Alternating-time temporal logic with resource bounds

Hoang Nga Nguyen*, Natasha Alechina*,
Brian Logan*, and Abdur Rakib**

*School of Computer Science, The University of Nottingham, UK,
{nza,bsl,hnn}@cs.nott.ac.uk
**School of Computer Science, The University of Nottingham Malaysia
Campus, Malaysia , Abdur.Rakib@nottingham.edu.my

June 1, 2015

Abstract

Many problems in AI and multi-agent systems research are most naturally formulated in terms of the abilities of a coalition of agents. There exist several excellent logical tools for reasoning about coalitional ability. However, coalitional ability can be affected by the availability of resources, and there is no straightforward way of reasoning about resource requirements in logics such as Coalition Logic (CL) and Alternating-time Temporal Logic (ATL). In this paper, we describe a logic for reasoning about coalitional ability under resource constraints. We extend ATL with costs of actions and hence of strategies. We give a complete and sound axiomatisation of the resulting logic, Resource-Bounded ATL (RB-ATL), and a model-checking algorithm for it.

1 Introduction

In many situations, a group of agents can cooperate to achieve an outcome which cannot be achieved by any agent in the group acting individually. For example, in the prisoners dilemma, a single prisoner cannot ensure the optimal outcome, while a coalition of two prisoners can. Similarly, it may be possible for a set of cooperating agents to solve a difficult computational problem by distributing it, while a single agent may not have sufficient memory or processor power to solve it. In the latter case, there is an interaction between the amount of resources available to the agents (or the amount of resources which they are willing to contribute), and their ability to jointly achieve a goal.

In this paper we describe a logic, Resource-Bounded ATL (RB-ATL), for reasoning about coalitional ability under resource constraints. RB-ATL allows us to express and verify properties such as

(1) 'a coalition of agents A has a strategy to achieve a property φ provided they have resources b, but they cannot enforce φ under a tighter resource bound b_1',

(2) 'A has a strategy to maintain the property φ, provided they have resources b',

(3) 'A has a strategy to maintain φ until ψ becomes true, provided A has resources b'.

In Section 2.4, we illustrate the expressive power of RB-ATL on a simple example of a sensor network, where the agents (sensor nodes) require two resources: energy and memory.

In previous work, we studied a version of Coalition Logic with resource bounds, RBCL [5]. RBCL can express properties of the form (1) above, but not of the form (2) and (3). Other work on temporal logics and logics of coalitional ability with resource constraints includes [7, 8, 10, 11, 2]. However this work concentrates on model-checking complexity, rather than axiomatisation, which is the focus of this paper.

This paper is a revised and extended version of [4]. In [4], we gave a sound and complete axiomatisation and a model-checking algorithm for a version of RB-ATL without infinite resource bounds. The main differences from [4] are the addition of an infinite resource bound (to make the logic a conservative extension of ATL), and the addition of complete proofs and an illustrative example.[1] The remainder of this paper is organised as follows. In section 2, we present the syntax and semantics of RB-ATL and show how RB-ATL can be used to express properties of a simple sensor network. In section 3 we provide a sound and complete axiomatisation of RB-ATL. In section 4, we give a model-checking algorithm for RB-ATL. Finally, we survey related work in section 5 and conclude in section 6.

2 Syntax and semantics of RB-ATL

Consider a system of agents which can perform actions to change the state (we assume concurrent execution of actions by all agents). We denote the set of agents

[1] A preliminary version of RB-ATL with infinite bounds was introduced in [14].

by N. In order to reason about resources, we assume that actions have costs. Let R be a set of resources (such as money, energy, or anything else which may be required by an agent for performing an action). We assume that a cost of an action, for each of the resources, is a non-negative integer. The set of resource bounds $\mathbb{B}$ over R is defined as $\mathbb{B} = (\mathbb{N} \cup \{\infty\})^r$, where $r = |R|$. We denote by $\bar{0}$ the smallest resource bound $(0, \ldots, 0)$ and $\bar{\infty}$ the greatest resource bound $(\infty, \ldots, \infty)$.

2.1 Syntax of RB-ATL

The syntax of RB-ATL is defined as follows, where A is a non-empty subset of N and $b \in \mathbb{B}$.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle\langle A^b \rangle\rangle \bigcirc \varphi \mid \langle\langle A^b \rangle\rangle \Box \varphi \mid \langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$$

Here, $\langle\langle A^b \rangle\rangle \bigcirc \varphi$ means that a coalition A can ensure that the next state satisfies φ under resource bound b. $\langle\langle A^b \rangle\rangle \Box \varphi$ means that A has a strategy to make sure that φ is always true, and the cost of this strategy is at most b. Similarly, $\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$ means that A has a strategy to enforce ψ while maintaining the truth of φ, and the cost of this strategy is at most b. Notice the meaning of these operators when $b = \bar{\infty}$ is the same as their counterparts in ATL; in other words, the ATL operator $\langle\langle A \rangle\rangle$ corresponds to $\langle\langle A^{\bar{\infty}} \rangle\rangle$ for $A \neq \emptyset$ and $\langle\langle \emptyset \rangle\rangle$ to the dual of $\langle\langle N^{\bar{\infty}} \rangle\rangle$.

2.2 Semantics of RB-ATL

To interpret this language, we extend the definition of concurrent game structures [6] with resource requirements for executing actions. For consistency with [6], in what follows we refer to agents as 'players' and actions as 'moves'.

Definition 1. *A Resource-bounded Concurrent Game Structure (RB-CGS) is a tuple $S = (n, r, Q, \Pi, \pi, d, c, \delta)$ where:*

- $n \geq 1$ *is the number of players (agents), we denote the set of players* $\{1, \ldots, n\}$ *by* N*;*
- r *is the number of resources;*
- Q *is a non-empty set of states;*
- Π *is a finite set of propositional variables;*
- $\pi : \Pi \to \wp(Q)$ *is a function which assigns to each variable in* Π *a subset of* Q*;*

- *$d : Q \times N \to \mathbb{N}$ is a function which indicates the number of available moves (actions) for each player $a \in N$ at a state $q \in Q$ such that $d(q, a) \geq 1$. At each state $q \in Q$, we denote the set of joint moves available for all players in N by $D(q)$. That is*

$$D(q) = \{1, \ldots, d(q, 1)\} \times \ldots \times \{1, \ldots, d(q, n)\}$$

- *$c : Q \times N \times \mathbb{N} \to \mathbb{B}$ is a partial function which indicates the minimal amount of resources required by each move available to each agent at a specific state;*

- *$\delta : Q \times \mathbb{N}^{|N|} \to Q$ is a partial function where $\delta(q, m)$ is the next state from q if the players execute the move $m \in D(q)$.*

We assume that each agent in each state has an available action with $\bar{0}$ cost (intuitively, it has the option of doing nothing).

Given a RB-CGS S, we denote by Q^* the set of finite sequences of states or finite computations and by Q^ω the set of infinite sequences of states or infinite computations. For a finite or infinite computation $\lambda = q_1 q_2 \ldots \in Q^* \cup Q^\omega$, we use the notation $\lambda[i] = q_i$ and $\lambda[i, j] = q_i \ldots q_j$. We denote the set of finite non-empty sequences of states by Q^+.

Definition 2. *Given a RB-CGS S and a state $q \in Q$, a move (or a joint action) for a coalition $A \subseteq N$ is a tuple $\sigma_A = (\sigma_a)_{a \in A}$ such that $1 \leq \sigma_a \leq d(q, a)$.*

By $D_A(q)$ we denote the set of all moves for A at state q. Given a move $m \in D(q)$, we denote by m_A the actions executed by A, $m_A = (m_a)_{a \in A}$. We define the set of all possible outcomes of a move $\sigma_A \in D_A(q)$ at state q as follows:

$$out(q, \sigma_A) = \{q' \in Q \mid \exists m \in D(q) : m_A = \sigma_A \wedge q' = \delta(q, m)\}$$

For convenience, we define the projection of ∞ components of a resource bound b on another bound d as $d \overset{\infty}{\leftleftarrows} b$ where for all $i \in \{1, \ldots, r\}$:

$$(d \overset{\infty}{\leftleftarrows} b)_i = \begin{cases} d_i & \text{if } b_i \neq \infty \\ \infty & \text{if } b_i = \infty \end{cases}$$

For example, $(2, 3, \infty, 6) \overset{\infty}{\leftleftarrows} (1, \infty, 3, \infty) = (2, \infty, \infty, \infty)$ and $\bar{0} \overset{\infty}{\leftleftarrows} (1, \infty, 3, \infty) = (0, \infty, 0, \infty)$. To compare costs and resource bounds, we use the usual pointwise vector comparison, that is, $(b_1, \ldots, b_r) \leq (d_1, \ldots, d_r)$ iff $b_i \leq d_i$ for $i \in \{1, \ldots, r\}$ where $n \leq \infty$ for all $n \in \mathbb{N}$. We also use pointwise vector addition: $(b_1, \ldots, b_r) + (d_1, \ldots, d_r) = (b_1 + d_1, \ldots, b_r + d_r)$ where $n + \infty = \infty$ for all $n \in \mathbb{N} \cup \{\infty\}$. Conversely, we also split a resource bound b into pairs of resource bounds (d, d') such that:

(i) $d + d' = b$,

(ii) $d_i = d'_i = \infty$ for all $i \in \{1, \dots, r\}$ such that $b_i = \infty$, and

(iii) $d \neq \bar{0} \overset{\infty}{\leftarrow} b$.

The set of all such pairs (d, d') is denoted by $split(b)$. Obviously, $split(b)$ is finite.

The cost of a move $\sigma_A \in D_A(q)$ is defined as $cost(q, \sigma_A) = \Sigma_{a \in A} c(q, a, \sigma_a)$. (Note that we use c for the cost of single actions and $cost$ for the cost of joint actions).

Definition 3. *Given a RB-CGS S, a strategy for a subset of players $A \subseteq N$ is a mapping F_A which associates each sequence $\lambda q \in Q^+$ to a move in $D_A(q)$.*

A computation $\lambda \in Q^\omega$ is consistent with F_A iff for all $i \geq 1$, $\lambda[i+1] \in out(\lambda[i], F_A(\lambda[1,i]))$. We denote by $out(q, F_A)$ the set of all such sequences λ starting from q, i.e. $\lambda[1] = q$. Given a non-empty finite prefix λ of a computation which is consistent with a strategy F_A, we define the cost of F_A with respect to λ as $cost(\lambda, F_A) = \sum_{i=1,\dots,|\lambda|-1} cost(\lambda[i], F_A(\lambda[1,i]))$.

Definition 4. *Given a bound b, a computation $\lambda \in out(q, F_A)$ is b-consistent with F_A iff, for every finite prefix λ' of λ, $cost(\lambda', F_A) \leq b$. We denote by $out(q_0, F_A, b)$ the set of all b-consistent computations. A strategy F_A is a b-strategy iff $out(q, F_A) = out(q, F_A, b)$ for any $q \in Q$.*

In other words, all executions of a b-strategy cost at most b resources. Note that this means that each computation of such a strategy starts with a finite prefix where some non-$(\bar{0} \overset{\infty}{\leftarrow} b)$ cost actions are executed, and continues with an infinite sequence of $(\bar{0} \overset{\infty}{\leftarrow} b)$-cost actions.

2.3 Truth definition for RB-ATL

Given a RB-CGS $S = (n, r, Q, \Pi, \pi, d, c, \delta)$, the truth definition for RB-ATL is given inductively as follows:

- $S, q \models p$ iff $q \in \pi(p)$;
- $S, q \models \neg\varphi$ iff $S, q \not\models \varphi$;
- $S, q \models \varphi \vee \psi$ iff $S, q \models \varphi$ or $S, q \models \psi$;
- $S, q \models \langle\langle A^b \rangle\rangle \bigcirc \varphi$ iff there exists a b-strategy F_A such that for all $\lambda \in out(q, F_A)$, $S, \lambda[2] \models \varphi$ iff there is a move $\sigma_A \in D_A(q)$ such that for all $q' \in out(\sigma_A)$, $S, q' \models \varphi$;

- $S, q \models \langle\langle A^b \rangle\rangle \Box \varphi$ iff there exists a b-strategy F_A for any $\lambda \in out(q, F_A)$, $S, \lambda[i] \models \varphi$ for all $i \geq 1$;
- $S, q \models \langle\langle A^b \rangle\rangle \varphi\, \mathcal{U} \psi$ iff there exists a b-strategy F_A such that for all $\lambda \in out(q, F_A)$, there exists $i \geq 1$ such that $S, \lambda[i] \models \psi$ and $S, \lambda[j] \models \varphi$ for all $j \in \{1, \ldots, i-1\}$.

Notice that the truth definition of $\langle\langle A^{\bar{\infty}} \rangle\rangle$ is the same as that of $\langle\langle A \rangle\rangle$ in ATL.

2.4 Example

To conclude this section, we describe a concrete scenario to illustrate the notions introduced above, and give some examples of the expressive power of RB-ATL.

Consider a sensor network consisting of two agents (sensor nodes), 1 and 2. The agents monitor for movement. If they detect movement, they can inform their neighbour. If an agent receives a communication from its neighbour, it can save it. If an agent has more than one record of movement, the agent can report this to the base station. We assume that 2 is closer to the base station than 1. We consider two resources, energy and memory. Sending a message requires energy (depending on the distance to the receiver) and saving a communication requires memory. Sending from 1 to 2 ($send12$) and from 2 to 1 ($send21$) both require 2 units of energy and 0 memory. Saving a record requires 0 units of energy and 1 unit of memory. Sending from 1 to the base station ($send1b$) requires 3 units of energy, and sending from 2 to the base ($send2b$) requires 1 unit of energy. The option of doing nothing ($idle$) is always available and costs nothing. In the initial state q_0, each agent has a record of having itself seen movement. The system is shown in Figure 1, where transitions between states are annotated with tuples of actions (the first element is an action by agent 1, and the second is an action by agent 2). We omit self-loops in each state by the joint action $(idle, idle)$ for readability.

In this scenario, both agents together can enforce the outcome q_6, where we assume that a proposition p (which means that the base station has been informed) holds. Moreover, they can achieve this by spending 3 units of energy and 1 unit of memory by choosing the following actions: $(send12, idle)$ in q_0, $(idle, save)$ in q_1, and $(idle, send2b)$ in q_4. This can be expressed in RB-ATL as $\langle\langle \{1,2\}^{(3,1)} \rangle\rangle \top \mathcal{U} p$. It is also the case that the agents cannot achieve this without using some memory, even if they use unlimited energy: $\neg\langle\langle \{1,2\}^{(\infty,0)} \rangle\rangle \top\, \mathcal{U} p$. Clearly, neither of the agents can single-handedly enforce q_6; however once the system is in q_6, either agent can trivially maintain p forever, since the only choice of action available to each agent there is $idle$. This can be expressed as $\langle\langle \{1,2\}^{(3,1)} \rangle\rangle \top \mathcal{U} \langle\langle \{1\}^{0,0} \rangle\rangle \Box p$.

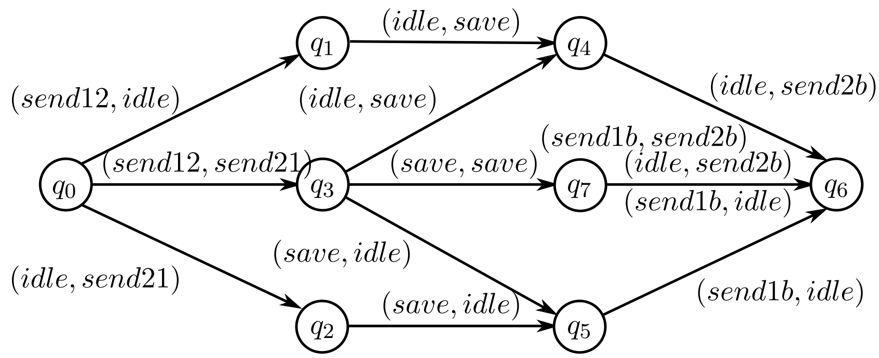

Figure 1: Sensor network example

3 Axiomatisation

In this section we present the axiomatic system for RB-ATL. To make the formulas below more readable, we define the following abbreviations:

$$\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi = \bigvee_{(d,d') \in split(b)} \langle\langle A^d \rangle\rangle \bigcirc \langle\langle A^{d'} \rangle\rangle \Box\varphi$$
$$\langle\langle A^b \rangle\rangle \bigcirc \varphi\mathcal{U}\psi = \bigvee_{(d,d') \in split(b)} \langle\langle A^d \rangle\rangle \bigcirc \langle\langle A^{d'} \rangle\rangle \varphi\mathcal{U}\psi$$

The axiomatic system consists of the following axiom schemas and rules of inference, where A, A_1 and A_2 are non-empty subsets of N, and b, $d \in \mathbb{B}$.

