Solution classes of the matrix second Painlevé hierarchy

P. R. Gordoa,[1] A. Pickering[1] and J. A. D. Wattis[2]

[1] Área de Matemática Aplicada, ESCET,
Universidad Rey Juan Carlos,
C/ Tulipán s/n, 28933 Móstoles,
Madrid, Spain

[2] School of Mathematical Sciences,
University of Nottingham,
University Park, Nottingham NG7 2RD,
United Kingdom

17 July 2021 (revised 4 April 2022)

Short title: Solutions of the matrix P_{II} hierarchy
Keywords: matrix Painlevé hierarchies, auto-Bäcklund transformations, solution classes
MSC2000 classification scheme numbers: 34M55, 37K35, 37K10, 33E17

Abstract

We explore the generation of classes of solutions of the matrix second Painlevé hierarchy. This involves the consideration of the application of compositions of auto-Bäcklund transformations to different initial solutions, with the number of distinct solutions obtained for each value of the parameter appearing in the hierarchy depending on the symmetry properties of the chosen initial solution. This paper not only extends our previous results for the matrix second Painlevé equation itself, given in a recent paper, to the matrix second Painlevé hierarchy, but also provides a more detailed account of the underlying process.

Corresponding author:
A. Pickering. email: andrew.pickering@urjc.es

1 Introduction

The discovery of a link between completely integrable partial differential equations (PDEs) and ordinary differential equations (ODEs) having the Painlevé property [1] led almost immediately to the derivation of the second Painlevé (P_{II}) hierarchy [1, 2] along with its auto-Bäcklund transformations (auto-BTs) [2]. This hierarchy, derived by similarity reduction using the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) hierarchies, consists of a sequence of ODEs of orders $2n,\ n = 1, 2, 3, \ldots,$ having as first member the second Painlevé equation [3, 4, 5, 6]. We refer to this hierarchy as the standard Painlevé hierarchy, in contrast to the generalised P_{II} hierarchy which includes terms corresponding to lower order mKdV flows [7] (see also [8], where in addition a generalised first Painlevé (P_I) hierarchy was given).

However, despite the initial results of Airault, and despite both classical and contemporary interest in higher order equations with the Painlevé property [9]—[15], it was to take some twenty years before interest in Painlevé hierarchies was to take off. In [16], Kudryashov rediscovered the standard P_{II} hierarchy, and also obtained the standard first Painlevé (P_I) hierarchy. In addition, an alternative form of the auto-BTs of the P_{II} hierarchy was given in [17]. Over the last quarter-century or so, a great many papers have been published on Painlevé hierarchies and their properties. This has included using and extending the ideas in [18] in order to use nonisospectral scattering problems to derive Painlevé hierarchies and their underlying linear problems [19], which has led to the derivation of differential, discrete and differential-delay Painlevé hierarchies [19]—[24]; see also [25]—[29] and the discussion in [30]. Auto-BTs for differential Painlevé hierarchies have been derived in [31] and, with the advent of a new more general approach given in [32], for discrete and differential-delay Painlevé equations and hierarchies [32, 33, 34], as well as for further examples of differential Painlevé hierarchies [35]. We refer also to [36]—[44] for higher order Painlevé equations and Painlevé hierarchies, and their properties.

The technique developed in [32] was also used in [45] (see also [46]) to obtain the auto-BTs of a matrix P_{II} equation. The derivation of matrix Painlevé hierachies using matrix PDE hierachies was, in fact, somewhat less straightforward than for the case of scalar dependent variables. The matrix P_{II} equation was introduced in [47, 48], and was known to pass the Painlevé test and to have an underlying linear problem [47]. However, as observed in [47], it was not known how to derive this equation from the matrix analogues of the mKdV equation; neither was it understood, as remarked in [48], how to obtain a second order matrix ODE from the third order matrix ODEs obtained as similarity reductions of the matrix mKdV equations. These problems were overcome in [49], where a matrix P_{II} hierarchy was obtained, along with properties thereof such as special integrals and auto-BTs, using a matrix mKdV hierarchy. In a more recent paper, the approach developed in [49] was used to derive matrix fourth Painlevé (P_{IV}) hierarchies, again along with related results such as special integrals and auto-BTs [50]. In both [49] and [50], alternative matrix P_{II} hierarchies were also discussed. It is also worth remarking that, to the best of our knowledge, the matrix dispersive water wave hierarchy constructed in [50], even in the isospectral case, is also new.

The matrix P_{II} hierarchy presented in [49] has led to a variety of related results. These include the derivation of Painlevé-style auto-BTs for a matrix PDE along with the derivation and study of related non-autonomous matrix lattice equations [51, 52, 53]. In particular, in [52], we considered the use of auto-BTs in order to derive solutions of the new matrix P_{II} equation derived in [49]. It was also foreseen that, in a subsequent paper, we would extend these results to the entire matrix P_{II} hierarchy, and it is that task to which we turn here.

The outline of the paper is as follows. In Section 2, we recall the definition of the matrix P_{II} hierarchy and its auto-BTs. We also give some preliminary results on the composition of auto-BTs in function of properties of the initial solution used. In Section 3, we further discuss the composition of auto-BTs, in the context of transforming to an equivalent equation in order to simplify the matrix coefficient appearing in our hierarchy. In addition, we give some results on upper-triangular solutions of our matrix hierarchy, as well as on extending the classes of initial solutions discussed in [52] for the matrix P_{II} equation to the case of the higher order equations of the P_{II} hierarchy. In Section 4 we present examples, with the aim of illustrating various points in the process of constructing sequences of solutions. We dedicate our final section to a discussion and conclusions.

2 The matrix second Painlevé hierarchy

2.1 Formulation of the matrix second Painlevé hierarchy

The matrix P_{II} hierarchy was derived in [49] using the matrix KdV and mKdV hierarchies, for the properties of which we refer to [54, 55, 56, 48, 57, 58, 59]. It consists of a sequence of ODEs in the $m \times m$ matrix dependent variable u and independent variable x, which can be written

$$\psi[u]\left(\tilde{\mathcal{R}}^{n-1}[u]u_x + \sum_{k=1}^{n-1} c_k \tilde{\mathcal{R}}^{k-1}[u]u_x\right) + c_0 u + uE + Eu + 2g_{n-1}xu - \alpha_n I = 0, \qquad n = 1, 2, 3, \dots, \tag{2.1}$$

where g_{n-1} ($\neq 0$), $c_0, c_1, \dots, c_{n-1}$ and α_n are scalar constants, E is a constant $m \times m$ matrix, I is the $m \times m$ identity matrix, $\psi[u]$ and $\phi[u]$ are defined as

$$\psi[u] = (\partial_x + A_u)\partial_x^{-1}(\partial_x - A_u) = \partial_x - A_u \partial_x^{-1} A_u, \tag{2.2}$$
$$\phi[u] = \partial_x - C_u \partial_x^{-1} C_u, \tag{2.3}$$

wherein

$$A_w = L_w + R_w, \qquad C_w = L_w - R_w, \tag{2.4}$$

and the left and right multiplication operators L_w and R_w are given by

$$L_w(z) = wz, \qquad R_w(z) = zw, \tag{2.5}$$

and

$$\tilde{\mathcal{R}}[u] = \phi[u]\psi[u] \tag{2.6}$$

is the recursion operator of the matrix mKdV hierarchy. We note that in [49], although we also allowed the autonomous case $g_{n-1} = 0$, it was the nonautonomous case $g_{n-1} \neq 0$ under consideration here that we defined as a matrix P_{II} hierarchy. In this case $g_{n-1} \neq 0$ we may assume without loss of generality that $g_{n-1} = -1/2$ and, shifting x via $x \to x + c_0$, that $c_0 = 0$. This then leads us to the choice of coefficients in the matrix P_{II} hierarchy (2.1) that we will be using in the present paper:

$$\psi[u]\left(\tilde{\mathcal{R}}^{n-1}[u]u_x + \sum_{k=1}^{n-1} c_k \tilde{\mathcal{R}}^{k-1}[u]u_x\right) + uE + Eu - xu - \alpha_n I = 0. \tag{2.7}$$

Corresponding results for (2.1) with $g_{n-1} = -1/2$ are readily obtained from the results derived herein by using the inverse of the shift on x used to arrive at (2.7), i.e., by shifting $x \to x - c_0$. The first two nontrivial members of the matrix P_{II} hierarchy (2.7) are

$$u_{xx} - 2u^3 + uE + Eu - xu - \alpha_1 I = 0, \tag{2.8}$$
$$\begin{aligned} &u_{xxxx} - 4u_{xx}u^2 - 4u^2u_{xx} - 2uu_{xx}u - 2u_x^2u - 2uu_x^2 - 6u_xuu_x + 6u^5 \\ &+c_1(u_{xx} - 2u^3) + uE + Eu - xu - \alpha_2 I = 0. \end{aligned} \tag{2.9}$$

The matrix P_{II} hierarchy (2.7) can also be written as follows [49]. First of all we define

$$K[w, E] = M_n + \sum_{k=1}^{n-1} c_k M_k + E - \frac{1}{2}xI, \tag{2.10}$$

where the quantities M_k are the variational derivatives of the Hamiltonian densities ($M_k = \delta\mathcal{H}_k$) of the matrix KdV hierarchy, defined recursively via

$$M_0 = \frac{1}{2}I, \qquad \mathcal{B}_0[w]M_{j+1} = \mathcal{B}_1[w]M_j, \quad j = 0, 1, 2, \ldots, \tag{2.11}$$

$\mathcal{B}_0[w]$ and $\mathcal{B}_1[w]$ being the two Hamiltonian operators of the matrix KdV hierarchy,

$$\mathcal{B}_0[w] = \partial_x, \qquad \mathcal{B}_1[w] = \partial_x^3 + A_w\partial_x + \partial_x A_w + C_w\partial_x^{-1}C_w. \tag{2.12}$$

Thus for example

$$M_1 = w, \qquad M_2 = w_{xx} + 3w^2, \qquad M_3 = w_{xxxx} + 5ww_{xx} + 5w_{xx}w + 5w_x^2 + 10w^3. \tag{2.13}$$

(In the current paper we will not need explicit expressions for the Hamiltonian densities $\mathcal{H}_n$.) The matrix P_{II} hierarchy (2.7) can then be written

$$(\partial_x + A_u)K[M[u], E] - \left(\alpha_n - \frac{1}{2}\right) I = 0, \tag{2.14}$$

where

$$M[u] = u_x - u^2. \tag{2.15}$$

Here $w = M[u] = u_x - u^2$ is just the well-known Miura map which relates the matrix KdV and matrix mKdV hierarchies (in dependent variables w and u, respectively). The equivalence of the two formulations of the matrix P_{II} hierarchy (2.7) and (2.14), and its relationship to the matrix KdV and matrix modified KdV hierarchies, can be found in [49].

Let us also remark that since the operators $\phi[u]$ and $\psi[u]$ depend on u quadratically, then they and also $\tilde{\mathcal{R}}[u]$ are invariant under $u \to -u$. It is then clear that $(u, \alpha_n) \to (-u, -\alpha_n)$ is a discrete symmetry of the matrix P_{II} hierarchy (2.7), and using this discrete symmetry in (2.14) then leads us to a third formulation of the matrix P_{II} hierarchy as

$$(\partial_x - A_u)K[M[-u], E] + \left(\alpha_n + \frac{1}{2}\right) I = 0. \tag{2.16}$$

This discrete symmetry is one of the auto-BTs of the matrix P_{II} hierarchy. Let us now consider these auto-BTs.

2.2 Auto-Bäcklund transformations of the matrix second Painlevé hierarchy

In order that our paper be self-contained, let us recall in the following two propositions some facts (see [49, 52]) about the auto-BTs of the matrix P_{II} hierarchy.

Proposition 2.1

The matrix P_{II} hierarchy has the following three auto-BTs, which map from solutions of the matrix P_{II} hierarchy in $(v, \tilde{\alpha}_n, F)$, i.e.,

$$(\partial_x + A_v)K[M[v], F] - \left(\tilde{\alpha}_n - \frac{1}{2}\right) I = 0, \tag{2.17}$$

to solutions of the matrix P_{II} hierarchy in (u, α_n, E), i.e., (2.14):

$$f: \qquad u = v + \frac{1}{2}(\alpha_n - \tilde{\alpha}_n)K[M[v], F]^{-1}, \qquad \alpha_n = -\tilde{\alpha}_n + 1, \qquad E = F, \tag{2.18}$$

$$g: \qquad u = -v, \qquad \alpha_n = -\tilde{\alpha}_n, \qquad E = F, \tag{2.19}$$

$$k: \qquad u = v^T, \qquad \alpha_n = \tilde{\alpha}_n, \qquad E = F^T. \tag{2.20}$$

The first of these auto-BTs requires that $K[M[v],F]$ be nonsingular.

Proof

To see that f is an auto-BT, we begin by observing that $M[u]=M[v]$:

$$\begin{aligned} u_x-u^2 &= v_x-v^2-\frac{1}{2}(\alpha_n-\tilde{\alpha}_n)K[M[v],F]^{-1}\left\{(\partial_x+A_v)\,K[M[v],F]+\frac{1}{2}(\alpha_n-\tilde{\alpha}_n)I\right\}K[M[v],F]^{-1} \\ &= v_x-v^2-\frac{1}{4}(\alpha_n-\tilde{\alpha}_n)(\alpha_n+\tilde{\alpha}_n-1)K[M[v],F]^{-2} \\ &= v_x-v^2. \end{aligned} \tag{2.21}$$

Since we also have $E=F$, it follows that

$$\begin{aligned} (\partial_x+A_u)\,K[M[u],E]-\left(\alpha_n-\frac{1}{2}\right)I &= (\partial_x+A_v)\,K[M[v],F]+(\alpha_n-\tilde{\alpha}_n)I-\left(\alpha_n-\frac{1}{2}\right)I \\ &= (\partial_x+A_v)\,K[M[v],F]-\left(\tilde{\alpha}_n-\frac{1}{2}\right)I=0, \end{aligned} \tag{2.22}$$

and so we see that (2.18) maps from solutions of (2.17) to solutions of (2.14).

