Commun. Math. Phys. 389, 1417–1443 (2022)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04304-7

Communications in Mathematical Physics

# Homotopical Analysis of 4d Chern-Simons Theory and Integrable Field Theories

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Received: 13 August 2020 / Accepted: 25 December 2021
Published online: 24 January 2022 – 

**Abstract:** This paper provides a detailed study of 4-dimensional Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{C}P^1$ for an arbitrary meromorphic 1-form $\omega$ on $\mathbb{C}P^1$. Using techniques from homotopy theory, the behaviour under finite gauge transformations of a suitably regularised version of the action proposed by Costello and Yamazaki is investigated. Its gauge invariance is related to boundary conditions on the surface defects located at the poles of $\omega$ that are determined by isotropic Lie subalgebras of a certain defect Lie algebra. The groupoid of fields satisfying such a boundary condition is proved to be equivalent to a groupoid that implements the boundary condition through a homotopy pullback, leading to the appearance of edge modes. The latter perspective is used to clarify how integrable field theories arise from 4-dimensional Chern-Simons theory.

## Contents



## 1. Introduction

Integrable field theories in 2 dimensions are characterised by the existence of an on-shell flat connection that depends meromorphically on an auxiliary Riemann surface, typically the Riemann sphere $\mathbb{C}P^1$. Such a *Lax connection* is often found by some clever guesswork, hence its origin is usually rather mysterious.

More recently, new approaches have been developed that provide very interesting algebraic and/or geometric explanations for the origin of Lax connections. From an algebraic perspective, 2-dimensional classical integrable field theories can be described in the Hamiltonian formalism as particular representations of Gaudin models associated with affine Kac-Moody algebras [Vic1]. From a geometric perspective, it was realised by Costello and Yamazaki [CY] that classical integrable field theories on a 2-dimensional manifold $\Sigma$ arise as specific solutions to a 4-dimensional generalisation of Chern-Simons theory, see also [Cos1,Cos2,Wit,CWY1,CWY2] for earlier works on this subject and [Vic2] for a relation to affine Gaudin models. The Lagrangian of the latter theory is given by $\omega \wedge \mathrm{CS}(A)$, where $\omega$ is a (fixed) meromorphic 1-form on $\mathbb{C}P^1$ and $\mathrm{CS}(A)$ is the Chern-Simons 3-form for a $\mathfrak{g}$-valued 1-form $A$ living on the product manifold $X = \Sigma \times C$, where $C$ is the Riemann sphere with the zeroes of $\omega$ removed to allow $A$ to have singularities there. In this approach, different integrable field theories on $\Sigma$ are obtained from different choices of meromorphic 1-forms $\omega$ together with suitable boundary conditions on the surface defects $\Sigma \times \{x\} \subset \Sigma \times \mathbb{C}P^1$ located at the poles $x$ of $\omega$. In particular, the equations of motion for the $\mathfrak{g}$-valued 1-form $A$ in the bulk, i.e. away from the poles of $\omega$, admit meromorphic solutions with poles at the zeroes of $\omega$, which correspond to the Lax connection of the integrable field theory.

The goal of the present paper is twofold. First, we provide a detailed and rigorous study of the 4-dimensional Chern-Simons action of [CY], its invariance under *finite* gauge transformations, and the structure of boundary conditions on the surface defects. For this we consider an arbitrary meromorphic 1-form $\omega$ on $\mathbb{C}P^1$, with an arbitrary finite set of poles $\boldsymbol{z} \subset \mathbb{C}P^1$ with each pole $x \in \boldsymbol{z}$ having an arbitrary order $n_x \in \mathbb{Z}_{\geq 1}$, which generalises considerably the cases of simple and double poles studied previously, see e.g. [CY,DLMV2]. (We would like to emphasise that, in the presence of higher order poles, the 4-dimensional Chern-Simons Lagrangian has to be regularised as in (3.3) in order to be locally integrable near each surface defect.) After a series of technical preparations in Sects. 2 and 3, our main result is Theorem 4.2, where we prove that the regularised 4-dimensional Chern-Simons action defines a gauge invariant function on the groupoid $\mathfrak{F}_{\mathrm{bc}}(X)$ of bulk fields $A$ and their gauge transformations $g : A \to {}^{g}A$, both subject to certain boundary conditions on the surface defects, cf. (4.2). The boundary conditions we consider are determined by a choice of Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\widehat{\boldsymbol{z}}}$ of the Lie algebra $\mathfrak{g}^{\widehat{\boldsymbol{z}}}$ of the product of jet groups $G^{\widehat{\boldsymbol{z}}} = \prod_{x \in \boldsymbol{z}} J^{n_x - 1} G$, where $n_x \geq 1$ is the order of the pole $x \in \boldsymbol{z}$ of $\omega$, that is isotropic with respect to a non-degenerate symmetric invariant bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega$ defined in terms of $\omega$. We note in passing that the appearance of jet groups has also been observed before in examples of conformal field theories, see [BR] and [Que].

The second goal of this paper is to clarify the passage from 4-dimensional Chern-Simons theory to 2-dimensional integrable field theories that was proposed in [CY]; see also [DLMV2] for some previous clarifications. The crucial new ingredient in our approach is Theorem 4.3, which proves that the groupoid $\mathfrak{F}_{\mathrm{bc}}(X)$ of bulk fields with boundary conditions in (4.2) is equivalent to the groupoid $\mathfrak{F}(X)$ in (4.5) whose objects are compatible pairs $(A, h)$ consisting of a bulk field $A$ and an *edge mode* $h : \Sigma \to G^{\widehat{\boldsymbol{z}}}$ on $\Sigma$ with values in the product of jet groups $G^{\widehat{\boldsymbol{z}}} = \prod_{x \in \boldsymbol{z}} J^{n_x - 1} G$. The groupoid $\mathfrak{F}(X)$ arises naturally by implementing the boundary conditions on the surface defects by a homotopy pullback (4.4) in the model category of groupoids, cf. [MMST]. Using this equivalence of groupoids, we can transfer the regularised 4-dimensional Chern-Simons action (3.3) to a gauge invariant action $S_{\omega}^{\text{ext}}$ on the groupoid $\mathfrak{F}(X)$, whose explicit form (4.7) justifies the interpretation of the edge mode $h : \Sigma \to G^{\widehat{z}}$ as the field content of a field theory on $\Sigma$.

The passage to a 2-dimensional integrable field theory consists of finding a specific solution $A = \mathcal{L}$ to the bulk equation of motion determined by (4.7) that qualifies as a Lax connection. Specifically, we introduce a subgroupoid $\mathfrak{F}_{\text{Lax}}(X)$ of $\mathfrak{F}(X)$ whose objects are compatible pairs $(\mathcal{L}, h)$, where the bulk field $\mathcal{L}$ is meromorphic with poles at the zeroes of $\omega$ on account of the bulk equation of motion and is *admissible* in the sense that the defect equation of motion can be lifted to a flatness condition for $\mathcal{L}$ on the whole of $X$, cf. (5.7). We also introduce in (5.8) a groupoid $\mathfrak{F}_{2\text{d}}(\Sigma)$ for the integrable field theory itself, whose objects consist only of an edge mode $h : \Sigma \to G^{\widehat{z}}$. We prove in Corollary 5.8 that the forgetful functor $\mathfrak{F}_{\text{Lax}}(X) \to \mathfrak{F}_{2\text{d}}(\Sigma)$ is an equivalence of groupoids if and only if, for each $h : \Sigma \to G^{\widehat{z}}$, there exists a unique connection $\mathcal{L}(h)$ such that the pair $(\mathcal{L}(h), h)$ belongs to $\mathfrak{F}_{\text{Lax}}(X)$. In this case one is able to transfer the action on $\mathfrak{F}(X)$ all the way down to $\mathfrak{F}_{2\text{d}}(\Sigma)$ to obtain the action for an integrable field theory on $\Sigma$ whose Lax connection is $\mathcal{L}(h)$. Unique solutions $\mathcal{L}$ to the compatibility condition on the pair $(\mathcal{L}, h)$ have been shown to exist in the case of single and double poles in [CY, DLMV2]. We do not address the issue of solvability of this condition in the general setting of the present work.

Let us briefly outline the content of this paper. In Sect. 2 we study 4-dimensional Chern-Simons theory and its gauge transformations for simple poles in $\omega$. This is generalised in Sect. 3 to the case of general poles. In Sect. 4 we link gauge invariance of the action to suitable boundary conditions and realise that an equivalent description involving also edge modes can be obtained. This equivalent perspective is used in Sect. 5 to explain how integrable field theories emerge from 4d Chern-Simons theory as particular partial solutions.

**Notations and conventions:**

Let $G$ be a simply connected matrix Lie group over $\mathbb{C}$ and let $\mathfrak{g}$ denote its Lie algebra. We fix a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

Let $\omega$ be a meromorphic 1-form on $\mathbb{C}P^1$. We denote by $\boldsymbol{\zeta} \subset \mathbb{C}P^1$ its finite subset of zeroes and by $\boldsymbol{z} \subset \mathbb{C}P^1$ its finite subset of poles. We shall assume that $\omega$ has at least one zero, namely $|\boldsymbol{\zeta}| \geq 1$. This implies that $\omega$ has at least three poles (counting multiplicities) and so, in particular, $|\boldsymbol{z}| \geq 1$.

Let $\Sigma := \mathbb{R}^2$ and $C := \mathbb{C}P^1 \setminus \boldsymbol{\zeta}$. We consider the 4-dimensional manifold

$$X := \Sigma \times C.$$

We fix a global holomorphic coordinate $z : C \to \mathbb{C}$ on $C$, which exists because it is assumed that $|\boldsymbol{\zeta}| \geq 1$. We can represent the 1-form $\omega$ in this coordinate as

$$\omega = \sum_{x \in \boldsymbol{z}} \sum_{p=0}^{n_x - 1} \frac{k_p^x \, dz}{(z - x)^{p+1}}, \tag{1.1}$$

where $k_p^x \in \mathbb{C}$, for each $p = 0, \dots, n_x - 1$, and $n_x \in \mathbb{Z}_{\geq 1}$ is the order of the pole $x \in \boldsymbol{z}$. By a slight abuse of notation, we shall denote by $\omega$ also the pullback along the projection $p_C : X \to C$ of the restriction of $\omega$ to $C$.

Using the Cartesian product structure of $X$ and the complex structure on $C$, we obtain a triple grading on the vector space of differential forms

$$\Omega^\bullet(X) = \bigoplus_{r=0}^{2} \bigoplus_{s,\bar{s}=0}^{1} \Omega^{r,s,\bar{s}}(X) \tag{1.2}$$

and the corresponding decomposition of the de Rham differential as $d_X = d_\Sigma + \partial + \bar{\partial}$. To simplify notation, we often denote $d_X$ simply by $d$.

## 2. Simple Poles in $\omega$

To begin with, we shall assume in this section that all the poles of $\omega$ are simple, i.e. we take $n_x = 1$ for all $x \in \mathfrak{z}$. The case with higher order poles in $\omega$ will require a regularisation of the action, which we shall return to in Sect. 3.

*2.1. Action.* Consider the 4-dimensional Chern-Simons action [CY]

$$S_\omega(A) = \frac{\mathrm{i}}{4\pi} \int_X \omega \wedge \mathrm{CS}(A), \tag{2.1}$$

where $A \in \Omega^1(X, \mathfrak{g})$ is a smooth $\mathfrak{g}$-valued 1-form on $X$ and $\mathrm{CS}(A) := \langle A, dA + \frac{1}{3}[A, A]\rangle \in \Omega^3(X)$ is the Chern-Simons 3-form.

Since $\omega$ is the pullback along $p_C : X \to C$ of a meromorphic 1-form on $C$ with poles in $\mathfrak{z}$, it is singular on the disjoint union of surface defects

$$D := \Sigma \times \mathfrak{z} = \bigsqcup_{x \in \mathfrak{z}} \Sigma_x, \tag{2.2}$$

where $\Sigma_x := \Sigma \times \{x\}$ for every pole $x \in \mathfrak{z}$. Later we shall make use of the embeddings of the individual surface defects $\Sigma_x$, for each $x \in \mathfrak{z}$, and of the disjoint union $D$, which we denote respectively by

$$\iota_x : \Sigma_x \hookrightarrow X, \qquad \boldsymbol{\iota} : D \hookrightarrow X. \tag{2.3}$$

The following lemma shows that the 4-form $\omega \wedge \mathrm{CS}(A) \in \Omega^4(X \setminus D)$ is locally integrable near $D$.

**Lemma 2.1.** *For any $\eta \in \Omega^3(X)$, the 4-form $\omega \wedge \eta \in \Omega^4(X \setminus D)$ is locally integrable near the surface defect $\Sigma_x$ associated with any simple pole $x \in \mathfrak{z}$ of $\omega$.*

*Proof.* We can write $\eta = \eta_{\bar{z}} \wedge d\bar{z} + \eta_z \wedge dz$, where $\eta_{\bar{z}} \in \Omega^{2,0,0}(X)$ and $\eta_z \in \Omega^2(X)$. Then $\omega \wedge \eta = \omega \wedge d\bar{z} \wedge \eta_{\bar{z}}$. Since $x$ is a simple pole of $\omega$, we can write $\omega = \frac{k_0^x}{z-x} dz + \widetilde{\omega}$, where the meromorphic 1-form $\widetilde{\omega}$ on $C$ is regular at $x$. In terms of polar coordinates $z = x + re^{\mathrm{i}\theta}$ we then have $\omega \wedge d\bar{z} = -2\mathrm{i}k_0^x e^{-\mathrm{i}\theta} dr \wedge d\theta + \widetilde{\omega} \wedge d\bar{z}$, which is locally integrable near $x$ and hence so is $\omega \wedge \eta$ near $\Sigma_x \subset X$. □

*2.2. Gauge transformations.* Consider the left action of the group $C^\infty(X, G)$ on $\Omega^1(X, \mathfrak{g})$ defined by

$$\begin{aligned} C^\infty(X, G) \times \Omega^1(X, \mathfrak{g}) &\longrightarrow \Omega^1(X, \mathfrak{g}), \\ (g, A) &\longmapsto {}^g A := -dgg^{-1} + gAg^{-1}. \end{aligned} \tag{2.4}$$

Under a gauge transformation $g : A \to {}^g A$, the action (2.1) transforms as

$$S_\omega({}^g A) = S_\omega(A) + \frac{\mathrm{i}}{4\pi} \int_X \omega \wedge d\langle g^{-1} dg, A\rangle + \frac{\mathrm{i}}{4\pi} \int_X \omega \wedge g^* \chi_G, \tag{2.5}$$

where $\chi_G := \frac{1}{6}\langle \theta_G, [\theta_G, \theta_G]\rangle \in \Omega^3(G)$ is the Cartan 3-form on $G$ and $\theta_G \in \Omega^1(G, \mathfrak{g})$ denotes the left Maurer-Cartan form on $G$, so that $g^* \chi_G = \frac{1}{6}\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle$.