Axioms

(PL) Tautologies of Propositional Logic

($\bot$) $\neg\langle\langle A^b \rangle\rangle \bigcirc \bot$

($\top$) $\langle\langle A^b \rangle\rangle \bigcirc \top$

(B) $\langle\langle A^b \rangle\rangle \bigcirc \varphi \rightarrow \langle\langle A^d \rangle\rangle \bigcirc \varphi$
where $b \leq d$

(S) $\langle\langle A_1^b \rangle\rangle \bigcirc \varphi \wedge \langle\langle A_2^d \rangle\rangle \bigcirc \psi \rightarrow \langle\langle (A_1 \cup A_2)^{b+d} \rangle\rangle \bigcirc (\varphi \wedge \psi)$
where $A_1 \cap A_2 = \emptyset$

($\mathbf{S}_N$) $\langle\langle A^b \rangle\rangle \bigcirc \varphi \wedge \neg\langle\langle N^{\bar{\infty}} \rangle\rangle \bigcirc \neg\psi \rightarrow \langle\langle A^b \rangle\rangle \bigcirc (\varphi \wedge \psi)$

($\mathbf{S}_{N+}$) $\langle\langle N^b \rangle\rangle \bigcirc \varphi \wedge \neg\langle\langle N^b \rangle\rangle \bigcirc \neg\psi \rightarrow \langle\langle N^b \rangle\rangle \bigcirc (\varphi \wedge \psi)$

($\mathbf{S}_{N-}$) $\neg\langle\langle N^b \rangle\rangle \bigcirc \neg\varphi \wedge \neg\langle\langle N^b \rangle\rangle \bigcirc \neg\psi \rightarrow \neg\langle\langle N^b \rangle\rangle \bigcirc \neg(\varphi \wedge \psi)$

(FP$_\Box$) $\langle\langle A^b\rangle\rangle\Box\varphi \leftrightarrow \varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc (\langle\langle A^b\rangle\rangle\Box\varphi)$

(FP$_\mathcal{U}$) $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \leftrightarrow \psi \vee (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \varphi\mathcal{U}\psi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi))$

Inference rules

(MP) $\dfrac{\varphi, \varphi \to \psi}{\psi}$

($\langle\langle A^b\rangle\rangle\bigcirc$-Monotonicity) $\dfrac{\varphi \to \psi}{\langle\langle A^b\rangle\rangle \bigcirc \varphi \to \langle\langle A^b\rangle\rangle \bigcirc \psi}$

($\langle\langle N^b\rangle\rangle\Box$-Necessitation) $\dfrac{\varphi}{\neg\langle\langle N^b\rangle\rangle\Box\neg\varphi}$

($\langle\langle A^b\rangle\rangle\Box$-Induction)

$$\frac{\theta \to (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \theta))}{\theta \to \langle\langle A^b\rangle\rangle\Box\varphi}$$

($\langle\langle A^b\rangle\rangle\mathcal{U}$-Induction)

$$\frac{(\psi \vee (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \varphi\mathcal{U}\psi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \theta))) \to \theta}{\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \to \theta}$$

Before proving soundness and completeness, i.e., every formula derived by the above system is valid and every valid formula can be derived by the above system, we give an intuitive explanation of the axioms and compare them with the axiomatic system for ATL given in [13].

First of all, observe that with the resource bounds removed, the axioms $(\bot)$, $(\top)$, $(\mathbf{S})$, and the inference rules ($\langle\langle A^b\rangle\rangle\bigcirc$-Monotonicity) and ($\langle\langle N^b\rangle\rangle\Box$-Necessitation) are identical to their ATL counterparts. Unlike ATL, we need several versions of $(\mathbf{S})$ since we do not have the $\langle\langle\emptyset^b\rangle\rangle$ modality and, as a result, $(\mathbf{S_N})$, $(\mathbf{S_{N+}})$ and $(\mathbf{S_{N-}})$ are not derivable from $(\mathbf{S})$. The axiom (B) says that if A can enforce φ under a resource bound b, then it can also enforce φ if it has more than b resources. The axiom (FP$_\Box$) is similar to its ATL counterpart. However, unlike in ATL, there are two ways to 'unwind' $\langle\langle A^b\rangle\rangle\Box\varphi$ in RB-ATL: one way is to make a move which costs a non-trivial amount of resources d, and then maintain φ with $b - d$ resources; the second way is to make a trivial $(\bar{0} \overset{\infty}{\leftarrow} b)$-cost move, and then maintain φ with b resources. Similarly for (FP$_\mathcal{U}$). Finally, the rules ($\langle\langle A^b\rangle\rangle\Box$-Induction) and ($\langle\langle A^b\rangle\rangle$ $\mathcal{U}$-Induction) correspond to the ATL axioms (GFP$_\Box$) and (LFP$_\mathcal{U}$); the first one says that $\Box$ corresponds to the greatest fixed point and the second that $\mathcal{U}$ corresponds to the least fixed point. This will be made more precise after we give fixed point characterisations of the temporal operators.

3.1 Fixed point characterisations of temporal operators

Consider an operation $[\langle\langle A^b \rangle\rangle \bigcirc]$ which, given a set of states X, returns the set of states from where A can enforce an outcome to be in X under resource bound b (this is the same as $Pre(A, X, b)$ defined in Section 4, which is in turn similar to *Pre* from [6]):

Definition 5. $[\langle\langle A^b \rangle\rangle \bigcirc] : \wp(Q) \to \wp(Q)$ *is defined as follows: given a set* $X \subseteq Q$, $[\langle\langle A^b \rangle\rangle \bigcirc](X)$ *is the set*

$$\{q \mid \exists \sigma \in D_A(q) : cost(q, \sigma) \leq b \wedge out(q, \sigma) \subseteq X\}$$

Let us define $\|\varphi\| = \{q \in Q \mid S, q \models \varphi\}$. It is straightforward that:

$$\|\langle\langle A^b \rangle\rangle \bigcirc \varphi\| = [\langle\langle A^b \rangle\rangle \bigcirc](\|\varphi\|)$$

Recall that if f is a monotone operator $2^Q \longrightarrow 2^Q$ (that is, $X \subseteq Y$ implies $f(X) \subseteq f(Y)$), then X is a fixed point of f if $f(X) = X$. By the Knaster-Tarski theorem, f has the least and the greatest fixed point. The least fixed point of f is denoted by $\mu X.f(X)$ and the greatest fixed point by $\nu X.f(X)$. We are going to show that the meanings of $\Box$ and $\mathcal{U}$ correspond to the greatest and the least fixed points of certain operations on sets of states.

Lemma 1. *For all $q \in Q$, the following fixed point characterisations hold:*

1. $q \in \|\langle\langle A^b \rangle\rangle \Box \varphi\|$ *iff* $q \in \nu X.\|\varphi\| \cap (\|\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi\| \cup [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X))$ *iff there is a b-strategy F_A for A such that for all $\lambda \in out(q, F_A)$, $\lambda[i] \in \|\varphi\|$ for all $i \geq 1$*

2. $q \in \|\langle\langle A^b \rangle\rangle \varphi \; \mathcal{U} \psi\|$ *iff* $q \in \mu X.\|\psi\| \cup (\|\varphi\| \cap (\|\langle\langle A^b \rangle\rangle \bigcirc \varphi \; \mathcal{U} \psi\| \cup [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X)))$ *iff there is a b-strategy F_A for A such that for all $\lambda \in out(q, F_A)$, there exists $i \geq 1$ such that $\lambda[i] \in \|\psi\|$ and $\lambda[j] \in \|\varphi\|$ for all $j < i$*

Proof. We will only provide the proof for the first case as the second can be done in a similar way. For convenience, let us denote $f(X) = \|\varphi\| \cap (\|\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi\| \cup [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X))$. We firstly show that $f(X)$ is monotone. Let $X_1 \subseteq X_2 \subseteq Q$. Let $q \in f(X_1)$, then $q \in \|\varphi\|$ and either $q \in \|\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi\|$ or $q \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X_1)$. From the definition of $[\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc]()$, we have that $q \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X_1)$ implies $q \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](X_2)$; hence $q \in f(X_2)$.

Therefore, $f(X)$ is monotone and there is the greatest fixed point $\nu X.f(X)$. We now show that $Y = \|\langle\langle A^b \rangle\rangle \Box \varphi\|$ is a post-fixed point of $f(X)$, i.e. $f(Y) \supseteq Y$. Let $q \in Y$, by the semantics definition, we have that there is a b-strategy F_A such that for any $\lambda \in out(q, F_A)$, $\lambda[i] \in \|\varphi\|$ for all $i \geq 1$. Then, $q = \lambda[1] \in \|\varphi\|$. Assume that $b' = cost(q, F_A(q))$, let b'' be a resource bound such that $(b' \overset{\infty}{\leftarrow} b, b'') \in split(b)$. For each $q' \in out(q, F_A(q))$, we define a b''-strategy $F_{q'}$ as the remainder of F_A from q', i.e., $F_{q',A}(q'\kappa) = F_A(qq'\kappa)$ for all $\kappa \in Q^*$. Then, for all $q'\kappa \in out(q', F_{q',A})$, we have that $qq'\kappa \in out(q, F_A)$. It is straightforward that any computation in $out(q', F_{q',A})$ costs at most b''. Then, $q' \in \|\langle\langle A^{b''}\rangle\rangle\Box\varphi\|$. Thus, $q \in [\langle\langle A^{b'}\rangle\rangle\bigcirc](\|\langle\langle A^{b''}\rangle\rangle\Box\varphi\|)$. If $b' \not\leq \bar{0} \overset{\infty}{\leftarrow} b$, we have that $q \in \|\langle\langle A^{b}\rangle\rangle \bigcirc \Box\varphi\|$, otherwise $q \in [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\bigcirc](\|\langle\langle A^{b}\rangle\rangle\Box\varphi\|)$. This means that $q \in f(\|\langle\langle A^{b}\rangle\rangle\Box\varphi\|)$.

In order to show that $Y = \|\langle\langle A^{b}\rangle\rangle\Box\varphi\|$ is in fact the greatest fixed point of $f(X)$, we show that, for every post-fixed point Z, $Z \subseteq Y$.

We have $f(X) = \|\varphi\| \cap (\|\langle\langle A^{b}\rangle\rangle \bigcirc \Box\varphi\| \cup [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\bigcirc](Z))$. Assume $q \in Z$, we have:

$$
\begin{aligned}
q \in Z \;\Rightarrow q \in \|\varphi\| \text{ and either } & q \in \|\langle\langle A^{b}\rangle\rangle \bigcirc \Box\varphi\| \text{ or} \\
& q \in [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\bigcirc](Z)
\end{aligned}
$$

We define a b-strategy F_A which will maintain φ by induction on the length of inputs for F_A. Let Λ^i denote the set of inputs of length $i \geq 1$ for F_A. Initially, $\Lambda^1 = \{q\}$. We will define F_A for input of length i and Λ^{i+1} inductively on $i \geq 1$ such that, for all $\lambda \in \Lambda^{i+1}$, either $cost(\lambda, F_A) \leq d$ and $\lambda[i+1] \in \|\langle\langle A^{d'}\rangle\rangle\Box\varphi\|$ for some $(d, d') \in split(b)$ or $cost(\lambda, F_A) \leq \bar{0} \overset{\infty}{\leftarrow} b$ and $\lambda[i+1] \in Z$.

- Case $i = 1$, recall that $q \in \|\varphi\|$ and either $q \in \|\langle\langle A^{b}\rangle\rangle \bigcirc \Box\varphi\|$ or $q \in [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\bigcirc](Z)$.

  - If $q \in \|\langle\langle A^{b}\rangle\rangle \bigcirc \Box\varphi\|$, then there exists $(d, d') \in split(b)$ such that $q \in \|\langle\langle A^{d}\rangle\rangle \bigcirc \langle\langle A^{d'}\rangle\rangle\Box\varphi\|$; then, we have:

$$
\begin{aligned}
q \in \|\langle\langle A^{d}\rangle\rangle\bigcirc\langle\langle A^{d'}\rangle\rangle\Box\varphi\| \;&\Rightarrow q \in [\langle\langle A^{d}\rangle\rangle\bigcirc](\|\langle\langle A^{d'}\rangle\rangle\Box\varphi\|) \\
&\Rightarrow \exists\sigma_A \in D_A(q) : \; cost(q, \sigma_A) \leq d \wedge \\
&\qquad\qquad out(q, \sigma_A) \subseteq \|\langle\langle A^{d'}\rangle\rangle\Box\varphi\| \\
&\Rightarrow \forall q' \in out(q, \sigma_A), \\
&\qquad \exists\; d'\text{-strategy } F_{A,q'}, \\
&\qquad \forall\lambda \in out(q', F_{A,q'}), \forall j \geq 1 : \lambda[j] \in \|\varphi\|
\end{aligned}
$$

  Then, we define $F_A(q) = \sigma_A$ and $\Lambda^2 = \{qq' \mid q' \in out(q, \sigma_A)\}$. Obviously, we have:

$$
\forall qq' \in \Lambda^2 : cost(qq, F_A) = cost(q, \sigma_A) \leq d \wedge q \in \|\langle\langle A^{d'}\rangle\rangle\Box\varphi\|
$$

– If $q \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](Z)$, we have:

$$q \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](Z) \Rightarrow \exists \sigma_A \in D_A(q) : \begin{array}{l} cost(q, \sigma_A) \leq \bar{0} \overset{\infty}{\leftarrow} b \wedge \\ out(q, \sigma_A) \subseteq Z \end{array}$$

Let $F_A(q) = \sigma_A$ and $\Lambda^2 = \{qq' \mid q' \in out(q, \sigma_A)\}$. Then, we have:

$$\begin{aligned} \forall qq' \in \Lambda^2 &\Rightarrow cost(qq', F_A) = cost(q, \sigma_A) \leq \bar{0} \overset{\infty}{\leftarrow} b \wedge q' \in out(q, \sigma_A) \\ &\Rightarrow q' \in Z \quad \text{ as } out(q, \sigma_A) \subseteq Z \end{aligned}$$

- Case $i > 1$, let us assume that F_A for inputs of length $i-1$ and Λ^i have been defined. By the induction hypothesis, we have, for all $\lambda \in \Lambda^i$, either $cost(\lambda, F_A) \leq d$ and $\lambda[i] \in \|\langle\langle A^{d'} \rangle\rangle \Box \varphi\|$ for some $(d, d') \in split(b)$ or $cost(\lambda, F_A) \leq \bar{0} \overset{\infty}{\leftarrow} b$ and $\lambda[i] \in Z$. Let us define F_A for each input $\lambda \in \Lambda^i$ and Λ^{i+1}.

  – If $cost(\lambda, F_A) \leq d$ and $\lambda[i] \in \|\langle\langle A^{d'} \rangle\rangle \Box \varphi\|$, we have:

$$\lambda[i] \in \|\langle\langle A^{d'} \rangle\rangle \Box \varphi\| \Rightarrow \begin{array}{l} \exists\ d'\text{-strategy } F_{A,\lambda}, \\ \forall \lambda' \in out(\lambda[i], F_{A,\lambda}), \\ \forall j \geq 1 : \lambda'[j] \in \|\varphi\| \end{array}$$

Then, we define $F_A(\lambda) = F_{A,\lambda}(\lambda[i])$ and $\Lambda^{i+1} = \{\lambda q' \mid \lambda \in \Lambda^i \wedge q' \in out(\lambda[i], F_{A,\lambda}(\lambda[i]))\}$. Let $cost(\lambda[i], F_{A,\lambda}(\lambda[i])) = d''$, we have, for all $\lambda q' \in \Lambda^{i+1}$ (where $\lambda \in \Lambda^i$ and $q' \in out(\lambda[i], F_{A,\lambda}(\lambda[i]))$:

$$\begin{aligned} cost(\lambda q', F_A) &= cost(\lambda, F_A) + cost(\lambda[i], F_A(\lambda)) \\ &\leq d + cost(\lambda[i], F_{A,\lambda}(\lambda[i])) \\ &= d + d'' \end{aligned}$$

Furthermore, by considering the $(d' - d'')$-strategy $F_{A,\lambda q'}$ where $F_{A,\lambda q'}(\lambda') = F_{A,\lambda}(q'\lambda')$ for all $\lambda' \in Q^+$, we have:

$$\forall \lambda' \in out(q', F_{A,\lambda q'}), \forall j \geq 1 : \lambda'[j] \in \|\varphi\| \Rightarrow q' \in \|\langle\langle A^{d'-d''} \rangle\rangle \Box \varphi\|$$

Finally, it is straightforward that $((d + d''), (d' - d'')) \in split(b)$.