The auto-BT g is the auto-BT corresponding to the discrete symmetry $(u,\alpha_n)\to(-u,-\alpha_n)$ of the matrix P_{II} hierarchy (2.14), or equivalently (2.7), as discussed at the end of Section 2.1.

To see that k is an auto-BT, let us begin by observing that if $\mathcal{Z}[u]$ satisfies $\mathcal{Z}[v^T]=(\mathcal{Z}[v])^T$, then $\psi[v^T]\mathcal{Z}[v^T]=(\psi[v]\mathcal{Z}[v])^T$ and $\phi[v^T]\mathcal{Z}[v^T]=(\phi[v]\mathcal{Z}[v])^T$; it then also follows that $\tilde{\mathcal{R}}[v^T]\mathcal{Z}[v^T]=(\tilde{\mathcal{R}}[v]\mathcal{Z}[v])^T$. Noting the identity

$$K[M[u],E]=(I-\partial_x^{-1}A_u)\left[\tilde{\mathcal{R}}^{n-1}[u]u_x+\sum_{k=1}^{n-1}c_k\tilde{\mathcal{R}}^{k-1}[u]u_x\right]+E-\frac{1}{2}xI, \tag{2.23}$$

we thus see, since $(v^T)_x=(v_x)^T$, that

$$K[M[v^T],F^T]=K[M[v],F]^T. \tag{2.24}$$

It then follows that

$$\begin{aligned} (\partial_x+A_u)\,K[M[u],E]-\left(\alpha_n-\frac{1}{2}\right)I &= (\partial_x+A_{v^T})\,K[M[v^T],F^T]-\left(\tilde{\alpha}_n-\frac{1}{2}\right)I \\ &= (\partial_x+A_{v^T})\,(K[M[v],F])^T-\left(\tilde{\alpha}_n-\frac{1}{2}\right)I \\ &= \left[(\partial_x+A_v)\,K[M[v],F]-\left(\tilde{\alpha}_n-\frac{1}{2}\right)I\right]^T=0, \end{aligned} \tag{2.25}$$

and so we see that (2.20) maps from solutions of (2.17) to solutions of (2.14).
□

Proposition 2.2

The auto-BTs f, g and k satisfy the following:
(a) each is an involution, i.e., $f^2=1$, $g^2=1$ and $k^2=1$;
(b) k commutes with both f and g, i.e., $kf=fk$ and $kg=gk$.

Proof

(a) In order to see that f is an involution, we consider a second iteration from (2.14) to a solution of the matrix P_{II} hierarchy in $(w, \widehat{\alpha}_n, G)$, i.e.,

$$(\partial_x + A_w)K[M[w], G] - \left(\widehat{\alpha}_n - \frac{1}{2}\right) I = 0, \tag{2.26}$$

given by

$$w = u + \frac{1}{2}(\widehat{\alpha}_n - \alpha_n)K[M[u], E]^{-1}, \qquad \widehat{\alpha}_n = -\alpha_n + 1, \qquad G = E. \tag{2.27}$$

We then express $(w, \widehat{\alpha}_n, G)$ in terms of $(v, \tilde{\alpha}_n, F)$: since we have u given by (2.18), and since $M[u] = M[v]$ and $E = F$, we obtain

$$\begin{aligned} w &= u + \frac{1}{2}(\widehat{\alpha}_n - \alpha_n)K[M[u], E]^{-1} = v + \frac{1}{2}(\alpha_n - \tilde{\alpha}_n)K[M[v], F]^{-1} + \frac{1}{2}(\widehat{\alpha}_n - \alpha_n)K[M[v], F]^{-1} \\ &= v + \frac{1}{2}(\widehat{\alpha}_n - \tilde{\alpha}_n)K[M[v], F]^{-1}. \end{aligned} \tag{2.28}$$

But

$$\widehat{\alpha}_n = -\alpha_n + 1 = -(-\tilde{\alpha}_n + 1) + 1 = \tilde{\alpha}_n \tag{2.29}$$

and so $w = v$. Also, $G = E = F$. Thus we have $w = v$, $\widehat{\alpha}_n = \tilde{\alpha}_n$ and $G = F$, i.e., f is an involution.

As for the auto-BTs g and k, these are clearly also involutions.

(b) To see that k commutes with f, we consider first of all mapping from (2.17) to (2.14) via f (2.18), and then from (2.14) to (2.26) via k, i.e., via $w = u^T$, $\widehat{\alpha}_n = \alpha_n$, $G = E^T$. We thus obtain:

$$w = u^T = v^T + \frac{1}{2}(1 - 2\tilde{\alpha}_n)\big(K[M[v], F]^{-1}\big)^T = v^T + \frac{1}{2}(1 - 2\tilde{\alpha}_n)\big(K[M[v], F]^T\big)^{-1}, \tag{2.30}$$

$$\widehat{\alpha}_n = \alpha_n = -\tilde{\alpha}_n + 1, \tag{2.31}$$

$$G = E^T = F^T. \tag{2.32}$$

On the other hand, mapping from (2.17) to (2.14) via k (2.20), and then from (2.14) to (2.26) via f (2.27) gives:

$$w = u + \frac{1}{2}(1 - 2\alpha_n)K[M[u], E]^{-1} = v^T + \frac{1}{2}(1 - 2\tilde{\alpha}_n)\big(K[M[v^T], F^T]\big)^{-1}, \tag{2.33}$$

$$\widehat{\alpha}_n = -\alpha_n + 1 = -\tilde{\alpha}_n + 1, \tag{2.34}$$

$$G = E = F^T. \tag{2.35}$$

Given that equation (2.24) holds, we see that the above expressions for w, $\widehat{\alpha}_n$ and G in terms of v, $\tilde{\alpha}_n$ and F coincide, i.e., k and f commute, $kf = fk$.

The auto-BTs k and g also clearly commute, $kg = gk$.

□

The group of auto-BTs of the matrix P_{II} hierarchy has generators f, g and k as given in Proposition 2.1. These generators are subject to the relations given in Proposition 2.2, i.e. $f^2 = 1$, $g^2 = 1$, $k^2 = 1$, $kf = fk$ and $kg = gk$. Given the first three relations, we see that the last two can also be written as $(fk)^2 = 1$ and $(gk)^2 = 1$, respectively. Thus the group of auto-BTs of the matrix P_{II} hierarchy has the presentation

$$G = \langle f, g, k \; ; \; f^2 = g^2 = k^2 = (fk)^2 = (gk)^2 = 1 \rangle, \tag{2.36}$$

and is isomorphic to the direct product of the affine Weyl group of type $A_1^{(1)}$ with the cyclic group $\mathbb{Z}_2$, i.e., $G \cong A_1^{(1)} \times Z_2$ [49, 52]. For the special case where all matrices (i.e., the dependent variable and the coefficient matrix) are symmetric, k is just the identity transformation and so the group of auto-BTs for this restricted case is, as it is for the scalar case, $A_1^{(1)}$. (From (2.24) we see that, if v and F are symmetric, so is the result of applying f and g; thus, for symmetric v and F, the result of applying any combination of f and g is symmetric.)

In order to discuss the iteration of the above three auto-BTs, let us consider the two composite auto-BTs $r = gf$ and $s = fg$:

$$r = gf: \qquad u = -v + \frac{1}{2}(\alpha_n + \tilde{\alpha}_n) K[M[v], F]^{-1}, \qquad \alpha_n = \tilde{\alpha}_n - 1, \qquad E = F, \tag{2.37}$$

$$s = fg: \qquad u = -v + \frac{1}{2}(\alpha_n + \tilde{\alpha}_n) K[M[-v], F]^{-1}, \qquad \alpha_n = \tilde{\alpha}_n + 1, \qquad E = F. \tag{2.38}$$

We note in passing that these transformations are inverse to each other: $rs = (gf)(fg) = gf^2g = g^2 = 1$. From the defining relations of the group G, it is clear that any composition of f, g and k can be written in one of the following forms:

$$k^{\epsilon_1} f^{\epsilon_2} (gf)^q = k^{\epsilon_1} f^{\epsilon_2} r^q, \qquad \epsilon_1, \epsilon_2 \in \{0, 1\}, q \in \{0, 1, 2 \ldots\}; \tag{2.39}$$

$$k^{\epsilon_1} g^{\epsilon_2} (fg)^q = k^{\epsilon_1} g^{\epsilon_2} s^q, \qquad \epsilon_1, \epsilon_2 \in \{0, 1\}, q \in \{0, 1, 2 \ldots\} \tag{2.40}$$

(which coincide when each has $\epsilon_2 = q = 0$ and the same ϵ_1). We note that the first of these compositions maps a solution of (2.17) for initial parameter value $\tilde{\alpha}_n = \beta$ to a solution of (2.17) for parameter value either $\tilde{\alpha}_n = \beta - q$ (if $\epsilon_2 = 0$) or $\tilde{\alpha}_n = -\beta + q + 1$ (if $\epsilon_2 = 1$), and the second maps a solution of (2.17) for initial parameter value $\tilde{\alpha}_n = \beta$ to a solution of (2.17) for parameter value either $\tilde{\alpha}_n = \beta + q$ (if $\epsilon_2 = 0$) or $\tilde{\alpha}_n = -\beta - q$ (if $\epsilon_2 = 1$).

Let us now consider the iteration of solutions of (2.17), beginning with seed solutions for initial parameter values $\tilde{\alpha}_n = \beta = 0$ and $\tilde{\alpha}_n = \beta = \frac{1}{2}$. The motivation for this lies in the corresponding possible classes of initial solutions discussed in Section 3.2 for initial parameter values $\tilde{\alpha}_n = \beta = 0$ and $\tilde{\alpha}_n = \beta = \frac{1}{2}$, and which are extensions to our matrix case of the initial solutions of scalar P_{II} for these same parameter values. First, we consider the case $\beta = 0$, and second, the case $\beta = \frac{1}{2}$.

We consider the iteration of solutions of (2.17), beginning with a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = 0$. The composition (2.39) with $\epsilon_2 = 1$ and $q = t - 1 \geq 0$ yields solutions $v_1 = fr^{t-1}v_0$ and kv_1 of (2.17) for integer parameter value $\tilde{\alpha}_n = t \geq 1$. The composition (2.40) with $\epsilon_2 = 0$ and $q = t \geq 0$ yields solutions $v_2 = s^t v_0$ and kv_2 of (2.17) for integer parameter value $\tilde{\alpha}_n = t \geq 0$.

Lemma 2.1

The solutions v_1, v_2, kv_1 and kv_2 obtained as described above for each positive integer parameter value $\tilde{\alpha}_n = t \geq 1$ satisfy:
(a) $v_1 = v_2 \iff v_0 = 0$;
(b) $v_1 = kv_1 \iff v_0 = kv_0 \iff v_2 = kv_2$;
(c) $v_2 = kv_1 \iff v_1 = kv_2 \iff v_0 = gkv_0$.
(The solutions $v_2 = v_0$ and $kv_2 = kv_0$, for parameter value $\tilde{\alpha}_n = 0$, are included in Lemma 2.2 below.)

Proof

(a) $v_1 = v_2 \iff fr^{t-1}v_0 = s^t v_0 \iff (fg)^{t-1} f v_0 = (fg)^t v_0 \iff v_0 = gv_0 \iff v_0 = -v_0 \iff v_0 = 0$
(b) $v_1 = kv_1 \iff fr^{t-1}v_0 = kfr^{t-1}v_0 \iff fr^{t-1}v_0 = fr^{t-1}kv_0 \iff v_0 = kv_0$, and
$v_0 = kv_0 \iff s^t v_0 = s^t kv_0 \iff s^t v_0 = ks^t v_0 \iff v_2 = kv_2$
(c) $v_2 = kv_1 \iff v_1 = kv_2$ since $k^2 = 1$, and
$v_1 = kv_2 \iff fr^{t-1}v_0 = ks^t v_0 \iff (fg)^{t-1} f v_0 = k(fg)^t v_0 \iff (fg)^{t-1} f v_0 = (fg)^{t-1} fgkv_0$
$\iff v_0 = gkv_0$
□

We consider again the iteration of solutions of (2.17), beginning with a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = 0$. The composition (2.39) with $\epsilon_2 = 0$ and $q = t \geq 0$ yields solutions $v_3 = r^t v_0$ and kv_3 of (2.17) for integer parameter value $\tilde{\alpha}_n = -t \leq 0$. The composition (2.40) with $\epsilon_2 = 1$ and $q = t \geq 0$ yields solutions $v_4 = gs^t v_0$ and kv_4 of (2.17) for integer parameter value $\tilde{\alpha}_n = -t \leq 0$.

Lemma 2.2

The solutions v_3, v_4 kv_3 and kv_4 obtained as described above for each non-positive integer parameter value $\tilde{\alpha}_n = -t \leq 0$ satisfy:
(a) $v_3 = v_4 \iff v_0 = 0$;
(b) $v_3 = kv_3 \iff v_0 = kv_0 \iff v_4 = kv_4$;
(c) $v_4 = kv_3 \iff v_3 = kv_4 \iff v_0 = gkv_0$.