Define the *defect group* $G^{\boldsymbol{z}}$ and its Lie algebra $\mathfrak{g}^{\boldsymbol{z}}$ as

$$G^{\boldsymbol{z}} := \prod_{x \in \boldsymbol{z}} G, \qquad \mathfrak{g}^{\boldsymbol{z}} := \prod_{x \in \boldsymbol{z}} \mathfrak{g}.$$

We endow $\mathfrak{g}^{\boldsymbol{z}}$ with the non-degenerate invariant symmetric bilinear form

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega : \mathfrak{g}^{\boldsymbol{z}} \times \mathfrak{g}^{\boldsymbol{z}} \longrightarrow \mathbb{C}, \qquad \langle\!\langle X, Y \rangle\!\rangle_\omega := \sum_{x \in \boldsymbol{z}} k_0^x \, \langle X_x, Y_x\rangle, \tag{2.6}$$

for every $X = (X_x)_{x \in \boldsymbol{z}},\ Y = (Y_x)_{x \in \boldsymbol{z}} \in \mathfrak{g}^{\boldsymbol{z}}$, where $k_0^x \in \mathbb{C}$ is the residue of $\omega$ at $x \in \boldsymbol{z}$. For $\mathfrak{g}$-valued 1-forms on $D$ and smooth $G$-valued maps on $D$, we have the isomorphisms

$$\Omega^1(D, \mathfrak{g}) \cong \prod_{x \in \boldsymbol{z}} \Omega^1(\Sigma_x, \mathfrak{g}) \cong \Omega^1(\Sigma, \mathfrak{g}^{\boldsymbol{z}}), \tag{2.7a}$$

$$C^\infty(D, G) \cong \prod_{x \in \boldsymbol{z}} C^\infty(\Sigma_x, G) \cong C^\infty(\Sigma, G^{\boldsymbol{z}}). \tag{2.7b}$$

The pullbacks by the second embedding in (2.3) of $\mathfrak{g}$-valued 1-forms on $X$ and of smooth $G$-valued maps on $X$ can therefore be thought of as maps

$$\iota^* : \Omega^1(X, \mathfrak{g}) \longrightarrow \Omega^1(\Sigma, \mathfrak{g}^{\boldsymbol{z}}), \qquad \iota^* : C^\infty(X, G) \longrightarrow C^\infty(\Sigma, G^{\boldsymbol{z}}).$$

**Lemma 2.2.** *For any $\eta \in \Omega^2(X)$, we have*

$$\int_X \omega \wedge d\eta = 2\pi \mathrm{i} \sum_{x \in \boldsymbol{z}} k_0^x \int_{\Sigma_x} \iota_x^* \eta.$$

*Proof.* Recalling our notations and conventions at the end of Section 1, we have

$$\int_X \omega \wedge d\eta = \int_X \omega \wedge (d_\Sigma + \bar{\partial})\eta = \int_X \omega \wedge \bar{\partial}\eta - \int_X d_\Sigma(\omega \wedge \eta),$$

where we used the decomposition of the de Rham differential $d\eta = d_\Sigma \eta + \partial\eta + \bar{\partial}\eta$ and the fact that $\omega$ is the pullback along $p_C : X \to C$ of a meromorphic 1-form on $C$, hence $\omega \wedge \partial\eta = 0$ and $d_\Sigma(\omega \wedge \eta) = -\omega \wedge d_\Sigma \eta$. The second term in the equation displayed above vanishes by Stokes' theorem on $\Sigma$. The result now follows by the Cauchy-Pompeiu integral formula. $\square$

**Proposition 2.3.** *For any $g \in C^\infty(X, G)$ and $A \in \Omega^1(X, \mathfrak{g})$, we have*

$$\int_X \omega \wedge d\langle g^{-1}dg, A\rangle = 2\pi \mathrm{i} \int_\Sigma \langle\!\langle (\boldsymbol{\iota}^* g)^{-1} d_\Sigma(\boldsymbol{\iota}^* g), \boldsymbol{\iota}^* A\rangle\!\rangle_\omega.$$

*Proof.* Applying Lemma 2.2, we obtain

$$\int_X \omega \wedge d\langle g^{-1}dg, A\rangle = 2\pi \mathrm{i} \sum_{x\in z} \int_{\Sigma_x} k_0^x \left\langle (\iota_x^* g)^{-1} d_{\Sigma_x}(\iota_x^* g), \iota_x^* A\right\rangle.$$

The result follows by definition (2.6) of the bilinear form on $\mathfrak{g}^z$. □

By Proposition 2.3, the second term on the right hand side of (2.5) now manifestly depends only on the defect fields $\boldsymbol{\iota}^* g \in C^\infty(\Sigma, G^z) \cong C^\infty(D, G)$ and $\boldsymbol{\iota}^* A \in \Omega^1(\Sigma, \mathfrak{g}^z) \cong \Omega^1(D, \mathfrak{g})$. We will show in Proposition 2.8 below that the same is true for the third term on the right hand side of (2.5). To prove this, we first need to introduce further notations and techniques.

For a manifold $M$ and a closed subset $S \subset M$ with embedding $\iota : S \hookrightarrow M$, let $C^0_S(M, G)$ (resp. $C^\infty_S(M, G)$) denote the set of continuous (resp. smooth) maps $g : M \to G$ such that $\iota^* g = e$, where by abuse of notation $e$ denotes the constant map $S \to G$ to the identity element $e \in G$.

Let $I := [0, 1] \subset \mathbb{R}$ denote the closed unit interval and define the maps $j_t : M \hookrightarrow M \times I,\ p \mapsto (p, t)$, for every $t \in I$. A *relative continuous* (resp. *smooth*) *homotopy* between two maps $g, g' \in C^0_S(M, G)$ (resp. $g, g' \in C^\infty_S(M, G)$) is a map $H \in C^0_{S\times I}(M \times I, G)$ (resp. $H \in C^\infty_{S\times I}(M \times I, G)$) such that

$$j_0^* H = g, \qquad j_1^* H = g'.$$

We write $g \sim_S g'$ (resp. $g \sim^\infty_S g'$) and say that $g$ and $g'$ are homotopic relative to $S$. This defines equivalence relations $\sim_S$ on $C^0_S(M, G)$ and $\sim^\infty_S$ on $C^\infty_S(M, G)$.

**Lemma 2.4.** *The canonical map*

$$C^\infty_D(X, G)\big/{\sim^\infty_D} \longrightarrow C^0_D(X, G)\big/{\sim_D}$$

*is a bijection.*

*Proof.* Let $g, g' \in C^\infty_D(X, G)$ be such that $g \sim_D g'$. By [Lee, Theorem 6.29], it follows that $g \sim^\infty_D g'$. Hence, the given map is injective.

Now let $g \in C^0_D(X, G)$. Then $\boldsymbol{\iota}^* g = e$ is smooth, so by [Lee, Theorem 6.26] it follows that $g \sim_D g'$ for some $g' \in C^\infty_D(X, G)$. Hence, the given map is surjective. □

Recall the projection $p_C : X \to C$. For any $a \in \Sigma$, we also consider the smooth embedding $i_a : C \hookrightarrow X,\ z \mapsto (a, z)$. We have that $p_C(D) = z$ and $i_a(z) \subset D$.

**Lemma 2.5.** *For any $a \in \Sigma$, the maps*

$$C^0_D(X, G)\big/{\sim_D} \underset{p_C^*}{\overset{i_a^*}{\rightleftarrows}} C^0_z(C, G)\big/{\sim_z}$$

*exhibit a bijection.*

*Remark 2.6.* The maps $p_C^*$ and $\iota_a^*$ are well-defined. Indeed, suppose more generally that $M$, $M'$ are topological spaces with closed subsets $S \subset M$, $S' \subset M'$ and corresponding embedding maps $\iota : S \hookrightarrow M$ and $\iota' : S' \hookrightarrow M'$. Let $f : M \to M'$ be a continuous map such that $f(S) \subset S'$. Then the pullback by $f$ induces a map $f^* : C^0_{S'}(M', G) \to C^0_S(M, G)$. Indeed, if $g \in C^0_{S'}(M', G)$ then $f^* g \in C^0_S(M, G)$ since

$$\iota^* f^* g = (f \circ \iota)^* g = (\iota' \circ f|_S)^* g = f|_S^* \, \iota'^* g = e,$$

where $f|_S : S \to S'$ is the restriction of $f$ to $S \subset M$ and in the final step we used the fact that $\iota'^* g = e$. Moreover, given any relative homotopy $H \in C^0_{S' \times I}(M' \times I, G)$, we have $(f \times \mathrm{id})^* H \in C^0_{S \times I}(M \times I, G)$ since $(\iota \times \mathrm{id})^*(f \times \mathrm{id})^* H = e$ by the same computation as above. We therefore obtain a well-defined map between the relative homotopy classes $f^* : C^0_{S'}(M', G)\big/\!\sim_{S'} \to C^0_S(M, G)\big/\!\sim_S$, as required.

*Proof of Lemma 2.5.* Let $a \in \Sigma$. We have to show that $i_a^*$ and $p_C^*$ are inverses of each other. Since $p_C \circ i_a = \mathrm{id}_C$, we have $i_a^* \, p_C^* = (p_C \circ i_a)^* = \mathrm{id}$.

Consider now $i_a \circ p_C : X \to X$. We have a continuous homotopy

$$H : X \times I \longrightarrow X, \qquad (p, z, t) \longmapsto \big((1 - t)p + ta, z\big)$$

between $\mathrm{id}_X$ and $i_a \circ p_C$. Note that $H(D \times I) \subset D$, in other words $H \circ (\boldsymbol{\iota} \times \mathrm{id}) = \boldsymbol{\iota} \circ H|_{D \times I}$. For any $g \in C^0_D(X, G)$, the continuous map $g \circ H : X \times I \to G$ belongs to $C^0_{D \times I}(X \times I, G)$ since

$$(\boldsymbol{\iota} \times \mathrm{id})^*(g \circ H) = g \circ H \circ (\boldsymbol{\iota} \times \mathrm{id}) = g \circ \boldsymbol{\iota} \circ H|_{D \times I} = (\boldsymbol{\iota}^* g) \circ H|_{D \times I} = e.$$

In the final equality we used the fact that $\boldsymbol{\iota}^* g = e$ since $g \in C^0_D(X, G)$. Moreover, $j_0^*(g \circ H) = g$ and $j_1^*(g \circ H) = g \circ i_a \circ p_C = p_C^* i_a^* g$ so that $g \circ H$ is a relative continuous homotopy between $g$ and $p_C^* i_a^* g$, i.e. $p_C^* i_a^* g \sim_D g$. Hence $p_C^* i_a^* = \mathrm{id}$, as required. □

**Lemma 2.7.** *$C^0_z(C, G)\big/\!\sim_z$ is a singleton.*

*Proof.* A relative continuous homotopy $H \in C^0_{z \times I}(C \times I, G)$ between two maps $g, g' \in C^0_z(C, G)$ is a continuous path in the mapping space $\mathrm{Map}_z(C, G)$ from $g$ to $g'$. Thus

$$C^0_z(C, G)\big/\!\sim_z \;\cong\; \pi_0\big(\mathrm{Map}_z(C, G)\big).$$

Now fix any point $x \in z$. The inclusion $i : z \hookrightarrow C$ induces a continuous map

$$i^* : \mathrm{Map}_{\{x\}}(C, G) \longrightarrow \mathrm{Map}_{\{x\}}(z, G)$$

between based mapping spaces, whose fibre over the constant map $e \in \mathrm{Map}_{\{x\}}(z, G)$ is $\mathrm{Map}_z(C, G)$. Hence, we get a fibre sequence

$$\mathrm{Map}_z(C, G) \hookrightarrow \mathrm{Map}_{\{x\}}(C, G) \xrightarrow{\;i^*\;} \mathrm{Map}_{\{x\}}(z, G).$$

Since $i : z \hookrightarrow C$ is a cofibration, it follows that $i^*$ is a fibration and hence we obtain a long exact sequence of homotopy groups

$$\ldots \longrightarrow \pi_1\big(\mathrm{Map}_{\{x\}}(z, G)\big) \longrightarrow \pi_0\big(\mathrm{Map}_z(C, G)\big) \longrightarrow \pi_0\big(\mathrm{Map}_{\{x\}}(C, G)\big) \longrightarrow \ldots$$

Observe that $\pi_1(\mathrm{Map}_{\{x\}}(z, G)) \cong \pi_1(G)^{|z|-1} \cong \{*\}$ is trivial since $G$ is assumed to be simply connected. To compute $\pi_0(\mathrm{Map}_{\{x\}}(C, G))$, we recall that $C = \mathbb{C}P^1 \setminus \boldsymbol{\zeta}$ is topologically a 2-sphere $S^2$ with $|\boldsymbol{\zeta}| \geq 1$ punctures. Hence, there exists a deformation retract from $C$ to a bouquet of circles $\bigvee^{|\boldsymbol{\zeta}|-1} S^1$, where $\vee$ denotes the wedge sum (i.e. categorical coproduct) of pointed topological spaces. It then follows that

$$\pi_0\big(\mathrm{Map}_{\{x\}}(C, G)\big) \cong \pi_0\big(\mathrm{Map}_{\{x\}}(S^1, G)\big)^{|\boldsymbol{\zeta}|-1} \cong \pi_1(G)^{|\boldsymbol{\zeta}|-1} \cong \{*\}$$

is trivial since $G$ is assumed to be simply connected. From the long exact sequence we conclude that $\pi_0(\mathrm{Map}_{\boldsymbol{z}}(C, G))$ is a singleton, which completes the proof. □

**Proposition 2.8.** *The integral $\int_X \omega \wedge g^*\chi_G$ depends on $g \in C^\infty(X, G)$ only through $\boldsymbol{\iota}^* g \in C^\infty(\Sigma, G^{\boldsymbol{z}})$.*

*Remark 2.9.* The present situation is to be contrasted with the usual WZ-term in the WZW model action. Indeed, to even write the latter down one has to extend a field $g \in C^\infty(S^2, G)$ to a field $\widetilde{g} \in C^\infty(B^3, G)$ on the 3-dimensional ball $B^3$ with $\partial B^3 = S^2$. This is possible as $\pi_2(G) = 0$ but the extension $\widetilde{g}$ is not unique. The set of homotopy classes of smooth maps $\widetilde{g} \in C^\infty(B^3, G)$ with $\widetilde{g}|_{S^2} = g$ is measured by $\pi_3(G)$, which for a simple Lie group $G$ is given by $\pi_3(G) \cong \mathbb{Z}$. For the extensions $\widetilde{g}$ in different homotopy classes, the integrals $\int_{B^3} \widetilde{g}^*\chi_G$ differ by integer multiples of a constant.

*Proof of Proposition 2.8.* For any $g, h \in C^\infty(X, G)$, we have the Polyakov-Wiegmann identity

$$(gh^{-1})^*\chi_G = g^*\chi_G - h^*\chi_G + d\langle g^{-1}dg, h^{-1}dh\rangle.$$

Using Lemma 2.2 and the definition of the bilinear form (2.6) on $\mathfrak{g}^{\boldsymbol{z}}$, we find

$$\begin{aligned} \int_X \omega \wedge d\langle g^{-1}dg, h^{-1}dh\rangle &= 2\pi\mathrm{i} \sum_{x\in \boldsymbol{z}} \int_{\Sigma_x} k_0^x \left\langle (\iota_x^* g)^{-1} d_{\Sigma_x}(\iota_x^* g), (\iota_x^* h)^{-1} d_{\Sigma_x}(\iota_x^* h)\right\rangle \\ &= 2\pi\mathrm{i} \int_\Sigma \langle\!\langle (\boldsymbol{\iota}^* g)^{-1} d_\Sigma(\boldsymbol{\iota}^* g), (\boldsymbol{\iota}^* h)^{-1} d_\Sigma(\boldsymbol{\iota}^* h)\rangle\!\rangle_\omega. \end{aligned}$$

In particular, if $\boldsymbol{\iota}^* g = \boldsymbol{\iota}^* h$ then this vanishes by the skew-symmetry of the bilinear pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\omega : \Omega^1(\Sigma, \mathfrak{g}^{\boldsymbol{z}}) \times \Omega^1(\Sigma, \mathfrak{g}^{\boldsymbol{z}}) \to \Omega^2(\Sigma)$ on 1-forms. It follows that

$$\int_X \omega \wedge (gh^{-1})^*\chi_G = \int_X \omega \wedge g^*\chi_G - \int_X \omega \wedge h^*\chi_G \tag{2.8}$$

for any $g, h \in C^\infty(X, G)$ such that $\boldsymbol{\iota}^* g = \boldsymbol{\iota}^* h$.