  – If $cost(\lambda, F_A) \leq \bar{0} \overset{\infty}{\leftarrow} b$ and $\lambda[i] \in Z$, we have

$$\begin{aligned} \lambda[i] \in Z &\Rightarrow \lambda[i] \in f(Z) \\ &\Rightarrow \lambda[i] \in \|\varphi\| \text{ and either } \begin{array}{l} \lambda[i] \in \|\langle\langle A^b \rangle\rangle \bigcirc \Box \varphi\| \text{ or} \\ \lambda[i] \in [\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc](Z) \end{array} \end{aligned}$$

* If $\lambda[i] \in \|\langle\!\langle A^b \rangle\!\rangle \bigcirc \Box\varphi\|$, then there exists $(d, d') \in split(b)$ such that $\lambda[i] \in \|\langle\!\langle A^d \rangle\!\rangle \bigcirc \langle\!\langle A^{d'} \rangle\!\rangle \Box\varphi\|$; then, we have:

$$\begin{aligned}
&\lambda[i] \in \|\langle\!\langle A^d \rangle\!\rangle \bigcirc \langle\!\langle A^{d'} \rangle\!\rangle \Box\varphi\| \\
&\qquad \Rightarrow \exists \sigma_A \in D_A(\lambda[i]) : \; cost(\lambda[i], \sigma_A) \leq d \wedge \\
&\qquad\qquad\qquad\qquad out(\lambda[i], \sigma_A) \subseteq \|\langle\!\langle A^{d'} \rangle\!\rangle \Box\varphi\| \\
&\qquad \Rightarrow \; \forall q' \in out(\lambda[i], \sigma_A), \\
&\qquad\quad\;\; \exists \; d'\text{-strategy } F_{A,q'}, \\
&\qquad\quad\;\; \forall \lambda' \in out(q', F_{A,q'}), \forall j \geq 1 : \lambda'[j] \in \|\varphi\|
\end{aligned}$$

Then, we define for every $\lambda \in \Lambda^i$ that $F_A(\lambda) = \sigma_A$ and $\Lambda^{i+1} = \{\lambda q' \mid \lambda \in \Lambda^i \wedge q' \in out(\lambda[i], \sigma_A)\}$. Obviously, we have:

$$\begin{aligned}
\forall \lambda q' \in \Lambda^{i+1} : \; & cost(\lambda, F_A) \; = cost(\lambda, F_A) + cost(\lambda[i], \sigma_A) \\
& \qquad\qquad\;\; \leq (\bar{0} \overset{\infty}{\leftarrow} b) + d = d \\
& \wedge \\
& q' \in \|\langle\!\langle A^{d'} \rangle\!\rangle \Box\varphi\|
\end{aligned}$$

* If $\lambda[i] \in [\langle\!\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\!\rangle \bigcirc](Z)$, we have:

$$\begin{aligned}
\lambda[i] \in [\langle\!\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\!\rangle \bigcirc](Z) \Rightarrow \exists \sigma_{A,\lambda} \in D_A(\lambda[i]) : \; & cost(\lambda[i], \sigma_{A,\lambda}) \leq \bar{0} \overset{\infty}{\leftarrow} b \wedge \\
& out(\lambda[i], \sigma_{A,\lambda}) \subseteq Z
\end{aligned}$$

Then, we define for every $\lambda \in \Lambda^i$ that $F_A(\lambda) = \sigma_{A,\lambda}$ and $\Lambda^{i+1} = \{\lambda q' \mid \lambda \in \Lambda^i q' \in out(\lambda[i], \sigma_{A,\lambda})\}$. Obviously, we have:

$$\begin{aligned}
\forall \lambda q' \in \Lambda^{i+1} \; \Rightarrow \; & cost(\lambda q', F_A) = cost(\lambda, F_A) + cost(\lambda[i], \sigma_{A,\lambda}) \wedge \\
& q' \in out(\lambda[i], \sigma_{A,\lambda}) \\
\Rightarrow \; & cost(\lambda q', F_A) \leq \bar{0} \overset{\infty}{\leftarrow} b \wedge \\
& q' \in Z \quad \text{ as } out(\lambda[i], \sigma_{A,\lambda}) \subseteq Z
\end{aligned}$$

Given the above construction of F_A, we have that

$$\begin{aligned}
\forall \lambda \in out(q, F_A) \text{ and } i \geq 1 \; & \Rightarrow \lambda \in \Lambda^i \\
& \Rightarrow \text{either } \exists (d, d') \in split(b) : \\
& \qquad\qquad\qquad cost(\lambda[1, i], F_A) \leq d \wedge \\
& \qquad\qquad\qquad \lambda[i] \in \|\langle\!\langle A^{d'} \rangle\!\rangle \Box\varphi\| \\
& \qquad\quad\;\; \text{or} \\
& \qquad\quad\;\; cost(\lambda[1, i] \leq \bar{0} \overset{\infty}{\leftarrow} b \wedge \lambda[i] \in Z \\
& \Rightarrow cost(\lambda[1, i], F_A) \leq b \wedge \lambda[i] \in \|\varphi\|
\end{aligned}$$

In other words, $q \in \|\langle\langle A^b \rangle\rangle \Box \varphi\|$, i.e., $q \in Y$.

Therefore, $Z \subseteq Y$; hence, Y is the greatest post-fixed point of $f(X)$, hence also the greatest fixed point of $f(X)$. $\Box$

3.2 Soundness of RB-ATL

First, we prove that the axioms of RB-ATL are valid.

(⊥) is valid because there is no b-strategy F_A such that for all $\lambda \in out(q, F_A)$, $\lambda[1]$ makes $\bot$ true.

(⊤) is valid because A has a $\bar{0}$-strategy F_A such that for all $\lambda \in out(q, F_A)$, $\lambda[1]$ makes $\top$ true.

(B) is valid because if there is a b-strategy F_A such that for all $\lambda \in out(q, F_A)$, $\lambda[1]$ makes φ true, then the same F_A is also a d-strategy which has the same property.

(S) is valid because if there exists a strategy F_{A_1} to enforce φ and a strategy F_{A_2} to enforce ψ, then there exists a joint strategy $F_{A_1 \cup A_2}$ (with the same moves for A_1 and A_2 as F_{A_1} and F_{A_2}, respectively) to enforce both φ and ψ.

(S$_N$) is valid because if there exists a b-strategy F_A to enforce φ, and, for all strategies of N, ψ is inevitable, then $\varphi \wedge \psi$ can be enforced in by F_A.

(S$_{N+}$) is valid because if there exists a b-strategy F_N to enforce φ, and, for all strategies of N which cost at most b, ψ is inevitable, then $\varphi \wedge \psi$ can be enforced in by F_N.

(S$_{N-}$) is valid because if, for all strategies of N, φ and ψ are inevitable, then so is $\varphi \wedge \psi$.

(FP$_\Box$) is valid by Lemma 1(1) and **(FP$_\mathcal{U}$)** by Lemma 1(2).

Then, we prove that the inference rules preserve validity (the proof for **(MP)** is standard, hence it is omitted):

($\langle\langle A^b \rangle\rangle \bigcirc$-Monotonicity), ($\langle\langle A^b \rangle\rangle \Box$-Monotonicity), and ($\langle\langle A^b \rangle\rangle \mathcal{U}$-Monotonicity) clearly preserve validity, since if $\|\varphi\| \subseteq \|\psi\|$ and an outcome in $\|\varphi\|$ can be enforced, then an outcome in $\|\psi\|$ can also be enforced by the same strategy.

($\langle\langle N^b \rangle\rangle \Box$-Necessitation) is valid since if φ is logically true, then it is inevitable in perpetuity.

($\langle\langle A^b \rangle\rangle \Box$-Induction) and ($\langle\langle A^b \rangle\rangle \mathcal{U}$-Induction) preserve validity by Lemma 1.

3.3 Completeness of RB-ATL

The proof of completeness is based on [13]. We construct a satisfying model for a formula φ_0 which is consistent with the axiomatic system for RB-ATL.

In the proof, we assume when convenient that all formulas are in negation normal form of RB-ATL. The syntax of negation normal form RB-ATL is as follows:

$$\begin{aligned}\varphi ::= p \mid \neg p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \; & \langle\langle A^b \rangle\rangle \bigcirc \varphi \mid \neg \langle\langle A^b \rangle\rangle \bigcirc \varphi \mid \\ & \langle\langle A^b \rangle\rangle \Box \varphi \mid \neg \langle\langle A^b \rangle\rangle \Box \varphi \mid \\ & \langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi \mid \neg \langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi\end{aligned}$$

where A is a non-empty coalition and $b \in \mathbb{B}$. Given a normal form formula φ of RB-ATL, we denote by $\sim \varphi$ the normal form negation of φ. Given an RB-CGS S and a state q, the semantics of normal form RB-ATL is the same as RB-ATL, except for formulas $\neg\langle\langle A^b \rangle\rangle \bigcirc \varphi$, $\neg\langle\langle A^b \rangle\rangle \Box \varphi$ and $\neg\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$ which are defined as follows:

- $S, q \models \neg\langle\langle A^b \rangle\rangle \bigcirc \varphi$ iff for every b-strategy F_A, there exists $\lambda \in out(q, F_A)$ such that $S, \lambda[1] \models \sim \varphi$ iff $\forall \sigma_A \in D_A(q) : cost(q, \sigma_A) \leq b \rightarrow \exists q' \in out(q, \sigma) : S, q' \models \sim \varphi$.
- $S, q \models \neg\langle\langle A^b \rangle\rangle \Box \varphi$ iff for every b-strategy F_A, there exists $\lambda \in out(q, F_A)$ and $i \geq 1$ such that $S, \lambda[i] \models \sim \varphi$,
- $S, q \models \neg\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$ iff for every b-strategy F_A, there exists $\lambda \in out(q, F_A)$ such that if there exists $i \geq 1$ with $S, \lambda[i] \models \psi$ then there is $j \in \{1, \ldots, i-1\}$ where $S, \lambda[j] \models \sim \varphi$.

The model is constructed in a way very similar to the construction in [13]. It is assembled from finite trees where nodes are labelled by sets of formulas. First we define the set of formulas used in the labelling.

Definition 6. *The closure $cl(\varphi_0)$ is the smallest set of formulas satisfying the following closure conditions:*

- *all sub-formulas of φ_0 including φ_0 itself are in $cl(\varphi_0)$;*
- *if $\langle\langle A^b \rangle\rangle \Box \varphi$ is in $cl(\varphi_0)$, then so are $\langle\langle A^d \rangle\rangle \bigcirc \langle\langle A^{d'} \rangle\rangle \Box \varphi$ for all $(d, d') \in split(b)$ and also $\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b} \rangle\rangle \bigcirc \langle\langle A^b \rangle\rangle \Box \varphi$;*
- *if $\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$ is in $cl(\varphi_0)$, then so are $\langle\langle A^d \rangle\rangle \bigcirc \langle\langle A^{d'} \rangle\rangle \varphi \mathcal{U} \psi$ for all $(d, d') \in split(b)$ and also $\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b} \rangle\rangle \bigcirc \langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi$;*
- *if φ is in $cl(\varphi_0)$, then so is $\sim \varphi$; and*

- *$cl(\varphi_0)$ is also closed under finite positive boolean operators ($\vee$ and $\wedge$) up to tautology equivalence.*

Note that $cl(\varphi_0)$ is finite. Let Γ be the set of maximal consistent subsets of $cl(\varphi_0)$. We define trees (T, V, C) over Γ in a similar way as [13] where

- $T \subseteq (\mathbb{N}^n)^*$ is the set of nodes;
- $V : T \longrightarrow \Gamma$ is a labelling function which assigns to each node a consistent set; and
- $C : T \times N \times \mathbb{N} \to \mathbb{N}^r$ is a (partial) cost function which assigns a cost to each action available at a node.

Intuitively, nodes in a tree are identified with finite words corresponding to the sequence of joint actions by the grand coalition which leads to that node. The root is the empty word ϵ and each node t corresponds to a finite computation the last state of which is t. An interior node of the tree is a node but not a leaf. A formula is in $V(t)$ intuitively means that the formula is true in t. Finally, the cost of an action j of an agent i at a node t is given by $C(t, i, j)$. As in [13], the construction proceeds in three stages. The first stage is producing locally consistent trees, namely trees where the labelling satisfies conditions on successor nodes which makes it possible to prove a truth lemma for the next step modalities. The second stage is proving the existence of trees which realise eventualities (essentially, make the labelling consistent with the truth conditions for the $\Box$ and $\mathcal{U}$ modalities). Finally, the finite trees realising eventualities are combined into one infinite tree model.

Definition 7. *A tree (T, V, C) is locally consistent iff for any interior node $t \in T$:*

1. *If $\langle\langle A^b \rangle\rangle \bigcirc \varphi$ in $V(t)$, then there is a move σ_A such that $cost(t, \sigma_A) \leq b$ and for any $t' \in out(t, \sigma_A)$ we have $\varphi \in V(t')$; and*
2. *If $\neg\langle\langle A^b \rangle\rangle \bigcirc \varphi$ in $V(t)$, then for any move σ_A with $cost(t, \sigma_A) \leq b$, there exists $t' \in out(t, \sigma_A)$ where $\neg\varphi \in V(t')$.*

Two following lemmas are used as a crucial step in the local consistency proof.

Lemma 2. *Let $\Phi = \{\langle\langle A_1^{b_1} \rangle\rangle \bigcirc \varphi_1, \ldots, \langle\langle A_k^{b_k} \rangle\rangle \bigcirc \varphi_k, \neg\langle\langle N^{\bar{\infty}} \rangle\rangle \bigcirc \chi_1, \ldots, \neg\langle\langle N^{\bar{\infty}} \rangle\rangle \bigcirc \chi_m, \neg\langle\langle A^b \rangle\rangle \bigcirc \psi\}$ be a consistent set of formulas in which:*

- *all A_i are both non-empty and pairwise disjoint*
- $\bigcup_i A_i \subseteq A$
- $\Sigma_i b_i \leq b$

Then, $\Psi = \{\varphi_1, \ldots, \varphi_k, \sim\chi_1, \ldots, \sim\chi_m, \sim\psi\}$ *is also consistent.*

Proof. When $k = 0$ (or $m = 0$), we can always add the axiom $\langle\langle A^{\bar{0}}\rangle\rangle \bigcirc \top$ (or $\neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \bot$) into Φ. Hence, it is sufficient to prove this lemma with $k > 0$ and $m > 0$.

Let $A' = \bigcup_i A_i$, $b' = \sum_i b_i$ and $\varphi = \bigwedge_i \varphi_i$ and $\chi = \bigwedge_j \sim\chi_j$.

