y)\right) \times$$
$$\times \delta\left(\dot{\sigma}_X^x - \mathcal{F}_{\sigma^x}(n_X, \sigma_X^x, \sigma_X^y)\right)\delta\left(\dot{\sigma}_X^y - \mathcal{F}_{\sigma^y}(n_X, \sigma_X^x, \sigma_X^y)\right), \tag{S30}$$

where the integral $\int \mathcal{D}[\vec{\xi}]P(\vec{\xi})$ averages over all noise configurations described by the Gaussian noise distribution $P(\vec{\xi}) = \exp\left\{-\frac{1}{2}\int_X \left(\vec{\xi}_X\right)^\mathsf{T} M^{-1}\vec{\xi}_X\right\}$. The factor $J[n_X, \sigma_X^x, \sigma_X^y]$ is a Jacobian which, for our present

purposes, can be conveniently set to 1 after choosing a proper, retarded regularization [2, 3]. Introducing three sets of imaginary response fields $\tilde{n}_X, \tilde{\sigma}_X^x, \tilde{\sigma}_X^y$, exploiting the Fourier transform $\delta(f(n)) = \int \mathcal{D}\tilde{n} \exp(-\tilde{n}f(n))$ and integrating over the noise variables $\xi^{n/x/y}$, Z can be cast into a path-integral form

$$Z = \int \mathcal{D}[\tilde{n}_X, n_X, \tilde{\sigma}_X^x \sigma_X^x, \tilde{\sigma}_X^y \sigma_X^y] \, e^{-S}. \quad \text{(S31)}$$

where, up to leading order in a (spatial) derivative expansion, the action reads

$$S = \int_X \tilde{n}_X \left[\partial_t - D\nabla^2 - (z\kappa - 1) - \frac{1}{2}n\tilde{X}\right] n_X + 2z\kappa\tilde{n}_X n_X^2$$

$$+ \int_X \tilde{\sigma}_X^y \left[\partial_t + \frac{z\kappa+1}{2} + z\kappa n_X + \frac{1}{2}\tilde{n}_X\right] \sigma_X^y - \frac{1}{2}\left(\tilde{\sigma}_X^y\right)^2 + \tilde{\sigma}_X^y(z\Omega(4n_X^2 - 2n_X)) - \sigma_X^y(z\Omega\tilde{n}_X n_X) \quad \text{(S32)}$$

$$+ \int_X \tilde{\sigma}_X^x \left[\partial_t + \frac{z\kappa+1}{2} + z\kappa n_X + z\Omega\sigma_X^y + \frac{1}{2}\tilde{n}_X\right] \sigma_X^x - \frac{1}{2}\left(\tilde{\sigma}_X^x\right)^2 - z\Omega\tilde{\sigma}_X^y \left(\sigma_X^x\right)^2.$$

Since $\frac{z\kappa+1}{2} \geq 1/2$ throughout the physical parameter region $\kappa \geq 0$, both the σ^x and the σ^y fields remain gapped and one can therefore neglect the subleading derivative and fluctuating terms within square brackets. This yields an action which is separately quadratic in $(\sigma_X^x, \tilde{\sigma}_X^x)$ and $(\sigma_X^y, \tilde{\sigma}_X^y)$. These modes can be actually integrated out exactly and, up to RG-irrelevant terms, one obtains action (5) in the main text. The couplings correspond to the following combinations of miscroscopic parameters:

$$\Delta = 1 - z\kappa - \frac{8z^2\Omega^2}{(z\kappa+1)^3}, \quad \text{(S33a)}$$

$$u_3 = 2z\left(\kappa - \frac{2z\Omega^2}{z\kappa+1}\right), \quad \text{(S33b)}$$

$$u_4 = \frac{8z^2\Omega^2}{z\kappa+1}, \quad \text{(S33c)}$$

$$\mu_4 = \frac{2z^2\Omega^2}{(z\kappa+1)^2} + \frac{128z^4\Omega^4}{(z\kappa+1)^6}. \quad \text{(S33d)}$$

Our procedure differs conceptually from the approach advocated in [4]. We found it necessary to accurately capture the short distance physics of the problem.

ADDITIONAL DETAILS ON THE NATURE OF THE OBSERVED PHASE TRANSITIONS

Except for the α point, in the proximity of the transitions $u_3 \neq 0$ and one can rescale the fields according to $n \to Kn$, $\tilde{n} \to \tilde{n}/K$ with the choice $K = 1/\sqrt{2u_3}$. Thus, one finds

$$S = \int_X \tilde{n}_X \left[\partial_t - D\nabla^2 + \Delta + \kappa_3 n_X + \frac{u_4}{2u_3}n_X^2\right] n_X$$

$$- \int_X \tilde{n}_X^2 \left[\kappa_3 n_X + \mu_4 n_X^2\right]. \quad \text{(S34)}$$