The latter condition can be equivalently stated as $\boldsymbol{\iota}^*(gh^{-1}) = e$, or in other words $gh^{-1} \in C^\infty_D(X, G)$. By Lemmas 2.4, 2.5 and 2.7 we deduce that $C^\infty_D(X, G)\big/\!\sim^\infty_D$ is a singleton. Hence, there exists a relative smooth homotopy $H \in C^\infty_{D\times I}(X \times I, G)$ between $gh^{-1}$ and $e \in C^\infty_D(X, G)$, i.e. $j_0^* H = gh^{-1}$ and $j_1^* H = e$. Then

$$\begin{aligned} d\bigg(\int_I H^*\chi_G\bigg) &= \int_I dH^*\chi_G = \int_I \big(d_{X\times I} - d_I\big) H^*\chi_G \\ &= -\int_I d_I H^*\chi_G = j_0^* H^*\chi_G - j_1^* H^*\chi_G = (gh^{-1})^*\chi_G, \end{aligned} \tag{2.9}$$

where in the second step $d_{X\times I} = d + d_I$ is the differential on $\Omega^\bullet(X \times I)$. In the third equality we used the fact that $H^*\chi_G \in \Omega^3(X \times I)$ is closed, i.e. $d_{X\times I} H^*\chi_G = 0$, and in the second last step we used Stokes' theorem. In the final step we used the fact that $e^*\chi_G = 0$.

Taking the wedge product of (2.9) with $\omega$ and integrating over $X$ we obtain, using again Lemma 2.2,

$$\begin{aligned}\int_X \omega \wedge (gh^{-1})^*\chi_G &= \int_X \omega \wedge d\bigg(\int_I H^*\chi_G\bigg)\\ &= 2\pi\mathrm{i}\sum_{x\in\mathbf{z}} k_0^x \int_{\Sigma_x\times I} (\iota_x \times \mathrm{id})^* H^*\chi_G = 0. \end{aligned} \tag{2.10}$$

In the last equality we used the fact that $(\iota_x \times \mathrm{id})^* H = e \in C^\infty(\Sigma_x \times I, G)$, for every $x \in \mathbf{z}$, and again that $e^*\chi_G = 0$. Finally, by combining (2.10) with (2.8), it follows that $\int_X \omega \wedge g^*\chi_G = \int_X \omega \wedge h^*\chi_G$ for any $g, h \in C^\infty(X, G)$ such that $\boldsymbol{\iota}^*g = \boldsymbol{\iota}^*h$. This completes the proof. □

Recall the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega$ on the Lie algebra $\mathfrak{g}^{\mathbf{z}}$ introduced in (2.6). We let $\chi_{G^{\mathbf{z}}} := \frac{1}{6}\langle\!\langle \theta_{G^{\mathbf{z}}}, [\theta_{G^{\mathbf{z}}}, \theta_{G^{\mathbf{z}}}]\rangle\!\rangle_\omega \in \Omega^3(G^{\mathbf{z}})$ denote the corresponding Cartan 3-form on $G^{\mathbf{z}}$, where $\theta_{G^{\mathbf{z}}} \in \Omega^1(G^{\mathbf{z}}, \mathfrak{g}^{\mathbf{z}})$ is the left Maurer-Cartan form on $G^{\mathbf{z}}$. Since $\Omega^1(G^{\mathbf{z}}, \mathfrak{g}^{\mathbf{z}}) \cong \prod_{x\in\mathbf{z}} \Omega^1(G^{\mathbf{z}}, \mathfrak{g})$, we can express $\theta_{G^{\mathbf{z}}}$ as a tuple $(\theta_G^x)_{x\in\mathbf{z}}$ of $\mathfrak{g}$-valued 1-forms on $G^{\mathbf{z}}$. Here, for each $x \in \mathbf{z}$, $\theta_G^x = \pi_x^*\theta_G \in \Omega^1(G^{\mathbf{z}}, \mathfrak{g})$ is the pullback of the left Maurer-Cartan form $\theta_G$ on $G$ along the canonical projection $\pi_x : G^{\mathbf{z}} \to G$ onto the $x$-factor of $G^{\mathbf{z}}$. It then also follows that $\chi_{G^{\mathbf{z}}} = \sum_{x\in\mathbf{z}} k_0^x\, \chi_G^x$, where $\chi_G^x := \frac{1}{6}\langle \theta_G^x, [\theta_G^x, \theta_G^x]\rangle = \pi_x^*\chi_G \in \Omega^3(G^{\mathbf{z}})$.

**Proposition 2.10.** *For any $g \in C^\infty(X, G)$, we have*

$$\int_X \omega \wedge g^*\chi_G = -2\pi\mathrm{i}\int_{\Sigma\times I} \widehat{g}^*\chi_{G^{\mathbf{z}}},$$

*where $\widehat{g} \in C^\infty(\Sigma \times I, G^{\mathbf{z}})$ is any lazy homotopy between $\boldsymbol{\iota}^*g \in C^\infty(\Sigma, G^{\mathbf{z}})$ and the constant map $e \in C^\infty(\Sigma, G^{\mathbf{z}})$.*

*Proof.* First we note that since $\Sigma = \mathbb{R}^2$ is contractible and $G^{\mathbf{z}}$ is connected, as $G$ is, there exists a lazy homotopy $\widehat{g} \in C^\infty(\Sigma \times I, G^{\mathbf{z}})$ between $\boldsymbol{\iota}^*g$ and $e$, namely such that $\widehat{g}(-, t) = \boldsymbol{\iota}^*g$ for $t$ near 0 and $\widehat{g}(-, t) = e$ for $t$ near 1.

Let us denote by $\Delta$ the unit disc and by $\varrho : \Delta \to I$ the radial coordinate. Let $\Delta_x \subset C$ be disjoint discs around each $x \in \mathbf{z}$. We then have the following isomorphism

$$C^\infty\bigg(\bigsqcup_{x\in\mathbf{z}} \Sigma \times \Delta_x, G\bigg) \cong \prod_{x\in\mathbf{z}} C^\infty(\Sigma \times \Delta_x, G) \cong C^\infty(\Sigma \times \Delta, G^{\mathbf{z}}).$$

Consider $(\mathrm{id}_\Sigma \times \varrho)^*\widehat{g} \in C^\infty(\Sigma \times \Delta, G^{\mathbf{z}})$, regard it as a smooth map $\bigsqcup_{x\in\mathbf{z}} \Sigma \times \Delta_x \to G$ under the above isomorphism and extend the latter to the whole of $X$ by the identity $e \in G$. By construction, this defines a smooth map $\widetilde{g} \in C^\infty(X, G)$ such that $\boldsymbol{\iota}^*\widetilde{g} = \boldsymbol{\iota}^*g \in C^\infty(\Sigma, G^{\mathbf{z}})$. (Note that $\widetilde{g}$ is smooth because $\widehat{g}$ is lazy.) By Proposition 2.8, we deduce

$$\int_X \omega \wedge g^*\chi_G = \int_X \omega \wedge \widetilde{g}^*\chi_G.$$

It remains to compute the integral on the right hand side. This can be done directly as in [DLMV2, §3.3]. In view of generalising this computation to the higher order pole case later in Sect. 3, it is useful to repeat it in the present language. We find

$$\int_X \omega \wedge \widetilde{g}^* \chi_G = \sum_{x\in z} \int_{\Sigma\times\Delta_x} \omega \wedge \widetilde{g}^* \chi_G = \sum_{x\in z} \int_{\Sigma\times\Delta_x} \omega \wedge d\Big( - \int_{\gamma_z} \widetilde{g}^* \chi_G \Big)$$
$$= -2\pi\mathrm{i} \sum_{x\in z} k_0^x \int_{\Sigma_x} \int_{\gamma_x} \widetilde{g}^* \chi_G = -2\pi\mathrm{i} \sum_{x\in z} k_0^x \int_{\Sigma\times I} \widehat{g}^* \chi_G^x = -2\pi\mathrm{i} \int_{\Sigma\times I} \widehat{g}^* \chi_{G^z}.$$

The first equality follows from noting that $\widetilde{g}^* \chi_G$ vanishes outside of $\bigsqcup_{x\in z} \Sigma\times\Delta_x \subset X$. In the second step, we used the fact that $\widetilde{g}^* \chi_G$ is closed, hence exact on the contractible subspaces $\Sigma\times\Delta_x \subset X$. In particular, the value of an explicit primitive $-\int_{\gamma_z} \widetilde{g}^* \chi_G$ at the point $(p, z) \in \Sigma\times\Delta_x$ is given by the integral along a radial path $\gamma_z : I \to \Delta_x$ from $(p, z)$ to a point $(p, z_0)$ lying on the boundary of $\Sigma\times\Delta_x$. In the third equality we used Lemma 2.2. In the second last step we used the identification of $\Sigma_x \times \gamma_x(I)$ with $\Sigma\times I$ and that of $\widetilde{g} : \Sigma_x \times \gamma_x(I) \to G$ with $\pi_x \circ \widehat{g} : \Sigma\times I \to G$. The last equality follows from the identity $\chi_{G^z} = \sum_{x\in z} k_0^x \, \chi_G^x$. □

## 3. Higher Order Poles in $\omega$

We would like to extend the constructions of Sect. 2 to the case when the meromorphic 1-form $\omega$ has higher order poles. The immediate problem we face is that $\omega \wedge \mathrm{CS}(A)$ is not locally integrable around such a higher order pole $x$. We will therefore begin by introducing a regularisation of the action (2.1). A closely related approach to the regularisation of integrals on Riemann surfaces appeared in [LZ] shortly after the first version of this paper became available.

*3.1. Regularised action.* Let $n := \max\, \{n_x\}_{x\in z}$ denote the maximal order among all the poles of $\omega$. Consider the Weil algebra $\mathcal{T}^n := \mathbb{C}[\varepsilon]/(\varepsilon^n)$ of order $n$. (If $n_x = 1$ for all $x \in z$ then $n = 1$ and hence $\mathcal{T}^n \cong \mathbb{C}$.)

For each $\mathcal{T}^n$-valued $r$-form $\zeta = \sum_{p=0}^{n-1} \zeta_p \otimes \varepsilon^p \in \Omega^r(X) \otimes_{\mathbb{C}} \mathcal{T}^n$, we define the *regularised wedge product* with $\omega$ (cf. (1.1)) as

$$(\omega \wedge \zeta)_{\mathrm{reg}} := \sum_{x\in z} \sum_{p=0}^{n_x-1} \frac{k_p^x \, dz}{z - x} \wedge \zeta_p \in \Omega^{r+1}(X \setminus D), \tag{3.1}$$

where $D = \bigsqcup_{x\in z} \Sigma_x \subset X$ is defined in (2.2) as the disjoint union of the surface defects $\Sigma_x = \Sigma \times \{x\}$. As a consequence of Lemma 2.1, we obtain

**Corollary 3.1.** *For any $\zeta \in \Omega^3(X) \otimes_{\mathbb{C}} \mathcal{T}^n$, the 4-form $(\omega\wedge\zeta)_{\mathrm{reg}} \in \Omega^4(X \setminus D)$ is locally integrable near $D$.*

We have the morphism of vector spaces (or $C^\infty$-rings in the case $r = 0$)

$$j_X^* : \Omega^r(X) \longrightarrow \Omega^r(X) \otimes_{\mathbb{C}} \mathcal{T}^n, \qquad \eta \longmapsto \sum_{p=0}^{n-1} \frac{1}{p!} \partial_z^p \eta \otimes \varepsilon^p, \tag{3.2}$$

given by the holomorphic part of the $(n-1)$-jet prolongation of smooth $r$-forms on $X$, for any $r = 0, \ldots, 4$. The regularised wedge product (3.1) can be related as follows to the ordinary wedge product.

**Lemma 3.2.** *For any $\eta \in \Omega^3(X)$, we have a decomposition*

$$\omega \wedge \eta = (\omega \wedge j_X^* \eta)_{\text{reg}} + d\psi,$$

*where $\psi \in \Omega^3(X \setminus D)$ is singular on $\Sigma_x$ for $x \in z$ if $n_x > 1$ and $\psi = 0$ if $n = 1$.*

*Proof.* We can rewrite (1.1) as

$$\omega = \sum_{x \in z} \sum_{p=0}^{n_x - 1} \frac{(-1)^p}{p!} \partial_z^p \left( \frac{k_p^x}{z - x} \right) dz.$$

Taking the wedge product with $\eta$, it then follows from the Leibniz rule that

$$\omega \wedge \eta = \sum_{x \in z} \sum_{p=0}^{n_x - 1} \sum_{r=0}^{p} \frac{(-1)^{p-r}}{r!(p-r)!} dz \wedge \partial_z^{p-r} \left( \frac{k_p^x}{z - x} \partial_z^r \eta \right).$$

The terms with $r = p$ yield $(\omega \wedge j_X^* \eta)_{\text{reg}}$. All of the remaining terms in the sum over $r$ can be written as the de Rham differential of

$$\psi = \sum_{x \in z} \sum_{p=0}^{n_x - 1} \sum_{r=0}^{p-1} \frac{(-1)^{p-r}}{r!(p-r)!} \partial_z^{p-1-r} \left( \frac{k_p^x}{z - x} \partial_z^r \eta^{2,0,1} \right),$$

where $\eta^{2,0,1} \in \Omega^{2,0,1}(X)$ denotes the $(2, 0, 1)$-component of $\eta \in \Omega^3(X)$ with respect to the grading in (1.2). This $\psi$ is singular on $\Sigma_x$ if $n_x > 1$ and vanishes if $n_x = 1$ for all $x \in z$. □

In the case when $\omega$ has higher order poles, Lemma 3.2 and Corollary 3.1 motivate the following definition of the *regularised action*

$$S_\omega(A) := \frac{\mathrm{i}}{4\pi} \int_X \big( \omega \wedge j_X^* \mathrm{CS}(A) \big)_{\text{reg}}. \tag{3.3}$$

This reduces to the action (2.1) of [CY] in the case when $\omega$ only has simple poles.

*3.2. Gauge transformations.* Under a gauge transformation $g : A \to {}^g A$ as in (2.4), the regularised action (3.3) transforms as (cf. (2.5))

$$S_\omega({}^g A) = S_\omega(A) + \frac{\mathrm{i}}{4\pi} \int_X \big( \omega \wedge j_X^* d\langle g^{-1} dg, A \rangle \big)_{\text{reg}} + \frac{\mathrm{i}}{4\pi} \int_X \big( \omega \wedge j_X^* (g^* \chi_G) \big)_{\text{reg}}, \tag{3.4}$$

where the Cartan 3-form $\chi_G \in \Omega^3(G)$ on $G$ was defined in Sect. 2.2.