Assume to the contrary that Ψ is inconsistent, we have:

(1) $\vdash \bigwedge\limits_i \langle\langle A_i^{b_i}\rangle\rangle \bigcirc \varphi_i \to \langle\langle A^b\rangle\rangle \bigcirc \varphi$

by (**S**), (**B**), $A' \subseteq A$ and $b' \leq b$

(2) $\vdash \bigwedge\limits_j \neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \chi_j \to \neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \neg\chi$

by ($\mathbf{S}_{N-}$)

(3) $\vdash \langle\langle A^b\rangle\rangle \bigcirc \varphi \wedge \neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \neg\chi \to \langle\langle A^b\rangle\rangle \bigcirc (\varphi \wedge \chi)$

by ($\mathbf{S}_N$)

(4) $\vdash \bigwedge\limits_i \langle\langle A_i^{b_i}\rangle\rangle \bigcirc \varphi_i \wedge \bigwedge\limits_j \neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \chi_j \to \langle\langle A^b\rangle\rangle \bigcirc (\varphi \wedge \chi)$

by (1), (2) and (3)

(5) $\vdash \varphi \wedge \chi \to \psi$

as Ψ is inconsistent

(6) $\vdash \langle\langle A^b\rangle\rangle \bigcirc (\varphi \wedge \chi) \to \langle\langle A^b\rangle\rangle \bigcirc \psi$

by (5) and $\langle\langle A^b\rangle\rangle\bigcirc$-monotonicity

(7) $\vdash \bigwedge\limits_i \langle\langle A_i^{b_i}\rangle\rangle \bigcirc \varphi_i \wedge \bigwedge\limits_j \neg\langle\langle N^{\bar{\infty}}\rangle\rangle \bigcirc \chi_j \to \langle\langle A^b\rangle\rangle \bigcirc \psi$

by (4) and (6)

Therefore, $\Phi \cup \{\langle\langle A^b\rangle\rangle \bigcirc \psi\}$ is consistent, which is a contradiction $\square$

Similarly, we have the following lemma:

Lemma 3. *Let* $\Phi = \{\langle\langle A_1^{b_1}\rangle\rangle \bigcirc \varphi_1, \ldots, \langle\langle A_k^{b_k}\rangle\rangle \bigcirc \varphi_k, \neg\langle\langle N^{d_1}\rangle\rangle \bigcirc \chi_1, \ldots, \neg\langle\langle N^{d_m}\rangle\rangle \bigcirc \chi_m\}$ *be a consistent set of formulas in which:*

- *The* A_i*'s are both non-empty and pairwise disjoint*
- $\Sigma_i b_i \leq d_j$ *for all* j

Then, $\Psi = \{\varphi_1, \ldots, \varphi_k, \sim\chi_1, \ldots, \sim\chi_m\}$ *is also consistent.*

Proof. When $k = 0$ (or $m = 0$), we can always add the axiom $\langle\langle N^{\bar{0}} \rangle\rangle \bigcirc \top$ (or $\neg\langle\langle N^{\bar{\infty}} \rangle\rangle \bigcirc \bot$) into Φ. Hence, it is sufficient to prove this lemma with $k > 0$ and $m > 0$.

Let $A = \bigcup_i A_i$, $b = \sum_i b_i$ and $\varphi = \bigwedge_i \varphi_i$ and $\chi = \bigwedge_j \sim\chi_j$.

Assume to the contrary that Ψ is inconsistent, we have:

$$(1) \vdash \bigwedge_i \langle\langle A_i^{b_i} \rangle\rangle \bigcirc \varphi_i \to \langle\langle A^b \rangle\rangle \bigcirc \varphi$$

by (**S**)

$$(2) \vdash \neg\langle\langle N^{d_j} \rangle\rangle \bigcirc \chi_j \to \neg\langle\langle N^b \rangle\rangle \bigcirc \chi_j$$

by **B** and $b \leq d_j$

$$(3) \vdash \bigwedge_j \neg\langle\langle N^b \rangle\rangle \bigcirc \chi_j \to \neg\langle\langle N^b \rangle\rangle \bigcirc \neg\chi$$

by ($\mathbf{S}_{N-}$)

$$(4) \vdash \langle\langle A^b \rangle\rangle \bigcirc \varphi \wedge \neg\langle\langle N^b \rangle\rangle \bigcirc \neg\chi \to \langle\langle A^b \rangle\rangle \bigcirc (\varphi \wedge \chi)$$

by ($\mathbf{S}_{N+}$)

$$(5) \vdash \bigwedge_i \langle\langle A_i^{b_i} \rangle\rangle \bigcirc \varphi_i \wedge \bigwedge_j \neg\langle\langle N^b \rangle\rangle \bigcirc \chi_j \to \langle\langle A^b \rangle\rangle \bigcirc (\varphi \wedge \chi)$$

by (1), (2), (3) and (4)

$$(6) \vdash \varphi \wedge \chi \to \bot$$

as Ψ is inconsistent

$$(7) \vdash \langle\langle A^b \rangle\rangle \bigcirc (\varphi \wedge \chi) \to \langle\langle A^b \rangle\rangle \bigcirc \bot$$

by (6) and $\langle\langle A^b \rangle\rangle\bigcirc$-monotonicity

$$(8) \vdash \bigwedge_i \langle\langle A_i^{b_i} \rangle\rangle \bigcirc \varphi_i \wedge \bigwedge_j \neg\langle\langle N^b \rangle\rangle \bigcirc \chi_j \to \langle\langle A^b \rangle\rangle \bigcirc \bot$$

by (5) and (7)

Therefore, $\Phi \cup \{\langle\langle A^b \rangle\rangle \bigcirc \bot\}$ is consistent, which is a contradiction ☐

Lemma 4. *Let Φ be a finite consistent set of formulas. Let $\Phi_{\bigcirc}$ be the subset of Φ which contains all formulas of the form $\langle\langle A^b \rangle\rangle \bigcirc \varphi$ or their negations. Let $k \in \mathbb{N}$ be such that $|\Phi_{\bigcirc}| < k$, then there is a locally consistent tree (T, V, C) of height one where $T = \{\epsilon\} \cup \{1, \dots, k\}^n$ and $V(\epsilon) = \Phi$.*

Proof. Denote $T' = \{1, \dots, k\}^n$; hence $T = \{\epsilon\}\dot{\cup}T'$ where we denote by $\dot{\cup}$ the disjoint union operator. Furthermore, we assume that

$$\Phi_{\bigcirc} = \Phi_{\bigcirc}^{+} \,\dot{\cup}\, \Phi_{\bigcirc}^{-} \,\dot{\cup}\, \Phi_{N\bigcirc}^{-}$$

where

$$\Phi^+_{\bigcirc} = \{\langle\!\langle A_1^{b_1}\rangle\!\rangle \bigcirc \varphi_1, \ldots, \langle\!\langle A_m^{b_m}\rangle\!\rangle \bigcirc \varphi_m\},$$
$$\Phi^-_{\bigcirc} = \{\neg\langle\!\langle B_1^{d_1}\rangle\!\rangle \bigcirc \psi_1, \ldots, \neg\langle\!\langle B_l^{d_l}\rangle\!\rangle \bigcirc \psi_l\} \quad \text{s.t. } \forall i : B_i \neq N$$

and

$$\Phi^-_{N\bigcirc} = \{\neg\langle\!\langle N^{e_1}\rangle\!\rangle \bigcirc \chi_1, \ldots, \neg\langle\!\langle N^{e_h}\rangle\!\rangle \bigcirc \chi_h\}$$

First, let $f \in \mathbb{N}$ be the maximal number which occurs in $e_1, \ldots, e_h$; we define $\overline{f+1} = \{f+1\}^n$. It is straightforward that for all $e \in \{e_1, \ldots, e_h\}$, if $e \neq \bar{\infty}$ then $\overline{f+1} \not\leq e$. We define a function *deinf* : $\mathbb{B}^\infty \to \mathbb{B}$ which removes infinity from a bound as follows: *deinf*$(b) = b'$ where for all $i = 1, \ldots, r$

$$b'_i = \begin{cases} b_i & \text{if } b_i \neq \infty \\ f+1 & \text{otherwise} \end{cases}$$

It is also straightforward that for all $e \in \{e_1, \ldots, e_h\}$ and $b \in \{b_1, \ldots, b_m\}$, if $e \neq \bar{\infty}$ and $b \notin \mathbb{N}^r$ (i.e., b contains some ∞) then $deinf(b) \not\leq e$.

Let us construct a tree with a root labelled by Φ and k^n children denoted by $t = (a_1, \ldots, a_n) \in \{1, \ldots, k\}^n$. Intuitively, we allow each agent to perform k different actions where the special action k always costs $\bar{0}$. For convenience, we denote the action of agent i in t by $t_i = a_i$ and the joint move by a coalition A in t by $t_A = (t_i)_{i \in A}$. In the following, we define the labelling function $V(t)$ for each leaf t and the cost function $C(\epsilon, i, a)$ for each agent $i \in N$ and action $a \in \{1, \ldots, k\}$:

(a) For each $\langle\!\langle A_p^{b_p}\rangle\!\rangle \bigcirc \varphi_p \in \Phi^+_{\bigcirc}$ where $A_p \neq \emptyset$, φ_p is added to $V(t)$ for all t such that $\forall i \in A_p : t_i = p$. Let $\min_{A_p}$ be the smallest number in A_p, we assign the cost of action p performed by $\min_{A_p}$ to be *deinf*(b_p), i.e. $C(\epsilon, \min_{A_p}, p) =$ *deinf*(b_p). For other agents i in $A_p \setminus \{\min_{A_p}\}$, we assign $C(\epsilon, i, p) = \bar{0}$. For other unassigned-cost actions, their costs are assigned as follows:

$$C(\epsilon, i, p) = \begin{cases} \bar{0} & \text{if } p \leq m \text{ or } p = k \\ \overline{f+1} & \text{if } m < p < k \end{cases}$$

We define $C(t, A) = \sum_{i \in A} C(\epsilon, i, t_i)$ as the cost of the joint action by the coalition A and $C(t) = C(t, N)$ as the cost of the joint action by the grand coalition.

(b) For each $\neg\langle\!\langle N^{e_p}\rangle\!\rangle \bigcirc \chi_p \in \Phi^-_{N\bigcirc}$, $\sim\chi_p$ is added to $V(t)$ whenever $C(t) \leq e_p$.

(c) Finally, we will add at most one formula from $\Phi^-_{\bigcirc}$ to $V(t)$. Let

$$\begin{aligned}\Phi^-_{\bigcirc}(t) &= \{\neg\langle\!\langle B^d\rangle\!\rangle \bigcirc \psi \in \Phi^-_{\bigcirc} \mid C(t,B) \leq d\}\\ &= \{\neg\langle\!\langle B^{d_{i_1}}_{i_1}\rangle\!\rangle \bigcirc \psi_{i_1}, \ldots, \neg\langle\!\langle B^{d_{l_t}}_{i_{l_t}}\rangle\!\rangle \bigcirc \psi_{l_t}\}\end{aligned}$$

where $1 \leq i_1 < i_2 < \ldots < i_{l_t} \leq l$. Let $I = \{i' \mid t_{i'} \in \{m+1,\ldots,m+l\}\}$ and $j = \sum_{i\in I}(t_i - m - 1) \mod l_t + 1$. Consider $\neg\langle\!\langle B^{d_{i_j}}_{i_j}\rangle\!\rangle \bigcirc \psi_{i_j}$: if $N \backslash B_{i_j} \subseteq I$, then $\sim\psi_{i_j}$ is added into $V(t)$.

We now prove that the constructed tree (T, V, C) is locally consistent. First, we show that all labels are consistent. It is obvious that $V(\epsilon) = \Phi$ is consistent. For each child $t \in T'$, since at most one formula $\sim\psi_p$ such that $\neg\langle\!\langle B^{d_p}_p\rangle\!\rangle \bigcirc \psi_p \in \Phi^-_{\bigcirc}$ is added into $V(t)$, we consider the following two cases:

Case $\forall p \in \{1,\ldots,l\} : \sim\psi_p \notin V(t)$:

Let us assume that

$$\begin{aligned}V(t) = \ &\{\varphi_{i_1},\ldots,\varphi_{i_{m_t}}\} \,\dot\cup\\ &\{\sim\chi_{j_1},\ldots,\sim\chi_{j_{h_t}}\}\end{aligned}$$

where $1 \leq i_1 \leq \ldots \leq i_{m_t} \leq m$ and $1 \leq j_1 \leq \ldots \leq j_{h_t} \leq m$. Then, we have:

$$\begin{aligned}\text{(a)}\ &\Rightarrow \forall p \in \{i_1,\ldots,i_{m_t}\}, \forall i \in A_p : t_i = p\\ &\Rightarrow \forall p,p' \in \{i_1,\ldots,i_{m_t}\} : p \neq p' \rightarrow A_p \cap A_{p'} = \emptyset\end{aligned}$$

and if $\forall p \in \{i_1,\ldots,i_{m_t}\} : b_p \in \mathbb{N}^r$, then

$$\begin{aligned}\text{(a)}\ &\Rightarrow \forall p \in \{i_1,\ldots,i_{m_t}\} : deinf(b_p) = b_p = C(t,A_p)\\ &\Rightarrow \sum_{p\in\{i_1,\ldots,i_{m_t}\}} b_p = \sum_{p\in\{i_1,\ldots,i_{m_t}\}} C(t,A_p) \leq C(t)\\ &\Rightarrow \forall j \in \{j_1,\ldots,j_{h_t}\} \sum_{p\in\{i_1,\ldots,i_{m_t}\}} b_p \leq e_j \text{ by (b)}\end{aligned}$$

otherwise, if $\exists p \in \{i_1, \ldots, i_{m_t}\} : b_p \notin \mathbb{N}^r$, then

$$
\begin{aligned}
\text{(a)} \ &\Rightarrow \forall p \in \{i_1, \ldots, i_{m_t}\} : \mathit{deinf}(b_p) = C(t, A_p) \\
&\Rightarrow \forall j \in \{1, \ldots, h\} : e_j \neq \bar{\infty} \rightarrow \\
&\quad \sum_{p \in \{i_1, \ldots, i_{m_t}\}} \mathit{deinf}(b_p) = \sum_{p \in \{i_1, \ldots, i_{m_t}\}} C(t, A_p) \not\leq e_j \\
&\qquad \text{as } \mathit{deinf}(b_p) \not\leq e_j \\
&\Rightarrow \forall j \in \{j_1, \ldots, j_{h_t}\} : e_j = \bar{\infty} \text{ as } C(t, A_p) \leq C(t) \leq e_j \text{ by (b)} \\
&\Rightarrow \forall j \in \{j_1, \ldots, j_{h_t}\} : \sum_{p \in \{i_1, \ldots, i_{m_t}\}} b_p \leq e_j
\end{aligned}
$$

Therefore, by Lemma 3, $V(t)$ is consistent.

Case $\exists! q \in \{1, \ldots, l\} : \sim\psi_q \in V(t)$:
Let us assume that

$$
\begin{aligned}
V(t) = \ &\{\varphi_{i_1}, \ldots, \varphi_{i_{m_t}}\} \ \dot{\cup} \\
&\{\sim\psi_q\} \ \dot{\cup} \\
&\{\sim\chi_{j_1}, \ldots, \sim\chi_{j_{h_t}}\}
\end{aligned}
$$

where $1 \leq i_1 \leq \ldots \leq i_{m_t} \leq m$ and $1 \leq j_1 \leq \ldots \leq j_{h_t} \leq m$. Recall that:

$$\Phi^-_{\bigcirc}(t) = \{\neg\langle\langle B^{d_{q_1}}_{q_1}\rangle\rangle \bigcirc \psi_{q_1}, \ldots, \neg\langle\langle B^{d_{l_t}}_{q_{l_t}}\rangle\rangle \bigcirc \psi_{l_t}\}$$

hence, $q \in \{q_1, \ldots, q_{l_t}\}$, and also

$$I = \{i' \mid t_{i'} \in \{m+1, \ldots, m+l\}\}$$

hence, $\forall i \in \{i_1, \ldots, i_{m_t}\}$: $I \cap A_i = \emptyset$ since $t_{i'} = i \leq m$ for all $i' \in A_i$.

Similar to the previous case, we have:

$$
\begin{aligned}
\text{(a)} \ &\Rightarrow \forall p \in \{i_1, \ldots, i_{m_t}\}, \forall i \in A_p : t_i = p \\
&\Rightarrow \forall p, p' \in \{i_1, \ldots, i_{m_t}\} : p \neq p' \rightarrow A_p \cap A_{p'} = \emptyset
\end{aligned}
$$

Then, we have:

$$
\begin{aligned}
\text{(c)} \ &\Rightarrow N \setminus B_q \subseteq I \\
&\Rightarrow I \neq \emptyset \quad \text{as } B_q \neq N \\
&\Rightarrow C(t) \geq C(t, I) \geq \overline{f+1} \quad \text{by (a)} \\
&\Rightarrow \forall j \in \{1, \ldots, h\} : e_j \neq \bar{\infty} \rightarrow C(t) \not\leq e_j \quad \text{as } \overline{f+1} \not\leq e_j \\
&\Rightarrow \forall j \in \{j_1, \ldots, j_{h_t}\} : e_j = \bar{\infty} \quad \text{by (b)}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(c)} \;&\Rightarrow N \setminus B_q \subseteq I \\
&\Rightarrow B_q \supseteq N \setminus I \\
&\Rightarrow \forall i \in \{i_1, \ldots, i_{m_t}\} : B_q \supseteq A_i \quad \text{ as } I \cap A_i = \emptyset \\
&\Rightarrow B_q \supseteq A_{i_1} \dot\cup \ldots \dot\cup A_{i_{m_t}} \\
&\Rightarrow C(t, B_q) \geq C(t, A_{i_1} \dot\cup \ldots \dot\cup A_{i_{m_t}}) \\
&\Rightarrow C(t, B_q) \geq C(t, A_{i_1}) + \ldots + C(t, A_{i_{m_t}}) \\
&\Rightarrow d_q \geq \sum_{i \in \{i_1, \ldots, i_{m_t}\}} b_i
\end{aligned}
$$

Therefore, by Lemma 2, $V(t)$ is consistent.