Proof

(a) $v_3 = v_4 \iff r^t v_0 = gs^t v_0 \iff (gf)^t v_0 = (gf)^t g v_0 \iff v_0 = gv_0 \iff v_0 = -v_0 \iff v_0 = 0$
(b) $v_3 = kv_3 \iff r^t v_0 = kr^t v_0 \iff r^t v_0 = r^t k v_0 \iff v_0 = kv_0$, and
$v_0 = kv_0 \iff gs^t v_0 = gs^t k v_0 \iff gs^t v_0 = kgs^t v_0 \iff v_4 = kv_4$
(c) $v_4 = kv_3 \iff v_3 = kv_4$ since $k^2 = 1$, and
$v_3 = kv_4 \iff r^t v_0 = kgs^t v_0 \iff (gf)^t v_0 = k(gf)^t g v_0 \iff (gf)^t v_0 = (gf)^t gk v_0 \iff v_0 = gkv_0$
□

Remark 2.1

In the compositions (2.39) and (2.40), when acting with k (i.e., when $\epsilon_1 = 1$), the solutions obtained are in fact solutions of (2.17) for coefficient matrix F^T. Thus, if we assume $F = F^T$, then the solutions obtained are all solutions of the *same equation*, since in (2.17) we always have the same coefficient matrix F.

Taking Remark 2.1 into account, Lemmas 2.1 and 2.2 lead us to the following:

Proposition 2.3

Given a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = 0$:
(1) for each integer parameter value α_n the auto-BTs f and g yield either: exactly one solution of (2.17), when $v_0 = 0$; or two distinct solutions of (2.17), when $v_0 \neq 0$.
(2) when $F = F^T$, then for each integer parameter value α_n the auto-BTs f, g and k yield: exactly one solution of (2.17), when $v_0 = 0$; two distinct solutions of (2.17), when v_0 is nonzero symmetric or nonzero antisymmetric; or four distinct solutions of (2.17), when v_0 is neither symmetric nor antisymmetric.

We now consider the iteration of solutions of (2.17), beginning with a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = \frac{1}{2}$. The composition (2.39) with $\epsilon_2 = 1$ and $q = t \geq 0$ yields solutions $v_1 = fr^t v_0$ and kv_1 of (2.17) for half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$. The composition (2.40) with $\epsilon_2 = 0$ and $q = t \geq 0$ yields solutions $v_2 = s^t v_0$ and kv_2 of (2.17) for half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$.

Lemma 2.3

The solutions v_1, v_2, kv_1 and kv_2 obtained as described above for each positive half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2}$, $t \geq 0$, satisfy:
(a) $v_1 = v_2$;
(b) $v_1 = kv_1 \iff v_0 = kv_0$.

Proof

(a) $v_1 = v_2 \iff fr^t v_0 = s^t v_0 \iff (fg)^t f v_0 = (fg)^t v_0 \iff fv_0 = v_0$, which is satisfied since if in (2.18) $\tilde{\alpha}_n = \frac{1}{2}$, then $\alpha_n = \frac{1}{2}$ and $u = v$ (even if $K[M[v], F]$ in (2.18) is singular, if $\tilde{\alpha}_n = \frac{1}{2}$ we may define $u = v$)

(b) $v_1 = kv_1 \iff fr^t v_0 = kfr^t v_0 \iff fr^t v_0 = fr^t kv_0 \iff v_0 = kv_0$
□

We now consider again the iteration of solutions of (2.17), beginning with a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = \frac{1}{2}$. The composition (2.39) with $\epsilon_2 = 0$ and $q = t+1 \geq 1$ yields solutions $v_3 = r^{t+1}v_0$ and kv_3 of (2.17) for half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$. The composition (2.40) with $\epsilon_2 = 1$ and $q = t \geq 0$ yields solutions $v_4 = gs^t v_0$ and kv_4 of (2.17) for half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$.

Lemma 2.4

The solutions v_3, v_4, kv_3 and kv_4 obtained as described above for each negative half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2}$, $t \geq 0$, satisfy:
(a) $v_3 = v_4$;
(b) $v_3 = kv_3 \iff v_0 = kv_0$.
(The solutions $v_3 = v_0$ and $kv_3 = kv_0$, for parameter value $\tilde{\alpha}_n = \frac{1}{2}$, are included in Lemma 2.3 above.)

Proof

(a) $v_3 = v_4 \iff r^{t+1}v_0 = gs^t v_0 \iff (gf)^t gfv_0 = (gf)^t gv_0 \iff fv_0 = v_0$, which is satisfied since if in (2.18) $\tilde{\alpha}_n = \frac{1}{2}$, then $\alpha_n = \frac{1}{2}$ and $u = v$ (even if $K[M[v], F]$ in (2.18) is singular, if $\tilde{\alpha}_n = \frac{1}{2}$ we may define $u = v$)
(b) $v_3 = kv_3 \iff r^{t+1}v_0 = kr^{t+1}v_0 \iff r^{t+1}v_0 = r^{t+1}kv_0 \iff v_0 = kv_0$
□

Taking Remark 2.1 into account, Lemmas 2.3 and 2.4 lead us to the following:

Proposition 2.4

Given a seed solution v_0 for initial parameter value $\tilde{\alpha}_n = \beta = \frac{1}{2}$ (and recalling that we may define $fv_0 = v_0$):
(1) for each half-odd-integer parameter value α_n the auto-BTs f and g yield exactly one solution of (2.17).
(2) when $F = F^T$, then for each half-odd-integer parameter value α_n the auto-BTs f, g and k yield either: exactly one solution of (2.17), when v_0 is symmetric; or two distinct solutions of (2.17), when v_0 is not symmetric.

Remark 2.2

With respect to Propositions 2.3 and 2.4, we note that when F is symmetric:
(1) the assumption that v is symmetric gives a consistent reduction of equation (2.17), from equations for m^2 scalar variables to equations for $\frac{1}{2}m(m+1)$ scalar variables.
(2) the assumption that v is antisymmetric gives, for $\tilde{\alpha}_n = 0$, a consistent reduction of equation (2.17), from equations for m^2 scalar variables to equations for $\frac{1}{2}m(m-1)$ scalar variables.

3 Classes of solutions of the matrix second Painlevé hierarchy

3.1 On the iteration of auto-BTs

3.1.1 Mappings of the coefficient matrix

We now turn to the question of the iteration of auto-BTs in order to generate solutions of the matrix P_{II} hierarchy,

$$\psi[\hat{v}]\left(\tilde{\mathcal{R}}^{n-1}[\hat{v}]\hat{v}_x + \sum_{k=1}^{n-1} c_k \tilde{\mathcal{R}}^{k-1}[\hat{v}]\hat{v}_x\right) + \hat{v}\hat{F} + \hat{F}\hat{v} - x\hat{v} - \tilde{\alpha}_n I = 0. \tag{3.1}$$

This equation is polynomial in $\hat{F}$, $\hat{v}$ and derivatives of $\hat{v}$, with an additive terms a multiple of I, so the substitution

$$p: \qquad \hat{v} = PvP^{-1}, \qquad \hat{F} = PFP^{-1}, \tag{3.2}$$

where P is a nonsingular constant matrix, yields

$$\psi[v]\left(\tilde{\mathcal{R}}^{n-1}[v]v_x + \sum_{k=1}^{n-1} c_k \tilde{\mathcal{R}}^{k-1}[v]v_x\right) + vF + Fv - xv - \tilde{\alpha}_n I = 0. \tag{3.3}$$

We will refer (3.2) as the mapping p. This is a mapping $\hat{v} = pv$ from solutions of (3.3) to solutions of (3.1).

The above means that we can always make a transformation from (3.1) to an equation (3.3) with coefficient matrix F similar to $\hat{F}$ and the same parameter $\tilde{\alpha}_n$. The most obvious choice is to take F in Jordan canonical form. In particular, if $\hat{F}$ is a normal matrix (i.e., $\hat{F}^*\hat{F} = \hat{F}\hat{F}^*$, where $\hat{F}^*$ is the conjugate transpose of $\hat{F}$), then we may take F to be diagonal. Normal matrices include Hermitian and real symmetric matrices (where for such $\hat{F}$ the diagonal matrix F is real), and skew-Hermitian and real skew-symmetric matrices (where for such $\hat{F}$ the diagonal matrix F is pure imaginary). However, F may in fact be taken to be any matrix similar to $\hat{F}$, e.g., we may always assume F to be upper-triangular, or symmetric. Here we prefer to think of p as a transformation which allows us to fix a form of the coefficient matrix, rather than as an auto-BT; see also Appendix A.

It is straightforward to show that the transformation p commutes with the auto-BTs f (2.18) and g (2.19), and so also with the auto-BTs $r = gf$ (2.37) and $s = fg$ (2.38). Thus the result of acting with any composition of the four auto-BTs f, g, r and s can be calculated either at the level of equation (3.1) or at the level of equation (3.3). However, the transformation p does not commute with the auto-BT k (2.20) unless P^TP commutes with both F and v, which requires, in the general case, that $P^TP = \gamma I$ for some constant $\gamma \neq 0$. (If, for real symmetric $\hat{F}$ we map onto a diagonal F using an orthogonal P, this condition is satisfied since then $P^TP = I$.)

3.1.2 Compositions of auto-BTs

From the above we see that we may undertake the generation of solutions of equation (3.1) using the iteration of auto-BTs as follows. Given an initial solution $\hat{v}_0$ of (3.1), we obtain a corresponding solution $v_0 = p^{-1}\hat{v}_0$ of (3.3), for some F similar to $\hat{F}$. Alternatively, in the absence of a solution of (3.1), we may transform to (3.3) in order to simplify the search for an initial solution using an ansatz: if we take F to be in Jordan canonical form, or simply upper-triangular, then we may use as an ansatz that the initial solution v_0 of (3.3) is upper-triangular; if we take F to be symmetric, then we may use as an ansatz that v_0 is symmetric or, for $\tilde{\alpha}_n = 0$, antisymmetric (see Remark 2.2). In either case, whether $\hat{v}_0$ is given in advance or not, we begin with initial solutions v_0 (for some F similar to $\hat{F}$) and $\hat{v}_0 = pv_0$. We undertake the iteration at the level of equation (3.3), and then return to (3.1), rather than iterating at the level of (3.1) itself, noting that for the compositions (2.39) and (2.40),

$$k^{\epsilon_1} f^{\epsilon_2} r^q \hat{v}_0 = k^{\epsilon_1} f^{\epsilon_2} r^q p v_0 = k^{\epsilon_1} p f^{\epsilon_2} r^q v_0 \quad \text{and} \quad k^{\epsilon_1} g^{\epsilon_2} s^q \hat{v}_0 = k^{\epsilon_1} g^{\epsilon_2} s^q p v_0 = k^{\epsilon_1} p g^{\epsilon_2} s^q v_0. \tag{3.4}$$

Here we act with k (when $\epsilon_1 = 1$) at the level of equation (3.1). We expect this scheme to simplify the process of generating solutions, as well as the use of an ansatz to obtain an initial solution, as we expect F in (3.3) to have been chosen to be simpler than $\hat{F}$ in (3.1). (We note that the case where $\hat{F}$ is already of a suitable form is included here as $P = I$, for which choice $\hat{v}_0 = v_0$, $\hat{F} = F$, and (3.1) and (3.3) coincide.)

3.1.3 Upper-triangular matrices

As indicated above, the case of upper-triangular matrices is of particular interest, and will be of great use in our discussion of the iteration of solutions. Since we may always take F in (3.3) to be in Jordan canonical form or even just upper-triangular, if using an ansatz to obtain a corresponding initial solution v_0 of (3.3), we may then ask that this initial solution also be upper-triangular. For upper-triangular v_0 and F, since xI is also upper triangular, the auto-BTs f, g, r and s yield upper-triangular matrices: any solution v of (3.3) generated from such an initial solution v_0 using these auto-BTs will also be upper-triangular.

Let us denote the components of v and F by v_{ij} and F_{ij} respectively. We note that if v and F in (3.3) are upper-triangular, then the equations for the $\frac{1}{2}m(m+1)$ elements v_{ij} of v are nonlinear if $j=i$ and linear if $j>i$. Let us now consider these nonlinear equations for v_{ii}, as this will prove useful later.

Given a scalar function $\bar{v}$ of x, and a scalar constant $\bar{F}$, we define the scalar quantities $G[\bar{v},\bar{F}]$ and $H[\bar{v},\bar{F}]$ via

$$G[\bar{v},\bar{F}]I = K[M[\bar{v}I],\bar{F}I], \qquad H[\bar{v},\bar{F}]I = K[M[-\bar{v}I],\bar{F}I]. \tag{3.5}$$

In the case where both v and F are upper-triangular, the diagonal elements of $K[M[v],F]$ and $K[M[-v],F]$ are then given by $G[v_{ii},F_{ii}]$ and $H[v_{ii},F_{ii}]$ respectively, and the diagonal elements of the matrix P_{II} hierarchy (3.3) by

$$(\partial_x + 2v_{ii})G[v_{ii},F_{ii}] - \left(\tilde{\alpha}_n - \frac{1}{2}\right) = 0, \qquad i = 1,2,\ldots,m, \tag{3.6}$$

or alternatively

$$(\partial_x - 2v_{ii})H[v_{ii},F_{ii}] + \left(\tilde{\alpha}_n + \frac{1}{2}\right) = 0, \qquad i = 1,2,\ldots,m \tag{3.7}$$

(see the formulations (2.14)) and (2.16)) of the matrix P_{II} hierarchy.)