Consider the Weil algebra $\mathcal{T}_x^{n_x} := \mathbb{C}[\varepsilon_x]/(\varepsilon_x^{n_x})$ of order $n_x$, the order of the pole $x \in z$ of $\omega$. (Note that for a simple pole $n_x = 1$ and thus $\mathcal{T}_x^{n_x} \cong \mathbb{C}$.) We denote by $\ell \mathcal{T}_x^{n_x}$ the locus of the Weil algebra, which is a formal manifold in the context of synthetic differential geometry [Koc]. Loosely speaking, one should think of $\ell \mathcal{T}_x^{n_x}$ as an infinitesimal thickening of the point $x \in z$. In the present setting, the surface defects $\Sigma_x$ of Sect. 2.1 are replaced by formal manifolds

$$\Sigma_x^{n_x} := \Sigma \times \ell \mathcal{T}_x^{n_x},$$

for each $x \in z$. The disjoint union of the surface defects $\Sigma_x$ in (2.2) is then replaced by the disjoint union of their infinitesimal thickenings $\Sigma_x^{n_x}$, namely

$$\widehat{D} := \bigsqcup_{x \in z} \Sigma_x^{n_x}.$$

For each $x \in z$, we have a morphism of $C^\infty$-rings

$$j_x^* : C^\infty(X) \longrightarrow C^\infty(\Sigma_x) \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}, \qquad f \longmapsto \sum_{p=0}^{n_x - 1} \frac{1}{p!} \iota_x^*(\partial_z^p f) \otimes \varepsilon_x^p, \tag{3.5a}$$

given by pulling back along $\iota_x : \Sigma_x \to X$ the holomorphic part of the $(n_x - 1)$-jet prolongation of the smooth function $f$. It defines a morphism of formal manifolds

$$j_x : \Sigma_x^{n_x} \longrightarrow X. \tag{3.5b}$$

The canonical induced morphism

$$\boldsymbol{j}^* : C^\infty(X) \to \prod_{x \in z} C^\infty(\Sigma_x) \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x} \tag{3.6a}$$

to the product of $C^\infty$-rings defines a morphism of formal manifolds

$$\boldsymbol{j} : \widehat{D} \longrightarrow X. \tag{3.6b}$$

The pair of morphisms (3.5b) and (3.6b) play an analogous role to the embeddings (2.3) in the higher pole case.

We generalise the definition of the defect group $G^z$ and its Lie algebra $\mathfrak{g}^z$ from Sect. 2.2 to the case of higher order poles as follows. Recall that, for each $k \geq 1$, the mapping space $C^\infty(\ell\mathcal{T}^k, M)$ from $\ell\mathcal{T}^k$ to a manifold $M$ is a manifold, namely the total space of the bundle of $(k-1)$-jets of curves in $M$. We define the *defect group* $G^{\widehat{z}}$ and its Lie algebra $\mathfrak{g}^{\widehat{z}}$ as

$$G^{\widehat{z}} := \prod_{x \in z} C^\infty(\ell\mathcal{T}_x^{n_x}, G), \qquad \mathfrak{g}^{\widehat{z}} := \prod_{x \in z} \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}. \tag{3.7}$$

Since $G \subseteq \mathrm{GL}_N(\mathbb{C})$ is assumed to be a matrix Lie group, the defect group $G^{\widehat{z}}$ admits a presentation as a subgroup of the product $\prod_{x \in z} \mathrm{GL}_N(\mathcal{T}_x^{n_x})$ of general linear groups with entries in the Weil algebras $\mathcal{T}_x^{n_x}$.

We endow $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}$ with the non-degenerate invariant symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \big(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}\big) \times \big(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}\big) \longrightarrow \mathbb{C}, \qquad \langle X \otimes \varepsilon_x^p, Y \otimes \varepsilon_x^q \rangle := k_{p+q}^x \langle X, Y \rangle. \tag{3.8}$$

Non-degeneracy follows from the fact that $k_{n_x - 1}^x \neq 0$, by definition of $n_x$. This then extends to a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}^{\widehat{z}}$, which we denote by

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega : \mathfrak{g}^{\widehat{z}} \times \mathfrak{g}^{\widehat{z}} \longrightarrow \mathbb{C}. \tag{3.9}$$

In the case when $\omega$ has only simple poles this definition reduces to (2.6).

We have the isomorphisms

$$\Omega^1(\widehat{D}, \mathfrak{g}) := \prod_{x \in z} \Omega^1(\Sigma, \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{T}_x^{n_x}) \cong \Omega^1(\Sigma, \mathfrak{g}^{\widehat{z}}), \tag{3.10a}$$

$$C^\infty(\widehat{D}, G) := \prod_{x \in z} C^\infty\big(\Sigma, C^\infty(\ell\mathcal{T}_x^{n_x}, G)\big) \cong C^\infty(\Sigma, G^{\widehat{z}}). \tag{3.10b}$$

By virtue of the isomorphism (3.10a), the pullback of smooth $\mathfrak{g}$-valued 1-forms on $X$ by the morphism (3.6b) induces a map, cf. (3.5a),

$$\boldsymbol{j}^* : \Omega^r(X, \mathfrak{g}) \longrightarrow \Omega^r(\Sigma, \mathfrak{g}^{\widehat{z}}), \qquad \eta \longmapsto \left( \sum_{p=0}^{n_x-1} \frac{1}{p!} \iota_x^*(\partial_z^p \eta) \otimes \varepsilon_x^p \right)_{x \in z}, \tag{3.11}$$

for each $r = 0, \ldots, 4$. Likewise, the pullback of smooth $G$-valued maps on $X$ by (3.6b) induces a map

$$\boldsymbol{j}^* : C^\infty(X, G) \longrightarrow C^\infty(\Sigma, G^{\widehat{z}}), \qquad g \longmapsto \left( \sum_{p=0}^{n_x-1} \frac{1}{p!} \iota_x^*(\partial_z^p g) \otimes \varepsilon_x^p \right)_{x \in z}, \tag{3.12}$$

where the presentation $G^{\widehat{z}} \subseteq \prod_{x \in z} \mathrm{GL}_N(\mathcal{T}_x^{n_x})$ as a matrix Lie group is understood. Using the Leibniz rule, one easily proves that $\boldsymbol{j}^*$ is a group homomorphism, i.e.

$$\boldsymbol{j}^*(g' g) = (\boldsymbol{j}^* g')\,(\boldsymbol{j}^* g), \tag{3.13}$$

for all $g, g' \in C^\infty(X, G)$.

The following result extends Lemma 2.2 to the case of higher order poles.

**Lemma 3.3.** *For any $\zeta = \sum_{p=0}^{n-1} \zeta_p \otimes \varepsilon^p \in \Omega^2(X) \otimes_{\mathbb{C}} \mathcal{T}^n$, we have*

$$\int_X (\omega \wedge d\zeta)_{\mathrm{reg}} = 2\pi \mathrm{i} \sum_{x \in z} \sum_{p=0}^{n_x-1} k_p^x \int_{\Sigma_x} \iota_x^* \zeta_p.$$

*Proof.* Since $d\zeta = \sum_{p=0}^{n-1} d\zeta_p \otimes \varepsilon^p$, using the definition (3.1) we find

$$\int_X (\omega \wedge d\zeta)_{\mathrm{reg}} = \sum_{x \in z} \sum_{p=0}^{n_x-1} \int_X \frac{k_p^x\, dz}{z - x} \wedge d\zeta_p = 2\pi \mathrm{i} \sum_{x \in z} \sum_{p=0}^{n_x-1} k_p^x \int_{\Sigma_x} \iota_x^* \zeta_p,$$

where in the second equality we used Lemma 2.2. □

We may now rewrite the second term on the right hand side of (3.4) as follows.

**Proposition 3.4.** *For any $g \in C^\infty(X, G)$ and $A \in \Omega^1(X, \mathfrak{g})$, we have*

$$\int_X \big(\omega \wedge j_X^* d\langle g^{-1} dg, A\rangle\big)_{\mathrm{reg}} = 2\pi \mathrm{i} \int_\Sigma \langle\!\langle (\boldsymbol{j}^* g)^{-1} d_\Sigma (\boldsymbol{j}^* g), \boldsymbol{j}^* A \rangle\!\rangle_\omega.$$

*Proof.* It follows from Lemma 3.3 that

$$\int_X \big(\omega \wedge j_X^* d\langle g^{-1}dg, A\rangle\big)_{\rm reg} = 2\pi\mathrm{i} \sum_{x\in z} \sum_{p=0}^{n_x-1} k_p^x \int_{\Sigma_x} \iota_x^*\bigg(\frac{1}{p!}\partial_z^p \big\langle g^{-1}dg, A\big\rangle\bigg).$$

Applying the Leibniz rule to the right hand side, we find

$$\begin{aligned}
&2\pi\mathrm{i} \sum_{x\in z} \sum_{p=0}^{n_x-1} \sum_{r=0}^{p} \int_{\Sigma_x} k_p^x \Big\langle \frac{1}{r!}\iota_x^*\big(\partial_z^r(g^{-1}dg)\big), \frac{1}{(p-r)!}\iota_x^*\big(\partial_z^{p-r}A\big)\Big\rangle \\
&= 2\pi\mathrm{i} \sum_{x\in z} \sum_{p=0}^{n_x-1} \sum_{r=0}^{p} \int_{\Sigma_x} \Big\langle \frac{1}{r!}\iota_x^*\big(\partial_z^r(g^{-1}dg)\big)\otimes \varepsilon_x^r, \frac{1}{(p-r)!}\iota_x^*\big(\partial_z^{p-r}A\big)\otimes \varepsilon_x^{p-r}\Big\rangle \\
&= 2\pi\mathrm{i} \int_\Sigma \langle\!\langle \boldsymbol{j}^*(g^{-1}dg), \boldsymbol{j}^*A\rangle\!\rangle_\omega = 2\pi\mathrm{i}\int_\Sigma \langle\!\langle (\boldsymbol{j}^*g)^{-1}d_\Sigma(\boldsymbol{j}^*g), \boldsymbol{j}^*A\rangle\!\rangle_\omega,
\end{aligned}$$

where in the first equality we used the definition of the bilinear form (3.8) on $\mathfrak{g}\otimes_{\mathbb{C}}\mathcal{T}_x^{n_x}$ and in the second equality the definition of the bilinear form (3.9). The last equality follows from the identity $\boldsymbol{j}^*(g^{-1}dg) = (\boldsymbol{j}^*g)^{-1}d_\Sigma(\boldsymbol{j}^*g)$, which is proved similarly to (3.13) by a simple Leibniz rule argument. □

We now turn to the third term on the right hand side of (3.4), which requires some preparation. We denote by $\widehat{G} := C^\infty(\ell\mathcal{T}^n, G)$ the mapping space from the locus of the Weil algebra $\mathcal{T}^n = \mathbb{C}[\varepsilon]/(\varepsilon^n)$ to $G$, where we recall that $n = \max\,\{n_x\}_{x\in z}$ is the maximal order of all poles. Note that $\widehat{G}$ is the Lie group of $(n-1)$-jets of curves in $G$ and that its Lie algebra is $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes_{\mathbb{C}}\mathcal{T}^n$. Analogously to (3.2), we introduce the map

$$j_X^* : C^\infty(X,G) \longrightarrow C^\infty(X,\widehat{G}), \qquad g \longmapsto \sum_{p=0}^{n-1}\frac{1}{p!}\partial_z^p g\otimes\varepsilon^p,$$

which describes the holomorphic part of the $(n-1)$-jet prolongation of smooth $G$-valued maps on $X$. Using the Leibniz rule, one easily proves that $j_X^*$ is a group homomorphism, i.e. $j_X^*(g'\,g) = (j_X^*g')\,(j_X^*g)$, for all $g, g' \in C^\infty(X,G)$. Using again the Leibniz rule, one further shows that the $\mathcal{T}^n$-valued form $j_X^*(g^*\chi_G) \in \Omega^3(X)\otimes_{\mathbb{C}}\mathcal{T}^n$ in (3.4) can be expressed as

$$j_X^*(g^*\chi_G) = (j_X^*g)^*\overline{\chi}_{\widehat{G}}, \tag{3.14}$$

where $\overline{\chi}_{\widehat{G}} := \frac{1}{6}\langle\theta_{\widehat{G}}, [\theta_{\widehat{G}},\theta_{\widehat{G}}]\rangle \in \Omega^3(\widehat{G})\otimes_{\mathbb{C}}\mathcal{T}^n$ is the $\mathcal{T}^n$-valued Cartan 3-form defined by the $\mathcal{T}^n$-bilinear extension $\langle\cdot,\cdot\rangle : \widehat{\mathfrak{g}}\times\widehat{\mathfrak{g}}\to\mathcal{T}^n$ of the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$.

The generalisation of Proposition 2.8 to the higher pole case requires a suitable modification of the Lemmas 2.4, 2.5 and 2.7 to maps with values in the jet group $\widehat{G} = C^\infty(\ell\mathcal{T}^n, G)$. Let us start by highlighting the commutative diagram

$$\begin{tikzcd}
C^\infty(X,G) \arrow[r, "j_X^*"] \arrow[d, "\boldsymbol{j}^*"'] & C^\infty(X,\widehat{G}) \arrow[d, "\boldsymbol{\iota}^*"] \\
C^\infty(\Sigma, G^{\widehat{z}}) & C^\infty(\Sigma,\widehat{G}^z) \arrow[l, "\mathrm{trunc}"]
\end{tikzcd} \tag{3.15}$$

where the map trunc is given by post-composition with the map

$$\widehat{G}^{z} = \prod_{x\in z} C^{\infty}(\ell\mathcal{T}^{n}, G) \longrightarrow \prod_{x\in z} C^{\infty}(\ell\mathcal{T}_x^{n_x}, G) = G^{\widehat{z}} \tag{3.16}$$

that truncates the orders of jets. (Recall that by definition $n_x \leq n$, for all $x \in z$.) We generalise the concepts of relative maps and relative homotopies from Sect. 2.2 by

$$C^{\infty}_{\widehat{D}}(X, \widehat{G}) := \big\{ g \in C^{\infty}(X, \widehat{G}) \ : \ \mathrm{trunc}\, \boldsymbol{\iota}^{*} g = e \big\} \tag{3.17a}$$

and

$$C^{\infty}_{\widehat{D}\times I}(X\times I, \widehat{G}) := \big\{ H \in C^{\infty}(X\times I, \widehat{G}) \ : \ \mathrm{trunc}\, (\boldsymbol{\iota}\times \mathrm{id})^{*} H = e \big\}, \tag{3.17b}$$

where $I = [0,1]$ is the unit interval. We denote by $C^{\infty}_{\widehat{D}}(X, \widehat{G})\big/{\sim}^{\infty}_{\widehat{D}}$ the corresponding set of homotopy classes.

