

The definition of the scalar product: An analysis and critique of a classroom episode

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Abstract

In this paper we present, analyse and critique an episode from a secondary school lesson involving an introduction to the definition of the scalar product. Although the teacher attempted to be explicit about the difference between a definition and a theorem, emphasizing that a definition was just an arbitrary assumption, a student rejected the teacher's definition in favour of his own alternative. With reference to this particular case, we seek to explore some ways in which teachers can introduce mathematical definitions to students so as to support, rather than attempt to circumvent, their mathematical sense making. In this regard, we believe that it is important to develop learning opportunities for students that help them to gain some appreciation of important structural and historical reasons that underpin the definitional choices made.

Keywords

Definitions

Scalar product

Sense making

1. Introduction

The teaching of definitions in school mathematics is recognised as challenging for those who see the learning of mathematics in terms of sense making (Lakatos, 1976; Vinner, 1991; Foster, 2014a; Tall, 2013). Definitions make no logical claims, so require no mathematical justification, and yet students may be reluctant to accept them, or may appear to accept them but do so only superficially. Rabin, Fuller and Harel (2013) have shown that students may expect too much from a definition by demanding that the teacher demonstrate that it is true. It might be thought that this could be prevented if the teacher were to be explicit about the difference between a definition and a theorem, and that a definition may just be seen as something that is assumed true. In this paper we examine a classroom episode in which the teacher attempted to do exactly that when introducing the definition of the scalar product to a class of 17-18-year-old secondary school students. However, a student rejected the standard definition in favour of his own alternative. We analyse this episode and draw out ways in which teachers can introduce formal mathematical definitions to students so as to support their mathematical sense making rather than attempt to circumvent it.

2. Mathematical definitions

Definitions and axioms are needed in mathematics mainly to avoid issues of logical circularity and infinite regression (Human & Nel, 1989; De Villiers, 1995). If a concept consists of a network of various logical relations and properties, attempts to prove all of the properties will inevitably lead to a circular argument of the kind: $A \Rightarrow B \Rightarrow C \Rightarrow \dots \Rightarrow A$. The only way to avoid this is to choose one of the properties as the definition (an unproved assumption) from which to derive all the others. Several different equivalent definitions can usually be formulated simply by selecting different combinations of necessary and sufficient properties. For example, “lines are parallel if and only if corresponding angles are equal” is

equivalent to “lines are parallel if and only if alternate angles are equal”, and it is arbitrary which we take as the definition and which as the theorem (Human & Nel, 1989; Rabin, Fuller & Harel, 2013; Tall, 2013). Mathematicians often choose their definitions as a matter of convenience.¹

There are two broad ways of defining concepts in mathematics, which may be termed *descriptive (a posteriori)* and *constructive (a priori)* (Edwards & Ward, 2004; Human, 1978, pp. 164-165; Krygowska, 1971). Descriptive definitions systematise existing knowledge, whereas constructive definitions produce new knowledge (De Villiers, 2009). A crucial distinction is that in descriptive defining the concept image *precedes* the eventual concept definition (Vinner & Hershkowitz, 1980; Vinner, 1983), whereas in constructive defining the concept image is developed or explored *after* the concept definition.

The scalar product of two vectors that is the topic of this paper might appear to be an example of a constructive definition; however, historically that was not the case. The definition arose in a descriptive manner from the applied context of physics, and was abstracted from concepts such as mechanical work done, defined in terms of force and displacement as $\mathbf{F} \cdot \mathbf{s} = Fs \cos \theta$. In this view, mechanical work is defined as the projection of the applied force onto

¹ For example, it is customary in the definition of groups to interpret the conditions of having an identity and an inverse as implying both “left” and “right” identities and inverses. However, a left identity together with a left inverse logically imply a right identity and a right inverse, but it requires a short detour to prove this. In the same vein, though Boolean Algebra is usually defined by five axioms, these can be reduced to only three axioms, but it requires substantial deductive work to establish that. Likewise, by choosing definitions based on the symmetry properties of quadrilaterals the deductive derivations of other properties are substantially simplified in comparison with the traditional proofs based on congruency (De Villiers, 2011). For the same reason of convenience and deductive simplification, the partition definitions of Euclid for the quadrilaterals (i.e., excluding squares from the rectangles and rhombi, etc.) are no longer used in school geometry, but instead inclusive definitions are generally preferred (De Villiers, 1994; Foster, 2014b).