The actions of the auto-BTs f, g, $r=gf$ and $s=fg$ on (3.3) induce mappings of these diagonal elements given by

$$\begin{aligned}
u_{ii} &= v_{ii} + \frac{1}{2}(\alpha_n - \tilde{\alpha}_n)G[v_{ii},F_{ii}]^{-1}, & \alpha_n &= -\tilde{\alpha}_n + 1, & (3.8)\\
u_{ii} &= -v_{ii}, & \alpha_n &= -\tilde{\alpha}_n, & (3.9)\\
u_{ii} &= -v_{ii} + \frac{1}{2}(\alpha_n + \tilde{\alpha}_n)G[v_{ii},F_{ii}]^{-1}, & \alpha_n &= \tilde{\alpha}_n - 1, & (3.10)\\
u_{ii} &= -v_{ii} + \frac{1}{2}(\alpha_n + \tilde{\alpha}_n)H[v_{ii},F_{ii}]^{-1}, & \alpha_n &= \tilde{\alpha}_n + 1, & (3.11)
\end{aligned}$$

respectively. Equation (3.6) is just the scalar generalized P_{II} hierarchy, and (3.8)—(3.11) are its well-known auto-BTs: see [40, 42], as well of course as [2] for the standard case with $c_k = 0$, $k = 1,2,\ldots,n-1$, and $F_{ii} = 0$ (an alternative formulation being given in [17]). These auto-BTs map from solutions v_{ii} of (3.6) for parameter value $\tilde{\alpha}_n$ to solutions u_{ii} of the same equation for parameter value α_n, i.e.,

$$(\partial_x + 2u_{ii})G[u_{ii},F_{ii}] - \left(\alpha_n - \frac{1}{2}\right) = 0. \tag{3.12}$$

Taking into account the above considerations, we obtain:

Lemma 3.1

Let v be an upper-triangular solution of (3.3) for upper-triangular F. Then, if the matrix $K[M[v],F]$ is singular, we must have $\tilde{\alpha}_n = 1/2$. Similarly, if the matrix $K[M[-v],F]$ is singular, then we must have $\tilde{\alpha}_n = -1/2$.

Proof

If v and F are upper-triangular, then $\det\left(K[M[v],F]\right) = \prod_{i=1}^{m} G[v_{ii},F_{ii}]$, and if for some $i = 1,2,\ldots m$ we have $G[v_{ii},F_{ii}] = 0$ then, since v_{ii} satisfies (3.6), we must have $\tilde{\alpha}_n = 1/2$. Similarly, if v and F are upper triangular, then $\det\left(K[M[-v],F]\right) = \prod_{i=1}^{m} H[v_{ii},F_{ii}]$, and if for some $i = 1,2,\ldots m$ we have $H[v_{ii},F_{ii}] = 0$ then, since v_{ii} satisfies (3.7), we must have $\tilde{\alpha}_n = -1/2$.
□

Remark 3.1

There is a further simplification that we may make in equations (3.1) and (3.3). We recall that in Section 2.1 we used a shift in x to set $c_0 = 0$ in our matrix P_{II} hierarchy. We may further shift x in (3.1) via $x \to x + 2\lambda_k$, where λ_k is any one of the eigenvalues of $\hat{F}$: absorbing this shift in the coefficient matrix $\hat{F}$, we see that the result is to replace $\hat{F}$ in (3.1) by $\hat{F} - \lambda_k I$. This then means replacing F in (3.3) by $F - \lambda_k I$. Thus, when F is upper triangular, we can always use a shift on x to set at least one of its diagonal elements equal to zero (perhaps making others nonzero): see the examples in Section 4.

3.2 Classes of initial solutions

Let us now turn to the iterative generation of solutions of equation (3.1) by means of the auto-BTs f, g and k. We thus consider the problem of finding initial solutions of equation (3.3), or equivalently (2.17), including through the use of an ansatz, where F is similar to some original $\hat{F}$ appearing in (3.1). This represents the extension foreseen in [52] of our results therein for the matrix P_{II} equation to the matrix P_{II} hierarchy. We discuss here four classes of initial solution of (3.3). The first two classes of initial solution that we consider are direct analogs of the initial solutions used in the scalar P_{II} case. We assume throughout this section, and without loss of generality, that F in equation (3.3) has been taken to be (in Jordan canonical form or) upper triangular.

3.2.1 Initial solution A

As our first class of initial solution of the matrix hierarchy (3.3) we take:

$$v_0 = 0 \qquad \text{for parameter value} \qquad \tilde{\alpha}_n = \beta = 0. \tag{3.13}$$

Since $v_0 = 0$, F and xI are upper-triangular, then so are the solutions $v_1 = fr^{t-1}v_0$ and $v_2 = s^t v_0$ of (3.3) obtained as in Lemma 2.1 for each positive integer parameter value $\tilde{\alpha}_n = t = 1, 2, 3, \ldots$, as well as the solutions $v_3 = r^t v_0$ and $v_4 = gs^t v_0$ of (3.3) obtained as in Lemma 2.2 for each non-positive integer parameter value $\tilde{\alpha}_n = -t = 0, -1, -2, -3, \ldots$. Since in the construction of these solutions the auto-BTs f, r and s are applied to solutions of (3.3) corresponding to integer values of the parameter $\tilde{\alpha}$, we see from Lemma 3.1 that the matrices $K[M[v], F]$ and $K[M[-v], F]$ appearing in these auto-BTs are always non-singular.

The first part of Proposition 2.3 applied to equation (3.3) then tells us that $v_1 = v_2$ and $v_3 = v_4$, i.e., using the auto-BTs f and g, we obtain exactly one solution of equation (3.3) for each integer value of the parameter $\tilde{\alpha}_n$.

Proceeding as described in Section 3.1.2, the transformation p then yields exactly one solution $pv_1 = pv_2$ of (3.1) for each positive integer value of the parameter $\tilde{\alpha}_n$, as well as exactly one solution $pv_3 = pv_4$ of (3.1) for each non-positive integer value of the parameter $\tilde{\alpha}_n$: these solutions, exactly one for each integer value of the parameter $\tilde{\alpha}_n$, correspond to the action of compositions of the auto-BTs f and g on the initial solution $\hat{v}_0 = pv_0 = 0$ of (3.1).

We now assume that $\hat{F}$ is symmetric and apply the second part of Proposition 2.3 to equation (3.1): since $\hat{v}_0 = 0$, the auto-BT k does not yield additional solutions of (3.1). Compositions of the auto-BTs f, g and k yield, for each integer value of the parameter $\tilde{\alpha}_n$, exactly one solution of (3.1) with $\hat{F}$ symmetric.

The solutions of (3.1) described here are matrix analogues of the rational solutions of the scalar P_{II} hierarchy (the iterative construction of rational solutions of members of the scalar P_{II} hierarchy has been considered in [60, 61, 62, 40]). In the special case $\hat{F} = 0$, they reduce to the form (rational solution of scalar hierarchy)$\times I$.

3.2.2 Initial solution B

As our second class of initial solution of the matrix hierarchy (3.3) we take:

$$v_0 \text{ general solution of } K[M[v],F]=0 \qquad \text{for parameter value} \qquad \tilde{\alpha}_n=\beta=\frac{1}{2}, \tag{3.14}$$

i.e., the general solution of the basic special integral $K[M[v],F]=0$ of the matrix P_{II} hierarchy (3.3) (see (2.17)). This equation can be written as the system

$$K[w,F]=0, \qquad\qquad w=M[v]=v_x-v^2, \tag{3.15}$$

from where, linearizing the second of these equations, we see that the general solution of (3.14) can be obtained from that of the system

$$K[w,F] = 0, \tag{3.16}$$

$$y_{xx}+wy = 0 \tag{3.17}$$

as $v=-y_xy^{-1}$. Recall that F is upper-triangular.

The equation

$$K[w,F]=M_n+\sum_{k=1}^{n-1}c_kM_k+F-\frac{1}{2}xI=0, \tag{3.18}$$

for $n\geq 2$, is just the matrix P_I hierarchy [49]. Our initial solution v_0 is to be obtained by solving the equation $y_{xx}+wy=0$, where w is the general solution of (3.18), and then setting $v=-y_xy^{-1}$. Of course, we do not expect explicit solutions of (3.18): even the simplest case $n=2$ with w and F scalar, equivalent to the first Painlevé equation, does not have any solutions expressible in terms of classical functions. For $n=1$, equation (3.18) reads $w+F-\frac{1}{2}xI=0$, and so y is to be obtained as the solution of the matrix Airy equation

$$y_{xx}=\left(F-\frac{1}{2}xI\right)y. \tag{3.19}$$

This case $n=1$ was discussed in [52].

Beginning with the initial solution defined by (3.14), we obtain, as in Lemma 2.3, the solutions $v_1=fr^tv_0$ and $v_2=s^tv_0$ of (3.3) for each positive half-odd-integer parameter value $\tilde{\alpha}_n=t+\frac{1}{2}=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$, and also, as in Lemma 2.4, the solutions $v_3=r^{t+1}v_0$ and $v_4=gs^tv_0$ of (3.3) for each negative half-odd-integer parameter value $\tilde{\alpha}_n=-t-\frac{1}{2}=-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots$. Here we define $fv_0=v_0$, which implies $rv_0=-v_0$, even though $K[M[v_0],F]=0$.

In the construction of these solutions, we require that the matrices $K[M[v],F]$ and $K[M[-v],F]$ appearing in the auto-BTs f, r and s are non-singular. In order to show that this is indeed so, let us consider the special case of upper-triangular matrices. That is, instead of the general solution of $K[M[v],F]=0$, we ask that v_0 be the general upper-triangular matrix solution of this equation. This solution can be obtained as $v=-y_xy^{-1}$, where y is the general upper-triangular matrix solution of (3.17) and w the general upper-triangular matrix solution of (3.18) (we recall that F has been taken to be upper-triangular). For this special choice of v_0, all solutions constructed as described above are also upper-triangular (since xI is also upper-triangular). In the construction of v_1, for $t=0$ we obtain $v_1=v_0$, and for $t>0$ the first application of r yields $-v_0$, solution for parameter value $\tilde{\alpha}_n=-\frac{1}{2}$. Similarly, in the construction of v_3, for all $t\geq 0$, the first application of r again yields $-v_0$, solution for parameter value $\tilde{\alpha}_n=-\frac{1}{2}$. These results are by definition, and are the only applications of f and r in v_1 and v_3 to a solution of (3.3) for parameter value $\tilde{\alpha}_n=\frac{1}{2}$. Furthermore, the auto-BT s used in the construction of v_2 and v_4 is never applied to a solution of (3.3) for parameter value $\tilde{\alpha}_n=-\frac{1}{2}$. From Lemma

3.1 it then follows that, in the construction of these upper-triangular solutions, the matrices $K[M[v],F]$ and $K[M[-v],F]$ appearing in the auto-BTs f, r and s are always non-singular (except for three special instances, where the result of the application of f and r on v_0 has been defined). Since taking v_0 to be the general upper-triangular matrix solution of $K[M[v],F]=0$ represents making a choice of particular solution of this equation, it follows that the matrices $K[M[v],F]$ and $K[M[-v],F]$ appearing in the auto-BTs f, r and s are always non-singular where our initial solution is instead defined as in (3.14) (again, except for three special instances, where the result of the application of f and r on v_0 has been defined).

The first part of Proposition 2.4 applied to equation (3.3) then tells us that, with our initial solution defined as in (3.14), $v_1=v_2$ and $v_3=v_4$, i.e., using the auto-BTs f and g we obtain exactly one solution of equation (3.3) for each half-odd-integer value of the parameter $\tilde{\alpha}_n$.

Proceeding as described in Section 3.1.2, the transformation p then yields exactly one solution $pv_1=pv_2$ of (3.1) for each positive half-odd-integer value of the parameter $\tilde{\alpha}_n$, as well as exactly one solution $pv_3=pv_4$ of (3.1) for each negative half-odd-integer value of the parameter $\tilde{\alpha}_n$: these solutions, exactly one for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, correspond to the action of compositions of the auto-BTs f and g on the initial solution $\hat{v}_0=pv_0$ of (3.1).

We now assume that $\hat{F}$ is symmetric and apply the second part of Proposition 2.4 to equation (3.1): the auto-BT k yields an additional solution of (3.1) if and only if $\hat{v}_0$ is non-symmetric. That is, compositions of the auto-BTs f, g and k yield, for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, exactly one or two solutions of (3.1) with $\hat{F}$ symmetric, depending on whether $\hat{v}_0$ is symmetric or non-symmetric, respectively.

The solutions of (3.1) described here are matrix analogues of the iterated special integral solutions of the scalar P_{II} hierarchy (we note that for $n\geq 2$ the explicit generation of such sequences of solutions in the scalar case has not in fact been much considered, as in order to start this process a solution of a member of the P_I hierarchy is required; a remark on the structure of such iterated solutions for $n=2$ if a generic solution of P_I is assumed, with a comparison being made to the iterated Airy case for $n=1$, can, however, be found in [2] (some brief comments can also be found in [61] for $n=2$, as well as in [62])). For $n=1$, they are matrix analogues of the iterated special integral solutions of scalar P_{II}, these last being expressible in terms of Airy functions.

Finally we remark that choosing as initial solution v_0 of (3.3) the general solution of $K[M[-v],F]=0$ for parameter value $\tilde{\alpha}_n=\beta=-\frac{1}{2}$ yields the same results as choosing (3.14) (since g maps between these two initial solutions).

3.2.3 Initial solution C

Let us now consider upper triangular solutions of (3.3) (recall that F is upper-triangular). If v is upper-triangular then its diagonal elements v_{ii} satisfy the equations (3.6), and its non-diagonal elements v_{ij}, $j>i$, satisfy linear equations.

We may then take the diagonal elements of an upper-triangular initial solution v_0 to be $v_{ii}=0$ for parameter value $\alpha_n=\beta=0$, and the non-diagonal elements v_{ij}, $j>i$, to be the general solutions of the resulting linear equations. These linear equations may be solved recursively along diagonals parallel to the leading diagonal, with the last linear equation to be solved being that for the upper-right-hand corner element v_{1m}.