Let us now prove that (T, V, C) satisfies the two local consistency conditions of Definition 7.

1. Assume that $\langle\!\langle A_p^{b_p} \rangle\!\rangle \bigcirc \varphi_p \in V(\epsilon)$. Consider the joint action σ for A_p such that $\sigma_i = p$ for all $i \in A_p$. We have:

   $$out(\epsilon, \sigma) = \{t \in T' \mid \forall i \in A_p : t_i = p\}$$

   and

   $$
   \begin{aligned}
   \langle\!\langle A_p^{b_p} \rangle\!\rangle \bigcirc \varphi_p \in V(\epsilon) \;&\Rightarrow \langle\!\langle A_p^{b_p} \rangle\!\rangle \bigcirc \varphi_p \in \Phi^+_\bigcirc \\
   &\Rightarrow \forall t \in T' : (\forall i \in A_p : t_i = p) \rightarrow \varphi_p \in V(t) \quad \text{ by (a)} \\
   &\Rightarrow \forall t \in out(\epsilon, \sigma) : \varphi_p \in V(t)
   \end{aligned}
   $$

2. If $\neg\langle\!\langle N^{e_p} \rangle\!\rangle \bigcirc \chi_p \in V(\epsilon)$, let us consider the joint action t such that $C(t) \leq e_p$. Obviously, $out(\epsilon, t) = \{t\}$. Hence, by (b), $\sim\chi_p \in V(t)$.

   If $\neg\langle\!\langle B_p^{d_p} \rangle\!\rangle \bigcirc \psi_p \in V(\epsilon)$ where $B_p \neq N$, let σ be an arbitrary joint move for the coalition B_p such that $cost(\epsilon, \sigma) \leq d_p$. We will determine an outcome $t \in out(\epsilon, \sigma)$, such that $\sim \psi_p \in V(t)$. As $t \in out(\epsilon, \sigma)$, $t_i = \sigma_i$ for all $i \in B_p$; it remains to determine t_i for $i \notin B_p$.

   Let $t' \in T'$ such that

   $$t'_i = \begin{cases} \sigma_i & \text{if } i \in B_p \\ m+1 & \text{otherwise} \end{cases}$$

   and

   $$\Phi^-_\bigcirc(\sigma) = \Phi^-_\bigcirc(t')$$

$$= \{\neg\langle\langle B_{i_1}^{d_{i_1}}\rangle\rangle \bigcirc \psi_{i_1}, \ldots, \neg\langle\langle B_{i_{l_\sigma}}^{d_{i_{l_\sigma}}}\rangle\rangle \bigcirc \psi_{i_{l_\sigma}}\}$$

$$I_\sigma = \{i \in B_p \mid \sigma_i \in \{m+1, \ldots, m+l\}\}$$

$$j_\sigma = \sum_{i \in I_\sigma} (\sigma_i - m - 1)$$

Since $C(t', B_p) \leq d_p$, $\neg\langle\langle B_p^{d_p}\rangle\rangle \bigcirc \psi_p \in \Phi_{\bigcirc}^{-}(\sigma)$. Let $p = i_{j_*}$ for some $j_* \in \{1, \ldots, l_\sigma\}$.

Let ι be an arbitrary agent in $N \setminus B_p \neq \emptyset$. We define:

$$t_i = \begin{cases} m + (j_* - j_\sigma - 1) \mod l_\sigma + 1 & \text{if } i = \iota \\ m + 1 & \text{if } i \in N \setminus B_p \setminus \{\iota\} \end{cases}$$

Let us prove that $\sim\psi_p \in V(t)$.

We have:

$$\forall i \in B_p : t_i = \sigma_i = t'_i \Rightarrow C(\epsilon, i, t_i) = C(\epsilon, i, t'_i)$$

and

$$\begin{aligned} \forall i \in N \setminus B_p : t_i \in \{m+1, \ldots, m+l\} &\Rightarrow C(\epsilon, i, t_i) = \overline{f+1} \quad \text{by (a)} \\ &\Rightarrow C(\epsilon, i, t_i) = C(\epsilon, i, t'_i) \end{aligned}$$

Therefore, $\Phi_{\bigcirc}^{-}(t) = \Phi_{\bigcirc}^{-}(t') = \Phi_{\bigcirc}^{-}(\sigma)$; hence, $l_t = l_\sigma$. Then,

$$\begin{aligned} I &= \{i \mid t_i \in \{m+1, \ldots, m+l\}\} \\ &= I_\sigma \dot{\cup} (N \setminus B_p) \\ j &= \sum_{i \in I} (t_i - m - 1) \mod l_t + 1 \\ &= (\sum_{i \in I_\sigma} (t_i - m - 1) + \\ &\quad (t_\iota - m - 1) + \\ &\quad \sum_{i \in N \setminus B_p \setminus \{\iota\}} (t_i - m - 1)) \mod l_t + 1 \\ &= (j_\sigma + (j_* - j_\sigma - 1) \mod l_t) \mod l_t + 1 \\ &= (j_* - 1) \mod l_t + 1 \\ &= j_* \end{aligned}$$

Recall that $i_{j_*} = p$. Then, we have:

$$N \setminus B_p \subseteq N \setminus B_p \cup I_\sigma = I$$

Therefore, according to (c), $\sim\psi_p \in V(t)$.

$\square$

The next stage of the proof is to consider what conditions on tree labelling we need to be able to prove the truth lemma for other temporal modalities. The definition of what it means to 'realise' formulas of the form $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$, $\neg\langle\langle A^b\rangle\rangle\Box\varphi$, $\langle\langle A^b\rangle\rangle\Box\varphi$, $\neg\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ is similar to the one in [13] (essentially the truth conditions for the formulas with 'satisfied' replaced by 'in the labelling of').

The following lemma and its proof are similar to the correspondings in [13], but for formulas of RB-ATL.

Lemma 5. *For any subset $Y \subseteq \Gamma$, there is a formula $\chi_Y \in cl(\varphi_0)$, called the characteristic formula of Y, such that for every $y \in \Gamma$, $\chi_Y \in y$ iff $y \in Y$.*

Proof. For any maximally consistent subset y of $cl(\varphi_0)$, we define:

$$\chi_{\{y\}} = \bigwedge_{\varphi \in y} \varphi \quad \text{and} \quad \chi_Y = \bigvee_{y \in Y} \chi_{\{y\}}$$

First, since $cl(\varphi_0)$ is closed under finite positive boolean operators, $\chi_{\{y\}} \in y$.

$(\Rightarrow)$: Assume that $\chi_Y \in y$. Then, we have:

$$\begin{aligned} \chi_Y \in y &\Rightarrow \vdash \chi_{\{y\}} \to \chi_Y \quad \text{a PL tautology} \\ &\Rightarrow \chi_Y \in y \quad \text{as } \chi_{\{y\}} \in y \text{ and } y \text{ is maximal} \end{aligned}$$

$(\Rightarrow)$: Assume that $y \notin Y$. Then, for any $y' \in Y$, we have:

$$\begin{aligned} \exists\theta \in cl(\varphi_0) : \theta \in y' &\text{ and } \sim\theta \in y \\ &\Rightarrow \chi_{\{y'\}} \wedge \sim\theta \text{ is inconsistent} \\ &\Rightarrow \chi_{\{y'\}} \notin y \end{aligned}$$

Therefore, $\chi_Y \notin y$.

$\square$

In what follows, $\Psi_{\bigcirc}$ is the set of formulas of the form $\langle\langle A^b\rangle\rangle\bigcirc\varphi$ or $\neg\langle\langle A^b\rangle\rangle\bigcirc\varphi$ from $cl(\varphi_0)$.

Lemma 6. *Given $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ and $x \in \Gamma$, there is finite tree (T, V, C) over Γ such that:*

- *every interior node of (T, V, C) has k^n children where $k = |\Psi_{\bigcirc}| + 1$,*
- *(T, V, C) is locally consistent,*
- *$V(\epsilon) = x$, and*
- *if $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \in x$ then (T, V, C) realises $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ from ϵ*

Proof. The proof is similar to the corresponding proof in [13], but also uses induction on the bound b.

Let $Z \subseteq \Gamma$ such that, for any $x \in Z$, there is a finite tree obeying all conditions of Lemma 6. We shall prove the lemma by showing that $Z = \Gamma$. In the following, assume that $x \in \Gamma$.

- If $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \notin x$, let us construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = x$. Since (T, V, C) satisfies all conditions of Lemma 6, $x \in Z$.
- If $\langle\langle A^b\rangle\rangle\varphi\,\mathcal{U}\psi \in x$, we first show that $\eta = (\psi \vee (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \varphi\,\mathcal{U}\psi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \chi_Z))) \to \chi_Z$ is a theorem. Then, we have that:

$$(1) \vdash (\psi \vee (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \varphi\mathcal{U}\psi \vee (\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \chi_Z))) \to \chi_Z$$

$$(2) \vdash \langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \to \chi_Z \qquad \text{by } \textbf{(}\langle\langle A^b\rangle\rangle\mathcal{U}\textbf{-Induction)}$$

  Therefore,

$$\begin{aligned} \langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \in x &\Rightarrow \chi_Z \in x \quad \text{as } x \text{ is maximal} \\ &\Rightarrow x \in Z \quad \text{by Lemma 5} \end{aligned}$$

However, it remains to prove that η is a theorem. This is done by showing that η belongs to any maximal consistent set q of RB-ATL in three cases:

- If $\langle\langle A^b\rangle\rangle\varphi\,\mathcal{U}\psi \notin q$, let us construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = q \cap cl(\varphi_0)$. Since (T, V, C) satisfies all conditions of the lemma, $q \cap cl(\varphi_0) \in Z$. Then, we have:

$$\begin{aligned} q \cap cl(\varphi_0) \in Z &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\ &\Rightarrow \chi_Z \in q \\ &\Rightarrow \eta \in q \quad \text{as } q \text{ is maximal} \end{aligned}$$

- If $\psi \vee (\varphi \wedge (\langle\langle A^b \rangle\rangle \bigcirc \varphi \mathcal{U} \psi \vee (\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z)) \notin q$, then it is straightforward that $\eta \in q$ since q is maximal.
- If $\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi \in q$ and $\psi \vee (\varphi \wedge (\langle\langle A^b \rangle\rangle \bigcirc \varphi \mathcal{U} \psi \vee (\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z))) \in q$, we prove that $\eta \in q$ by induction on b.

  Base case: Assume that $b = \bar{0} \overset{\infty}{\leftarrow} b$. As $\psi \vee (\varphi \wedge \langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z) \in q$, we have either $\psi \in q$ or $\varphi \wedge \langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z \in q$. Let us consider the following two sub-cases:

  - if $\psi \in q$, let us construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = q \cap cl(\varphi_0) \ni \psi$. Then, (T, V, C) satisfies all conditions of Lemma 5; hence, $q \cap cl(\varphi_0) \in Z$; therefore, similar to the above argument of the case $\langle\langle A^b \rangle\rangle \varphi \mathcal{U} \psi \notin q$, $\eta \in Z$.
  - if $\varphi \wedge \langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z \in q$, then both φ and $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z \in q$. Let $\Phi = q \cap cl(\varphi_0)$. Obviously, $\Phi_{\bigcirc} \subseteq \Psi_{\bigcirc}$; therefore, $|\Phi_{\bigcirc}| < k$. Then, by Lemma 4, there exists a locally consistent tree (T_0, V_0, C_0) of height one where $T_0 = \{\epsilon\} \dot{\cup} \{1, \ldots, k\}^n$ and $V_0(\epsilon) = q \cap cl(\varphi_0)$.

    For each $c \in \{1, \ldots, k\}^n$, let Φ_c be an arbitrary set from Γ such that $\Phi_c \supseteq V_0(c)$. Let (T_1, V_1, C_1) be a finite tree such that $T_1 = T_0$, $C_1 = C_0$, $V_1(\epsilon) = V_0(\epsilon)$ and $V_1(c) = \Phi_c$ for all $c \in \{1, \ldots, k\}^n$. Since (T_0, V_0, C_0) is locally consistent, so is (T_1, V_1, C_1).

    For every child $c \in \{1, \ldots, k\}^n$ such that $\chi_Z \in V_1(c)$, we have:

    $$\begin{aligned} \chi_Z \in V_1(c) &\Rightarrow V_1(c) \in Z \qquad \text{by Lemma 5} \\ &\Rightarrow \exists \text{ a local consistent tree } (T_c, V_c, C_c) \\ &\qquad \text{which satisfies all conditions of Lemma 6} \end{aligned}$$

    Let us consider a finite tree (T, V, C) where

    $$\begin{aligned} T = \{\epsilon\} \cup \{&c \in \{1, \ldots, k\}^n \mid \chi_Z \notin V_1(c)\} \\ &\{ct \mid c \in \{1, \ldots, k\}^n, t \in T_c, \chi_Z \in V_1(c)\} \end{aligned}$$

    for all $t \in T$:

    $$V(t) = \begin{cases} V_1(\epsilon) & \text{if } t = \epsilon \\ V_1(c) & \text{if } t = c \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, \chi_Z \notin V_1(c) \\ V_c(ct') & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, t' \in T_c, \chi_Z \in V_1(c) \end{cases}$$

for all $t \in T$, $i \in N$ and $j \in \{1, \dots, k\}$:

$$C(t,i,j) = \begin{cases} C_1(\epsilon, i, j) & \text{if } t = \epsilon \\ C_c(ct', i, j) & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \dots, k\}^n, t' \in T_c, \chi_Z \in V_1(c) \end{cases}$$

It is straightforward that (T, V, C) is also locally consistent and all of its interior nodes have k^n children.

Let us show that (T, V, C) realises $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \varphi \ \mathcal{U} \psi$ at ϵ. Let $c \in \{1, \dots, k\}^n$ such that $C(c, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$, i.e., c_A is a joint move which costs at most $\bar{0} \overset{\infty}{\leftarrow} b$,we have:

$$\begin{aligned}
&\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z \in V_1(\epsilon) = V(\epsilon) \\
&\quad \Rightarrow \forall c' \in \{1, \dots, k\}^n : c'_A = c_A \rightarrow \chi_Z \in V(c) \\
&\quad \Rightarrow \forall c' \in \{1, \dots, k\}^n : c'_A = c_A \rightarrow V(c') = V_{c'}(\epsilon) \in Z \\
&\quad \Rightarrow \forall c' \in \{1, \dots, k\}^n : c'_A = c_A \rightarrow \langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \varphi \mathcal{U} \psi \\
&\qquad \text{is realised at the root of } (T_{c'}, V_{c'}, C_{c'}) \\
&\quad \Rightarrow \forall c' \in \{1, \dots, k\}^n : c'_A = c_A \rightarrow \exists \bar{0} \overset{\infty}{\leftarrow} b\text{-strategy} F_{A,c'} \\
&\qquad \text{realises } \langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \varphi \mathcal{U} \psi \text{ at the root of } (T_{c'}, V_{c'}, C_{c'})
\end{aligned}$$

Let us consider a $\bar{0} \overset{\infty}{\leftarrow} b$-strategy F_A where

$$F_A(\lambda) = \begin{cases} c_A & \text{if } \lambda = \epsilon \\ F_{A,c'}(c't) & \text{if } c' \in \{1, \dots, k\}^n, t \in T_{c'}, \chi_Z \in V_1(c') \end{cases}$$

It is straightforward that F_A realises $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \varphi \mathcal{U} \psi$ from the root ϵ of (T, V, C). Hence, as $T(\epsilon) = q \cap cl(\varphi_0)$, we have:

$$\begin{aligned}
q \cap cl(\varphi_0) \in Z &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\
&\Rightarrow \chi_Z \in q \\
&\Rightarrow \eta \in q \quad \text{as } q \text{ is maximal}
\end{aligned}$$

Induction step: Assume that $b > \bar{0} \overset{\infty}{\leftarrow} b$. Similar to the base case, as $\psi \vee (\varphi \wedge (\langle\langle A^b \rangle\rangle \bigcirc \varphi \mathcal{U} \psi \vee (\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z))) \in q$, let us consider the following three sub-cases:

- if $\psi \in q$, the proof is the repetition of that for the base case.
- if φ and $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \chi_Z \in q$, the proof is the repetition of that for the base case.