the displacement (or vice versa), and the $\cos \theta$ arises to account for the component of the force effective in the direction of the displacement (or, equivalently, the actual displacement made in the direction of the force). In other words, historically the intuitive concept image of the scalar product within physics preceded the formal definition within mathematics, and the further purely mathematical exploration of its properties and its mathematical applications by Hamilton, Grassmann, Maxwell and others with the general development of vector analysis from about 1830 onwards (Knott, 1978). Even if we completely disregard the earlier context of the scalar product in physics, the development of modern vector analysis was undoubtedly strongly influenced by the geometrical treatment of complex numbers proposed by Wessel and Argand in 1797 and 1806 respectively. Moreover, it is significant that Kramer (1982, p. 74) mentions that the Wessel-Argand diagram was primarily designed to represent a vector interpretation of a complex number for various, existing applications of complex numbers to physics.

Hewitt (1999, 2001a, 2001b) has advocated *telling* students things which are *arbitrary* in mathematics, but actively avoiding telling them things which are *necessary*, instead helping them to work these out for themselves. It might be assumed that this implies that students must simply be told definitions, and this could indeed be the case for *terminological* definitions (Human & Nel, 1989, pp. 58-59), which distinguish concepts from one another, such as “a number divisible by 2 is called even”. Clearly students need to be told the conventionally accepted names and notations for various mathematical objects and processes – it is impossible for them to “rediscover” or “reinvent” such terminology and notation on their own.

However, *propositional* definitions (similar to what Tall [2013] has called *formal* definitions) involve organising (systematising) the internal logical relationships among the various properties of the concept deductively, and with these it may be possible to engage students in the process of both constructively and descriptively defining some concepts themselves. In this way, they may better understand how definitions arise, that their choice is arbitrary, and yet that some definitions are more convenient than others (De Villiers, 2009). This could assist students in making sense of propositional definitions later when they are directly provided to them as their mathematics education progresses to a more formal style of presentation (Tall, 2013).

Since the definition of a scalar product essentially describes a particular computational procedure, it can be viewed more as a terminological definition than a propositional definition. On this basis, it would appear that one has no choice but to simply tell or give students the scalar product definition. However, as we will illustrate in presenting and discussing the classroom episode, this common teaching approach of just announcing the definition may be problematic for some students, especially if the intellectual needs of students are not considered.

Harel (2013, 2008a) stresses that students must have an intellectual need for the mathematics that is taught to them, and he distinguishes this from affective needs related to motivation, such as a need to use mathematics to transform the world in some way. In particular, Harel (2013) defines the following intellectual needs:

- (i) the need for *certainty* as the human desire to know whether something is true (p. 124);
- (ii) the need for *causality* as the desire to explain, to determine the cause of something (p. 126);

- (iii) the need for *computation* as the desire to quantify, calculate and compare (p. 131);
- (iv) the need for *communication* as the desire to unambiguously formulate and formalise (p. 137);
- (v) the need for *structure* as the desire to organise knowledge into a logical structure (p. 140).

In the context of mathematical definitions, our interpretation of Harel's model is that students need to see where a new definition comes from (causality) and what it might offer in terms of computation or insight (computation and communication). They also need to develop confidence (certainty) that the definition will lead to stable, certain outcomes that are consistent (structure) with the mathematics that they already know.

Harel (2013) points out that "since intellectual need depends on the learner's background and knowledge, what constitutes an intellectual need for one particular population of students may not be so for another population of students" (p. 145). Implicit here is the idea of students bringing different prior knowledge and experiences to the learning process and how these influence new learning (Ausubel, 1968). This prior knowledge has been aptly labelled as *met-befores* by Tall (2013). Such met-befores can be supportive of new learning or problematic, where concepts that may have fitted earlier thinking but are now inadequate for the new situation constitute misconceptions. When students encounter an unfamiliar mathematical situation, which requires the invention or utilisation of a new definition, they are likely to compare it with their previously encountered ideas in order to seek some resonance. This may result in students carrying over ideas inappropriately from one situation to another, but it may also assist their learning. Supportive met-befores give the experienced mathematician an advantage when approaching something new, but conversely extensive familiarity with one mathematical structure or organisation may make it harder to enter into a

contrasting one. If new definitions are not concordant with some of the mathematics that students know, or think that they know, some revision and reorganisation will be needed. Cognitive conflict will need to be resolved not through mere assimilation but through accommodation and cognitive restructuring.