**Lemma 3.5.** $C^{\infty}_{\widehat{D}}(X, \widehat{G})\big/{\sim}^{\infty}_{\widehat{D}}$ *is a singleton.*

*Proof.* We recall from [Viz] that there exists, for each $k \geq 1$, a diffeomorphism $C^{\infty}(\ell\mathcal{T}^{k}, G) \cong G \times \mathfrak{g}^{k-1}$ between the $(k-1)$-jet group and a Cartesian product of $G$ with $k-1$ copies of the Lie algebra $\mathfrak{g}$. Under these diffeomorphisms, the maps $\widehat{G} = C^{\infty}(\ell\mathcal{T}^{n}, G) \to C^{\infty}(\ell\mathcal{T}_x^{n_x}, G)$ truncating the jet orders are given by projection maps $G\times\mathfrak{g}^{n-1} \to G\times\mathfrak{g}^{n_x-1}$ onto the first $n_x$ factors. From the universal property of products and the definition (3.17), one obtains that

$$C^{\infty}_{\widehat{D}}(X, \widehat{G})\big/{\sim}^{\infty}_{\widehat{D}} \ \cong \ C^{\infty}_{D}(X, G)\big/{\sim}^{\infty}_{D} \times \prod_{i=1}^{n-1} C^{\infty}_{D_i}(X, \mathfrak{g})\big/{\sim}^{\infty}_{D_i} \tag{3.18}$$

is a product of sets of relative homotopy classes of maps as in Sect. 2.2, where $D = \bigsqcup_{x\in z} \Sigma_x$ is the non-thickened defect and

$$D_i := \bigsqcup_{x\in z\,:\, n_x - 1 \geq i} \Sigma_x$$

is the disjoint union of the non-thickened connected components of the defect $\widehat{D}$ that support $i$-jet data, for $i = 1, \dots, n-1$. By the same arguments as in the proofs of Lemmas 2.4, 2.5 and 2.7, one shows that each factor on the right hand side of (3.18) is a singleton. Hence, their product is a singleton too. □

The following result is the generalisation of Proposition 2.8 to the case of higher order poles.

**Proposition 3.6.** *The integral* $\int_X \big(\omega\wedge\widetilde{g}^{*}\overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}}$ *depends on* $\widetilde{g} \in C^{\infty}(X, \widehat{G})$ *only through* $\mathrm{trunc}\, \boldsymbol{\iota}^{*}\widetilde{g} \in C^{\infty}(\Sigma, G^{\widehat{z}})$. *In particular,* $\int_X \big(\omega\wedge j_X^{*}(g^{*}\chi_G)\big)_{\mathrm{reg}}$ *depends on* $g \in C^{\infty}(X, G)$ *only through* $\boldsymbol{j}^{*} g \in C^{\infty}(\Sigma, G^{\widehat{z}})$.

*Proof.* This is very similar to the proof of Proposition 2.8. We refer to the latter for certain details, highlighting only the parts of the proof which are different in the present higher order pole setting.

Let $\widetilde{g}, \widetilde{h} \in C^\infty(X, \widehat{G})$ be such that $\mathrm{trunc}\, \boldsymbol{\iota}^*\widetilde{g} = \mathrm{trunc}\, \boldsymbol{\iota}^*\widetilde{h}$. From the Polyakov-Wiegmann identity and an argument as in the proof of Proposition 3.4, we obtain

$$\int_X \big(\omega \wedge \widetilde{g}^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} - \int_X \big(\omega \wedge \widetilde{h}^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} = \int_X \big(\omega \wedge (\widetilde{g}\widetilde{h}^{-1})^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}}.$$

It remains to prove that the right hand side of this equation vanishes, provided that $\mathrm{trunc}\, \boldsymbol{\iota}^*\widetilde{g} = \mathrm{trunc}\, \boldsymbol{\iota}^*\widetilde{h}$, which by (3.17a) is equivalent to $\widetilde{g}\widetilde{h}^{-1} \in C^\infty_{\widehat{D}}(X, \widehat{G})$. It follows from Lemma 3.5 that there exists a homotopy $H \in C^\infty_{\widehat{D}\times I}(X \times I, \widehat{G})$ between $\widetilde{g}\widetilde{h}^{-1}$ and $e \in C^\infty_{\widehat{D}}(X, \widehat{G})$. We deduce that

$$(\widetilde{g}\widetilde{h}^{-1})^* \overline{\chi}_{\widehat{G}} = d\bigg(\int_I H^* \overline{\chi}_{\widehat{G}}\bigg)$$

by the same line of arguments as in (2.9). It then follows by using Lemma 3.3 that

$$\int_X \big(\omega \wedge (\widetilde{g}\widetilde{h}^{-1})^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} = 2\pi \mathrm{i} \sum_{x\in z} \sum_{p=0}^{n_x-1} k^x_p \int_{\Sigma_x \times I} (\iota_x \times \mathrm{id})^* (H^* \overline{\chi}_{\widehat{G}})_p = 0,$$

where the last equality follows from $\mathrm{trunc}\, (\boldsymbol{\iota} \times \mathrm{id})^* H = e \in C^\infty(\Sigma, G^{\widehat{z}})$ by definition of $H \in C^\infty_{\widehat{D}\times I}(X \times I, \widehat{G})$, cf. (3.17b).

The special case in the statement of this proposition is a consequence of (3.14) and (3.15). $\square$

We can now prove the generalisation of Proposition 2.10 to the present setting.

**Proposition 3.7.** *For any $g \in C^\infty(X, G)$, we have*

$$\int_X \big(\omega \wedge j_X^*(g^*\chi_G)\big)_{\mathrm{reg}} = -2\pi \mathrm{i} \int_{\Sigma \times I} \widehat{g}^* \chi_{G^{\widehat{z}}},$$

*where $\chi_{G^{\widehat{z}}} \in \Omega^3(G^{\widehat{z}})$ is the Cartan 3-form on $G^{\widehat{z}}$ and $\widehat{g} \in C^\infty(\Sigma \times I, G^{\widehat{z}})$ is any lazy homotopy between $\boldsymbol{j}^*g \in C^\infty(\Sigma, G^{\widehat{z}})$ and the constant map $e \in C^\infty(\Sigma, G^{\widehat{z}})$.*

*Proof.* The argument is an adaptation of the proof of Proposition 2.10 to the case of higher order poles. We thus refer to the latter for certain details and highlight only the features pertaining to the present case.

Since $G^{\widehat{z}}$ is connected and $\Sigma = \mathbb{R}^2$ is contractible, there exists a lazy homotopy $\widehat{g} \in C^\infty(\Sigma \times I, G^{\widehat{z}})$ between $\boldsymbol{j}^*g$ and $e$. Using the fact that the jet order truncation map $\widehat{G}^z \to G^{\widehat{z}}$ in (3.16) is a trivial fibre bundle [Viz], we can lift $\widehat{g}$ to a lazy homotopy $\overline{g} \in C^\infty(\Sigma \times I, \widehat{G}^z)$ between a lift of $\boldsymbol{j}^*g$ and the identity element $e \in C^\infty(\Sigma \times I, \widehat{G}^z)$. By construction, $\mathrm{trunc}\, \overline{g} = \widehat{g}$.

As in the proof of Proposition 2.10, let $\varrho : \Delta \to I$ denote the radial coordinate on the unit disc $\Delta$ and let $\Delta_x \subset C$ be disjoint discs around each pole $x \in z$. We define $\widetilde{g} \in C^\infty(X, \widehat{G})$ as the extension from $\bigsqcup_{x\in z} \Sigma \times \Delta_x$ to $X$ by the identity of the image of $(\mathrm{id}_\Sigma \times \varrho)^* \overline{g} \in C^\infty(\Sigma \times \Delta, \widehat{G}^z)$ under the isomorphism

$$C^\infty\bigg(\bigsqcup_{x\in z} \Sigma \times \Delta_x, \widehat{G}\bigg) \cong C^\infty(\Sigma \times \Delta, \widehat{G}^z).$$

By construction, we have that $\operatorname{trunc}\boldsymbol{\iota}^*\widetilde{g} = \boldsymbol{j}^* g = \operatorname{trunc}\boldsymbol{\iota}^* j_X^* g$, hence Proposition 3.6 implies that

$$\int_X \big(\omega \wedge j_X^*(g^*\chi_G)\big)_{\mathrm{reg}} = \int_X \big(\omega \wedge (j_X^* g)^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} = \int_X \big(\omega \wedge \widetilde{g}^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}}.$$

The integral on the right hand side can be computed by following the same steps as in the end of the proof of Proposition 2.10. Explicitly, we have

$$\begin{aligned}\int_X \big(\omega \wedge \widetilde{g}^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} &= \sum_{x\in z} \int_{\Sigma\times\Delta_x} \big(\omega \wedge \widetilde{g}^* \overline{\chi}_{\widehat{G}}\big)_{\mathrm{reg}} = \sum_{x\in z} \int_{\Sigma\times\Delta_x} \Big(\omega \wedge d\Big(-\int_{\gamma_z} \widetilde{g}^* \overline{\chi}_{\widehat{G}}\Big)\Big)_{\mathrm{reg}} \\ &= -2\pi\mathrm{i} \sum_{x\in z} \sum_{p=0}^{n_x-1} k_p^x \int_{\Sigma_x} \int_{\gamma_x} (\widetilde{g}^* \overline{\chi}_{\widehat{G}})_p = -2\pi\mathrm{i} \int_{\Sigma\times I} \widehat{g}^* \chi_{G^{\widehat{z}}}.\end{aligned}$$

In the third equality we used Lemma 3.3 and in the last step we used the definition of the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega$ on $\mathfrak{g}^{\widehat{z}}$ from (3.8) and (3.9). □

## 4. Boundary Conditions on Surface Defects

The results in this section are stated and proved for poles of arbitrary orders $n_x \geq 1$. We use our notational conventions from the higher order pole case in Sect. 3. The definitions and results in Sect. 3 reduce to the ones in Sect. 2 in the case when all poles are simple, i.e. $n_x = 1$, for all $x \in z$, and consequently $n = 1$.

*4.1. Bulk fields with boundary conditions.* We introduce a groupoid of bulk fields with boundary conditions at the (thickened) surface defect $\widehat{D}$. Imposing these boundary conditions will have the effect of making the action (3.3) gauge invariant.

To define the relevant groupoid, let us first observe that the action (3.3) is invariant under translations by $\mathfrak{g}$-valued $(0, 1, 0)$-forms, i.e.

$$S_\omega(A + \lambda) = S_\omega(A),$$

for all $A \in \Omega^1(X, \mathfrak{g})$ and $\lambda \in \Omega^{0,1,0}(X, \mathfrak{g})$, which is due to the fact that $\omega \in \Omega^{0,1,0}(X)$. Hence, the action descends to the quotient

$$\overline{\Omega}^1(X, \mathfrak{g}) := \frac{\Omega^1(X, \mathfrak{g})}{\Omega^{0,1,0}(X, \mathfrak{g})} \cong \Omega^{1,0,0}(X, \mathfrak{g}) \oplus \Omega^{0,0,1}(X, \mathfrak{g}), \tag{4.1}$$

where the last isomorphism is due to the direct sum decomposition (1.2) of forms on $X$. The gauge transformations in (2.4) also descend to the quotient, because for every $g \in C^\infty(X, G)$ and $\lambda \in \Omega^{0,1,0}(X, \mathfrak{g})$ we have

$${}^g(A + \lambda) = -dgg^{-1} + gAg^{-1} + g\lambda g^{-1} = {}^gA + g\lambda g^{-1}$$

and $g\lambda g^{-1} \in \Omega^{0,1,0}(X, \mathfrak{g})$. Abusing notation slightly, we will denote also by ${}^gA$ the action of a gauge transformation $g \in C^\infty(X, G)$ on a 1-form $A \in \overline{\Omega}^1(X, \mathfrak{g})$ under the isomorphism in (4.1), which explicitly reads

$${}^gA = -\overline{d}gg^{-1} + gAg^{-1},$$

where $\overline{d} := d_\Sigma + \bar{\partial}$.

We define the groupoid of *bulk fields* on $X$ by

$$\boldsymbol{B}G_{\mathrm{con}}(X) := \begin{cases} \mathrm{Obj}: \ A \in \overline{\Omega}^1(X, \mathfrak{g}), \\ \mathrm{Mor}: g : A \to {}^g A, \quad \text{with } g \in C^\infty(X, G), \end{cases}$$

and the groupoid of *defect fields* on $\widehat{D}$ by

$$\boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma) := \begin{cases} \mathrm{Obj}: \ a \in \Omega^1(\Sigma, \mathfrak{g}^{\widehat{z}}), \\ \mathrm{Mor}: k : a \to {}^k a, \quad \text{with } k \in C^\infty(\Sigma, G^{\widehat{z}}), \end{cases}$$

where $G^{\widehat{z}}$ is the defect group and $\mathfrak{g}^{\widehat{z}}$ its Lie algebra, cf. (3.7). We would like to emphasise that there is no need to introduce different bundles in these groupoids, because every principal $G$-bundle on $X$ and every principal $G^{\widehat{z}}$-bundle on $\Sigma$ is trivialisable. The latter follows from $\Sigma = \mathbb{R}^2$ being homotopic to a point, while the former follows from the existence of a deformation retract from $X$ to a bouquet of circles $\bigvee^{|\boldsymbol{\zeta}|-1} S^1$ and the short calculation

$$\pi_0\big(\mathrm{Map}_{\{a\}}(X, \boldsymbol{B}G)\big) \cong \pi_0\big(\mathrm{Map}_{\{a\}}(S^1, \boldsymbol{B}G)\big)^{|\boldsymbol{\zeta}|-1} \cong \pi_0(G)^{|\boldsymbol{\zeta}|-1} \cong \{*\},$$

where $a \in X$ is any choice of base point and $\boldsymbol{B}G$ denotes the classifying space of principal $G$-bundles. The last isomorphism follows since $G$ is connected.

Using (3.11) for $r = 1$ and (3.12), we introduce the functor

$$\boldsymbol{j}^* : \boldsymbol{B}G_{\mathrm{con}}(X) \longrightarrow \boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma)$$

that sends an object $A$ to $\boldsymbol{j}^* A$ (note that this is well-defined on the quotients in (4.1)) and a morphism $g : A \to {}^g A$ to $\boldsymbol{j}^* g : \boldsymbol{j}^* A \to \boldsymbol{j}^*({}^g A) = {}^{\boldsymbol{j}^* g}(\boldsymbol{j}^* A)$.

In order to impose boundary conditions for the field $A \in \overline{\Omega}^1(X, \mathfrak{g})$ on the surface defect $\widehat{D}$, we introduce a subgroupoid of $\boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma)$ as follows. Fix a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\widehat{z}}$, which is isotropic with respect to the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega$ in (3.9), and let $K \subset G^{\widehat{z}}$ denote the corresponding connected Lie subgroup. We define

$$\boldsymbol{B}K_{\mathrm{con}}(\Sigma) := \begin{cases} \mathrm{Obj}: \ a \in \Omega^1(\Sigma, \mathfrak{k}) \subset \Omega^1(\Sigma, \mathfrak{g}^{\widehat{z}}), \\ \mathrm{Mor}: k : a \to {}^k a, \quad \text{with } k \in C^\infty(\Sigma, K) \subset C^\infty(\Sigma, G^{\widehat{z}}), \end{cases}$$

and observe that, by definition, there is an inclusion functor

$$\boldsymbol{B}K_{\mathrm{con}}(\Sigma) \hookrightarrow \boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma).$$

Given such a choice of an isotropic Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\widehat{z}}$, we define the groupoid of *bulk fields with boundary conditions* by

$$\mathfrak{F}_{\mathrm{bc}}(X) := \begin{cases} \mathrm{Obj}: \ A \in \overline{\Omega}^1(X, \mathfrak{g}), \quad \text{s.t. } \boldsymbol{j}^* A \in \Omega^1(\Sigma, \mathfrak{k}), \\ \mathrm{Mor}: g : A \to {}^g A, \quad \text{with } g \in C^\infty(X, G) \text{ s.t. } \boldsymbol{j}^* g \in C^\infty(\Sigma, K). \end{cases} \tag{4.2}$$

Given any morphism $g : A \to {}^g A$ in $\mathfrak{F}_{\mathrm{bc}}(X)$, we have $(\boldsymbol{j}^* g)^{-1} d_\Sigma(\boldsymbol{j}^* g) \in \Omega^1(\Sigma, \mathfrak{k})$ and $\boldsymbol{j}^* A \in \Omega^1(\Sigma, \mathfrak{k})$. Hence, the second term on the right hand side of (3.4) vanishes on account of Proposition 3.4 and the isotropy of $\mathfrak{k} \subset \mathfrak{g}^{\widehat{z}}$. The proposition below shows that the last term on the right hand side of (3.4) also vanishes.