with $\kappa_3 = Ku_3 = 1/2K = \sqrt{u_3/2}$. In the absence of the u_4 coupling, i.e. for $\Omega = 0$, this is the action describing classical directed percolation. It features the characteristic rapidity inversion symmetry under $n \leftrightarrow -\tilde{n}$, $t \to -t$. For $\Omega > 0$, the relevance of the u_4 coupling has to be considered, which is determined by the scaling behavior of the fields $n, \tilde{n}$. In the absence of a thermal fluctuation dissipation relation, both fields $n, \tilde{n}$ typically have the same scaling dimension [2, 3, 5]. This leads to an upper critical dimension of $d_c = 4$ for the cubic coupling u_3 and an upper critical dimension of $d_c = 2$ for the quartic coupling u_4. In dimensions $d > 2$, u_4 renormalizes to zero in the RG flow and the rapidity inversion symmetry is restored in the infrared regime. Hence, the effective low frequency theory, and therefore the long time dynamics, is again described by the directed percolation class. On the other hand, for $d < 2$, u_4 is relevant in the renormalization group sense and the absence of rapidity inversion introduces a different non-equilibrium dynamics at the phase transition, which is not captured by the DP universality class. In $d = 2$, the quartic couplings are marginal and whether they become relevant or irrelevant in the RG flow has to be determined by a renormalization group analysis of the problem.

At the point α, u_3 vanishes microscopically and the rescaling leading to the action (S34) is not defined. In dimensions $d > 2$ this point features a second order phase transition in the absence of the rapidity inversion symmetry. Since the leading order term in the effective potential Γ (Eq. (6) in the main text) is RG irrelevant in $d > 2$, one expects mean-field scaling behavior at this point. On the other hand, in $d < 2$ the effective theory at the α-point corresponds to the new universality class in the absence of rapidity inversion.

In any experiments with cold atoms, the presence of small fluctuation-inducing terms $\sim \Delta_x\sigma^x$ or $\Delta_y\sigma^y$ is hardly avoidable at the microscopic level. These will

generate fluctuations on top of the absorbing state and lead to a temperature type term $\sim \tilde{n}_X^2 T$ in the action (S34), with $T \approx \Delta_x, \Delta_y$. The present discussion of the non-equilibrium phase transitions is then valid on length scales $l^{-1} \geq \sqrt{T}$.

DETAILS ON THE OPTIMAL PATH APPROXIMATION

In the present setting, the noise $\Xi_X \equiv \frac{1}{2} n_X + \mu_4 n_X^2$ increases monotonically with the density. As a consequence, it favors the (fluctuationless) zero density solution over the finite-density one. In order to determine the distribution function for the density variable in the vicinity of the active-to-inactive transition, we apply the optimal path approximation [2, 3] to the partition function. Note that the system remains gapped for $\Delta > 0$ (where the first-order transition is expected to take place) and therefore we can — as a first approximation — neglect spatial fluctuations and approximate $n_X, \tilde{n}_X$ by spatially homogeneous but temporally fluctuating fields $n_t, \tilde{n}_t$. This yields the action

$$S = V \int_t \left[\tilde{n}_t \partial_t n_t + \tilde{n}_t \Gamma'(n_t) - \tilde{n}_t^2 \Xi(n_t) \right] \quad \text{(S35)}$$

with the shorthand $\Gamma'(n) = \delta\Gamma/\delta n$. The optimal path for the configurations $n_t, \tilde{n}_t$ corresponds to the configurations for which the non-fluctuating part of the action vanishes, i.e. for which $\tilde{n}_t \Gamma'(n_t) - \tilde{n}_t^2 \Xi(n_t) = 0$. This equation shows the two trivial solutions $\tilde{n}_t = 0$ and n_t arbitrary as well as $n_t = 0$ and $\tilde{n}_t$ arbitrary. Apart from this, there exists the non-trivial solution

$$\tilde{n}_t^{\text{op}} = \frac{\Gamma'(n_t)}{\Xi(n_t)}. \quad \text{(S36)}$$

Considering only configurations which correspond to the optimal path the action becomes

$$W(n) = \int_0^\infty \tilde{n}_t^{op} \partial_t n_t \mathrm{d}t = \int_{n_0}^n \frac{\Gamma'(m)}{\Xi(m)} \mathrm{d}m, \quad \text{(S37)}$$

with the change of variable $\partial_t n_t \mathrm{d}t \to \mathrm{d}m$ and where n_0 is the initial condition — whose specific value is irrelevant — and n the steady-state value of the density. The corresponding density distribution function is

$$P(n) = \frac{1}{Z} e^{-VW(n)}, \quad \text{(S38)}$$

with $Z = \int_n e^{-VW(n)}$, if the integral exists. For a system in thermal equilibrium, $\Xi(n) \propto T$ is simply proportional to the temperature and one recovers the naive expectation $P(n) \sim \exp(-\Gamma(n)/T)$. For the present noise terms

$$W(n) = \frac{1}{\mu_4} \left[\Delta l + u_3 \left(n - \frac{l}{2\mu_4}\right) + u_4 \frac{4n\mu_4(n\mu_4 - 1) + 2l}{16\mu_4^3} \right], \text{(S39)}$$

with $l = \log[1 + 2n\mu_4]$.

The minima of $W(n)$ are $n_1 = 0$ and $n_2 = -\frac{u_3}{2u_4} + \sqrt{\frac{u_3^2}{4u_4^2} - \frac{\Delta}{u_4}}$ and coincide with the ones of $\Gamma(n)$. The two functionals, however, may differ significantly. In particular, the global minimum of W does not necessarily coincide with the global minimum of Γ, such that the presence of a non-equilibrium noise term can strongly modify the phase boundary as a function of the noise strength. Since $W(0) = 0$ for all parameters, the first order transition line separating the active from the inactive phase for $\Delta > 0$ is determined by the equation $W(n_2) = 0$.

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