Rabin, Fuller and Harel (2013) analysed two classroom episodes in which students also rejected their teachers' (standard) mathematical definitions in favour of their own. The authors argued that the teachers' approaches failed to address the students' intellectual needs and that students must be helped to see "enough advantages of the new ideas to make it rational to pursue them further" (Rabin, Fuller & Harel, 2013, p. 658). In agreement with Freudenthal (1973) and others, they suggest that a historical perspective on how the particular mathematical discourse in question has evolved can assist with doing this. In particular, they caution against teachers confounding definitions and theorems. When students are presented with an arbitrary mathematical definition, but are led by their teacher to believe that it must be accepted as "correct", they may be inclined to view their teacher's justifications for it as lacking.

3. Method

We now draw on a classroom episode involving the introduction of the definition of the scalar product to a class of secondary school students. We selected this episode from recent observations because the teacher in question made a particular effort to make the definition/theorem distinction especially plain. However, we shall see that this alone was not enough to support acceptance of the definition by all the students. Methodologically, we then go on to analyse, critique and discuss this episode in the light of the aforementioned perspectives on definitions. Specifically, we look at the episode epistemologically and

explore to what extent the lesson authentically illustrated the mathematical process of defining (constructively or descriptively), as well as how the lesson attended to the various intellectual needs identified by Harel (2008a, 2008b, 2013). We then further explore some ways in which teachers might introduce mathematical definitions to students. In particular we seek to answer the question: How might a formal definition be introduced to students in ways that support mathematical sense making?

The episode we describe took place in a UK secondary school (age 11-18) in a class of sixth-form students (age 17-18) studying the Core 4 unit of an A-level mathematics course. Thus they were older than the students in the episodes discussed by Rabin, Fuller and Harel (2013), were tackling a more advanced topic, and were presumably bringing to it more plentiful and well-developed met-befores (Tall, 2013). The students had not encountered the scalar product or matrix multiplication before, but naturally had had many experiences over the years of numerical and algebraic multiplication. For these students, the scalar product might be viewed as a substantial extension to their previous understandings of multiplication. The word “product” in the name (and the notation of the dot and the language of “dot product”) would be likely to draw on students’ met-befores relating to multiplying numbers and algebraic expressions. Such met-befores could be supportive in helping students to recognise a binary operation and to appreciate aspects of commutativity, associativity and distributivity implicit in the teacher’s presentation (see below). However, these met-befores might also be expected to be problematic where the scalar product differs from ordinary multiplication, such as, for example, in the direction of the vectors being critical in determining the value of the product, in the fact that scalar multiplication is not closed (the result is not a vector), and in associativity, for instance, being meaningless, since “ $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ ” is undefined.

The data presented below comes from detailed notes taken at the time the lesson was observed by one of the authors and during discussion with the teacher immediately afterwards. The teacher was experienced in teaching mathematics at this level and over the previous 18 months of teaching this class had sought to establish a classroom culture in which the mathematics presented was never accepted on the authority of the teacher but where everything was proved and built up in a logical sequence. To this end, he had made an attempt throughout to distinguish clearly and explicitly between definitions, which were presented as arbitrary, and theorems, which were seen as necessary. We use this episode illustratively and do not seek to make generalisations about how other teachers might introduce the topic or how other students might respond. Our interest lies in the pedagogical issues raised by this particular episode.

