RE-CONCEPTUALISING CONCEPTUAL UNDERSTANDING IN MATHEMATICS

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In this theoretical paper we explore interrelationships between conceptual and procedural understanding of mathematics in the context of individuals and groups. We question the enterprise of attempting to assess learners' mathematical understanding by inviting them to perform a (perhaps unfamiliar) procedure or offer an explanation. Would it be appropriate to describe a learner in possession of an algorithm for responding satisfactorily to such prompts as displaying conceptual understanding? We relate the discussion to Searle's "Chinese Room" thought experiment and draw on Habermas' Theory of Communicative Action to develop potential implications for addressing the problem of interpreting learners' mathematical understanding.

INTRODUCTION

The quest to help learners develop a deep and meaningful understanding of mathematics has become the holy grail for mathematics educators (Llewellyn, 2012), particularly since Skemp's (1976) seminal division of understanding into "instrumental" and "relational" categories. Relational (or conceptual) understanding is seen as more powerful, authentic and satisfying for the learner, representing true mathematical sense-making. But how can we know whether or not a learner has this relational understanding in any particular area of mathematics? The short, closed questions which dominate traditional paper-based assessments are unlikely to elicit this information. Hewitt (2009, p. 91) comments that "it is perfectly possible for a student to get right answers whilst not knowing about the mathematics within their work", and offers an example in which a learner aged 12–13 was finding the areas of triangles by multiplying the base by the height and dividing by 2, but admitted that he had no idea why he was multiplying or dividing by 2. This same example is used by Skemp (1976) to exemplify his distinction between instrumental and relational understanding of mathematics. Yet inviting learners to go further and explain their mathematics is also problematic. An invitation to "explain" an answer may be experienced as yet another request for "a performance": the "right" explanation that will satisfy a teacher or examiner may be memorised or produced algorithmically, just like the answer itself.

We might ask what it means for learners to have relational understanding of factorising a quadratic expression, for instance (Foster, 2014). If they can perform the procedure fluently (i.e., quickly, accurately, flexibly and confidently) then would we be satisfied (Foster, 2013)? We might argue that relational understanding involves adapting what is known to novel, non-straightforward problem-solving situations. Yet a robust enough algorithm will dispose of a very wide range of scenarios, including unanticipated ones, and a comprehensive enough set of algorithms might successfully deal with any situation likely to be encountered in any assessment (MacCormick, 2012). If the learner's performance continued to be faultless would we wish to probe their thinking further? To some extent mathematical fluency entails withdrawing attention from the details of why and how the procedure works so as to speed up the process and allow cognitive space for focusing on wider aspects of the problem (Hewitt, 1996; Foster, 2013). A mathematician does not want to have to differentiate $3x^2 - 2x + 4$ from first principles every time, although they are capable of doing so. Perhaps relational understanding involves an ability to deconstruct the procedure *if required* rather than an expectation that this is going on every time it is carried out? But deconstructing a procedure could *itself* be regarded as a procedure, and presumably one that can be prepared for – even memorised, just as proofs can be memorised. So is there something more to relational understanding than expert procedural fluency, and if so how might this be conceptualised? Is there a difference between being able to manipulate syntax and being able to understanding meaning?

PROCEDURAL AND CONCEPTUAL KNOWLEDGE

Skemp's (1976) famous distinction between instrumental and relational understanding characterises relational understanding as "knowing both what to do and why" (p. 20), whereas instrumental understanding is merely "rules without reasons" (p. 20). While acknowledging that "one can often get the right answer more quickly and reliably by instrumental thinking than relational" (p. 23), he nonetheless criticises instrumental learning as a proliferation of little rules to remember rather than fewer general principles with wider application. More recently, the terms procedural and conceptual learning have been widely adopted, and theoretical interpretations of these in mathematics education have increasingly highlighted their interweaving and iterative relationship (Star, 2005; Baroody, Feil & Johnson, 2007; Star, 2007; Kieran, 2013; Star & Stylianides, 2013; Foster, 2014).

The most commonly-used definitions of procedural and conceptual knowledge in the context of mathematics are those due to Hiebert and Lefevre (1986). They see conceptual knowledge as knowledge that is rich in relationships, where the connections between facts are as important as the facts themselves, whereas procedural knowledge is rules for solving mathematical problems. This distinction parallels Skemp's (1976) conclusion that there are really two kinds of *mathematics* – instrumental and relational – dealing with different kinds of knowledge. More recently, Star (2005, 2007) distinguishes between *types* of knowledge (knowledge about procedures or knowledge about concepts) and *qualities* of knowledge (superficial or deep), and complains that these are frequently confounded. He highlights the way in which "procedural" is often equated with "superficial", and "conceptual" with "deep", and draws attention to the possibility of "deep procedural knowledge" and "superficial conceptual knowledge" as valid categories. Kieran (2013) goes further in declaring the dichotomy between conceptual understanding and procedural skills a fundamentally false one. Other researchers have also explored the interplay between procedural and conceptual knowledge (Sfard, 1991), with Gray and Tall (1994) integrating processes and concepts into what they term "procepts" (Tall, 2013). But there remains the question of what precisely it is that conceptual knowledge consists of beyond confident procedural knowledge.