Employing the same reasoning as in Section 3.2.1, and using Lemma 3.1, we see that in the subsequent construction of the upper-triangular solutions $v_1=fr^{t-1}v_0$ and $v_2=s^tv_0$ of (3.3), as in Lemma 2.1, for each positive integer parameter value $\tilde{\alpha}_n=t=1,2,3,\ldots$, and also of the upper-triangular solutions $v_3=r^tv_0$ and $v_4=gs^tv_0$ of (3.3), as in Lemma 2.2, for each non-positive integer parameter value $\tilde{\alpha}_n=-t=0,-1,-2,-3,\ldots$, the matrices $K[M[v],F]$ and $K[M[-v],F]$ appearing in the auto-BTs f, r and s are always non-singular.

Since we now have $v_0\neq 0$, the first part of Proposition 2.3 applied to equation (3.3) tells us that $v_1\neq v_2$ and $v_3\neq v_4$, i.e., using the auto-BTs f and g, we obtain two distinct solutions of equation (3.3) for each integer value of the parameter $\tilde{\alpha}_n$. The diagonal elements of these solutions, which give rational solutions of the members of the scalar Painlevé hierarchies (3.6), coincide, and are identical to the diagonal elements of the solutions obtained in Section 3.2.1.

Proceeding as described in Section 3.1.2, the transformation p then yields two solutions pv_1 and pv_2 of (3.1) for each positive integer value of the parameter $\tilde{\alpha}_n$, as well as two solutions pv_3 and pv_4 of (3.1) for each non-positive integer value of the parameter $\tilde{\alpha}_n$: these pairs of distinct solutions for each integer value of the parameter $\tilde{\alpha}_n$ correspond to the action of compositions of the auto-BTs f and g on the initial solution $\hat{v}_0 = pv_0$ of (3.1).

We now assume that $\hat{F}$ is symmetric and apply the second part of Proposition 2.3 to equation (3.1): if our nonzero $\hat{v}_0$ is either symmetric or antisymmetric, the auto-BT k does not yield additional solutions of (3.1); otherwise, it provides a further two distinct solutions of (3.1). Compositions of the auto-BTs f, g and k thus yield, for each integer value of the parameter $\tilde{\alpha}_n$, either two or four distinct solutions of (3.1) with $\hat{F}$ symmetric.

The solutions of (3.1) described here are generalizations of those obtained in Section 3.2.1, i.e., they are generalizations of our matrix analogues of the rational solutions of the scalar P_{II} hierarchy.

3.2.4 Initial solution D

Once again we consider upper triangular solutions of (3.3) (F is upper-triangular): the diagonal elements v_{ii} of an upper-triangular solution v of (3.3) satisfy the equations (3.6), and its non-diagonal elements v_{ij}, $j > i$, satisfy linear equations.

We may then take the diagonal elements v_{ii} of an upper-triangular initial solution v_0 to be given as

$$v_{ii} \text{ general solution of } G[v_{ii}, F_{ii}] = 0 \qquad \text{for parameter value} \qquad \tilde{\alpha}_n = \beta = \frac{1}{2}, \tag{3.20}$$

i.e., the general solution of the basic special integral $G[v_{ii}, F_{ii}] = 0$ of the scalar generalized P_{II} hierarchy (3.6) (see [40, 43], as well as [16, 37, 44] for the case of the standard P_{II} hierarchy), and the non-diagonal elements v_{ij}, $j > i$, to be the general solutions of the resulting linear equations. Again, these linear equations may be solved recursively along diagonals parallel to the leading diagonal, with the last linear equation to be solved being that for the upper-right-hand corner element v_{1m}. In Section 3.2.2 we discussed taking as initial solution the general upper-triangular matrix solution of $K[M[v], F] = 0$ for $\alpha_n = \beta = \frac{1}{2}$, itself a particular case of the choice of initial solution (3.14). The choice of initial solution proposed here is a generalization of this particular case.

We define $fv_0 = v_0$, which implies $rv_0 = -v_0$, even though $K[M[v_0], F]$ is singular (note that in Section 3.2.2 we had $K[M[v_0], F] = 0$, but now we have that $K[M[v_0], F]$ is strictly upper-triangular). Using the same arguments as used with regard to the upper-triangular matrices discussed in Section 3.2.2, as well as Lemma 3.1, we then see that in the subsequent construction of the upper-triangular solutions $v_1 = fr^t v_0$ and $v_2 = s^t v_0$ of (3.3), as in Lemma 2.3, for each positive half-odd-integer parameter value $\tilde{\alpha}_n = t + \frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and also of the upper-triangular solutions $v_3 = r^{t+1} v_0$ and $v_4 = gs^t v_0$ of (3.3), as in Lemma 2.4, for each negative half-odd-integer parameter value $\tilde{\alpha}_n = -t - \frac{1}{2} = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$, the matrices $K[M[v], F]$ and $K[M[-v], F]$ appearing in the auto-BTs f, r and s are always non-singular (except for the same three special instances as discussed in Section 3.2.2, where the result of the application of f and r on v_0 is as has been defined above).

The first part of Proposition 2.4 applied to equation (3.3) then tells us that, with our initial solution defined as above, $v_1 = v_2$ and $v_3 = v_4$, i.e., using the auto-BTs f and g we obtain exactly one solution of equation (3.3) for each half-odd-integer value of the parameter $\tilde{\alpha}_n$. The diagonal elements of these solutions are identical to the diagonal elements of the solutions obtained in Section 3.2.2 in the above-mentioned particular case with initial solution the general upper-triangular matrix solution of $K[M[v], F] = 0$ for $\alpha_n = \beta = \frac{1}{2}$.

Proceeding as described in Section 3.1.2, the transformation p then yields exactly one solution $pv_1 = pv_2$ of (3.1) for each positive half-odd-integer value of the parameter $\tilde{\alpha}_n$, as well as exactly one solution $pv_3 = pv_4$ of (3.1) for each negative half-odd-integer value of the parameter $\tilde{\alpha}_n$: these solutions, exactly one for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, correspond to the action of compositions of the auto-BTs f and g on the initial solution $\hat{v}_0 = pv_0$ of (3.1).

We now assume that $\hat{F}$ is symmetric and apply the second part of Proposition 2.4 to equation (3.1): the auto-BT k yields an additional solution of (3.1) if and only if $\hat{v}_0$ is non-symmetric. That is, compositions of the auto-BTs f, g and k yield, for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, exactly one or two solutions of (3.1) with $\hat{F}$ symmetric, depending on whether $\hat{v}_0$ is symmetric or non-symmetric, respectively.

The solutions of (3.1) described here are generalizations of the matrix analogues of iterated special integral solutions of the scalar P_{II} hierarchy obtained in Section 3.2.2 in the above-mentioned particular case where the initial solution v_0 of (3.3) is taken to be the general upper-triangular matrix solution of $K[M[v], F] = 0$ for $\alpha_n = \beta = \frac{1}{2}$.

We also remark that, in the same way as in Section 3.2.2, choosing as initial solution of (3.3) an upper-triangular v_0 with v_{ii} the general solution of $H[v_{ii}, F_{ii}] = 0$ and with non-diagonal elements v_{ij}, $j > i$, the general solutions of the resulting linear equations, for parameter value $\tilde{\alpha}_n = \beta = -\frac{1}{2}$, yields the same results as those obtained from the choice of initial solution made here (since g maps between these two initial solutions).

Remark 3.2

With respect to the application of auto-BTs discussed above, for the classes of initial solutions B, C and D, we note that when $\hat{F}$ is symmetric, a condition that an initial solution $\hat{v}_0 \neq 0$ of (3.1) be symmetric or (for $\tilde{\alpha}_n = 0$) antisymmetric will in general mean imposing restrictions on this initial solution: see Section 4 for examples.

4 Examples

4.1 Example One

As a first example, we consider the construction of solutions of (3.1) by transforming to (3.3) and then seeking initial solutions of this last equation of class C. We assume that the matrix F in (3.3) is upper-triangular with nonzero elements appearing only on the leading diagonal and the adjacent upper diagonal. This class of matrices then includes the Jordan canonical form of $\hat{F}$. We may, moreover, use a shift on x as described in Remark 3.1 in order to set at least one of the diagonal elements of F equal to zero (perhaps making others nonzero). The elements of the leading diagonal we label as a_j, $j = 1, 2, \ldots, m$, and the elements of the adjacent upper diagonal as b_j, $j = 1, 2, \ldots, m-1$. We seek an initial solution $v = v_0$ of (3.3) as an upper-triangular matrix, where the diagonal elements are taken to be $v_{ii} = 0$ and the parameter value to be $\tilde{\alpha}_n = \beta = 0$, and the non-diagonal elements v_{ij}, $j > i$, are taken to be the general solutions of the resulting linear equations. These linear equations may be solved recursively along diagonals parallel to the leading diagonal.

It is clear from the form of $\psi[u]$ (2.2) and $\phi[u]$ (2.3), and so also of the recursion operator (2.6), that all nonlinearities in (3.3) are of odd order, and so in particular are of order greater than or equal to three. Thus, since we are assuming as initial solution v_0 a strictly upper-triangular matrix, we see that for $m \leq 3$ the nonlinear terms in (3.3) make no contribution to the linear equations for v_{ij}, $j > i$. For $m \geq 4$, the nonlinear terms in (3.3) lead to inhomogeneous terms in these equations, involving the solutions of equations arising along lower diagonals. The terms $vF + Fv$ provide linear terms, as well as such inhomogeneous terms.

Let us consider the cases $m = 2, 3, 4$. We label the elements of our upper-triangular initial solutions v_0 as $V_1, \ldots, V_{m-1}$, $W_1, \ldots, W_{m-2}$ and Z. That is, for $m = 2$ we take

$$F = \begin{pmatrix} a_1 & b_1 \\ 0 & a_2 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

for $m = 3$

$$F = \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & a_2 & b_2 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 0 & V_1 & W_1 \\ 0 & 0 & V_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and for $m = 4$

$$F = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ 0 & a_2 & b_2 & 0 \\ 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & a_4 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} 0 & V_1 & W_1 & Z \\ 0 & 0 & V_2 & W_2 \\ 0 & 0 & 0 & V_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

For $n = 1$ in (3.3) and $m = 2, 3, 4$, we obtain that the above elements of our initial solutions $v = v_0$ for parameter value $\tilde{\alpha}_1 = \beta = 0$ of the second order member of our matrix P_{II} hierarchy, i.e.,

$$v_{xx} - 2v^3 + vF + Fv - xv - \alpha_1 I = 0, \tag{4.4}$$

satisfy the linear equations

$$\begin{aligned} V_{k,xx} - xV_k + (a_k + a_{k+1})V_k &= 0, \qquad k = 1, \ldots, m-1, & (4.5)\\ W_{k,xx} - xW_k + (a_k + a_{k+2})W_k + b_{k+1}V_k + b_k V_{k+1} &= 0, \qquad k = 1, \ldots, m-2, & (4.6)\\ Z_{xx} - xZ + (a_1 + a_4)Z + b_3 W_1 + b_1 W_2 - 2V_1 V_2 V_3 &= 0. & (4.7) \end{aligned}$$

For $n = 2$ in (3.3) and $m = 2, 3, 4$, we obtain that the above elements of our initial solutions $v = v_0$ for parameter value $\tilde{\alpha}_2 = \beta = 0$ of the fourth order member of the matrix P_{II} hierarchy, i.e.,

$$v_{xxxx} - 4v_{xx}v^2 - 4v^2 v_{xx} - 2vv_{xx}v - 2v_x^2 v - 2vv_x^2 - 6v_x v v_x + 6v^5 + c_1(v_{xx} - 2v^3) + vF + Fv - xv - \alpha_2 I = 0, \tag{4.8}$$

satisfy the linear equations

$$\begin{aligned} V_{k,xxxx} + c_1 V_{k,xx} - xV_k + (a_k + a_{k+1})V_k &= 0, \qquad k = 1, \ldots, m-1, & (4.9)\\ W_{k,xxxx} + c_1 W_{k,xx} - xW_k + (a_k + a_{k+2})W_k + b_{k+1}V_k + b_k V_{k+1} &= 0, \qquad k = 1, \ldots, m-2, & (4.10)\\ Z_{xxxx} + c_1 Z_{xx} - xZ + (a_1 + a_4)Z + b_3 W_1 + b_1 W_2 - 4V_{1,xx}V_2 V_3 - 2V_1 V_{2,xx} V_3 & \\ -4V_1 V_2 V_{3,xx} - 2V_{1,x}V_{2,x}V_3 - 6V_{1,x}V_2 V_{3,x} - 2V_1 V_{2,x} V_{3,x} - 2c_1 V_1 V_2 V_3 &= 0. & (4.11) \end{aligned}$$

Solving the above systems of linear equations, and similar such systems arising for greater values of m and n, then provides us with our initial solutions v_0. As described in Section 3.2.3, we may then use our auto-BTs to obtain, for each integer value of the parameter $\tilde{\alpha}_n$, solutions of our matrix P_{II} hierarchy (3.1) (reversing also any shift in x, of the kind discussed in Remark 3.1, that we may have used).