4. The classroom episode

The teacher began this introductory lesson to the scalar product by saying: “In maths you can define anything you like, so long as it doesn’t contradict anything you’ve already got and so long as it’s useful”. He explained to the students that although they knew how to multiply a vector by a scalar, they currently had no meaning for multiplying a vector by a vector. He said, “So let’s define something. So, right off the top of my head, here’s a definition”, and he wrote on the board:

$$\underline{a \cdot b} = ab \cos \theta$$

saying, as he wrote, “vector a multiplied by vector b – we say ‘ a dot b ’ – is equal to the magnitude of a times the magnitude of b timesed by \cos of θ , where θ is the angle between the two vectors”. The teacher drew the diagram shown in Figure 1 in order to indicate the definition of θ .

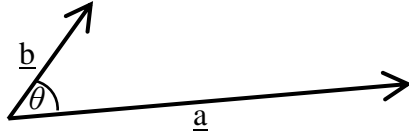


Figure 1. Two vectors and the angle between them.

He then said, with a smile, “So this is made up! It’s just a definition. You’re not allowed to ask ‘Where did the cos theta come from?’!” As a joke, one student immediately asked this and everyone laughed. Another student asked why the teacher had, in handwriting, underlined the “*a*” and the “*b*” on the left-hand side, but not those on the right, and the teacher explained that this *scalar* product takes two vectors as the input and produces a scalar number as the output. The teacher said, “Are you prepared to accept this arbitrary definition and see what happens with it? It turns out to be really useful – obviously, otherwise we wouldn’t have bothered defining it. If it didn’t do anything interesting I wouldn’t be teaching it to you!”

The teacher then explained that it was “easy to work out dot products, even though it might not look like it at first”. He wrote:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$$

and then expanded the brackets as:

$$\begin{aligned} & \mathbf{i} \cdot 4\mathbf{i} + \mathbf{i} \cdot 5\mathbf{j} + \mathbf{i} \cdot 6\mathbf{k} \\ & + 2\mathbf{j} \cdot 4\mathbf{i} + 2\mathbf{j} \cdot 5\mathbf{j} + 2\mathbf{j} \cdot 6\mathbf{k} \\ & + 3\mathbf{k} \cdot 4\mathbf{i} + 3\mathbf{k} \cdot 5\mathbf{j} + 3\mathbf{k} \cdot 6\mathbf{k} \end{aligned}$$

He did not address any of the assumptions lying behind these steps. By showing that \mathbf{i} , \mathbf{j} and \mathbf{k} are mutually orthogonal, the teacher persuaded the students that for perpendicular vectors θ

would be 90° , so $\cos \theta$ would be zero, so six of these nine terms would disappear, leaving just $\mathbf{i} \cdot 4\mathbf{i} + 2\mathbf{j} \cdot 5\mathbf{j} + 3\mathbf{k} \cdot 6\mathbf{k}$. In a similar way, he explained that for parallel vectors θ would be 0° , so $\cos \theta$ would be 1, reducing the expression to $4 + 10 + 18 = 32$. Since the left-hand side of $\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$ was 32, and the students already knew how to calculate the magnitudes a and b , he showed that this enabled them to find $\cos \theta$, and hence θ , which he stressed would have been very hard to do in three dimensions using the conventional trigonometrical methods, such as the cosine rule, which they had at their disposal. The teacher commented: “Isn’t that nice? You make up a definition – just out of nowhere – and when you look at what happens with it suddenly you find you can work out angles between lines in three d[imensions]. Isn’t that great!”

The class began working on exercises finding θ for given \mathbf{a} and \mathbf{b} . While the teacher was circulating around the classroom, a student asked him:

If you can define $\mathbf{a} \cdot \mathbf{b}$ to be anything you like, why not define it to be just $\cos \theta$? The magnitudes are just a hassle – you have to divide by them to get rid of them – so why not just make it $\cos \theta$?

The teacher stated afterwards that he was unprepared for this question, but at the time he seemed pleasantly surprised by it. He had anticipated that students might propose the definition $\mathbf{a} \cdot \mathbf{b} = ab$, leaving out the $\cos \theta$, but not $\mathbf{a} \cdot \mathbf{b} = \cos \theta$, leaving out the ab , which he saw as being fundamental to the idea that this was supposed to be some kind of “product”.