THE CHINESE ROOM

Searle's (1980) famous thought experiment about a "Chinese Room" was an attack on the "strong" artificial intelligence claim that a computer is a mind, having cognitive states such as "understanding". Searle imagined a native English speaker who knew no Chinese locked in a room with a book of instructions for manipulating Chinese symbols. Messages in Chinese are posted through the door and the English speaker follows the instructions in the book to produce new messages in Chinese, which they post out of the room. Unknown to them, they are having a conversation in Chinese, a language which they do not speak a word of. Searle argued that syntax does not add up to semantics; behaving "as if" you understand is not the same as understanding. But it is very difficult to pinpoint exactly where the difference lies (Gavalas, 2007). Searle does acknowledge that "The rules are in English, and I understand these rules as well as any other native speaker of English" (1980, p. 418), but it remains mysterious exactly what test could distinguish a competently performing machine from a real mathematician. A learner performing a mathematical procedure may be making mathematical sense to an observing mathematician, such as a teacher, without apparently knowing much themselves about what they are doing.

The focus here has now changed from whether the computer (or the mind as a computer) understands mathematics to the question of whether some computer could be such that it is indistinguishable from a real mathematician. It may be that, whether or not you could tell them apart, they would perform the tasks of producing syntactically correct mathematics in importantly different manners. Thus the issue becomes the *sense* in which rules are being followed. If rules are followed in a meaningful sense and their semantic content is well defined and connected within constellations of schemas, then test item responses could be strong evidence of mathematical understanding. But this requires that those items are designed so that they engage procedural knowledge in a sophisticated manner which takes into account all of the aspects of the concept image that is the object of assessment. We could specify an additional requirement that the test be administered to a human being and not a computer. While this may seem flippant, it points to the heart of Searle's argument, which is that humans follow rules through semantic causality that is more or less part of the "hardware" of our brains; that there is no (or minimal) "software" layer (Searle, 1984). So does this imply that truly instrumental understanding is an impossibility for a human being?

MATHEMATICAL UNDERSTANDING

Searle's later articulation of social theory addresses how language can be used to create a social reality which is iterative and generative (Searle 1995, 2010). Further, Searle articulates an analysis of language that points towards strong connections between the structure of language and the structure of intentional states. In some ways this leads us back to the idea of the mathematician as performing *as though* merely in command of a complex constellation of algorithms that are triggered and brought to bear in a purely syntactical manner. In light of the argument put forth by Searle, we should rather say that the mathematician employs an array of mathematical understandings which have semantic content. While this seems unsatisfying, as though Searle is saying "it is semantic when humans do it", it bears strong connections with Sierpinska's articulation of procedural understanding and its relationship to conceptual understanding. Procedural understandings, according to Sierpinska (1994):

> are representations based on some sort of schema of actions, procedures. There must be a conceptual component in them – these procedures serve to manipulate abstract objects, symbols, and they are sufficiently general to be applied in a variety of cases. Without the conceptual component they would not become procedures. We may only say that the conceptual component is stronger or weaker. (p. 51)

Hence, it is reasonable for a mathematician to see many elements of their understanding as arrays of algorithms that allow them to address wide categories of mathematical problems. Yet this is fundamentally different from how a digital computer would operate in a purely syntactical approach.

Gordon, Achiman and Melman (1981, p. 2) define *rules* as "statements of the logical form 'In type-*Y* situations one does ... *X*'". For Wittgenstein (1953), it is not possible to *choose* to follow a rule: "When I obey a rule, I do not choose. I obey the rule *blindly*" (p. 85, original emphasis). Otherwise it is not a rule. It is in this sense that Searle raises a question fundamental to this discussion: Should understanding mathematics be understood as sophisticated algorithmic arrays which are akin to complex computer programs? Searle's (1984) critique of this and related ideas has several facets, the most pertinent of which is that there is an ambiguity in what is meant by rule following and that humans and computers do not follow rules in the same sense. In essence, Searle argues that humans follow rules in as much as they understand the *meaning* of the rules (which is thus semantic and about intentional states), whereas computers are purely syntactical in their rule following; they can be said to "*act in accord with formal procedures*" (ibid. p. 45, original emphasis).

Returning to the question of relational versus instrumental understanding, it seems that if we follow Searle's arguments we can say that mathematical understanding is probably not effective human understanding if it is primarily instrumental (in the sense of syntactical rule following). However, it is clear that procedural, syntactical and algorithmic practices and concepts form an important part of the background to meaningful mathematical understanding. Thus from a perspective of assessment we would expect it to be important to assess algorithmic fluency while also seeking to assess the strength of the conceptual content associated with the procedural performance.