As a concrete example let us consider the case $m = 3$ and

$$\hat{F} = \begin{pmatrix} d - bi - c + 2a & b + di - ci + ai & b - ci + ai \\ b + di - ci - ei + 2ai & bi - d + c + e - a & bi + c - a \\ di + ei - ci & c - e - d & c \end{pmatrix}, \tag{4.12}$$

where a, b, c, d, e are constant. This matrix has eigenvalues $\lambda = a, c, e$. A shift $x \to x + 2a$ leads us to the consideration of the system (3.1) but now with $\hat{F} \to \hat{F} - aI$, i.e., with

$$\hat{F} = \begin{pmatrix} d - bi - f & b + di - fi & b - fi \\ b + di - fi - gi & bi - d + f + g & f + bi \\ di + gi - fi & f - g - d & f \end{pmatrix}, \tag{4.13}$$

where $f = c - a$ and $g = e - a$. We now take

$$P = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \tag{4.14}$$

and thus transform from (3.1) with (4.13) to (3.3) with

$$F = \begin{pmatrix} 0 & b & 0 \\ 0 & f & d \\ 0 & 0 & g \end{pmatrix}. \tag{4.15}$$

This is of the form considered in (4.2). Our shift on x has resulted in setting the first element on the leading diagonal of F equal to zero. For $n=1$ and $n=2$ the systems of linear equations satisfied by the elements V_1, V_2 and $W=W_1$ of the initial solution v_0 as given in (4.2) are

$$V_{1,xx} - xV_1 + fV_1 = 0, \tag{4.16}$$
$$V_{2,xx} - xV_2 + (f+g)V_2 = 0, \tag{4.17}$$
$$W_{xx} - xW + gW + dV_1 + bV_2 = 0, \tag{4.18}$$

and

$$V_{1,xxxx} + c_1 V_{1,xx} - xV_1 + fV_1 = 0, \tag{4.19}$$
$$V_{2,xxxx} + c_1 V_{2,xx} - xV_2 + (f+g)V_2 = 0, \tag{4.20}$$
$$W_{xxxx} + c_1 W_{xx} - xW + gW + dV_1 + bV_2 = 0, \tag{4.21}$$

respectively. Solving the above systems of linear equations in order to obtain a nonzero initial solution v_0, our auto-BTs f and g and the transformation p then allow us to obtain, for each integer value of the parameter $\tilde{\alpha}_n$, two distinct solutions of our second and fourth order matrix P_{II} equations given by (3.1) for $n=1$ and $n=2$, in this case $m=3$ and with $\hat{F}$ given by (4.13). Reversing the shift made on x then yields two distinct solutions of these equations with $\hat{F}$ given by (4.12).

Let us briefly consider the process of recursively solving such linear equations. As illustrative examples, we discuss first of all the system (4.16)—(4.18), and secondly the system (4.19)—(4.21) where, in order to shorten the discussion, we take $c_1 = 0$. Equations (4.16) and (4.17) are Airy equations, and have general solutions

$$V_1 = \nu_1 Ai(x-f) + \nu_2 Bi(x-f) \quad \text{and} \quad V_2 = \nu_3 Ai(x-f-g) + \nu_4 Bi(x-f-g) \tag{4.22}$$

respectively, where all ν_i are arbitrary constants and $Ai(z)$ and $Bi(z)$ are the usual Airy functions (linearly independent solutions of the Airy equation $\mathcal{Y}_{zz} = z\mathcal{Y}$). The general solution of equation (4.18) can be obtained using variation of parameters. The corresponding homogeneous equation $W_{xx} - xW + gW = 0$ is again an Airy equation, with general solution W_1 given by

$$W_1 = \nu_5 Ai(x-g) + \nu_6 Bi(x-g) \tag{4.23}$$

where ν_5 and ν_6 are arbitrary constants, and variation of parameters thus leads to the particular solution W_2 of (4.18) given by

$$W_2 = Ai(x-g)\int \frac{(dV_1 + bV_2)Bi(x-g)}{\mathcal{W}}dx - Bi(x-g)\int \frac{(dV_1 + bV_2)Ai(x-g)}{\mathcal{W}}dx \tag{4.24}$$

where V_1 and V_2 are as given by (4.22) and $\mathcal{W}$ is the Wronskian of $Ai(x-g)$ and $Bi(x-g)$. The general solution of the system (4.16)—(4.18) then consists of V_1 and V_2 as given by (4.22) along with $W = W_1 + W_2$.

We recall that the general solution of the Airy equation $\mathcal{Y}_{zz} = z\mathcal{Y}$ can be obtained via everywhere-convergent series solutions about the ordinary point $z=0$:

$$\mathcal{Y} = \mu_0 \mathcal{Y}_0 + \mu_1 \mathcal{Y}_1 = \mu_0 \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2}{3})z^{3n}}{9^n n! \Gamma(n+\frac{2}{3})} + \mu_1 z \sum_{n=0}^{\infty} \frac{\Gamma(\frac{4}{3})z^{3n}}{9^n n! \Gamma(n+\frac{4}{3})} \tag{4.25}$$

where μ_0 and μ_1 are arbitrary constants. (see, e.g., [63]). The two linearly independent solutions $\mathcal{Y}_0$ and $\mathcal{Y}_1$ form a basis of solutions of the Airy equation. In particular, making in (4.25) the choices $\mu_0 = 3^{-\frac{2}{3}}/\Gamma(\frac{2}{3})$ and $\mu_1 = -3^{-\frac{4}{3}}/\Gamma(\frac{4}{3})$, and $\mu_0 = 3^{-\frac{1}{6}}/\Gamma(\frac{2}{3})$ and $\mu_1 = 3^{-\frac{5}{6}}/\Gamma(\frac{4}{3})$, yields $Ai(z)$ and $Bi(z)$ respectively. We now turn to the system (4.19)—(4.21) with $c_1 = 0$. Each equation in this system is either a homogeneous or inhomogeneous fourth order Airy equation. The general solution of the fourth order Airy equation $\mathcal{Y}_{zzzz} = z\mathcal{Y}$ can, just as for the Airy equation itself, be expressed using everywhere-convergent series solutions about the ordinary point $z = 0$. Indeed, such series solutions for the fourth order Airy equation have been constructed in [64, 65], about the ordinary point $\tau = 0$ for the fourth order Airy equation in the form $\mathcal{Y}_{\tau\tau\tau\tau} + \tau\mathcal{Y} = 0$ (i.e., with $z = -\tau$). The resulting series solutions are found to be of the form

$$\mathcal{Y} = \sum_{k=0}^{3} \mu_k \mathcal{Y}_k, \quad \mathcal{Y}_k = \tau^k \sum_{n=0}^{\infty} a_{k,n}\tau^{5n}, \quad \text{where all} \quad a_{k,0} = 1, \tag{4.26}$$

where all μ_k are arbitrary constants. Similarly to the case of the Airy equation itself, the linearly independent solutions $\mathcal{Y}_k$, $k = 0, 1, 2, 3$, form a basis of solutions of the fourth order Airy equation, and appropriate linear combinations allow the construction of further (equivalent) sets of four linearly independent solutions. One such set [64, 65] of four linearly independent solutions of $\mathcal{Y}_{\tau\tau\tau\tau} + \tau\mathcal{Y} = 0$ consists of the fourth order Airy function of the first kind $Ai_4(\tau)$ — this function is an analogue of the usual Airy function $Ai(z)$ and is obtained from (4.26) by making a specific choice of the constants μ_k — along with three other functions, denoted by $\widetilde{Ai}_4(\tau)$, $G_3(\tau)$ and $G_4(\tau)$, which may also be defined via appropriate choices of the constants μ_k in (4.26). Using this set of four functions, we may, for our case $c_1 = 0$, write the general solutions of (4.19) and (4.20) as

$$V_1 = \nu_1 Ai_4(-(x-f)) + \nu_2 \widetilde{Ai}_4(-(x-f)) + \nu_3 G_3(-(x-f)) + \nu_4 G_4(-(x-f)) \tag{4.27}$$

and

$$V_1 = \nu_5 Ai_4(-(x-f-g)) + \nu_6 \widetilde{Ai}_4(-(x-f-g)) + \nu_7 G_3(-(x-f-g)) + \nu_8 G_4(-(x-f-g)) \tag{4.28}$$

respectively, where all ν_i are arbitrary constants. The general solution of equation (4.21) with $c_1 = 0$ can be obtained using variation of parameters: the corresponding homogeneous equation $W_{xxxx} - xW + gW = 0$ is again a fourth order Airy equation, with general solution W_1 given by

$$W_1 = \nu_9 Ai_4(-(x-g)) + \nu_{10} \widetilde{Ai}_4(-(x-g)) + \nu_{11} G_3(-(x-g)) + \nu_{12} G_4(-(x-g)) \tag{4.29}$$

where once again all ν_i are arbitrary constants; variation of parameters then leads to the particular solution W_2 of the inhomogeneous equation given by

$$W_2 = Ai_4(-(x-g)) \int \frac{\mathcal{W}_1}{\mathcal{W}} dx + \widetilde{Ai}_4(-(x-g)) \int \frac{\mathcal{W}_2}{\mathcal{W}} dx + G_3(-(x-g)) \int \frac{\mathcal{W}_3}{\mathcal{W}} dx + G_4(-(x-g)) \int \frac{\mathcal{W}_4}{\mathcal{W}} dx, \tag{4.30}$$

where $\mathcal{W}$ is the Wronskian of the four functions $Ai_4(-(x-g))$, $\widetilde{Ai}_4(-(x-g))$, $G_3(-(x-g))$ and $G_4(-(x-g))$, and where each $\mathcal{W}_k$ is this same Wronskian but with the k-th column (i.e., the column with entries defined in terms of the k-th of the four functions $Ai_4(-(x-g))$, $\widetilde{Ai}_4(-(x-g))$, $G_3(-(x-g))$ and $G_4(-(x-g))$) replaced by the column $(0, 0, 0, -dV_1 - bV_2)^T$, with V_1 and V_2 being as in (4.27) and (4.28). The general solution of the system (4.19)—(4.21) for $c_1 = 0$ then consists of V_1 and V_2 as given by (4.27) and (4.28), along with $W = W_1 + W_2$. [The steps described here can also be followed for $c_1 \neq 0$: the general solution of the homogeneous equation $\mathcal{Y}_{\tau\tau\tau\tau} + c_1\mathcal{Y}_{\tau\tau} + \tau\mathcal{Y} = 0$ can be expressed using everywhere-convergent series solutions; linear combinations of the series obtained can be used to define four linearly independent solutions; using any four such functions, we can give expressions for the general solutions of (4.19) and (4.20), as well as of (4.21) by variation of parameters.]

Let us now consider the case where $\hat{F}$ in (4.13) is symmetric. This requires $g = 0$ and $b = di$, i.e., $\hat{F} = \hat{F}_s$ where

$$\hat{F}_s = \begin{pmatrix} 2d - f & 2di - fi & di - fi \\ 2di - fi & -2d + f & f - d \\ di - fi & f - d & f \end{pmatrix}. \tag{4.31}$$

In particular, we note that V_1 and V_2 must then satisfy the same homogeneous linear equations. We also note that, corresponding to v_0 as given in (4.2), we have the initial solution $\hat{v}_0$ of (3.1) for $n = 1$ and $n = 2$, with $\hat{F} = \hat{F}_s$, given by

$$\hat{v}_0 = P v_0 P^{-1} = \begin{pmatrix} -iV_1 + V_2 + iW & V_1 + iV_2 - W & V_1 \\ V_1 + iV_2 - W & iV_1 - V_2 - iW & iV_1 \\ iV_2 & -V_2 & 0 \end{pmatrix}. \tag{4.32}$$

In the case where $\hat{v}_0$ is neither symmetric nor antisymmetric, our auto-BTs f, g and k allow us to obtain, for each integer value of the parameter $\tilde{\alpha}_n$, four distinct solutions of our second and fourth order matrix P_{II} equations given by (3.1) for $n = 1$ and $n = 2$, in this case $m = 3$ and with $\hat{F} = \hat{F}_s$. For nonzero $\hat{v}_0$ either symmetric or antisymmetric, the auto-BTs f, g and k yield, for each integer value of the parameter $\tilde{\alpha}_n$, two distinct solutions of these equations. Reversing the shift made on x then provides respectively four or two distinct solutions of these equations with $\hat{F} = \hat{F}_s + aI$.

We see that in order that the initial solution $\hat{v}_0$ as given by (4.32) be symmetric, we must impose the restiction $V_1 = iV_2$: we recall that V_1 and V_2 satisfy the same homogeneous linear equations. In order that the initial solution $\hat{v}_0$ as given by (4.32) be antisymmetric, we must impose the restictions $V_1 = -iV_2$ and $W = 0$: we note again that V_1 and V_2 satisfy the same homogeneous linear equations, and also that for $b = di$ and $V_1 = -iV_2$ the linear equations satisfied by W are homogeneous, so we may take $W = 0$ as a solution.