The teacher wrote $\mathbf{a} \cdot \mathbf{b} = \cos \theta$ on the board and stood thinking about it for several seconds. Talking aloud to himself, with the student listening, he said “It isn’t going to work, ’cause you’re going to get numbers bigger than 1, which \cos can’t be. It’s dividing by the magnitudes that gets it down to less than 1 ... If you could do it like that we wouldn’t bother

doing it the other way, so there must be some reason”. The student replied that if it was just a definition then we could choose it to be however we wanted it to be, to make life easy, and the teacher replied that he had no answer at that point and said that he would have to take away the student’s question to think about it.

5. Analysis and Discussion

It is firstly noteworthy that in the episode described here, no historical or epistemological justification of how or why the scalar product arose was offered, and the lesson therefore does not really attend to students’ need for causality. Moreover, the definition was presented *a priori*, yet poorly and un-authentically imitating constructive defining as a process, since it was not discussed as an extension or variation of the definition of multiplication of numbers to that of vectors. This is a clear example of “didactical inversion”, as described by Freudenthal (1973), and illustrates how the directly-introduced definition has become empty by ignoring its historical origin (Lakatos, 1976). Indeed, the teacher’s phrase “just out of nowhere” could be taken to indicate a rather naïve view of the history of mathematics. The teacher presumably felt that the pragmatic justification that “It turns out to be really useful – obviously, otherwise we wouldn’t have bothered defining it” should be enough.

A more practically-rooted, historically-informed introduction of the scalar product following a descriptive (*a posteriori*) approach to defining would have been to begin by drawing on, for example, the physical notion of mechanical work done, firstly when the force and the distance moved are aligned and subsequently when they are not. This could have motivated the inclusion of the $\cos \theta$ more naturally, without the implication that the scalar product exists primarily in order to calculate the angles between vectors.

Whereas the teachers in the episodes discussed by Rabin, Fuller and Harel (2013) did not seem to appreciate or stress the distinction between a definition and a theorem, the teacher in this episode was explicit about the fact that he was offering the class an arbitrary, terminological definition that they should not be able to question (Hewitt, 1999, 2001a, 2001b), for example, by stating at the outset “You’re not allowed to ask ‘Where did the $\cos \theta$ come from?’” Yet, as in Rabin, Fuller and Harel’s cases, a student rejected the definition being offered in favour of his own alternative. In our episode, the student’s rejection of the definition did not seem to be caused by a failure to see enough evidence that it was “true”. Instead, we argue that the pragmatic, computational aspect of the scalar product definition was overemphasised by the teacher relative to other important needs, especially the needs for certainty and for structure. Since the teacher in our episode is overt from the start about presenting an arbitrary definition, he was consequently in some difficulty when a student offered an alternative definition that seemed to the teacher wrong. The teacher avoided adopting an authority stance in which he simply told the student that his definition was “right” and the student’s was “wrong”. Having stressed to the class the arbitrariness of his definition, which he seemed to justify purely on pragmatic grounds of what might be accomplished with it later on, the teacher could not with consistency now say that it had any absolute truth about it. But this left him unable to argue against the student’s apparently easier alternative definition. The teacher stated in discussion after the lesson that he was prepared for students to question his definition (“You’re not allowed to ask ‘Where did the $\cos \theta$ come from?’”) and intended to justify it solely on the grounds of what it could achieve computationally. However, he was attacked on exactly this ground when the student questioned why his own simpler definition would not be *easier* to compute with.

In discussion after the lesson, the teacher indicated that he had an answer of sorts to the question “Why $\cos \theta$?”, which he *had* anticipated:

I would have said that it could be $\sin \theta$ instead, but that’s a product with a different name – the cross product. I guess you could define a third one with $\tan \theta$ but I’ve never heard of anyone doing that!

It is apparent that the teacher had considered dimensions of possible variation (Watson and Mason, 2005) in the definition of the scalar product but had failed to anticipate the suggested omission of the factor ab . The sense that the cosine could just as well be the sine supported the teacher in his view that the definition was arbitrary, but did not help him to answer the student’s question about the alternative, which the teacher had not considered, of having just $\mathbf{a} \cdot \mathbf{b} = \cos \theta$. When it was suggested to him by one of the authors that it would be possible to be even more reductive and write $\mathbf{a} \cdot \mathbf{b} = \theta$, which would be even easier computationally, as it would save having to carry out the inverse cosine operation, other than feeling that this would be “dimensionally suspect”, he could not explain what would be wrong with this.