- if φ and $\langle\langle A^{b_1}\rangle\rangle \bigcirc \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in q$ for some $(b_1, b_2) \in split(b)$, let $\Phi = q \cap cl(\varphi_0)$. Obviously, $\Phi_\bigcirc \subseteq \Psi_\bigcirc$; therefore, $\Phi_\bigcirc < k$. By Lemma 4, there exists a locally consistent tree (T_0, V_0, C_0) of height one where $T_0 = \{\epsilon\}\dot\cup\{1, \ldots, k\}^n$ and $V_0(\epsilon) = \Phi$.

  For each $c \in \{1, \ldots, k\}^n$, let Φ_c be an arbitrary set from Γ such that $\Phi_c \supseteq V_0(c)$. Let (T_1, V_1, C_1) be a finite tree such that $T_1 = T_0$, $C_1 = C_0$, $V_1(\epsilon) = V_0(\epsilon)$ and $V_1(c) = \Phi_c$ for all $c \in \{1, \ldots, k\}^n$. Since (T_0, V_0, C_0) is locally consistent, so is (T_1, V_1, C_1)

  For each $c \in \{1, \ldots, k\}^n$ such that $\langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V(c)$, as $b_2 < b$, we have:

$$\begin{aligned}\langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V(c) \Rightarrow \exists\ &\text{a locally consistent tree } (T_c, V_c, C_c)\\ &\text{which realises } \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi\\ &\text{by induction hypothesis}\end{aligned}$$

  Let us consider a finite tree (T, V, C) where

$$\begin{aligned}T = \{\epsilon\} \cup \{&c \in \{1, \ldots, k\}^n \mid \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \notin V_1(c)\}\\ &\{ct \mid c \in \{1, \ldots, k\}^n, t \in T_c, \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V_1(c)\}\end{aligned}$$

  for all $t \in T$:

$$V(t) = \begin{cases} V_1(\epsilon) & \text{if } t = \epsilon \\ V_1(c) & \text{if } t = c \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \notin V_1(c) \\ V_c(ct') & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, t' \in T_c, \\ & \qquad \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V_1(c) \end{cases}$$

  for all $t \in T$, $i \in N$ and $j \in \{1, \ldots, k\}$:

$$C(t, i, j) = \begin{cases} C_1(\epsilon, i, j) & \text{if } t = \epsilon \\ C_c(ct', i, j) & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, t' \in T_c, \\ & \qquad \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V_1(c) \end{cases}$$

  It is straightforward that (T, V, C) is also locally consistent and all of its interior nodes have k^n children.

Let us show that (T, V, C) realises $\langle\langle A^b \rangle\rangle \varphi \ \mathcal{U} \psi$ at ϵ. Let $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq b_1$, i.e., c_A is a joint move which costs at most b_1,we have:

$$
\begin{aligned}
&\langle\langle A^{b_1} \rangle\rangle \bigcirc \langle\langle A^{b_2} \rangle\rangle \varphi \mathcal{U} \psi \in V_1(\epsilon) = V(\epsilon) \\
&\quad \Rightarrow \forall c' \in \{1, \ldots, k\}^n : c'_A = c_A \to \langle\langle A^{b_2} \rangle\rangle \varphi \mathcal{U} \psi \in V(c') \\
&\quad \Rightarrow \forall c' \in \{1, \ldots, k\}^n : c'_A = c_A \to V(c') = V_{c'}(\epsilon) \in Z \\
&\quad \Rightarrow \forall c' \in \{1, \ldots, k\}^n : c'_A = c_A \to \langle\langle A^{b_2} \rangle\rangle \varphi \mathcal{U} \psi \\
&\qquad \text{is realised at the root of } (T_{c'}, V_{c'}, C_{c'}) \\
&\quad \Rightarrow \forall c' \in \{1, \ldots, k\}^n :, \exists \bar{b}_2\text{-strategy} F_{A,c'} \\
&\qquad \text{realises } \langle\langle A^{b_2} \rangle\rangle \varphi \mathcal{U} \psi \text{ at the root of } (T_{c'}, V_{c'}, C_{c'})
\end{aligned}
$$

Let us consider a b-strategy F_A where

$$
F_A(\lambda) = \begin{cases} c_A & \text{if } \lambda = \epsilon \\ F_{A,c'}(c't) & \text{if } c' \in \{1, \ldots, k\}^n, t \in T_{c'}, \chi_Z \in V_1(c') \end{cases}
$$

It is straightforward that F_A realises $\langle\langle A^b \rangle\rangle \varphi \ \mathcal{U} \psi$ from the root ϵ of (T, V, C). Hence, as $T(\epsilon) = q \cap cl(\varphi_0)$, we have:

$$
\begin{aligned}
q \cap cl(\varphi_0) \in Z \ &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\
&\Rightarrow \chi_Z \in q \\
&\Rightarrow \phi \in q \quad \text{as } q \text{ is maximal}
\end{aligned}
$$

□

Similarly, we have the following result:

Lemma 7. *Given $\neg\langle\langle A^b \rangle\rangle \Box \varphi$ and $x \in \Gamma$, there is finite tree (T, V, C) over Γ such that:*

- *every interior node of (T, V, C) has k^n children where $k = |\Psi_{\bigcirc}| + 1$*
- *(T, V, C) is locally consistent*
- *$V(\epsilon) = x$*
- *if $\neg\langle\langle A^b \rangle\rangle \Box \varphi \in x$ then (T, V, C) realises $\neg\langle\langle A^b \rangle\rangle \Box \varphi$ from ϵ*

Proof. The proof is similar to that of the previous lemma. Let $Z \subseteq \Gamma$ such that, for any $x \in Z$, there is a finite tree obeying all conditions of Lemma 7. We shall prove the lemma by showing that $Z = \Gamma$. In the following, assume that $x \in \Gamma$.

- If $\neg\langle\langle A^b\rangle\rangle\Box\varphi \notin x$, construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = x$. Since (T, V, C) satisfies all conditions of Lemma 7, $x \in Z$.

- If $\neg\langle\langle A^b\rangle\rangle\Box\varphi \in x$, we first show that $\eta = (\neg\varphi \vee (\neg\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z)) \to \chi_Z$ is a theorem. Then, we have:

$$(1) \vdash (\neg\varphi \vee (\neg\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z)) \to \chi_Z$$
$$(2) \vdash \neg\langle\langle A^b\rangle\rangle\Box\varphi \to \chi_Z \qquad \textbf{by } (\langle\langle A^b\rangle\rangle\Box\textbf{-Induction})$$

  Therefore,

$$\begin{aligned} \neg\langle\langle A^b\rangle\rangle\Box\varphi \in x &\Rightarrow \chi_Z \in x \quad \text{as } x \text{ is maximal} \\ &\Rightarrow x \in Z \quad \text{by Lemma 5} \end{aligned}$$

However, it remains to prove that η is a theorem. Again, this is done by showing that η belongs to any maximal consistent set q of RB-ATL in three cases:

- If $\neg\langle\langle A^b\rangle\rangle\Box\varphi \notin q$, let us construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = q \cap cl(\varphi_0)$. Since (T, V, C) satisfies all conditions of the lemma, $q \cap cl(\varphi_0) \in Z$. Then, we have:

$$\begin{aligned} q \cap cl(\varphi_0) \in Z &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\ &\Rightarrow \chi_Z \in q \\ &\Rightarrow \eta \in q \quad \text{as } q \text{ is maximal} \end{aligned}$$

- If $\neg\varphi \vee (\neg\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z) \notin q$, then it is straightforward that $\eta \in q$ since q is maximal.

- If $\neg\langle\langle A^b\rangle\rangle\Box\varphi$ and $\neg\varphi \vee (\neg\langle\langle A^b\rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z) \in q$, we prove that $\eta \in q$ by induction on b.

  Base case: Assume that $b = \bar{0} \overset{\infty}{\leftarrow} b$. As $\neg\varphi \vee \neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z \in q$, we have either $\neg\varphi \in q$ or $\neg\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\rangle \bigcirc \neg\chi_Z \in q$. Let us consider the following two sub-cases:

  - if $\neg\varphi \in q$, let us construct a simple tree (T, V, C) where $T = \{\epsilon\}$ and $V(\epsilon) = q \cap cl(\varphi_0) \ni \neg\varphi$. Then, (T, V, C) satisfies all conditions of Lemma 5; hence, $q \cap cl(\varphi_0) \in Z$; therefore, similar to the above argument of the case $\neg\langle\langle A^b\rangle\rangle\Box\varphi \notin q$, $\eta \in Z$.

– if $\neg\langle\!\langle A^{\bar{0}\overset{\infty}{\leftarrow}b}\rangle\!\rangle \bigcirc \neg\chi_Z$, let $\Phi = q \cap cl(\varphi_0)$. Obviously, $\Phi_{\bigcirc} \subseteq \Psi_{\bigcirc}$; therefore, $|\Phi_{\bigcirc}| < k$. Then, by Lemma 4, there exists a locally consistent tree (T_0, V_0, C_0) of height one where $T_0 = \{\epsilon\}\dot{\cup}\{1, \ldots, k\}^n$ and $V_0(\epsilon) = q \cap cl(\varphi_0)$.

For each $c \in \{1, \ldots, k\}^n$, let Φ_c be an arbitrary set from Γ such that $\Phi_c \supseteq V_0(c)$. Let (T_1, V_1, C_1) be a finite tree such that $T_1 = T_0$, $C_1 = C_0$, $V_1(\epsilon) = V_0(\epsilon)$ and $V_1(c) = \Phi_c$ for all $c \in \{1, \ldots, k\}^n$. Since (T_0, V_0, C_0) is locally consistent, so is (T_1, V_1, C_1). Then, for every $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$, i.e., c_A is a joint move which costs at most $\bar{0} \overset{\infty}{\leftarrow} b$, we have:

$$\begin{aligned}
C(c, A) \leq \bar{0} \overset{\infty}{\leftarrow} b \;&\Rightarrow \exists c'\{1, \ldots, k\}^n : \; c'_A = c_A \wedge \\
&\qquad\qquad\qquad \neg\neg\chi_Z = \chi_Z \in V_1(c') \\
&\Rightarrow V_1(c') \in Z \qquad \text{by Lemma 5} \\
&\Rightarrow \exists \text{ a locally consistent tree } (T_{c'}, V_{c'}, C_{c'}) \\
&\qquad \text{which satisfies all conditions of Lemma 7}
\end{aligned}$$

Let us consider a finite tree (T, V, C) where

$$\begin{aligned}
T = \{\epsilon\} \cup \; &\{c \in \{1, \ldots, k\}^n \mid \chi_Z \notin V_1(c)\} \\
&\{ct \mid c \in \{1, \ldots, k\}^n, t \in T_c, \chi_Z \in V_1(c)\}
\end{aligned}$$

for all $t \in T$:

$$V(t) = \begin{cases} V_1(\epsilon) & \text{if } t = \epsilon \\ V_1(c) & \text{if } t = c \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, \chi_Z \notin V_1(c) \\ V_c(ct') & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, t' \in T_c, \chi_Z \in V_1(c) \end{cases}$$

for all $t \in T$, $i \in N$ and $j \in \{1, \ldots, k\}$:

$$C(t, i, j) = \begin{cases} C_1(\epsilon, i, j) & \text{if } t = \epsilon \\ C_c(ct', i, j) & \text{if } t = ct' \text{ where} \\ & \qquad c \in \{1, \ldots, k\}^n, t' \in T_c, \chi_Z \in V_1(c) \end{cases}$$

It is straightforward that (T, V, C) is also locally consistent and all of its interior nodes have k^n children.

Let us show that (T, V, C) realises $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi$ at ϵ. Let $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$, i.e., a joint move which costs at most $\bar{0} \overset{\infty}{\leftarrow} b$, we have:

$$
\begin{aligned}
&\neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \neg \chi_Z \in V_1(\epsilon) = V(\epsilon) \\
&\quad \Rightarrow \exists c' \in \{1, \ldots, k\}^n : c'_A = c_A \rightarrow \chi_Z \in V(c') \\
&\quad \Rightarrow V(c') = V_{c'}(\epsilon) \in Z \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \\
&\qquad \text{is realised at the root of } (T_{c'}, V_{c'}, C_{c'}) \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \\
&\qquad \text{is realised at the root of } (T, V, C)
\end{aligned}
$$

Hence, as $T(\epsilon) = q \cap cl(\varphi_0)$, we have:

$$
\begin{aligned}
q \cap cl(\varphi_0) \in Z &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\
&\Rightarrow \chi_Z \in q \\
&\Rightarrow \eta \in q \quad \text{as } q \text{ is maximal}
\end{aligned}
$$

Induction step: Assume that $b > \bar{0} \overset{\infty}{\leftarrow} b$. Similar to the base case, as $\neg\varphi \vee (\neg\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \neg\chi_Z) \in q$, let us consider the following two sub-cases:

- if $\neg\varphi \in q$, the proof is the repetition of that for the base case.
- if $\neg\langle\langle A^b \rangle\rangle \bigcirc \Box\varphi \wedge \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \neg\chi_Z \in q$, then $\neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \neg\chi_Z \in q$ and $\neg\langle\langle A^{b_1} \rangle\rangle \bigcirc \langle\langle A^{b_2} \rangle\rangle \Box\varphi \in q$ for all $(b_1, b_2) \in split(b)$. Let $\Phi = q \cap cl(\varphi_0)$. Obviously, $\Phi_\bigcirc \subseteq \Psi_\bigcirc$; therefore, $\Phi_\bigcirc < k$. By Lemma 4, there exists a local consistent tree (T_0, V_0, C_0) of height one where $T_0 = \{\epsilon\} \dot{\cup} \{1, \ldots, k\}^n$ and $V_0(\epsilon) = \Phi$.

  For each $c \in \{1, \ldots, k\}^n$, let Φ_c be an arbitrary set from Γ such that $\Phi_c \supseteq V_0(c)$. Let (T_1, V_1, C_1) be a finite tree such that $T_1 = T_0$, $C_1 = C_0$, $V_1(\epsilon) = V_0(\epsilon)$ and $V_1(c) = \Phi_c$ for all $c \in \{1, \ldots, k\}^n$. Since (T_0, V_0, C_0) is locally consistent, so is (T_1, V_1, C_1). Then, for every $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq \bar{b}$, i.e., c_A is a joint move which costs at most $\bar{b}$, we have:

if $C(c,A) \leq \bar{0} \overset{\infty}{\leftarrow} b$:

$$\begin{aligned}
C(c,A) \leq \bar{b} &\Rightarrow \exists c'\{1,\ldots,k\}^n : c'_A = c_A \wedge \neg\neg\chi_Z = \chi_Z \in V_1(c') \\
&\Rightarrow V_1(c') \in Z \qquad \text{by Lemma 5} \\
&\Rightarrow \exists \text{ a locally consistent tree } (T_{c'}, V_{c'}, C_{c'}) \\
&\qquad \text{which satisfies all conditions of Lemma 7}
\end{aligned}$$

if $\bar{0} \overset{\infty}{\leftarrow} b < C(c,A) \leq b_1$ for some $(b_1,b_2) \in \mathit{split}(b)$:

$$\begin{aligned}
C(c,A) \leq \bar{b} &\Rightarrow \exists c'\{1,\ldots,k\}^n : c'_A = c_A \wedge \neg\langle\!\langle A^{b_2}\rangle\!\rangle\Box\varphi \in V_1(c') \\
&\Rightarrow V_1(c') \in Z \qquad \text{induction hypothesis} \\
&\Rightarrow \exists \text{ a locally consistent tree } (T_{c'}, V_{c'}, C_{c'}) \\
&\qquad \text{which satisfies all conditions of Lemma 7}
\end{aligned}$$

Let us consider a finite tree (T,V,C) where

$$\begin{aligned}
T = \{\epsilon\} \cup \{c \in \{1,\ldots,k\}^n \mid \chi_Z \notin V_1(c)\} \cup& \\
\{ct \mid c \in \{1,\ldots,k\}^n, t \in T_c, \chi_Z \in V_1(c) \text{ or}& \\
\exists(b_1,b_2) \in \mathit{split}(b) : \neg\langle\!\langle A^{b_2}\rangle\!\rangle\Box\varphi \in V_1(c)\}&
\end{aligned}$$

for all $t \in T$:

$$V(t) = \begin{cases} V_1(\epsilon) & \text{if } t = \epsilon \\ V_1(c) & \text{if } t = c \text{ where} \\ & \quad c \in \{1,\ldots,k\}^n, \chi_Z \notin V_1(c) \\ V_c(ct') & \text{if } t = ct' \text{ where} \\ & \quad c \in \{1,\ldots,k\}^n, t' \in T_c, \chi_Z \in V_1(c) \text{ or} \\ & \quad \exists(b_1,b_2) \in \mathit{split}(b) : \neg\langle\!\langle A^{b_2}\rangle\!\rangle\Box\varphi \in V_1(c) \end{cases}$$

and for all $t \in T$, $i \in N$ and $j \in \{1,\ldots,k\}$:

$$C(t,i,j) = \begin{cases} C_1(\epsilon,i,j) & \text{if } t = \epsilon \\ C_c(ct',i,j) & \text{if } t = ct' \text{ where} \\ & \quad c \in \{1,\ldots,k\}^n, t' \in T_c, \chi_Z \in V_1(c) \text{ or} \\ & \quad \exists(b_1,b_2) \in \mathit{split}(b) : \neg\langle\!\langle A^{b_2}\rangle\!\rangle\Box\varphi \in V_1(c) \end{cases}$$

It is straightforward that (T,V,C) is also locally consistent and all of its interior nodes have k^n children.