**Proposition 4.1.** $\int_X \big(\omega \wedge j_X^*(g^*\chi_G)\big)_{\mathrm{reg}} = 0$*, for every morphism* $g : A \to {}^gA$ *in* $\mathfrak{F}_{\mathrm{bc}}(X)$.

*Proof.* By Proposition 3.7, we have

$$\int_X \big(\omega \wedge j_X^*(g^*\chi_G)\big)_{\mathrm{reg}} = -2\pi\mathrm{i} \int_{\Sigma\times I} \widehat{g}^*\chi_{G^{\widehat{z}}},$$

where $\widehat{g} \in C^\infty(\Sigma \times I, G^{\widehat{z}})$ is any lazy homotopy between $\boldsymbol{j}^*g \in C^\infty(\Sigma, K)$ and $e \in C^\infty(\Sigma, K)$. Since $K$ is connected, we can choose a lazy homotopy $\widehat{g}$ with values in the Lie subgroup $K \subset G^{\widehat{z}}$, i.e. $\widehat{g} \in C^\infty(\Sigma \times I, K)$. It then follows that $\widehat{g}^{-1} d_{\Sigma\times I}\widehat{g} \in \Omega^1(\Sigma \times I, \mathfrak{k})$ and therefore $\widehat{g}^*\chi_{G^{\widehat{z}}} = 0$ since $\mathfrak{k} \subset \mathfrak{g}^{\widehat{z}}$ is an isotropic Lie subalgebra. □

Summing up, we obtain

**Theorem 4.2.** *The regularised* 4*-dimensional Chern-Simons action* $S_\omega$ *given in* (3.3) *defines a gauge invariant action on the groupoid* $\mathfrak{F}_{\mathrm{bc}}(X)$.

To conclude, we would like to note that the groupoid $\mathfrak{F}_{\mathrm{bc}}(X)$ in (4.2) is a model for the pullback

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{bc}}(X) \arrow[r, dashed] \arrow[d, dashed] & \boldsymbol{B}G_{\mathrm{con}}(X) \arrow[d, "j^*"] \\ \boldsymbol{B}K_{\mathrm{con}}(\Sigma) \arrow[r, hook] & \boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma) \end{tikzcd} \tag{4.3}$$

in the category of groupoids. This fact motivates our construction in the next subsection.

*4.2. Bulk fields with edge modes.* The category of groupoids is a category with weak equivalences, where the latter are given by equivalences of groupoids, i.e. fully faithful and essentially surjective functors. In general, pullbacks fail to preserve weak equivalences. This means that if we were to replace the pullback diagram in (4.3) by a weakly equivalent one, in general its pullback will not be weakly equivalent to $\mathfrak{F}_{\mathrm{bc}}(X)$. To solve this issue one considers homotopy pullbacks, instead of ordinary categorical pullbacks, which do preserve weak equivalences. We refer to [Hov,Rie] for an introduction to the frameworks of model and homotopical category theory that underlies the study of homotopy pullbacks.

Motivated by the above discussion, we define the *field groupoid* $\mathfrak{F}(X)$ as the homotopy pullback

$$\begin{tikzcd} \mathfrak{F}(X) \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "h"] & \boldsymbol{B}G_{\mathrm{con}}(X) \arrow[d, "j^*"] \\ \boldsymbol{B}K_{\mathrm{con}}(\Sigma) \arrow[r, hook] & \boldsymbol{B}G^{\widehat{z}}_{\mathrm{con}}(\Sigma) \end{tikzcd} \tag{4.4}$$

in the model category of groupoids. Computing this homotopy pullback by a standard construction (see e.g. [MMST, Appendix A] for a review), we obtain

$$\mathfrak{F}(X) := \begin{cases} \text{Obj}: \ (A,h) \in \overline{\Omega}^1(X,\mathfrak{g}) \times C^\infty(\Sigma, G^{\widehat{z}}), \quad \text{s.t. } {}^{h^{-1}}(\boldsymbol{j}^*A) \in \Omega^1(\Sigma,\mathfrak{k}), \\ \text{Mor}: \ (g,k) : (A,h) \to ({}^gA, (\boldsymbol{j}^*g)hk^{-1}), \\ \qquad\quad \text{with } \ g \in C^\infty(X,G) \ \text{ and } \ k \in C^\infty(\Sigma, K). \end{cases} \tag{4.5}$$

This is to be compared with the (strict) pullback $\mathfrak{F}_{\mathrm{bc}}(X)$ in (4.2).

**Theorem 4.3.** *The functor*

$$\Phi : \mathfrak{F}_{\mathrm{bc}}(X) \longrightarrow \mathfrak{F}(X)$$

*that sends an object $A$ to $(A, e)$ and a morphism $g : A \to {}^g A$ to $(g, \boldsymbol{j}^* g) : (A, e) \to ({}^g A, e)$ is an equivalence of groupoids.*

*Proof.* $\Phi$ is obviously faithful. To show that it is also full, consider objects $A, A' \in \mathfrak{F}_{\mathrm{bc}}(X)$ and let $(g, k) : \Phi(A) = (A, e) \to \Phi(A') = (A', e)$ be a morphism in $\mathfrak{F}(X)$. By definition, $A' = {}^g A$ and $(\boldsymbol{j}^* g)k^{-1} = e$, i.e. $\boldsymbol{j}^* g = k \in C^\infty(\Sigma, K)$. This shows that $g : A \to A'$ is a morphism in $\mathfrak{F}_{\mathrm{bc}}(X)$ and, indeed, $\Phi(g) = (g, k)$.

To conclude the proof, we have to show that $\Phi$ is essentially surjective. Let $(A, h) \in \mathfrak{F}(X)$. Recall that the jet order truncation map $\widehat{G}^{\boldsymbol{z}} \to G^{\widehat{\boldsymbol{z}}}$ in (3.16) is a trivial fibre bundle [Viz] and consider a lift $\widehat{h} \in C^\infty(\Sigma, \widehat{G}^{\boldsymbol{z}})$ of $h \in C^\infty(\Sigma, K) \subset C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$. By the construction in the proof of Proposition 2.10 (just consider the Lie group $\widehat{G}$ instead of $G$, noting that $\widehat{G}$ is connected since $G$ is), we obtain an extension $\widetilde{h} \in C^\infty(X, \widehat{G})$ of $\widehat{h}$, i.e. such that $\boldsymbol{\iota}^* \widetilde{h} = \widehat{h}$, with the following properties:

(a) the restriction of $\widetilde{h}$ to $\bigsqcup_{x \in \boldsymbol{z}} \Sigma \times \Delta_x \subset X$ is constant along $C$, where each $\Delta_x \subset C$ is a sufficiently small open disc centred at $x \in \boldsymbol{z}$, and
(b) $\widetilde{h}$ takes the constant value $e \in \widehat{G}$ on an open neighbourhood of $X \setminus \bigsqcup_{x \in \boldsymbol{z}} \Sigma \times \Delta'_x$, where each $\Delta'_x \supsetneq \Delta_x$ is a strictly larger open disc centred at $x \in \boldsymbol{z}$.

Using the diffeomorphism $\widehat{G} \cong G \times \mathfrak{g}^{n-1}$ from [Viz], $\widetilde{h} \in C^\infty(X, \widehat{G})$ can also be regarded as a tuple of maps on $X$, where $\widetilde{h}_0 \in C^\infty(X, G)$ is $G$-valued and $\widetilde{h}_i \in C^\infty(X, \mathfrak{g})$ is $\mathfrak{g}$-valued, for $i = 1, \dots, n-1$. Below we use these data to construct $g \in C^\infty(X, G)$ such that $\boldsymbol{\iota}^* j^*_X g = \widehat{h}$. For each $x \in \boldsymbol{z}$, consider the local coordinate $z - x$ on $\Delta'_x$ centred at $x$ and define $g$ on $\Sigma \times \Delta'_x$ by

$$g := \exp\left(\sum_{i=1}^{n-1} \frac{(z-x)^i}{i!}\, \xi_i\right) \widetilde{h}_0,$$

where each $\xi_i \in C^\infty(\Sigma \times \Delta'_x, \mathfrak{g})$, for $i = 1, \dots, n-1$, is a linear combination of the $\widetilde{h}_i$'s and of their Lie brackets. Arguing by induction on $i$, the explicit expression of the $\xi_i$'s is obtained by imposing the condition $\boldsymbol{\iota}^* j^*_X g = \widehat{h}$. (Explicitly, using (a) one finds $\xi_1 := \widetilde{h}_1$, $\xi_2 := \widetilde{h}_2$, $\xi_3 := \widetilde{h}_3 + \frac{1}{2}[\xi_1, \xi_2]$, ..., see [Viz].) So far, we defined $g$ only on $\bigsqcup_{x \in \boldsymbol{z}} \Sigma \times \Delta'_x \subset X$. Recalling (b), $g$ can be extended smoothly by $e \in G$ outside of $\bigsqcup_{x \in \boldsymbol{z}} \Sigma \times \Delta'_x \subset X$. This extension provides the desired $g \in C^\infty(X, G)$ such that $\boldsymbol{\iota}^* j^*_X g = \boldsymbol{\iota}^* \widetilde{h} = \widehat{h}$. In particular, by (3.15) we find $\boldsymbol{j}^* g = h$, from which it follows that $\boldsymbol{j}^*({}^{g^{-1}}A) = {}^{h^{-1}}(\boldsymbol{j}^* A) \in \Omega^1(\Sigma, \mathfrak{k})$, i.e. ${}^{g^{-1}}A$ is an object of $\mathfrak{F}_{\mathrm{bc}}(X)$, and that $(g, e) : \Phi({}^{g^{-1}}A) \to (A, h)$ is a morphism in $\mathfrak{F}(X)$. This completes the proof. □

In other words, Theorem 4.3 expresses the fact that the gauge field theories described by the two groupoids $\mathfrak{F}_{\mathrm{bc}}(X)$ and $\mathfrak{F}(X)$ are equivalent. That is, one may either use fields $A \in \overline{\Omega}^1(X, \mathfrak{g})$ satisfying the strict boundary condition $\boldsymbol{j}^* A \in \Omega^1(\Sigma, \mathfrak{k})$, or alternatively one may use pairs of fields $(A, h) \in \overline{\Omega}^1(X, \mathfrak{g}) \times C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ such that $\boldsymbol{j}^* A$ lies in $\Omega^1(\Sigma, \mathfrak{k})$ only up to a gauge transformation determined by the given additional field $h$ on the surface defect $\widehat{D}$. The additional field $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ living on the surface defect $\widehat{D}$ is called the *edge mode*.

Using the equivalence $\Phi$ from Theorem 4.3, we extend the gauge invariant action $S_\omega$ on the groupoid $\mathfrak{F}_{\mathrm{bc}}(X)$ to the field groupoid $\mathfrak{F}(X)$ including the edge modes. The extended action $S_\omega^{\mathrm{ext}}$ on $\mathfrak{F}(X)$ is uniquely determined by

$$S_\omega^{\mathrm{ext}} \circ \Phi = S_\omega. \tag{4.6a}$$

Explicitly, for each $(A, h) \in \mathfrak{F}(X)$, we use that $\Phi$ is essentially surjective to choose an object $\widetilde{A} \in \mathfrak{F}_{\mathrm{bc}}(X)$ and a morphism $(g, k) : \Phi(\widetilde{A}) \to (A, h)$ in $\mathfrak{F}(X)$ and set

$$S_\omega^{\mathrm{ext}}(A, h) := S_\omega(\widetilde{A}). \tag{4.6b}$$

Using that $\Phi$ is also full, one checks that the above definition actually gives a gauge invariant action $S_\omega^{\mathrm{ext}}$ on the field groupoid $\mathfrak{F}(X)$. In particular, we can choose $k = e$ and $g \in C^\infty(X, G)$ such that $\boldsymbol{j}^* g = h$ as in the proof of Theorem 4.3 and, using also (3.4) and Propositions 3.4 and 3.7, we compute $S_\omega^{\mathrm{ext}}$ explicitly as

$$S_\omega^{\mathrm{ext}}(A, h) = S_\omega({}^{g^{-1}}A) = S_\omega(A) + \frac{1}{2}\int_\Sigma \langle\!\langle d_\Sigma h h^{-1}, \boldsymbol{j}^* A\rangle\!\rangle_\omega - \frac{1}{2}\int_{\Sigma\times I} \widehat{h}^* \chi_{G^{\widehat{z}}}, \tag{4.7}$$

where $\widehat{h} \in C^\infty(\Sigma \times I, G^{\widehat{z}})$ is any lazy homotopy between $h \in C^\infty(\Sigma, G^{\widehat{z}})$ and the constant map $e \in C^\infty(\Sigma, G^{\widehat{z}})$. The action (4.7) is to be compared with the action of ordinary 3-dimensional (abelian) Chern-Simons theory given in [MMST, (5.1)].