Harel (2008a) comments that intellectual need is “largely ignored in teaching” (p. 900).

Certainly the teacher here does not follow the four-step process which Harel (2013) advocates in which (i) a specific intellectual need for these students and the scalar product is identified and then (ii) used to prompt questions that make sense for students to investigate, leading to (iii) a sequence of problems whose solutions help to answer those questions, and finally (iv) emergence of the concept of the scalar product. Such an approach might have been possible if the teacher were to have first asked students to use their prior knowledge to find the angles between given vectors and allowed them some time to struggle with approaches based on three-dimensional trigonometry.

However, the teacher did seek to convince his students that the definition he was proposing would have wonderful consequences, which appeared to relate primarily to the need for computation (Harel, 2013). This was not explicit at the start of the lesson, but emerged later. Students did not see, at least at the start of the lesson, “a reason – separate from the teacher’s authority – for extending definitions in particular ways” (Rabin, Fuller & Harel, 2013, p. 656). The teacher explicitly invited the class to trust him that something of value will emerge when he commented: “If it didn’t do anything interesting I wouldn’t be teaching it to you!”

Alternatively, in a more authentically constructive defining approach, the teacher could have reviewed the multiplication properties of real numbers before asking students to consider a vector \mathbf{v} written as $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and to think about the relationship between $r = |\mathbf{v}|$ (the length of the vector) and the components x , y and z . In exploring this, the idea that $x = r\cos\theta$ could have arisen naturally, since the dot product of \mathbf{v} and \mathbf{i} is x , the dot product of \mathbf{v} and \mathbf{j} is y and the dot product of \mathbf{v} and \mathbf{k} is z .

As in one of Rabin, Fuller and Harel’s (2013) examples, preserving the distributive property $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ is fundamental to the vector algebra being developed in the episode described here. In the student’s definition, $\mathbf{a} \cdot \mathbf{b} = \cos\theta$, all parallel vectors must have a dot product of 1, meaning that $\mathbf{i} \cdot \mathbf{i} = (2\mathbf{i}) \cdot \mathbf{i} = 1$, yet $2(\mathbf{i} \cdot \mathbf{i}) = 2$, so we lose associativity. Likewise, $\mathbf{i} \cdot (\mathbf{i} + \mathbf{i}) = \mathbf{i} \cdot 2\mathbf{i} = 1$, but $\mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{i} = 1 + 1 = 2$, so distributivity over addition is also lost. Thus, although this definition would allow us to compute θ more easily given $\mathbf{a} \cdot \mathbf{b}$, it would not allow us to compute $\mathbf{a} \cdot \mathbf{b}$ at all easily from \mathbf{a} and \mathbf{b} . The teacher did not mention during his presentation that the scalar product as he defined it was distributive over vector addition, although he tacitly assumed this when evaluating $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$. When it suited his purposes, he drew on convenient met-before of “multiplication” without comment. The

teacher's approach was entirely symbolic, looking at properties of operational symbolism, rather than seeking to base the argument on conceptual embodiment – namely the relationship between a vector and its components seen geometrically.

Rabin, Fuller and Harel (2013) regard it as legitimate when engaging in constructive (*a priori*) defining “to ‘look forward’, exploring the consequences of alternative possible conventions before deciding which one to adopt” (p. 650). In some cases of definitional ambiguity, no single resolution is made (Foster, 2011). For example, as the teacher was aware, there is more than one possible way to define multiplication of vectors; hence we have the scalar (dot) product and the vector (cross) product, each preserving some aspects of the notion of “multiplication” but having different properties and uses. In such circumstances the teacher who knows this cannot treat the definition of the scalar product as having an inevitability about it, since they are aware that other possibilities are in fact common. Perhaps as a consequence of this, in this case the teacher chose to direct the class's attention entirely to the possible computational benefits rather than to the need to preserve a robust algebraic structure.