4.2 Example Two

As a second example, we consider, for the case $m = 2$, the construction of solutions of (3.1) with

$$\hat{F} = \begin{pmatrix} 3a - bi - 2c & 2ai + b - 2ci \\ 3ai + b - 3ci & -2a + bi + 3c \end{pmatrix}, \tag{4.33}$$

where we seek initial solutions of corresponding equations (3.3) of class D. The matrix (4.33) has eigenvalues $\lambda = a, c$. A shift $x \to x + 2a$ leads us to consider the system (3.1) but now with $\hat{F} \to \hat{F} - aI$, i.e., with

$$\hat{F} = \begin{pmatrix} -2d - bi & -2di + b \\ -3di + b & 3d + bi \end{pmatrix}, \tag{4.34}$$

where $d = c - a$. Let us take

$$P = \begin{pmatrix} 1 & -2i \\ i & 3 \end{pmatrix}, \tag{4.35}$$

thus transforming from (3.1) with (4.34) to (3.3) with

$$F = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}. \tag{4.36}$$

For an initial solution

$$v_0 = \begin{pmatrix} U_1 & V \\ 0 & U_2 \end{pmatrix}, \tag{4.37}$$

U_1, U_2 and V must satisfy, in the case $n=1$ of a second order matrix P_{II} equation (4.4), the system

$$\begin{aligned} U_{1,xx}-2U_1^3-xU_1-\alpha_1 &= 0, &(4.38)\\ U_{2,xx}-2U_2^3-xU_2+2dU_2-\alpha_1 &= 0, &(4.39)\\ V_{xx}-2(U_1^2+U_1U_2+U_2^2)V-xV+dV+b(U_1+U_2) &= 0, &(4.40)\end{aligned}$$

and in the case $n=2$ of a fourth order matrix P_{II} equation (4.8), the system

$$\begin{aligned} U_{1,xxxx}-10U_1^2U_{1,xx}-10U_1U_{1,x}^2+6U_1^5+c_1(U_{1,xx}-2U_1^3)-xU_1-\alpha_2 &= 0, &(4.41)\\ U_{2,xxxx}-10U_2^2U_{2,xx}-10U_2U_{2,x}^2+6U_2^5+c_1(U_{2,xx}-2U_2^3)-xU_2+2dU_2-\alpha_2 &= 0, &(4.42)\\ V_{xxxx}-2(2U_1^2+U_1U_2+2U_2^2)V_{xx}-2(2U_1^2+U_1U_2+2U_2^2)_xV_x-2(U_{1,x}^2+3U_{1,x}U_{2,x}+U_{2,x}^2)V & \\ -2(3U_1U_{1,xx}+2U_2U_{1,xx}+2U_1U_{2,xx}+3U_2U_{2,xx})V+6(U_1^4+U_1^3U_2+U_1^2U_2^2+U_1U_2^3+U_2^4)V & \\ +c_1[V_{xx}-2(U_1^2+U_1U_2+U_2^2)V]-xV+dV+b(U_1+U_2) &= 0. &(4.43)\end{aligned}$$

For our initial solution v_0 of class D, we assume that U_1 satisfies the basic special integral $G[U_1,0]=0$, and that U_2 satisfies the basic special integral $G[U_2,d]=0$, for parameter value $\alpha_n=\beta=\frac{1}{2}$, i.e., in the case $n=1$,

$$\begin{aligned} U_{1,x}-U_1^2-\frac{1}{2}x &= 0, &(4.44)\\ U_{2,x}-U_2^2+d-\frac{1}{2}x &= 0, &(4.45)\end{aligned}$$

for $\alpha_1=\beta=\frac{1}{2}$, and in the case $n=2$,

$$\begin{aligned} U_{1,xxx}-2U_1U_{1,xx}+U_{1,x}^2-6U_1^2U_{1,x}+3U_1^4+c_1(U_{1,x}-U_1^2)-\frac{1}{2}x &= 0, &(4.46)\\ U_{2,xxx}-2U_2U_{2,xx}+U_{2,x}^2-6U_2^2U_{2,x}+3U_2^4+c_1(U_{2,x}-U_2^2)+d-\frac{1}{2}x &= 0, &(4.47)\end{aligned}$$

for $\alpha_2=\beta=\frac{1}{2}$. Solutions U_1 and U_2 of the basic special integrals (4.44) and (4.45) then give solutions of (4.38) and (4.39); equations (4.44) and (4.45) are linearisable and their solutions can be expressed using Airy functions. Likewise, solutions U_1 and U_2 of the basic special integrals (4.46) and (4.47) give solutions of (4.41) and (4.42); equations (4.46) and (4.47) correspond to the $k=3$ case of the Chazy XI equation, and may be solved using the first Painlevé transcendent as follows [9]. Equation (4.47) may be written

$$W_{xx}+3W^2+c_1W+d-\frac{1}{2}x=0, \qquad W=U_{2,x}-U_2^2 \tag{4.48}$$

(and similarly for equation (4.46)). The first of these is just the first Painlevé equation; the second can be linearised via $U_2=-\psi_x/\psi$ onto $\psi_{xx}+W\psi=0$. (For $n\geq 3$, the equations $G[v_{ii},F_{ii}]=0$ can be solved similarly using the solutions W of higher order members of the first Painlevé hierarchy and this linear equation.) Obtaining as above the solutions U_1 and U_2 of equations (4.44) and (4.45), or (4.46) and (4.47), the third element V of our initial solution v_0 is then obtained by solving the corresponding linear equations, i.e., (4.40) or (4.43), respectively. It is in this way that we construct our initial solution of class D. Our auto-BTs f and g and the transformation p then yield, for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, exactly one solution of our second and fourth order matrix P_{II} equations given by (3.1) for $n=1$ and $n=2$, in this case $m=2$ and with $\hat{F}$ given by (4.34). Reversing the shift made on x then yields exactly one solution of these equations with $\hat{F}$ given by (4.33).

Let us now consider the case where $\hat{F}$ in (4.34) is symmetric. This requires $d = 0$, i.e., $\hat{F} = \hat{F}_s$ where

$$\hat{F}_s = \begin{pmatrix} -bi & b \\ b & bi \end{pmatrix}. \tag{4.49}$$

In particular, we note that U_1 and U_2 must then satisfy the same first or third order equations. We also note that, corresponding to v_0 as given in (4.37), we have the initial solution $\hat{v}_0$ of (3.1) for $n = 1$ and $n = 2$, with $\hat{F} = \hat{F}_s$, given by

$$\hat{v}_0 = P v_0 P^{-1} = \begin{pmatrix} 3U_1 - 2U_2 - iV & 2iU_1 - 2iU_2 + V \\ 3iU_1 - 3iU_2 + V & -2U_1 + 3U_2 + iV \end{pmatrix}. \tag{4.50}$$

In the case where $\hat{v}_0$ is non-symmetric, our auto-BTs f, g and k allow us to obtain, for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, two distinct solutions of our second and fourth order matrix P_{II} equations given by (3.1) for $n = 1$ and $n = 2$, in this case $m = 2$ and with $\hat{F} = \hat{F}_s$. For symmetric $\hat{v}_0$, the auto-BTs f, g and k yield, for each half-odd-integer value of the parameter $\tilde{\alpha}_n$, exactly one solution of these equations. Reversing the shift made on x then provides respectively two distinct solutions or exactly one solution of these equations with $\hat{F} = \hat{F}_s + aI$. We see that in order that the initial solution $\hat{v}_0$ as given by (4.50) be symmetric, we must impose the restriction $U_1 = U_2$: we recall that U_1 and U_2 satisfy the same first or third order equations.

5 Discussion and conclusions

In this paper we have described how the auto-BTs of our matrix P_{II} hierarchy (2.14) can be used to obtain sequences of solutions, starting with one of four classes of initial solution, for parameter values $\alpha_n = \beta = 0$ or $\alpha_n = \beta = 1/2$. This is an extension of our previous results, presented in [52], where we discussed the use of auto-BTs to generate sequences of solutions, with four classes of initial solution, for our matrix P_{II} equation (2.8).

As explicit examples, we have considered initial solutions of class C, for $\alpha_n = \beta = 0$, and of class D, for $\alpha_n = \beta = 1/2$: we seek upper triangular initial solutions, and as solutions v_{ii} of members of the scalar generalised P_{II} hierarchy on the leading diagonal we take respectively $v_{ii} = 0$ or v_{ii} the general solution of a basic special integral of the scalar hierarchy, with non-diagonal elements v_{ij}, $j > i$ of our initial solutions being obtained, for both classes, as the general solutions of recursively-solved linear equations. We note that solving basic special integrals of the scalar generalised second Painlevé hierarchy requires use of the solutions of the scalar generalised first Painlevé hierarchy, which seems natural in the context of a study of our matrix P_{II} hierarchy. It would also seem natural to allow the use of the solutions of the scalar generalised second Painlevé hierarchy, which would then permit us to further generalize the classes of upper-triangular initial solutions for $\alpha_n = \beta = 0$ and $\alpha_n = \beta = 1/2$ (or more generally) by allowing us to assume the diagonal elements v_{ii} to be solutions of this scalar hierarchy other than $v_{ii} = 0$ or the general solution of a basic special integral. Within this context, we believe the linear equations for non-diagonal elements thus derived to be worthy of further study (see examples in Section 4.2); we will return to this topic in future papers.

The application of auto-BTs to an initial solution is not, of course, the only way of obtaining solutions of our matrix P_{II} hierarchy (2.14). One alternative, for example, is that used in [66] for the "fully noncommutative" P_{II} hierarchy presented therein: it is shown that each member of this hierarchy has a solution, connected to the Fredholm determinant of the n-th Airy matrix Hankel operator, which has a certain asymptotic behaviour.

In order to understand the relationship between solutions of the hierarchy presented in [66] and our matrix P_{II} hierarchy (2.14), let us rewrite the latter in the form

$$(\partial_x + A_u)\tilde{K}[M[u]] + uE + Eu - xu - \alpha_n I = 0, \tag{5.1}$$

where $\tilde{K}[M[u]]$ is given by

$$\tilde{K}[w] = M_n + \sum_{k=1}^{n-1} c_k M_k, \tag{5.2}$$

$w = M[u] = u_x - u^2$, and the quantities M_k are defined recursively as described in Section 2.1. This is to be compared to the hierarchy given in [66], for the matrix function W of the variables $s_1, s_2, \ldots, s_m$, defined as

$$\left(\frac{d}{dS} + A_W\right) M_n[M[W]] + (-4)^n(WS + SW) = 0, \quad \text{where} \quad \frac{d}{dS} = \sum_{i=1}^{m} \partial_{s_i} \tag{5.3}$$

and S is the diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_m)$. We now observe that, contrary to the claim made in [66], the hierarchy (5.3) corresponds in fact to a special case of our matrix P_{II} hierarchy (5.1). In order to see this, in (5.3) we make the change of variables[1] $s_i = \sum_{j=1}^m a_{ij}x_j$, where the nonsingular matrix $A = (a_{ij})$ is such that $a_{i1} = 1$, $i = 1, 2, \ldots, m$. It then follows that $\partial_{x_1} = \sum_{i=1}^m \partial_{s_i}$ and $S = x_1 I + G$, where G is the diagonal matrix $G = \mathrm{diag}\,(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)$ with $\tilde{x}_i = \sum_{j=2}^m a_{ij}x_j$, $i = 1, 2, \ldots, m$. Thus we obtain the nonautonomous ordinary differential equation hierarchy

$$(\partial_{x_1} + A_W)\, M_n[M[W]] + 2(-4)^n x_1 W + (-4)^n(WG + GW) = 0, \tag{5.4}$$

for $W = W(x_1)$, wherein the variables $x_2, x_3, \ldots, x_m$ appear as constant parameters in the matrix G. Finally, the rescaling $W(x_1) = 2(-1)^{n+1}u(y)$, $x_1 = \frac{1}{2}(-1)^{n+1}y$, allows us to set the coefficient of the nonautonomous term to -1. The resulting equation is then precisely of the form (5.1), but with independent variable y, in the special case where c_k=0, $k = 1, 2, \ldots, n-1$, $\alpha_n = 0$, and E is diagonal: $E = (-1)^n G$ with G as above.

Alternatively, let us consider (5.1) in the special case where $c_k = 0$, $k = 1, 2, \ldots, n-1$, $\alpha_n = 0$ and E is diagonalisable onto a matrix $\mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$ for parameters σ_k, $k = 1, 2, \ldots, m$, which means that we may always assume $E = (-1)^n \mathrm{diag}(t_1, t_2, \ldots, t_m)$ for parameters $t_k = (-1)^n\sigma_k$, $k = 1, 2, \ldots, m$. In (5.1) we make the change of variables $u(x) = \frac{1}{2}(-1)^{n+1}W(x_1)$, $x = 2(-1)^{n+1}x_1 - 2(-1)^{n+1}t_1$, which gives

$$(\partial_{x_1} + A_W)\, M_n[M[W]] + (-4)^n(WH + HW) = 0, \tag{5.5}$$

where $H = \mathrm{diag}(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_m)$ and $x_k = t_k - t_1$, $k = 2, 3, \ldots, m$. Setting $s_1 = x_1$ and $s_k = x_1 + x_k$, $k = 2, 3, \ldots, m$, and noting that $\partial_{x_1} = \sum_{i=1}^m \partial_{s_i}$ and $H = S$, we then obtain (5.3). We note that asymptotic results are given in [66] in the regime $s \to +\infty$ where $s = \frac{1}{m}\sum_{j=1}^m s_j$. Under the above transformation from (5.1), with $E = (-1)^n \mathrm{diag}(t_1, t_2, \ldots, t_m)$, to (5.3), we have the correspondence $s = \frac{1}{2}(-1)^{n+1}x + \frac{1}{m}\sum_{j=1}^m t_j$.

The fact that the hierarchy (5.3) corresponds to a special case of the matrix P_{II} hierarchy (5.1) then means that the result given in [66] provides a solution also of this special case of (5.1), or equivalently of (3.3), i.e., the special case $\alpha_n = 0$, all $c_k = 0$, $k = 1, 2, \ldots, m$, and E diagonal as discussed above. This then provides a further choice of initial solution v_0 of (3.3) for $\tilde{\alpha}_n = \beta = 0$, to which we can apply our auto-BTs and thus obtain solutions for integer values of $\tilde{\alpha}_n$ other $\tilde{\alpha}_n = 0$. We note here that our proposed initial solutions of class C, whose construction is discussed in Section 3.2.3, whilst lower triangular, are in the general case solutions of a broader class of equations than the subcase of (5.1) equivalent to (5.3), i.e., are solutions of equations to which the results in [66], unless further restrictions are made, do not apply. The reason for this (apart from not assuming that all $c_k = 0$, $k = 1, 2, \ldots, m$) is that we do not assume that F, taken to be upper-triangular in Section 3.2.3, corresponds to a diagonalisable matrix $\hat{F}$ in (3.1). This is the case, for example, for the concrete example discussed in Section 4.1 where the matrix $\hat{F}$ given by (4.12) is not assumed to be diagonalisable: it is non-diagonalisable, for example, when $f = 0$ and $bd \neq 0$, or in the symmetric case (4.31) when $d \neq 0$.