Let us show that (T, V, C) realises $\langle\langle A^b \rangle\rangle \Box \varphi$ at ϵ. Let $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq \bar{b}$, i.e., a joint move which costs at most b, we have:

if $C(c, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$:

$$
\begin{aligned}
&\neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \bigcirc \neg\chi_Z \in V_1(\epsilon) = V(\epsilon) \\
&\quad \Rightarrow \exists c' \in \{1, \ldots, k\}^n : c'_A = c_A \rightarrow \chi_Z \in V(c') \\
&\quad \Rightarrow V(c') = V_{c'}(\epsilon) \in Z \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \text{ is realised at the root of } (T_{c'}, V_{c'}, C_{c'}) \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \text{ is realised at the root of } (T, V, C)
\end{aligned}
$$

if $\bar{0} \overset{\infty}{\leftarrow} b < C(c, A) \leq b_1$ for some $(b_1, b_2) \in split(b)$:

$$
\begin{aligned}
&\neg\langle\langle A^{b_1} \rangle\rangle \bigcirc \langle\langle A^{b_2} \rangle\rangle \Box \varphi \in V_1(\epsilon) = V(\epsilon) \\
&\quad \Rightarrow \exists c' \in \{1, \ldots, k\}^n : c'_A = c_A \rightarrow \neg\langle\langle A^{b_2} \rangle\rangle \Box \varphi \in V(c') \\
&\quad \Rightarrow V(c') = V_{c'}(\epsilon) \in Z \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \text{ is realised at the root of } (T_{c'}, V_{c'}, C_{c'}) \\
&\quad \Rightarrow \neg\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b} \rangle\rangle \Box \varphi \text{ is realised at the root of } (T, V, C)
\end{aligned}
$$

Hence, as $T(\epsilon) = q \cap cl(\varphi_0)$, we have:

$$
\begin{aligned}
q \cap cl(\varphi_0) \in Z &\Rightarrow \chi_Z \in q \cap cl(\varphi_0) \quad \text{by Lemma 5} \\
&\Rightarrow \chi_Z \in q \\
&\Rightarrow \eta \in q \quad \text{as } q \text{ is maximal}
\end{aligned}
$$

$\Box$

Now we have almost all the ingredients for constructing the model for φ_0. For each consistent set x in Γ and an eventuality φ of $cl(\varphi_0)$, we have a finite tree $(T_{x,\varphi}, V_{x,\varphi}, C_{x,\varphi})$ with the root having label x which realises φ. Let the eventualities in $cl(\varphi_0)$ be listed as $\varphi^e_1, \ldots, \varphi^e_m$. Next, we define the final tree.

Definition 8. *The final tree $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$ is constructed inductively as follows.*

- *Initially, select an arbitrary $x \in \Gamma$ such that $\varphi_0 \in x$. As that formula is consistent, such a set exists. Let $(T_{x,\varphi^e_1}, V_{x,\varphi^e_1}, C_{x,\varphi^e_1})$ be the initial tree.*

- *Given the tree constructed so far and the last used eventuality φ_i^e. We replace every leaf labelled by $y \in \Gamma$ of the currently constructed tree with the tree $(T_{y,\varphi_j^e}, V_{y,\varphi_j^e}, C_{y,\varphi_j^e})$ where $j = i \mod m+1$.*

Let S_{φ_0} be the model which is based on $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$. (It is easy to define the assignment π using V.)

Lemma 8. *If $\langle\langle A^b\rangle\rangle\varphi\ \mathcal{U}\psi$ (or $\neg\langle\langle A^b\rangle\rangle\Box\varphi$) is in the label of some t of $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$, $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ (or $\neg\langle\langle A^b\rangle\rangle\Box\varphi$) is realised from t.*

Proof. Let us consider the first case when $\varphi_i^e = \langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi \in V(t)$ where t is a node of $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$. The proof for the case of $\varphi_i^e = \neg\langle\langle A^b\rangle\rangle\Box\varphi$ is also done similarly.

- If t happens to be the root of the sub-tree $(T_{t,\varphi_i^e}, V_{t,\varphi_i^e}, C_{t,\varphi_i^e})$, then the proof is done as φ_i^e is realised within this sub-tree at t, hence also in the final tree.
- Otherwise, we define inductively on b a b-strategy as follows:

  Base case: Assume that $b = \bar{0} \overset{\infty}{\leftarrow} b$, since $\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\varphi\ \mathcal{U}\psi \in V(t)$, as $V(t)$ is a maximally consistent set, we have that $\psi \vee (\varphi \wedge \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\varphi\mathcal{U}\psi) \in V(t)$:

  - If $\psi \in V(t)$, the proof is done as $\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\varphi\mathcal{U}\psi$ is immediately realised at t.
  - Otherwise, we have $\varphi \wedge \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\varphi\ \mathcal{U}\psi \in V(t)$. Then $\varphi \in V(t)$ and by Lemma 4, there exists $c \in \{1, \ldots, k\}^n$ such that $C(tc, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$ and for all $c \in \{1, \ldots, k\}^n$ with $c'_A = c_A$, we have $\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle\varphi\mathcal{U}\psi \in V(tc')$. Let $F_A(t) = c_A$. Then, we can continue with the same argument to define the strategy F_A until a node t' labelled by y in $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$ is reached where t' is the root of some sub-tree $(T_{y,\varphi_i^e}, V_{y,\varphi_i^e}, C_{y,\varphi_i^e})$. Such a node must exist because, according to the construction of the $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$, eventual formulas in $cl(\varphi_0)$ are cycled through. As $(T_{y,\varphi_i^e}, V_{y,\varphi_i^e}, C_{y,\varphi_i^e})$ realises φ_i^e, we can extend F_A to a b-strategy to realise φ_i^e.

  Induction Step: Assume that $b > \bar{0} \overset{\infty}{\leftarrow} b$, since $\langle\langle A^b\rangle\rangle\varphi\,\mathcal{U}\psi \in V(t)$, and $V(t)$ is a maximally consistent set, we have that $\psi \vee (\varphi \wedge (\langle\langle A^b\rangle\rangle \bigcirc \varphi\mathcal{U}\psi \vee \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi)) \in V(t)$.

  - If $\psi \in V(t)$, the proof is done as $\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ is immediately realised at t.

- If φ and $\langle\langle A^{b_1}\rangle\rangle \bigcirc \langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi \in V(t)$ for some $(b_1, b_2) = b$ (hence, $b_2 < b$), we have that $\varphi \in V(t)$ and by Lemma 4 and there exists $c \in \{1, \ldots, k\}^n$ such that $C(tc, A) \leq b_1$ and for all $c' \in \{1, \ldots, k\}^n$ with $c'_A = c_A$, we have $\langle\langle A^{b_2}\rangle\rangle\varphi\,\mathcal{U}\psi \in V(tc')$. Let $F_A(t) = c_A$. As $b_2 < b$, by the induction hypothesis, there is a strategy $F_{A,c}$ which realises $\langle\langle A^{b_2}\rangle\rangle\varphi\mathcal{U}\psi$ from tc. Hence, we just need to define $F_A(tc\lambda) = F_{A,c}(c\lambda)$. This simply gives us a b-strategy which realises $\langle\langle A^{b}\rangle\rangle\varphi\mathcal{U}\psi$ from t.
- Otherwise, we have φ and $\langle\langle A^{\bar{0} \overset{\infty}{\leftarrow} b}\rangle\rangle \bigcirc \langle\langle A^{b}\rangle\rangle\varphi\ \mathcal{U}\psi \in V(t)$. Let us repeat the argument in the base case where $\varphi \in V(t)$ and by Lemma 4 and we have that there exists $c \in \{1, \ldots, k\}^n$ such that $C(tc, A) \leq \bar{0} \overset{\infty}{\leftarrow} b$ and for all $c \in \{1, \ldots, k\}^n$ with $c'_A = c_A$, we have $\langle\langle A^{b}\rangle\rangle\varphi\,\mathcal{U}\psi \in V(tc)$. Let $F_A(t) = c_A$. Then, we can continue with the same argument to define the strategy F_A until a node t' labelled by y in $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$ is reached where t' is the root of some sub-tree $(T_{y,\varphi_i^e}, V_{y,\varphi_i^e}, C_{y,\varphi_i^e})$. Such a node must exist because, according to the construction of the $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$, eventualities in $cl(\varphi_0)$ are cycled through. As $(T_{y,\varphi_i^e}, V_{y,\varphi_i^e}, C_{y,\varphi_i^e})$ realises φ_i^e, we can extend F_A to a b-strategy to realise φ_i^e.

$\square$

Lemma 9. *If $\neg\langle\langle A^b\rangle\rangle\varphi\ \mathcal{U}\psi$ (or $\langle\langle A^b\rangle\rangle\Box\varphi$) is in the label of some t of $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$, $\neg\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ (or $\langle\langle A^b\rangle\rangle\Box\varphi$) is realised from t.*

Proof. Let us consider the case $\langle\langle A^b\rangle\rangle\Box\varphi \in V(t)$. The proof for $\neg\langle\langle A^b\rangle\rangle\varphi\mathcal{U}\psi$ is similar.

In the following, let $T' = \{1, \ldots, k\}^n$. We shall define a b-strategy F_A which realises $\langle\langle A^b\rangle\rangle\Box\varphi$ by induction on the length of inputs. Let Λ^i denote the set of inputs of length i. Initially, $\Lambda^1 = \{t\}$. We define $F(A)$ for inputs of length i and Λ^{i+1} of inputs of length $i+1$ inductively on $i \geq 1$ such that for all $\lambda \in \Lambda^{i+1}$, $cost(\lambda, F_A) \leq b_1$ and $\langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(\lambda[i+1])$ for some $(b_1, b_2) \in split(b) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b, b)\}$.

Base case: Assume that $i = 1$. We have

$$\begin{aligned}
\langle\langle A^b\rangle\rangle\Box\varphi \in V(t) &\Rightarrow \varphi \in V(t) \wedge \langle\langle A^{b_1}\rangle\rangle \bigcirc \langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(t) \\
&\qquad\quad \text{for some } (b_1, b_2) \in split(b) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b, b)\} \\
&\Rightarrow \exists c \in T' : C(tc, A) \leq b_1 \\
&\Rightarrow \forall c' \in T' : c'_A = c_A \rightarrow \langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(tc')
\end{aligned}$$

Then, we define $F_A(t) = c_A$ and $\Lambda^2 = \{tc' \mid c' \in T' : c'_A = c_A\}$. Obviously, we have $cost(tc', F_A) = C(tc', A) \leq b_1$ and $\langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(tc')$ for all $tc' \in \Lambda^2$.

Induction step: Assume that $i > 1$ and we have defined F_A for inputs of length i and Λ^{i+1} such that for all $\lambda \in \Lambda^{i+1}$, $cost(\lambda, F_A) \leq b_1$ and $\langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(\lambda[i+1])$ for some $(b_1, b_2) \in split(b) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b, b)\}$. Let us define F_A for inputs of length $i+1$ and Λ^{i+2}. We have, for all $\lambda \in \Lambda i + 1$

$$\begin{aligned}
\langle\langle A^{b_2}\rangle\rangle\Box\varphi \in V(\lambda[i+1]) &\Rightarrow \varphi \in V(\lambda[i+1]) \wedge \langle\langle A^{b_3}\rangle\rangle \bigcirc \langle\langle A^{b_4}\rangle\rangle\Box\varphi \in V(t) \\
&\qquad \text{for some } (b_3, b_4) \in split(b_2) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b_2, b_2)\} \\
&\Rightarrow \exists c_\lambda \in T' : C(\lambda c, A) \leq b_3 \\
&\Rightarrow \forall c' \in T' : c'_A = c_{\lambda A} \to \langle\langle A^{b_4}\rangle\rangle\Box\varphi \in V(\lambda c')
\end{aligned}$$

Then, we define $F_A(\lambda) = c_{\lambda A}$ for all $\lambda \in \Lambda^{i+1}$ and $\Lambda^{i+2} = \{\lambda c' \mid \lambda \in \Lambda^{i+1}, c' \in T' : c'_A = c_A\}$. Obviously, we have, $(b_1 + b_3, b_4) \in split(b) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b, b)\}$, $cost(\lambda c', F_A) = cost(\lambda, F_A) + C(\lambda c', A) \leq b_1 + b_3$ and $\langle\langle A^{b_4}\rangle\rangle\Box\varphi \in V(\lambda c')$ for all $\lambda c' \in \Lambda^2$.

Let $\Lambda = \bigcup_{i \geq 0} \Lambda^i$, we have

$$\begin{aligned}
\forall\lambda \in \Lambda &\Rightarrow \varphi \in V(\lambda[i]) \wedge \exists i \geq 0 : \lambda \in \Lambda^i \\
&\Rightarrow \varphi \in V(\lambda[i]) \wedge \\
&\qquad \exists (b_1, b_2) \in split(b) \cup \{(\bar{0} \overset{\infty}{\leftarrow} b, b)\} cost(\lambda, F_A) \leq b_1 \\
&\Rightarrow \varphi \in V(\lambda[i]) \wedge cost(\lambda, F_A) \leq b
\end{aligned}$$

Given the constructed strategy F_A, we have that $out(t, F_A) = \Lambda$. As $cost(\lambda, F_A) \leq b$ for all $\lambda \in \Lambda$, F_A is a b-strategy. Furthermore, as $\varphi \in V(\lambda[|\lambda|])$ for all $\lambda \in \Lambda$, it is straightforward that F_A realises $\langle\langle A^b\rangle\rangle\Box\varphi$.

$\square$

Finally, we show the following truth lemma.

Lemma 10. *For every node t of $(T_{\varphi_0}, V_{\varphi_0}, C_{\varphi_0})$ and every formula $\varphi \in cl(\varphi_0)$, if $\varphi \in V_{\varphi_0}(t)$ then $S_{\varphi_0}, t \models \varphi$.*

Proof. The proof is done by induction on the structure of φ.

- For the cases of propositions, negative proposition and disjunction, the proofs are trivial.

- Assume $\varphi = \langle\langle A^b\rangle\rangle \bigcirc \psi$, Lemma 4 ensures that there is a move c_A for some $c \in \{1, \ldots, k\}^n$ where $C(c, A) \leq b$ such that for all $c' \in \{1, \ldots, k\}^n$, we have $\psi \in V(tc')$. Then by the induction hypothesis, we have that $S_{\varphi_0}, tc' \models \psi$. Then, $S_{\varphi_0}, t \models \langle\langle A^b\rangle\rangle \bigcirc \psi$.

- Assume $\varphi = \neg\langle\langle A^b \rangle\rangle \bigcirc \psi$, Lemma 4 ensures that for all $c \in \{1, \ldots, k\}^n$ such that $C(c, A) \leq b$, there exists $c' \in \{1, \ldots, k\}^n$ such that $c'_A = c_A$ and $\sim\psi \in V(tc')$. Then by the induction hypothesis, we have that $S_{\varphi_0}, tc' \models\sim \psi$. Then, $S_{\varphi_0}, t \models \neg\langle\langle A^b \rangle\rangle \bigcirc \psi$.
- For the cases of $\langle\langle A^b \rangle\rangle \psi_1 \, \mathcal{U} \psi_2$, $\neg\langle\langle A^b \rangle\rangle \Box \psi$, $\neg\langle\langle A^b \rangle\rangle \psi_1 \, \mathcal{U} \psi_2$ and $\langle\langle A^b \rangle\rangle \Box \psi$, the proofs are trivial due to Lemmas 8 and 9.