The teacher did not attune the class to the possibility of other kinds of intellectual need beyond computation: the definition was presented solely as a pragmatic tool with the promise that students will later see its value. Other categories of need, particularly those of certainty, causality, communication and structure, might have usefully been brought into the discussion with the students. In their example of directed numbers, Rabin, Fuller and Harel (2013) suggest that students might carry out calculations using the distributive property to explore whether it is well-defined in the extended context of directed numbers. If it is, then it might make sense to adopt a convention for multiplication of negative numbers that preserves it.

Distributivity from a structural point of view is important in the episode described here too, and some way of drawing students' attention to it would seem to be important here as well. For example, students might have explored forming scalar products of various vectors with other vectors, and with the sums of other vectors, in order to suggest the conjecture that the scalar product as defined is distributive over addition. A proof of this might have followed so as to set the development of the idea of the scalar product on a firmer basis.

Mathematical definitions in new domains do not always seek to retain as much as possible of that which was true in the old domain. For instance, when defining the *cross* product we do not worry about losing commutativity, even though that is an important part of students' notions of multiplication, and something that the scalar product preserves. This is in a sense an example of constructive defining, i.e., a variation of a definition by changing some properties. When losing a property such as commutativity, there needs to be a compensating benefit. In fact, the non-commutativity of matrix multiplication, for instance, is extremely valuable, since it enables an isomorphism between this and the composition of transformations, where the order makes a difference to the outcome. Such an isomorphism requires certain structural properties to be preserved – the object of study of abstract algebra. In expanding algebraic terms and polynomials through “multiplication”, we seek to define them to correspond as closely as possible with the prototype examples, e.g., the field of real numbers. However, some properties of the prototype, such as commutativity of multiplication, do not always correspond in the most general examples. Tall (2013), for example, mentions that when a particular context is extended and ideas in the extension are blended with the originals, some will fit with the original and some will not. He terms this an “extensional blend” (pp. 128-130). When concepts and their definitions are abstracted from

other contexts, they almost never possess all the properties of the original context (Niss, 2006), and in some cases can be surprisingly different.

It is interesting to observe that the student with his proposed alternative definition easily and naturally engaged in constructive defining and displayed a clear sense that definitions are arbitrary and are often chosen for convenience. However, the student was restricted by the one-sided focus on only the computational aspects of the definition and the calculation procedure, without seeing the bigger picture. Indeed without a bigger picture of practical relevance or structural integrity, this kind of completely arbitrary constructive defining as done by the student is usually not very productive. Kline (1977, p. 48) termed it “postulate piddling”, and quoted the research mathematician Rolf Nevanlinna as saying that “The setting up of entirely arbitrary axiom systems as a starting point for logical research has never led to significant results”.

6. Conclusion

Although students’ “intellectual need” may be seen as the driving force for learning (Harel, 2008a, 2008b), in the episode described here there appears to have been an overemphasis on one aspect of intellectual need, namely computation, which blinded the students (and the teacher) to other important elements of the mathematical definition under consideration. Thus the epistemological justification (Harel, 2008a) for its creation was weak. Tall (2013, p. 113) comments that “even if one wishes to encourage learners to make sense of mathematics at every stage, the full crystalline structure of mathematics may lie in a later level that is not available to the learner at the time.” Even so, in this episode, the teacher’s approach of inviting students simply to trust the definition and see what happened appeared to hamper this student in drawing on what he had met-before in a productive way. Relevant met-befores for

these students would include not only the various notions of multiplication discussed above but also conceptualisations of the ways in which mathematics is done in school – the rules of the game, which Harel (2008a, 2008b) calls “ways of thinking”. The notation and language of the scalar product is clearly based on an analogy with the multiplication of real numbers, and the fact that the student’s suggestion $\mathbf{a} \cdot \mathbf{b} = \cos \theta$ involved no multiplication at all suggests that unfortunately no link was being made with the “product” aspect of the definition.

Rabin, Fuller and Harel (2013) identify the major problem in the two cases they discuss as the teacher presenting a situation that involves the adoption of a definition, but framing the task for students as “one of determining a provably correct answer” (p. 656). In this paper we have seen that even when that pedagogical error is not made, the different aspects of students’ needs have to be carefully balanced so that one does not assume disproportionate importance. In the case of the episode described here, fulfilling the need for computation by the teacher outweighed addressing the students’ need for certainty as well as that for explanation (which Harel treats under causality) and structure, leading to a problematic suggested definition from the student.