## 5. Passage to Integrable Field Theories

In order to link 4-dimensional Chern-Simons theory to integrable field theories, we introduce the full subgroupoid $\mathfrak{F}^{1,0,0}(X) \subset \mathfrak{F}(X)$ whose objects $(\mathcal{L}, h) \in \mathfrak{F}^{1,0,0}(X)$ are those objects of $\mathfrak{F}(X)$ (cf. (4.5)) which satisfy the additional condition that $\mathcal{L} \in \Omega^{1,0,0}(X, \mathfrak{g}) \subset \overline{\Omega}^1(X, \mathfrak{g})$ is a $(1, 0, 0)$-form on $X$, i.e. $\mathcal{L}$ has no $dz$ and $d\bar{z}$ components. Let us mention that morphisms $(g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, (\boldsymbol{j}^* g)hk^{-1})$ in $\mathfrak{F}^{1,0,0}(X)$ are then given by pairs of maps $(g, k) \in C^\infty(X, G) \times C^\infty(\Sigma, K)$ satisfying $\bar{\partial} g g^{-1} = 0$, which follows from the fact that, by definition, also $({}^g\mathcal{L}, (\boldsymbol{j}^* g)hk^{-1})$ lies in $\mathfrak{F}^{1,0,0}(X)$. Explicitly, the groupoid introduced above reads as

$$\mathfrak{F}^{1,0,0}(X) := \begin{cases} \text{Obj}: & (\mathcal{L}, h) \in \Omega^{1,0,0}(X, \mathfrak{g}) \times C^\infty(\Sigma, G^{\widehat{z}}), \quad \text{s.t. } {}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k}), \\ \text{Mor}: & (g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, (\boldsymbol{j}^* g)hk^{-1}), \\ & \text{with } \ g \in C^\infty(X, G) \text{ s.t. } \bar{\partial} g g^{-1} = 0 \ \text{ and } \ k \in C^\infty(\Sigma, K). \end{cases} \tag{5.1}$$

*Remark 5.1.* The inclusion functor $\mathfrak{F}^{1,0,0}(X) \hookrightarrow \mathfrak{F}(X)$ is by definition fully faithful. One might ask if it is also essentially surjective, hence an equivalence. By direct inspection, it is easy to realise that the answer is positive provided that, for each $(A, h) \in \mathfrak{F}(X)$, there exists $g \in C^\infty(X, G)$ such that $g^{-1}\bar{\partial} g = A^{0,0,1}$, where $A^{0,0,1} \in \Omega^{0,0,1}(X, \mathfrak{g})$ denotes the $(0, 0, 1)$-component of $A \in \overline{\Omega}^1(X, \mathfrak{g})$. In order to simplify the problem, suppose $G = \mathrm{GL}_N(\mathbb{C})$, let us fix a point $a \in \Sigma$ and consider the problem of finding such a $g$ on $C_a := \{a\} \times C \subset X$. Then an argument based on the inverse function theorem for Banach manifolds and elliptic regularity, cf. [AB, Section 5], shows that the above equation admits local solutions $\{g_\alpha\}$ subordinate to a cover $\{U_\alpha \subseteq C_a\}$ by sufficiently small open subsets of $C_a$. As a consequence, $\{g_{\alpha\beta}\} := \{g_\alpha g_\beta^{-1}\}$ is a Čech

1-cocycle on $C_a$ taking values in the sheaf of holomorphic $G$-valued functions. The latter is always trivial by [Fos, Theorem 30.5] because $C_a$ is a non-compact Riemann surface. This allows us to find a Čech 0-cochain $\{h_\alpha\}$ trivialising $\{g_{\alpha\beta}\}$. It follows that setting $g := h_\alpha^{-1} g_\alpha$ on each $U_\alpha$ defines $g \in C^\infty(C_a, G)$ such that $g^{-1}\bar{\partial} g = A^{0,0,1}$, as required. (Note that, in contrast to $\{g_\alpha\}$, $\{h_\alpha\}$ is holomorphic, which is crucial to check that $g$ indeed solves the above equation.) Extending this argument to the whole of $X$ and for arbitrary $G$ requires to establish smoothly $\Sigma$-parametrised analogues with target an arbitrary Lie group $G$ of the arguments in [AB, Section 5] and [Fos, Theorem 30.5]. Since essential surjectivity of $\mathfrak{F}^{1,0,0}(X) \hookrightarrow \mathfrak{F}(X)$ is not needed for our constructions below, we shall not further address this issue.

Since $\mathfrak{F}^{1,0,0}(X) \subset \mathfrak{F}(X)$ is a subgroupoid, we can restrict the action on $\mathfrak{F}(X)$ defined in (4.6) to $\mathfrak{F}^{1,0,0}(X)$. From the explicit expression (4.7), we obtain

$$S_\omega^{\mathrm{ext}}(\mathcal{L}, h) = \frac{\mathrm{i}}{4\pi} \int_X \big(\omega \wedge j_X^* \langle \mathcal{L}, \bar{\partial}\mathcal{L}\rangle\big)_{\mathrm{reg}} + \frac{1}{2}\int_\Sigma \langle\!\langle d_\Sigma h h^{-1}, \boldsymbol{j}^*\mathcal{L}\rangle\!\rangle_\omega - \frac{1}{2}\int_{\Sigma\times I} \widehat{h}^* \chi_{G^{\widehat{\mathfrak{z}}}}, \tag{5.2}$$

where the simplification in the first term follows from $\mathcal{L} \in \Omega^{1,0,0}(X, \mathfrak{g})$ by definition of the subgroupoid $\mathfrak{F}^{1,0,0}(X) \subset \mathfrak{F}(X)$, cf. (5.1).

Let us now derive the Euler-Lagrange equations corresponding to the action (5.2). For this we have to consider variations of objects $(\mathcal{L}, h) \in \mathfrak{F}^{1,0,0}(X)$, i.e. variations $(\mathcal{L}', h') = (\mathcal{L} + \epsilon \ell, e^{\epsilon\chi}\, h)$, with $\ell \in \Omega_c^{1,0,0}(X, \mathfrak{g})$ and $\chi \in C_c^\infty(\Sigma, \mathfrak{g}^{\widehat{\mathfrak{z}}})$, satisfying the condition ${}^{h'^{-1}}(\boldsymbol{j}^*\mathcal{L}') \in \Omega^1(\Sigma, \mathfrak{k})$. Expanding this condition to first order in $\epsilon$, one finds that the variations are constrained by

$$h^{-1}\big(d_\Sigma \chi + [\boldsymbol{j}^*\mathcal{L}, \chi] + \boldsymbol{j}^*\ell\big)h \in \Omega^1(\Sigma, \mathfrak{k}). \tag{5.3}$$

Varying the action (5.2) and using (5.3), one obtains

$$\delta_{(\ell,\chi)} S_\omega^{\mathrm{ext}}(\mathcal{L}, h) = \frac{\mathrm{i}}{2\pi}\int_X \big(\omega \wedge j_X^*\langle \ell, \bar{\partial}\mathcal{L}\rangle\big)_{\mathrm{reg}} - \int_\Sigma \big\langle\!\big\langle \chi, d_\Sigma(\boldsymbol{j}^*\mathcal{L}) + \tfrac{1}{2}\big[\boldsymbol{j}^*\mathcal{L}, \boldsymbol{j}^*\mathcal{L}\big]\big\rangle\!\big\rangle_\omega.$$

From bulk variations, i.e. $(\ell, \chi) = (\ell, 0)$ with $\operatorname{supp} \ell \subset X \setminus D$ (note that the constraint (5.3) is trivially satisfied), we obtain the equation of motion

$$\bar{\partial}\mathcal{L} = 0 \qquad \text{on } X \setminus D.$$

Because $\mathcal{L} \in \Omega^{1,0,0}(X, \mathfrak{g})$ is a smooth 1-form on $X$, this equation implies that $\mathcal{L}$ is holomorphic on all of $C$, i.e.

$$\bar{\partial}\mathcal{L} = 0 \qquad \text{on } X. \tag{5.4}$$

(Recall that $X = \Sigma \times C$, where $C = \mathbb{C}P^1 \setminus \boldsymbol{\zeta}$ is the Riemann sphere with the zeroes of $\omega$ removed. In particular, solutions to (5.4) on $X$ may have poles at $\boldsymbol{\zeta} \subset \mathbb{C}P^1$ with coefficients in $\Omega^1(\Sigma, \mathfrak{g})$, as required for Lax connections in integrable field theories.)

To study variations with support on the defect, we first observe that, given any $\chi \in C_c^\infty(\Sigma, \mathfrak{g}^{\widehat{\mathfrak{z}}})$, there exists $\ell \in \Omega_c^1(X, \mathfrak{g})$ such that the pair $(\ell, \chi)$ satisfies (5.3). Indeed, the equation $\boldsymbol{j}^*\ell = -d_\Sigma \chi - [\boldsymbol{j}^*\mathcal{L}, \chi]$ on the jets of $\ell$ can be solved for an arbitrary right hand side by the same method as in the proof of Theorem 4.3. Hence, we obtain the equation of motion

$$d_\Sigma(\boldsymbol{j}^*\mathcal{L}) + \tfrac{1}{2}\big[\boldsymbol{j}^*\mathcal{L}, \boldsymbol{j}^*\mathcal{L}\big] = 0 \qquad \text{on } \Sigma, \tag{5.5}$$

which means that $\boldsymbol{j}^*\mathcal{L} \in \Omega^1(\Sigma, \mathfrak{g}^{\widehat{\mathfrak{z}}})$ defines a flat $G^{\widehat{\mathfrak{z}}}$-connection on $\Sigma$.

To perform the passage to integrable field theories on $\Sigma$, we shall consider suitable solutions to the bulk equation of motion (5.4) with properties that resemble those of Lax connections. We will do this in two steps. First, we restrict attention to solutions that are meromorphic on $\mathbb{C}P^1$. Subsequently, we will further restrict attention to those solutions for which the defect equation of motion (5.5) can be lifted to a flatness condition for $\mathcal{L}$ on all of $X$. (Note that we do *not* solve the defect equation of motion (5.5) on $\Sigma$.)

More precisely, we introduce the following

**Definition 5.2.** Denoting by $m_y \in \mathbb{Z}_{\geq 1}$ the order of the zero $y \in \boldsymbol{\zeta}$ of $\omega$, we let $\Omega^{r,0,0}_{\mathcal{M}}(X, \mathfrak{g}) \subset \Omega^{r,0,0}(X, \mathfrak{g})$ be the subspace of those $\mathfrak{g}$-valued $(r, 0, 0)$-forms on $X$ that are meromorphic on $\mathbb{C}P^1$ with poles at each $y \in \boldsymbol{\zeta}$ of order at most $m_y$.

Note that, by definition, every $\mathcal{L} \in \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ is a solution to the bulk equation of motion (5.4). Furthermore, every $\mathcal{L} \in \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ can be written explicitly as

$$\mathcal{L} = \mathcal{L}_{\mathrm{c}} + \sum_{y \in \boldsymbol{\zeta} \setminus \{\infty\}} \sum_{q=0}^{m_y - 1} \frac{\mathcal{L}^y_q}{(z - y)^{q+1}} + \sum_{q=0}^{m_\infty - 1} \mathcal{L}^\infty_q z^{q+1}, \tag{5.6}$$

where $\mathcal{L}_{\mathrm{c}} \in \Omega^1(\Sigma, \mathfrak{g})$ and $\mathcal{L}^y_q \in \Omega^1(\Sigma, \mathfrak{g})$, for every $y \in \boldsymbol{\zeta}$ and $q = 0, \dots, m_y - 1$, are $\mathfrak{g}$-valued 1-forms on $\Sigma$. Note that the first term of (5.6) is constant on $\mathbb{C}P^1$, while the second and third terms describe the poles at $y \in \boldsymbol{\zeta} \setminus \{\infty\}$ and at the zero $y = \infty$ of $\omega$, respectively.

The 1-form $\mathcal{L} \in \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ is still too general to serve as a Lax connection for integrable field theories. The reason is that the flatness condition, which is encoded by the defect equation of motion (5.5), is a priori imposed only for the restriction via $\boldsymbol{j}^*$ to $\Sigma$ of (the jets of) the curvature $F_\Sigma(\mathcal{L}) := d_\Sigma \mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}] \in \Omega^{2,0,0}(X, \mathfrak{g})$. (Note that $\boldsymbol{j}^* F_\Sigma(\mathcal{L}) = d_\Sigma(\boldsymbol{j}^*\mathcal{L}) + \frac{1}{2}[\boldsymbol{j}^*\mathcal{L}, \boldsymbol{j}^*\mathcal{L}]$ because $\boldsymbol{j}^*$ given in (3.11) preserves both the differential $d_\Sigma$ and the Lie bracket $[\cdot, \cdot]$.) In order to upgrade the flatness condition from $\boldsymbol{j}^* F_\Sigma(\mathcal{L}) = 0$ on $\Sigma$ (cf. (5.5)) to $F_\Sigma(\mathcal{L}) = 0$ on $X$, i.e. prior to applying $\boldsymbol{j}^*$, we require the following

**Definition 5.3.** A form $\mathcal{L} \in \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ is called *admissible* if $F_\Sigma(\mathcal{L}) \in \Omega^{2,0,0}_{\mathcal{M}}(X, \mathfrak{g})$. We denote by $\Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g}) \subset \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ the subspace of admissible forms.

*Example 5.4.* Note that not every $\mathcal{L} \in \Omega^{1,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ is admissible, because the term $[\mathcal{L}, \mathcal{L}]$ in the curvature may have poles at $y \in \boldsymbol{\zeta}$ of order greater than $m_y$. A simple algebraic condition which ensures that $\mathcal{L}$, written in the form (5.6), is admissible is given by

$$\big[\mathcal{L}^y_q, \mathcal{L}^y_{q'}\big] = 0,$$

for all $y \in \boldsymbol{\zeta}$ and $q, q'$ with $q + q' + 2 > m_y$. One way to achieve this is the following: for each $y \in \boldsymbol{\zeta}$, we introduce a coordinate $\sigma_y : \Sigma \to \mathbb{R}$ on $\Sigma$ and take the 1-forms $\mathcal{L}^y_q \in \Omega^1(\Sigma, \mathfrak{g})$, for $q = 0, \dots, m_y - 1$, to be proportional to $d\sigma_y$. For example, to produce a Lorentzian integrable field theory, we fix a Minkowski metric on $\Sigma$, let $\sigma^\pm$ denote a corresponding pair of null coordinates, choose a subset $\boldsymbol{\zeta}^+ \subset \boldsymbol{\zeta}$ and then set $\sigma_y = \sigma^+$ for $y \in \boldsymbol{\zeta}^+$ and $\sigma_y = \sigma^-$ for $y \in \boldsymbol{\zeta} \setminus \boldsymbol{\zeta}^+$ in the complement, cf. [DLMV1].

**Lemma 5.5.** *For every $r = 0, 1, 2$, the restriction $\boldsymbol{j}^* : \Omega^{r,0,0}_{\mathcal{M}}(X, \mathfrak{g}) \to \Omega^r(\Sigma, \mathfrak{g}^{\widehat{\boldsymbol{z}}})$ of the morphism* (3.11) *to the subspace $\Omega^{r,0,0}_{\mathcal{M}}(X, \mathfrak{g}) \subset \Omega^r(X, \mathfrak{g})$ introduced in Definition* 5.2 *is injective.*

*Proof.* By definition, any $\eta \in \Omega^{r,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ is meromorphic on $\mathbb{C}P^1$ with poles at all $y \in \boldsymbol{\zeta}$ of order at most $m_y$ and with coefficients in $\Omega^r(\Sigma, \mathfrak{g})$. We need to show that if $\iota_x^*(\partial_z^p \eta) = 0$, for all $x \in \boldsymbol{z}$ and $p = 0, \dots, n_x - 1$, then $\eta = 0$.