[1] Compare with deriving d'Alembert's solution of the wave equation $(\partial_t + \partial_x)(\partial_t - \partial_x)z = 0$ by transforming to $\partial_\zeta \partial_\xi z = 0$ where $t = \zeta + \xi$, $x = \zeta - \xi$, but here with only one combination of derivatives $\sum_{i=1}^m \partial_{s_i}$, e.g., when solving $w_t + w_x = w_\zeta = 0$.

Finally, we remark that a matrix P_{II} hierarchy with non-commuting "independent" variable x, i.e., similar to the noncommutative second Painlevé equation presented in [67] in a more general algebraic setting, can be obtained by assuming such non-commutativity in the derivation of the matrix P_{II} hierarchy (2.1) given in [49], replacing xI with the non-commuting x ($x' = 1$) and not expanding $\mathcal{B}_1[w]x$, with the result that $2g_{n-1}xu$ in (2.1) is replaced by $g_{n-1}(ux+xu)$ (and $\alpha_n I$ by a scalar central parameter α_n). We will return to such examples in later papers. We will also continue our study of matrix hierarchies with scalar independent variable.

Acknowledgements

The work of PRG and AP is supported by the Ministry of Economy and Competitiveness of Spain under contract MTM2016-80276-P (AEI/FEDER, EU), and currently by the Ministry of Science and Innovation of Spain under contract PID2020-115273GB-100 (AEI/FEDER, EU).

A Solutions of classes of equations

In Section 3.1.1 we used the transformation p to permit a choice of the coeefficient matrix in the equations of our matrix P_{II} hierarchy. We prefer to think of the transformation p as one which allows us to fix a form of this coefficient matrix, rather than as an auto-BT. We could, however, in addition to the auto-BTs f, g and k, consider also the actions of a finite number of transformations of a similar form to p,

$$q_i : \qquad \hat{u} = Q_i \hat{v}\, Q_i^{-1}, \qquad \hat{E} = Q_i \hat{F}\, Q_i^{-1}, \tag{A.1}$$

with each Q_i a nonsingular constant matrix: these transformations map from solutions $\hat{v}$ of (3.1) with coefficient matrix $\hat{F}$ to solutions $\hat{u}$ of

$$\psi[\hat{u}]\left(\tilde{\mathcal{R}}^{n-1}[\hat{u}]\hat{u}_x + \sum_{k=1}^{n-1} c_k \tilde{\mathcal{R}}^{k-1}[\hat{u}]\hat{u}_x\right) + \hat{u}\hat{E} + \hat{E}\hat{u} - x\hat{u} - \tilde{\alpha}_n I = 0, \tag{A.2}$$

i.e., of the same equation but with coefficient matrix $\hat{E}$ (and the same parameter value $\tilde{\alpha}_n$).

Since compositions of such transformations give transformations of the same form ($q_i q_j \hat{v} = Q\,\hat{v}\,Q^{-1}$ with $Q = Q_i Q_j$), and since the mappings q_i and k commute with f and g, we have that any composition of transformations q_i, k, f and g acting on $\hat{v}_0$ can be written as

$$k^{\gamma} \prod_{i=1}^{j} (q_i k)\hat{v} \qquad \text{or} \qquad q_0^{\gamma} \prod_{i=1}^{j} (k q_i)\hat{v}, \qquad \gamma \in \{0,1\}, j \in \{0,1,2,\ldots\}, \tag{A.3}$$

where in each case

$$\hat{v} = f^{\epsilon_2} r^q \hat{v}_0 \qquad \text{or} \qquad \hat{v} = g^{\epsilon_2} s^q \hat{v}_0 \tag{A.4}$$

is a solution of (3.1) obtained as described in Sections 3.1.1 and 3.1.2. (In the above expressions the product with $j = 0$ is taken to be the identity transformation.) Similarly, the result of acting with the same composition of transformations q_i, k, f and g on $\hat{F}$ is

$$k^{\gamma} \prod_{i=1}^{j} (q_i k)\hat{F} \qquad \text{or} \qquad q_0^{\gamma} \prod_{i=1}^{j} (k q_i)\hat{F}, \qquad \gamma \in \{0,1\}, j \in \{0,1,2,\ldots\}, \tag{A.5}$$

since f and g leave $\hat{F}$ unchanged. It thus remains to consider the result of the above compositions of mappings q_i and k on a solution $\hat{v}$ of (3.1) and its corresponding coefficient matrix $\hat{F}$, $\hat{v}$ having been obtained from $\hat{v}_0$ using the auto-BTs f and g and so in general being a solution of (3.1) for some new parameter value $\tilde{\alpha}_n$.

It is, however, straightforward to show that the results of the above compositions on $\hat{v}$ and $\hat{F}$ can always be written either in the form

$$Q\,\hat{v}\,Q^{-1} \qquad \text{and} \qquad Q\,\hat{F}\,Q^{-1}, \tag{A.6}$$

for some nonsingular constant matrix Q (for $\gamma = j = 0$ in (A.3) and (A.5), $Q = I$), or in the form

$$Q\,\hat{v}^T\,Q^{-1} \qquad \text{and} \qquad Q\,\hat{F}^T\,Q^{-1}, \tag{A.7}$$

again for some nonsingular constant matrix Q (for $\gamma = 1$ and $j = 0$ in the first expressions in (A.3) and (A.5), i.e., for $k\hat{v}$ and $k\hat{F}$ as considered in Sections 3.1.1 and 3.1.2, $Q = I$).

Thus, when considering the result of a composition of transformations q_i, k, f and g on $\hat{v}_0$, $\hat{F}$ and $\tilde{\alpha}_n$, having obtained the solution $\hat{v}$ of (3.1) as given in (A.4), for the same coefficient matrix $\hat{F}$ but in general for some new parameter value $\tilde{\alpha}_n$, we then obtain either a solution similar to $\hat{v}$ for a coefficient matrix similar to $\hat{F}$, or a solution similar to $k\hat{v}$ for a coefficient matrix similar to $k\hat{F}$. The inclusion of transformations q_i thus leads us to the derivation of solutions of equivalence classes of equations, where we define two equations as equivalent if their coefficient matrices are similar. In the particular case where $\hat{F}$ is symmetric, we obtain solutions similar to $\hat{v}$ and $k\hat{v}$ for coefficient matrices similar to $\hat{F}$, i.e., solutions of equations equivalent to (3.1) for the above-mentioned new parameter value $\tilde{\alpha}_n$.

References

[1] M. J. Ablowitz and H. Segur, Phys. Rev. Lett. 38 (1977) 1103–1106.

[2] H. Airault, Stud. Appl. Math. 61 (1979) 31-53.

[3] P. Painlevé, Bull. Soc. Math. Fr. 28 (1900) 201–261.

[4] P. Painlevé, Acta Math. (1902) 25 1–85.

[5] B. Gambier, Acta Math. 33 (1910) 1-55.

[6] E. L. Ince, "Ordinary Differential Equations," (Dover, New York, 1956).

[7] H. Flaschka and A. C. Newell, Comm. Math. Phys. 76 (1980) 65-116.

[8] A. Pickering, Phys. Lett. A 301 (2002) 275-280.

[9] J. Chazy, Acta Math. 34 (1911) 317-385.

[10] R. Garnier, Ann. Sci. École Normale Sup. 48 (1912) 1-126.

[11] F. J. Bureau, Ann. Mat. Pura Appl. (IV) 66 (1964) 1-116.

[12] F. J. Bureau, Ann. Mat. Pura Appl. (IV) 64 (1964) 229-364.

[13] H. Exton, Rend. Mat. (6) 6 (1973) 419-462.

[14] I. P. Martynov, Differents. Uravn. 21 (1985) 764-771.

[15] I. P. Martynov, Differents. Uravn. 21 (1985) 937-946.

[16] N. A. Kudryashov, Phys. Lett. A 224 (1997) 353-360.

[17] N. A. Kudryashov and A. Pickering, J. Phys. A 31 (1998) 9505-9518.

[18] D. Levi, O. Ragnisco and M. A. Rodríguez, Theor. Math. Phys. 93 (1993) 1409-1414.

[19] P. R. Gordoa and A. Pickering, J. Math. Phys. 40 (1999) 5749-5786.

[20] P. R. Gordoa and A. Pickering, Europhys. Lett. 47 (1999) 21-24.

[21] P. R. Gordoa and A. Pickering, J. Phys. A 33 (2000) 557-567.

[22] P. R. Gordoa, N. Joshi and A. Pickering, Publ. Res. Inst. Math. Sci. (Kyoto) 37 (2001) 327-347.

[23] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Math. Phys. 46 (2005) 103509.

[24] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Nonlinear Math. Phys. 12, Suppl. 2 (2005) 180-196.

[25] P. R. Gordoa, A. Pickering and Z. N. Zhu, Chaos, Solitons and Fractals 29 (2006) 862-870.

[26] P. R. Gordoa, N. Joshi and A. Pickering, J. Math. Phys. 47 (2006) 073504.

[27] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Math. Phys. 48 (2007) 023515

[28] Z. N. Zhu, J. Phys. A 40 (2007) 7707-7719.

[29] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Math. Phys. 51 (2010) 053505.

[30] P. R. Gordoa, A. Pickering and J. A. D. Wattis, Appl. Math. Comp. 237 (2014) 77-84.

[31] P. R. Gordoa, N. Joshi, A. Pickering, J. Differential Equations 217 (2005) 124-153.

[32] A. Pickering, J. Math. Phys. 50 (2009) 013507.

[33] P. R. Gordoa and A. Pickering, J. Math. Phys. 54 (2013) 033502.

[34] Yu. Fedorov, P. R. Gordoa and A. Pickering, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 3462-3468.

[35] J. M. Conde, P. R. Gordoa and A. Pickering, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 448-458.

[36] U. Muğan and F. Jrad, J. Phys. A 32 (1999) 7933–7952.

[37] P. A. Clarkson, N. Joshi, A. Pickering, Inverse Problems 15 (1999) 175-187.

[38] C. M. Cosgrove, Stud. Appl. Math. 104 (2000) 1-65.

[39] U. Muğan and F. Jrad, J. Nonlinear Math. Phys. 9 (2002) 282–310.

[40] N. A. Kudryashov, J. Math. Phys. 44 (2003) 6160-6178.

[41] C. M. Cosgrove, Stud. Appl. Math. 116 (2006) 321-413.

[42] A. Sakka, Symmetry, Integrability and Geometry: Methods and Applications 5 (2009) 024.

[43] A. Pickering, Chaos, Solitons and Fractals 45 (2012) 935-941.

[44] N. A. Kudryashov, Regul. Chaotic Dyn. 19 (2014) 48-63.

[45] P. R. Gordoa, A. Pickering and Z. N. Zhu, Phys. Lett. A 374 (2010) 3422-3424.

[46] P. R. Gordoa, A. Pickering and Z. N. Zhu, Phys. Lett. A 377 (2013) 1345-1349.

[47] S. P. Balandin and V. V. Sokolov, Phys. Lett. A 246 (1998) 267-272.

[48] P. J. Olver and V. V. Sokolov, Commun. Math. Phys. 193 (1998) 245-268.

[49] P. R. Gordoa, A. Pickering and Z. N. Zhu, J. Differential Equations 261 (2016) 1128-1175.

[50] P. R. Gordoa and A. Pickering, J. Differential Equations 271 (2021) 499-532.

[51] P. R. Gordoa and A. Pickering, Phys. Lett. A 382 (2018) 1908–1915.

[52] A. Pickering, P. R. Gordoa and J. A. D. Wattis, Physica D 391 (2019) 72-86.

[53] P. R. Gordoa, A. Pickering and J. A. D. Wattis, Physica D 415 (2021) 132754.

[54] M. Wadati and T. Kamijo, Prog. Theor. Phys. 52 (1974) 397-414.

[55] F. Calogero and A. Degasperis, Nuovo Cimento B 39 (1977) 1-54.

[56] C. Athorne and A. P. Fordy, J. Phys. A 20 (1987) 1377-1386.

[57] M. Gürses, A. Karasu and V. V. Sokolov, J. Math. Phys. 40 (1999) 6473-6490.

[58] P. J. Olver and J. P. Wang, Proc. London. Math. Soc. (3) 81 (2000) 566-586.

[59] S. Carillo and C. Schiebold, J. Math. Phys. 50 (2009) 073510.

[60] N. A. Kudryashov and M. B. Soukharev, Phys. Lett. A 237 (1998) 206-216.

[61] A. N. W. Hone, Phys. D 118 (1998) 1-16.

[62] N. A. Kudryashov and A. Pickering, in "SIDE III — Symmetries and Integrability of Difference Equations," 245-253, eds. D. Levi and O. Ragnisco, CRM Proceedings and Lecture Notes 25 (American Mathematical Society, Providence, Rhode Island, 2000).

[63] C. M. Bender and S. A. Orszag, "Advanced Mathematical Methods for Scientists and Engineers," (Springer-Verlag, New York, 1999).

[64] S. O. Durugo, "Higher-Order Airy Functions of the First Kind and Spectral Properties of the Massless Relativistic Quartic Anharmonic Oscillator," (Ph. D. Thesis, Loughborough University, 2014).

[65] S. O. Durugo and J. Lőrinczi, J. Differential Equations 264 (2018) 3775-3809.

[66] S. Tarricone, Symmetry, Integrability and Geometry: Methods and Applications 17 (2021) 002.

[67] V. Retakh and V. Rubtsov, J. Phys. A: Math. Theor 43 (2010) 505204.