The mathematics teacher here effectively invites the students at the beginning of the lesson to trust him, or at least to suspend judgment. He justifies the definition of the scalar product pragmatically in terms of what it will enable the students to compute. However, in order to have certainty in the computation, students need to understand the properties which the definition is designed to preserve. Ideally students would be so aware of structural and historical matters that when encountering a new definition of a product they would spontaneously ask questions such as “Where does it come from?” or “Where is it applicable?”, and even technical questions such as “So is this product commutative?”

Students might be invited to “suppose” and to ask “what if” questions for themselves (compare Brown & Walter, 2005) whenever they encounter new concepts and definitions, and/or to have the teacher suggest some possibilities for them. It should not be expected that encountering a new definition will be trouble-free for the student, and nor should this be desirable; indeed, there is some evidence that when students engage in productive struggle against difficulties and are given opportunities to “fail” they learn more effectively (Heyd-Metzuyanim, Tabach, & Nachlieli, 2015).

Although a definition may be mathematically arbitrary (Hewitt, 1999, 2001a, 2001b), in the sense that it could be otherwise, it is not chosen completely at random: definitions are made pragmatically with an eye to their consequences (Freudenthal, 1973; Lakatos, 1976). The pedagogical challenge is to help students understand these pragmatic reasons. As mathematics becomes more formal, theoretic and axiomatic in the progression from school to university, there is an increasing tendency to introduce students to new concepts and structures via the direct presentation of definitions (Vinner, 1991). In such circumstances, it is impossible for students to anticipate fully the benefits or drawbacks of a definition beforehand. An important aspect of mathematical reasoning is to be prepared to “suppose” a statement and examine its consequences, without committing oneself in advance to its truth or falsity. Mathematical definitions are frequently constructed with purposes in mind that may be opaque before students have extensive experience of a topic. Definitions may be set up to avoid problems which the student cannot yet envisage or to make a distinction between the object being studied and something else that they have not yet encountered (Foster, 2014a).

Rather than the teacher or textbook merely providing students with ready-made definitions, it has been argued by Krygowska (1971), Freudenthal (1973), Human and Nel (1989), De Villiers (1998) and others, that the process of defining mathematical objects be made explicit for students so as to help them understand where definitions come from so that they may accept them as meaningful. Freudenthal (1973) has pointed out that, historically, formal axioms and definitions for mathematics were mostly put in place long after important results had been obtained. According to Freudenthal (1973) and Human (1978), good mathematics education should allow students, at least in part, to “re-invent” or “re-construct” mathematical content such as definitions and axioms. There is also a need to create suitable learning opportunities for students to occasionally experience their evolution over a period of time.

While we acknowledge that it is difficult or sometimes impossible for any teacher to simultaneously attend to all students’ intellectual needs when introducing a new definition and topic, we feel that a teacher should plan to attend to all of these needs at some point in order for learning to be meaningful. A teacher may also need to be alert to respond to students’ intellectual needs as they arise (Foster, 2015). For example, when the student offered his alternative, easier computational definition this might have been an opportune time to briefly deviate from the intended lesson plan and discuss the historical origin of the scalar product definition in relation to mechanical work. However, this particular teacher did not appear to have that perspective, and on the basis of his rationale that definitions were just chosen arbitrarily, the student’s alternative proposed definition was equally valid.

It would seem that balancing students’ various intellectual needs when introducing mathematical definitions is critical if they are to perceive the subject as one of sense making rather than accepting statements from the teacher as an authority (Foster, 2014a). The

approach taken by the teacher to introduce particular mathematical definitions has the potential to exert a powerful influence on students' views of mathematics. We hope that this classroom episode with the scalar product definition could be useful for initiating productive discussions in mathematics education at pre-service and in-service level, as well as for researchers in mathematics education to consider some of the possibilities and issues raised in this paper, not only when analysing other classrooms situations, but also when planning teaching experiments on the introduction of formal definitions.

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