Consider the polynomial $P(z) := \prod_{y \in \boldsymbol{\zeta} \setminus \{\infty\}} (z - y)^{m_y}$. Then $P\eta$ is a polynomial in $z$ of order at most $\sum_{y \in \boldsymbol{\zeta}} m_y$ with coefficients in $\Omega^r(\Sigma, \mathfrak{g})$. Since by assumption $\iota_x^*(\partial_z^p \eta) = 0$, for all $x \in \boldsymbol{z}$ and $p = 0, \dots, n_x - 1$, it follows by the Leibniz rule that $\iota_x^*(\partial_z^p (P\eta)) = 0$, for every $x \in \boldsymbol{z}$ and $p = 0, \dots, n_x - 1$. Since $\omega$ is a meromorphic 1-form on $\mathbb{C}P^1$, we have $\sum_{x \in \boldsymbol{z}} n_x = \sum_{y \in \boldsymbol{\zeta}} m_y + 2$, which is greater than the degree of the polynomial $P\eta$. It follows that $P\eta = 0$ and hence $\eta = 0$. □

**Proposition 5.6.** *For any admissible $\mathcal{L} \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$, the defect equation of motion (5.5), i.e. $\boldsymbol{j}^* F_\Sigma(\mathcal{L}) = 0$ on $\Sigma$, is equivalent to $F_\Sigma(\mathcal{L}) = 0$ on $X$.*

*Proof.* Suppose $\boldsymbol{j}^* F_\Sigma(\mathcal{L}) = 0$. Since $\mathcal{L}$ is admissible, $F_\Sigma(\mathcal{L}) \in \Omega^{2,0,0}_{\mathcal{M}}(X, \mathfrak{g})$ and hence $F_\Sigma(\mathcal{L}) = 0$ by Lemma 5.5. The converse is obvious. □

The above results motivate us to introduce a suitable subgroupoid of $\mathfrak{F}^{1,0,0}(X)$ whose objects $(\mathcal{L}, h)$ are such that $\mathcal{L} \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$ is admissible in the sense of Definition 5.3. In particular, such $\mathcal{L}$'s satisfy the bulk equation of motion (5.4), are meromorphic on $\mathbb{C}P^1$ with poles of the form (5.6) and, by Proposition 5.6, the defect equation of motion (5.5) is equivalent to flatness $F_\Sigma(\mathcal{L}) = 0$ on $X$. In other words, such $\mathcal{L}$'s satisfy all the necessary properties of Lax connections for integrable field theories. Concerning morphisms $(g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, (\boldsymbol{j}^* g) h k^{-1})$ between such objects, by definition of the groupoid $\mathfrak{F}^{1,0,0}(X)$ in (5.1) we have that $g \in C^\infty(X, G)$ is holomorphic on $C$. In order to preserve the pole structure (5.6) of admissible $\mathcal{L}$'s under gauge transformations, we further restrict our attention to those $g$ that are holomorphic on all of $\mathbb{C}P^1$, and hence constant along $\mathbb{C}P^1$. Summing up this discussion, we introduce the following (not necessarily full) subgroupoid of (5.1)

$$\mathfrak{F}_{\mathrm{Lax}}(X) := \begin{cases} \text{Obj}: \ (\mathcal{L}, h) \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g}) \times C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}}), \quad \text{s.t. } {}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k}), \\ \text{Mor}: (g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, (\boldsymbol{j}^* g) h k^{-1}), \\ \qquad\quad \text{with } \ g \in C^\infty(\Sigma, G) \ \text{ and } \ k \in C^\infty(\Sigma, K), \end{cases} \tag{5.7}$$

where we are implicitly identifying a map $g \in C^\infty(\Sigma, G)$ with its pullback along the projection $p_\Sigma : X \to \Sigma$. Under this identification, we have that $\boldsymbol{j}^* g = \Delta(g)$, where $\Delta : G \to G^{\boldsymbol{z}}\,,\ g \mapsto (g)_{x \in \boldsymbol{z}}$ is the diagonal map to the defect group (3.7).

With these preparations, we are now ready to describe how 2-dimensional integrable field theories arise from 4-dimensional Chern-Simons theory. Consider the groupoid

$$\mathfrak{F}_{2\mathrm{d}}(\Sigma) := \begin{cases} \text{Obj}: \ h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}}), \\ \text{Mor}: (g, k) : h \to \Delta(g) h k^{-1}, \\ \qquad\quad \text{with } \ g \in C^\infty(\Sigma, G) \ \text{ and } \ k \in C^\infty(\Sigma, K), \end{cases} \tag{5.8}$$

of $G^{\widehat{\boldsymbol{z}}}$-valued fields on $\Sigma$ and note that there exists a forgetful functor

$$\pi : \mathfrak{F}_{\mathrm{Lax}}(X) \longrightarrow \mathfrak{F}_{2\mathrm{d}}(\Sigma) \tag{5.9}$$

that sends an object $(\mathcal{L}, h)$ to $h$ and a morphism $(g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, (\boldsymbol{j}^* g) h k^{-1})$ to $(g, k) : h \to \Delta(g) h k^{-1}$. If this functor was fully faithful and essentially surjective, i.e. an equivalence, then we could transfer the action (5.2) to an action

$$S^{2\mathrm{d}}_\omega := S^{\mathrm{ext}}_\omega \circ \pi^{-1} \tag{5.10}$$

defined on the groupoid $\mathfrak{F}_{2\mathrm{d}}(\Sigma)$ in (5.8), where $\pi^{-1} : \mathfrak{F}_{2\mathrm{d}}(\Sigma) \to \mathfrak{F}_{\mathrm{Lax}}(X)$ denotes a quasi-inverse of $\pi$. Gauge invariance of $S_\omega^{\mathrm{ext}}$ entails that $S_\omega^{2\mathrm{d}}$ does not depend on the choice of quasi-inverse. While (5.9) is clearly a faithful functor, fullness and essential surjectivity do not appear to be automatic. These properties of the functor $\pi$ can be related to existence and uniqueness of solutions $\mathcal{L} \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$ for a fixed $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ to the condition ${}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k})$ on objects $(\mathcal{L}, h)$ of $\mathfrak{F}_{\mathrm{Lax}}(X)$, cf. (5.7).

**Proposition 5.7.** *The functor $\pi$ in (5.9) is essentially surjective if and only if it is surjective on objects, i.e. for each $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ there exists $\mathcal{L} \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$ such that $(\mathcal{L}, h) \in \mathfrak{F}_{\mathrm{Lax}}(X)$. It is full if and only if for each $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ there exists at most one object of the form $(\mathcal{L}, h)$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$.*

*Proof.* For the first statement, the implication "$\Leftarrow$" is obvious. To prove the implication "$\Rightarrow$", let us assume that $\pi$ is essentially surjective. Then there exists, for each $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$, an object $(\mathcal{L}', h')$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$ and a morphism $(g, k) : h \to h' = \pi(\mathcal{L}', h')$ in $\mathfrak{F}_{2\mathrm{d}}(\Sigma)$. Setting $\mathcal{L} := {}^{g^{-1}}\mathcal{L}' \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$, we obtain

$$ {}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) = {}^{h^{-1}\Delta(g^{-1})}(\boldsymbol{j}^*\mathcal{L}') = {}^{k^{-1}h'^{-1}}(\boldsymbol{j}^*\mathcal{L}') \in \Omega^1(\Sigma, \mathfrak{k}), $$

where in the second step we used $h' = \Delta(g)hk^{-1}$. The last step then follows from ${}^{h'^{-1}}(\boldsymbol{j}^*\mathcal{L}') \in \Omega^1(\Sigma, \mathfrak{k})$, as $(\mathcal{L}', h')$ is by hypothesis an object in $\mathfrak{F}_{\mathrm{Lax}}(X)$, and the fact that $k \in C^\infty(\Sigma, K)$ is a map to the subgroup $K \subset G^{\widehat{\boldsymbol{z}}}$.

Let us consider now the second statement. We prove the implication "$\Rightarrow$" by contraposition. Suppose that there exist objects $(\mathcal{L}, h)$, $(\mathcal{L}', h)$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$ such that $\mathcal{L}' \neq \mathcal{L}$. Then there does not exist a morphism $(\mathcal{L}, h) \to (\mathcal{L}', h)$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$ that maps under $\pi$ to the identity $\mathrm{id} : h \to h$ in $\mathfrak{F}_{2\mathrm{d}}(\Sigma)$, hence $\pi$ is not full. To prove the implication "$\Leftarrow$", let $(\mathcal{L}, h)$, $(\mathcal{L}', h')$ be arbitrary objects in $\mathfrak{F}_{\mathrm{Lax}}(X)$ and consider any morphism $(g, k) : h \to h'$ in $\mathfrak{F}_{2\mathrm{d}}(\Sigma)$. We define the morphism $(g, k) : (\mathcal{L}, h) \to ({}^g\mathcal{L}, h')$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$ and observe that by hypothesis ${}^g\mathcal{L} = \mathcal{L}'$. Hence, we obtain a morphism $(g, k) : (\mathcal{L}, h) \to (\mathcal{L}', h')$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$ and thereby prove that $\pi$ is full. □

**Corollary 5.8.** *The functor $\pi$ in (5.9) is an equivalence of groupoids if and only if for each $h \in C^\infty(\Sigma, G^{\widehat{\boldsymbol{z}}})$ there exists a unique $\mathcal{L} \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$ such that $(\mathcal{L}, h) \in \mathfrak{F}_{\mathrm{Lax}}(X)$, i.e. such that ${}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k})$.*

*Remark 5.9.* Let us note that whether or not the functor $\pi$ in (5.9) is an equivalence will depend on the choice of isotropic subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\widehat{\boldsymbol{z}}}$ used to impose boundary conditions at the surface defects in Sect. 4. Examples of suitable choices when $n_x \leq 2$ for all $x \in \boldsymbol{z}$ can be found in [CY,DLMV2]. In light of the present work, the problem of classifying isotropic subalgebras $\mathfrak{k} \subset \mathfrak{g}^{\widehat{\boldsymbol{z}}}$ for which the condition ${}^{h^{-1}}(\boldsymbol{j}^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k})$ admits a unique solution for $\mathcal{L}$ in terms of $h$ is an important one in view of the broader open problem of classifying 2-dimensional integrable field theories.

Suppose now that the functor $\pi$ in (5.9) is an equivalence. Using Corollary 5.8, we can then construct a *strict* inverse

$$ \pi^{-1} : \mathfrak{F}_{2\mathrm{d}}(\Sigma) \longrightarrow \mathfrak{F}_{\mathrm{Lax}}(X). $$

This functor sends an object $h$ to $(\mathcal{L}(h), h)$, where $\mathcal{L}(h) \in \Omega^{1,0,0}_{\mathrm{adm}}(X, \mathfrak{g})$ is the unique element such that $(\mathcal{L}(h), h)$ is an object in $\mathfrak{F}_{\mathrm{Lax}}(X)$. To a morphism $(g, k) : h \to h' = \Delta(g)hk^{-1}$ in $\mathfrak{F}_{2\mathrm{d}}(\Sigma)$, this functor assigns the morphism $(g, k) : (\mathcal{L}(h), h) \to$

$(\mathcal{L}(h'), h')$ in $\mathfrak{F}_{\mathrm{Lax}}(X)$, where $\mathcal{L}(h') = \mathcal{L}(\Delta(g)hk^{-1}) = {}^{g}\mathcal{L}(h)$ by the uniqueness of Corollary 5.8. Using this description of $\pi^{-1}$, we obtain an explicit expression for the action in (5.10)

$$S^{\mathrm{2d}}_{\omega}(h) = S^{\mathrm{ext}}_{\omega}(\mathcal{L}(h), h) = \frac{1}{2}\int_{\Sigma} \langle\!\langle d_{\Sigma}hh^{-1}, \boldsymbol{j}^{*}\mathcal{L}(h)\rangle\!\rangle_{\omega} - \frac{1}{2}\int_{\Sigma\times I} \widehat{h}^{*}\chi_{G^{\widehat{z}}}, \qquad (5.11)$$

where the first term in (5.2) vanishes because $\bar{\partial}\mathcal{L}(h) = 0$ by definition of the groupoid $\mathfrak{F}_{\mathrm{Lax}}(X)$ in (5.7). We would like to emphasise that the action (5.11) is for a $G^{\widehat{z}}$-valued field $h$ living on the 2-dimensional manifold $\Sigma$ and that it describes an integrable field theory with Lax connection $\mathcal{L}(h)$. Furthermore, the action $S^{\mathrm{2d}}_{\omega}$ is by construction gauge invariant under the morphisms of the groupoid $\mathfrak{F}_{\mathrm{2d}}(\Sigma)$ introduced in (5.8).

*Remark 5.10.* There is a more minimalistic procedure for transferring the action $S^{\mathrm{ext}}_{\omega}$ (cf. (4.6)) on the subgroupoid $\mathfrak{F}_{\mathrm{Lax}}(X) \subset \mathfrak{F}(X)$ to an action $S^{\mathrm{2d}}_{\omega}$ on $\mathfrak{F}_{\mathrm{2d}}(\Sigma)$ along the functor $\pi : \mathfrak{F}_{\mathrm{Lax}}(X) \to \mathfrak{F}_{\mathrm{2d}}(\Sigma)$ in (5.9), which only requires the latter to be essentially surjective and not necessarily full. This is based on the following observation. The datum of a gauge invariant action $S^{\mathrm{2d}}_{\omega}$ on the groupoid $\mathfrak{F}_{\mathrm{2d}}(\Sigma)$ is equivalent to the datum of a function $S^{\mathrm{2d}}_{\omega}$ on the set $\pi_0(\mathfrak{F}_{\mathrm{2d}}(\Sigma))$ of isomorphism classes of objects. Furthermore, essential surjectivity of the functor $\pi : \mathfrak{F}_{\mathrm{Lax}}(X) \to \mathfrak{F}_{\mathrm{2d}}(\Sigma)$ is equivalent to surjectivity of the induced map $\pi : \pi_0(\mathfrak{F}_{\mathrm{Lax}}(X)) \to \pi_0(\mathfrak{F}_{\mathrm{2d}}(\Sigma))$ between sets of isomorphism classes. Therefore, in order to transfer $S^{\mathrm{ext}}_{\omega}$ to $\mathfrak{F}_{\mathrm{2d}}(\Sigma)$, we can choose a section $\sigma$ of the surjective map $\pi : \pi_0(\mathfrak{F}_{\mathrm{Lax}}(X)) \to \pi_0(\mathfrak{F}_{\mathrm{2d}}(\Sigma))$ and define $S^{\mathrm{2d}}_{\omega} := S^{\mathrm{ext}}_{\omega} \circ \sigma$. More generally, we can choose a suitable measure $w$ on the set of sections $\sigma$ and define $S^{\mathrm{2d}}_{\omega}$ as the $w$-average over all sections $\sigma$ of $S^{\mathrm{ext}}_{\omega} \circ \sigma$. (For a fixed section $\sigma$, the Dirac measure $w = \delta_{\sigma}$ recovers the construction considered previously in this remark.) We stress, however, that this alternative construction of $S^{\mathrm{2d}}_{\omega}$ in general depends on the choice of measure $w$ on the set of sections $\sigma$. Whenever $\pi$ is both essentially surjective and full, $\pi : \pi_0(\mathfrak{F}_{\mathrm{Lax}}(X)) \to \pi_0(\mathfrak{F}_{\mathrm{2d}}(\Sigma))$ is actually bijective and hence $S^{\mathrm{2d}}_{\omega} := S^{\mathrm{ext}}_{\omega} \circ \pi^{-1}$ is uniquely determined (there is exactly one section $\sigma = \pi^{-1}$). In particular, the construction of $S^{\mathrm{2d}}_{\omega}$ presented before this remark agrees with the one considered here.

When $\pi$ is essentially surjective but not full, however, it becomes more difficult to interpret the output of our construction as an integrable field theory since the candidate Lax connection $\mathcal{L}$ in general fails to be *uniquely* determined by the field $h$ living on $\Sigma$.

*Acknowledgements.* B.V. would like to thank S. Lacroix and M. Magro for useful discussions. Furthermore, we are grateful to S. Bunk for suggesting the averaging construction mentioned in Remark 5.10. M.B. gratefully acknowledges the financial support of the National Group of Mathematical Physics GNFM-INdAM (Italy). A.S. gratefully acknowledges the financial support of the Royal Society (UK) through a Royal Society University Research Fellowship (UF150099), a Research Grant (RG160517) and two Enhancement Awards (RGF\EA\180270 and RGF\EA\201051).

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Communicated by C. Schweigert