$\Box$

Finally, we have the following theorem.

Theorem 1. *The axiom system for RB-ATL is sound and complete.*

4 Model-checking RB-ATL

In this section we describe a model-checking algorithm for RB-ATL which runs in time polynomial in the size of the formula (if resource bounds are encoded in unary) and the structure, and is exponential in the number of resources. The algorithm is similar to the model-checking algorithm for ATL given in [6]. The main differences from the algorithm for ATL are that we need to take the costs of strategies into account, and, instead of working with a straightforward set of subformulas $Sub(\varphi)$ of a given formula φ, we work with an extended set of subformulas $Sub^+(\varphi)$. $Sub^+(\varphi)$ includes $Sub(\varphi)$, and in addition:

- if $\langle\langle A^b \rangle\rangle \Box \psi \in Sub(\varphi)$, then $\langle\langle A^{d'} \rangle\rangle \Box \psi \in Sub^+(\varphi)$ for all d' such that $(d, d') \in split(b)$;
- if $\langle\langle A^b \rangle\rangle \psi_1 \, \mathcal{U} \psi_2 \in Sub(\varphi)$, then $\langle\langle A^{d'} \rangle\rangle \psi_1 \, \mathcal{U} \psi_2 \in Sub^+(\varphi)$ for all d' such that $(d, d') \in split(b)$.

We assume that $Sub^+(\varphi)$ is ordered in the increasing order of complexity and of resource bounds (so e.g., if $b \leq b'$, then $\langle\langle A^b \rangle\rangle \Box \psi$ precedes $\langle\langle A^{b'} \rangle\rangle \Box \psi$).

Theorem 2. *Given a structure $S = (n, r, Q, \Pi, \pi, d, c, \delta)$ and a formula φ, there is an algorithm which returns the set of states $[\varphi]_S$ satisfying φ: $[\varphi]_S = \{q \mid S, q \models \varphi\}$, which runs in time $O(|\varphi|^{2r+1} \times |S|)$, assuming resource bounds are encoded in unary.*

Proof. Consider the following model-checking algorithm:

for every φ' in $Sub^+(\varphi)$:

case $\varphi' == p$: $[\varphi']_S = \pi(p)$

case $\varphi' == \neg\psi$: $[\varphi']_S = Q \setminus [\psi]_S$

case $\varphi' == \psi_1 \wedge \psi_2$: $[\varphi']_S = [\psi_1]_S \cap [\psi_2]_S$

case $\varphi' == \langle\langle A^b \rangle\rangle \bigcirc \psi$: $[\varphi']_S = \mathit{Pre}(A, [\psi]_S, b)$

case $\varphi' == \langle\langle A^b \rangle\rangle \Box \psi$ for b where for all i, $b_i \in \{0, \infty\}$:
    $\rho := [true]_S; \tau := [\psi]_S;$
    while $\rho \not\subseteq \tau$ **do** $\rho := \tau; \tau := \mathit{Pre}(A, \rho, b) \cap [\psi]_S$ **od**;
    $[\varphi']_S := \rho$

case $\varphi' == \langle\langle A^b \rangle\rangle \Box \psi$ for b where for some i, $b_i \notin \{0, \infty\}$:
    $\rho := [false]_S; \tau := [false]_S;$
    foreach $d' \in \{d' \mid (d, d') \in split(b)\}$ **do**
        $\tau := \mathit{Pre}(A, [\langle\langle A^{d'} \rangle\rangle \Box \psi]_S, d) \cap [\psi]_S$
        while $\tau \not\subseteq \rho$ **do**
            $\rho := \rho \cup \tau; \tau := \mathit{Pre}(A, \rho, \bar{0} \overset{\infty}{\leftarrow} b) \cap [\psi]_S$
        od
    od;
    $[\varphi']_S := \rho$

case $\varphi' == \langle\langle A^b \rangle\rangle \psi_1 \mathcal{U} \psi_2$ for b where for all i, $b_i \in \{0, \infty\}$:
    $\rho := [false]_S; \tau := [\psi_2]_S;$
    while $\tau \not\subseteq \rho$ **do** $\rho := \rho \cup \tau; \tau := \mathit{Pre}(A, \rho, b) \cap [\psi_1]_S$ **od**;
    $[\varphi']_S := \rho$

case $\varphi' == \langle\langle A^b \rangle\rangle \psi_1 \mathcal{U} \psi_2$ where for some i, $b_i \notin \{0, \infty\}$:
    $\rho := [false]_S; \tau := [false]_S;$
    foreach $d' \in \{d' \mid (d, d') \in split(b)\}$ **do**
        $\tau := \mathit{Pre}(A, [\langle\langle A^{d'} \rangle\rangle \psi_1 \mathcal{U} \psi_2]_S, d) \cap [\psi_1]_S$
        while $\tau \not\subseteq \rho$ **do**
            $\rho := \rho \cup \tau; \tau := \mathit{Pre}(A, \rho, \bar{0} \overset{\infty}{\leftarrow} b) \cap [\psi_1]_S$
        od
    od;
    $[\varphi']_S := \rho$

Pre is a function which, given a coalition A, a set $\rho \subseteq Q$, and a bound b, returns a set of states q in which A has a move σ_A with cost $cost(q, \sigma_A) \leq b$ such that $out(q, \sigma_A) \subseteq \rho$. Observe that $\mathit{Pre}(A, \rho, \bar{\infty})$ is just $\mathit{Pre}(A, \rho)$ from [6].

The cases for propositional variables, negation, conjunction and $\langle\langle A^b \rangle\rangle \bigcirc \psi$ are straightforward. The cases where the resource bound consists of 0 and ∞ are also similar to [6]. However the cases for $\langle\langle A^b \rangle\rangle \Box \psi$ and $\langle\langle A^b \rangle\rangle \psi_1 \mathcal{U} \psi_2$ where b does not contain only 0 and ∞ have no counterpart in the ATL algorithm, and we explain these in some detail. First, note that the cases for $\langle\langle A^b \rangle\rangle \Box \psi$ and $\langle\langle A^b \rangle\rangle \psi_1 \mathcal{U} \psi_2$ where b consists of 0 and ∞ are the standard greatest and least fixed point computations respectively, which consider only 0 cost moves for i with $b_i = 0$ or any moves for i with $b_i = \infty$. In particular,

$$[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S = \mathit{Pre}(A, [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S, \bar{0} \overset{\infty}{\leftarrow} b) \cap [\psi]_S$$

and

$$[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \psi_1 \mathcal{U} \psi_2]_S = \mathit{Pre}(A, [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \psi_1 \mathcal{U} \psi_2]_S, \bar{0} \overset{\infty}{\leftarrow} b) \cap [\psi_2]_S$$

$[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S$ contains all states where A has a $\bar{0} \overset{\infty}{\leftarrow} b$-cost strategy to maintain ψ forever. Note that A has a b-cost strategy to maintain ψ forever if and only if it has a b-cost strategy to force the system into one of the $[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S$ states, while maintaining ψ. In other words, in order to compute $\langle\langle A^b \rangle\rangle \Box \psi$ for b containing $b_i \notin \{0, \infty\}$, we need to compute $\langle\langle A^b \rangle\rangle \psi\ \mathcal{U} \langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi$. This explains the similarity between the cases of $\langle\langle A^b \rangle\rangle \Box \psi$ and $\langle\langle A^b \rangle\rangle \psi_1\ \mathcal{U} \psi_2$ for the case of b not consisting solely of 0 and ∞. In the case of $\langle\langle A^b \rangle\rangle \Box \psi$, in the first execution of the **foreach** $d' \in \{d' \mid (d, d') \in \mathit{split}(b)\}$ loop, we have $d' = \bar{0} \overset{\infty}{\leftarrow} b$ and $\tau = \mathit{Pre}(A, [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S, b) \cap [\psi]_S$, which includes $\mathit{Pre}(A, [\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S, \bar{0} \overset{\infty}{\leftarrow} b) \cap [\psi]_S$, hence it also includes $[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \Box \psi]_S$. In the nested **while** loop, ρ accumulates the results and τ adds the ψ-states from where A has a $\bar{0} \overset{\infty}{\leftarrow} b$ strategy to enforce the outcome to be in ρ. In the outer loop, d' bounds are used in some order consistent with $<$, namely satisfying the condition that if $b_i < b_j$ then b_i is used before b_j.

In the case for $\langle\langle A^b \rangle\rangle \psi_1\ \mathcal{U} \psi_2$ where b does not consist only of 0 and ∞, after the first iteration of the **foreach** $d' \in \{d' \mid (d, d') \in \mathit{split}(b)\}$ loop, τ is $[\langle\langle A^{\bar{0}\overset{\infty}{\leftarrow} b} \rangle\rangle \psi_1\ \mathcal{U} \psi_2]_S$ which includes $[\psi_2]_S$. The rest is very similar to the case for $\langle\langle A^b \rangle\rangle \Box \psi$ where b does not consist solely of 0 and ∞.

Note that $|\mathit{split}(b)|$ is $O(b^r)$. If φ contains operators with bounds other than 0 and ∞, $|\mathit{Sub}^+(\varphi)|$ is $O(|\varphi| \times |\varphi|^r)$, assuming resource bounds are written in unary. In the $\langle\langle A^b \rangle\rangle \Box \psi$ and $\langle\langle A^b \rangle\rangle \psi_1 \mathcal{U} \psi_2$ cases, the outer loop is executed $O(|\varphi|^r)$ times and the inner loop is executed in total at most $|S|$ times. This gives us complexity $O(|\varphi| \times |\varphi|^r \times |\varphi|^r \times |S|)$, or $O(|\varphi|^{2r+1} \times |S|)$. Note that the lower bound for model-checking complexity is given by the model-checking complexity of ATL, which is polynomial time in the size of the model and the formula. $\Box$

5 Related work

Recent work on Alternating-Time Temporal Logic and Coalition Logic (for example, [15, 12, 16, 6, 13, 1]) has allowed the expression of many interesting properties of coalitions and strategies. However, there is no natural way of expressing resource requirements in these logics. The only work in this tradition that considered resources is [17], which introduced Coalitional Resource Games and studied complexity of decision problems for these games. A logic for describing Coalitional resource Games and a model-checking procedure for the logic were proposed in [3]; however the only modality that logic has is $\langle\langle A^b \rangle\rangle \bigcirc$ (only one step games were considered).

More recently, several extensions of temporal logics and logics of coalitional ability which are capable of expressing resource bounds have been proposed in the literature, for example, [7, 8, 10, 11, 2]. All of these papers consider only the model-checking problem, and some of the logics allow both consumption and production of resources by actions. There are many different proposals for the syntax and semantics of resource logics. In [8] several versions are given, for example, considering resource bounds both on the coalition A and the rest of the agents in the system, considering a fixed resource endowment of A in the initial state which affects their endowment after executing some actions, etc. In [10, 11] a different syntax and semantics are considered, also involving resource endowment of the whole system when evaluating a statement concerning a group of agents A. As observed in [8], subtle differences in truth conditions for resource logics result in the difference between decidability and undecidabiliity of the model-checking problem. In [8], undecidability of the model-checking problem for several versions of the logics is proved. The only decidable cases considered in [8] are an extension of Computation Tree Logic (CTL) [9] with resources (essentially one-agent ATL) and the version where on every path only a fixed finite amount of resources can be produced. Similarly, [10] gives a logic PRB-ATL (Priced Resource-Bounded ATL) with a decidable model-checking problem where the total amount of resources in the system has a fixed bound. The model-checking algorithm for PRB-ATL runs in time polynomial in the size of the model and exponential in the number of resources and the resource bound on the system. In [11] an EXPTIME lower bound in the number resources is shown. Recently, it has also been shown that if a zero-cost action is always available, the model-checking problem for RB-ATL with both production and consumption of resources is decidable, however it is EXPSPACE-hard [2].

6 Conclusions

We have provided a complete and sound axiomatisation of RB-ATL, a logic which extends ATL with resource bounds. The resulting logic can express interesting properties of coalitions of agents involving resource limitations. For example, it can express that a coalition can maintain the system in a φ-state indefinitely given a finite amount of resources (this essentially means that after a while φ can be maintained for free). We have also presented a model-checking algorithm for RB-ATL, which runs in time polynomial in the size of the model and the formula (assuming that resource bounds are encoded in unary) and exponential in the number of resources.

The semantics for RB-ATL presented in this paper, in particular the assumption that actions only consume but never produce resources, is motivated by verifying resource requirements for systems of agents where resources of interest are time, memory, bandwidth etc., which cannot be generated by agents. In future work, we plan to study axiomatisations of variants of RB-ATL where actions can have a negative cost, such as in [7, 2].

7 Acknowledgments

This work was supported by the Engineering and Physical Sciences Research Council [grant EP/E031226/1]. We thank the anonymous referees for their comments which helped to improve the paper.

References

[1] T. Ågotnes, W. van der Hoek, and M. Wooldridge. Reasoning about coalitional games. *Artificial Intelligence*, 173(1):45–79, 2009.

[2] N. Alechina, B. Logan, H. N. Nguyen, and F. Raimondi. Decidable model-checking for a resource logic with production of resources. In *Proceedings of the 21st European Conference on Artificial Intelligence (ECAI 2014)*, pages 9–14. IOS Press, 2014.

[3] N. Alechina, B. Logan, H. N. Nguyen, and A. Rakib. Expressing properties of coalitional ability under resource bounds. In X. He, J. F. Horty, and E. Pacuit, editors, *Logic, Rationality, and Interaction, Second International Workshop, LORI 2009, Proceedings*, volume 5834 of *Lecture Notes in Computer Science*, pages 1–14. Springer, 2009.

[4] N. Alechina, B. Logan, H. N. Nguyen, and A. Rakib. Resource-bounded alternating-time temporal logic. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2010)*, pages 481–488. IFAAMAS, 2010.

[5] N. Alechina, B. Logan, H. N. Nguyen, and A. Rakib. Logic for coalitions with bounded resources. *Journal of Logic and Computation*, 21(6):907–937, December 2011.

[6] R. Alur, T. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49(5):672–713, 2002.

[7] N. Bulling and B. Farwer. RTL and RTL*: Expressing abilities of resource-bounded agents. In J. Dix, M. Fisher, and P. Novák, editors, *Proceedings of the 10th International Workshop on Computational Logic in Multi-Agent Systems*, pages 2–19, 2009.

[8] N. Bulling and B. Farwer. On the (un-)decidability of model checking resource-bounded agents. In *Proceedings of the 19th European Conference on Artificial Intelligence (ECAI 2010)*, volume 215 of *Frontiers in Artificial Intelligence and Applications*, pages 567–572. IOS Press, 2010.

[9] E. M. Clarke, E. A. Emerson, and A. P. Sistla. Automatic verification of finite-state concurrent systems using temporal logic specifications. *ACM Transactions on Programming Languages and Systems*, 8(2):244–263, 1986.

[10] D. Della Monica, M. Napoli, and M. Parente. On a logic for coalitional games with priced-resource agents. *Electronic Notes in Theoretical Computer Science*, 278:215–228, 2011.

[11] D. Della Monica, M. Napoli, and M. Parente. Model checking coalitional games in shortage resource scenarios. In *Proceedings of the 4th International Symposium on Games, Automata, Logics and Formal Verification (GandALF 2013)*, volume 119 of *EPTCS*, pages 240–255, 2013.

[12] V. Goranko. Coalition games and alternating temporal logics. In *Proceeding of the Eighth Conference on Theoretical Aspects of R ationality and Knowledge (TARK VIII)*, pages 259–272. Morgan Kaufmann, 2001.

[13] V. Goranko and G. van Drimmelen. Complete axiomatization and decidability of alternating-time temporal logic. *Theor. Comput. Sci.*, 353(1-3):93–117, 2006.

[14] H. N. Nguyen. An extension of RB-ATL. In O. Boissier, A. E. Fallah-Seghrouchni, S. Hassas, and N. Maudet, editors, *Proceedings of The Multi-Agent Logics, Languages, and Organisations Federated Workshops (MALLOW 2010)*, volume 627 of *CEUR Workshop Proceedings*. CEUR-WS.org, 2010.

[15] M. Pauly. *Logic for Social Software*. Ph.D. thesis, ILLC, University of Amsterdam, 2001.

[16] M. Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 12(1):149–166, 2002.

[17] M. Wooldridge and P. E. Dunne. On the computational complexity of coalitional resource games. *Artificial Intelligence*, 170(10):835–871, 2006.