So in contrast to the kinds of digital computers that Searle and Hiebert and Lefevre are talking about, algorithms exist within a semantic framework. Perhaps it is as though a digital computer (syntactical machine environment) is being modelled using a semantic machine environment (the brain). If so, the potential problem for mathematics education relating to instrumental learning in mathematics may be that the seeming simplicity of rule following is made vastly more complicated by its need to run in a sort of virtual syntactic machine running on essentially semantic hardware. On the other hand, the generation of correct syntactical content is a power of certain constellations of semantic knowledge (relational knowledge). It seems that the teaching of algorithms and procedures is crucial for the development of sophisticated mathematical understanding, but also that *how* they are taught is critical to supporting the development in learners of mathematical understanding that goes beyond procedural understandings with weak conceptual content (Foster, 2014).

Habermas' theory of communication, partly based in and complementary to Searle's theories, can point towards models of understanding and how to assess it. In communicative action, as defined by Habermas (1984), action is coordinated intersubjectively through achieving understanding. The theory of communicative action (TCA) analyses communication as having an inherent rationality focused on the goal of achieving understanding. Using speech act theory and argumentation theory, Habermas identifies categories of validity claims that are raised in any communicative interaction and also identifies implicit preconditions for successful communication. The former is referred to by Habermas as 'discourse', but might better be termed 'validity-discourse', in order to differentiate it from other uses of that term in social sciences. The preconditions for communicative action are referred to collectively as the 'Ideal Speech Situation' by Habermas and constitute a set of counterfactual norms identified abductively as necessary for successful communication. These norms are focused on equitable conditions for participation in communication where the 'unforced force of the better argument' has the opportunity to motivate agreement. This is a bit tricky, as Habermas claims that such conditions must be assumed by participants as in operation in order to communicate, despite representing more of an ideal horizon that never completely obtains. Society is power-laden, and all communication occurs within a social context. Thus the breakdown of communication is all too common, and intersubjective understanding is seen as a fleeting and fallible goal that is ever approached but seldom attained.

The claim that Habermas's TCA and Searle's speech act theory are complementary and can be productively networked is based on the specific arguments made by Habermas in the TCA, his use of speech act theory to develop his ideas of communicative action and also upon analysis of similarities and departures between the principles, methodologies and questions of each author:

> Analytical philosophy, with the theory of meaning at its core, does offer a promising point of departure for a theory of communicative action that places understanding in language, as the medium for coordinating action at the focal point of interest. (Habermas 1984, p. 274)

While it might be possible to argue that Searle's theories depart somewhat from the kinds of analytic theories that Habermas wants to make use of, this is mistaken, since their focus is on incorporating theories of intentionality. Searle beings with the structure of linguistic expressions and then deals with intentionality, and importantly in his later work he introduces the idea of *collective* intentionality, which is focused on the coordination of speakers, and which is closely related to Habermas' ideas about the importance of intersubjectivity in communicative action:

> For a theory of communicative action only those analytic theories of meaning are instructive that start from the structure of linguistic expressions rather than from speakers' intentions. And the theory will need to keep in mind how the actions of several actors are linked to one another by means of the mechanism of reaching understanding. (Habermas 1984, p. 275)

Searle's ideas add rigour and detail at the level of social ontology and may allow for a more sophisticated operationalising of concepts and constructs based in Habermas' TCA. These ideas could be used to further network critical theory, cognitive science, neuroscience and other approaches to the study of mathematics education so that they may inform one another without reducing one to the other. Thus the issue of theoretical incommensurability may be navigated without theoretical insights becoming 'siloed' within various sub-cultures of theory which do not communicate with one another. A common theoretical language might allow researchers to disagree with greater clarity without running the risk of becoming an over-arching 'grand theory'. More broadly, Searle's ideas could serve as tools for building rigorous analysis of particular instances of theoretical networking, allowing productive discussion between theoretical perspectives.

These ideas can be operationalised to analyse small-group problem solving and in this manner interpret the mathematical understanding of participants (Kent, 2013), which could serve as the basis for the development of interactive assessment techniques, activities and protocols. Understanding from this perspective is about being able to identify what reasons, arguments and evidence could be legitimately raised to justify a claim. This emphasis on the identification of shared bases for validity can serve as a pragmatic approach to the analysis of human understanding in mathematics. Thus when we speak of assessing mathematical understanding we can begin to identify as a community of mathematicians and mathematics educators (with due consideration of developmental and disciplinary appropriateness) the claims and the appropriate reasons that justify these claims. We can consider how to engage participants in communicative actions around mathematical goals that require the articulation of arguments and justifications that show evidence that the participants can explain why certain mathematical claims are true.

Returning to the Chinese Room, this turn to the social does not suggest that there need be two people in the room, but rather that the person in the room must share requisite background knowledge or be able to develop it contextually with the Chinese speakers outside the room. The idea of communicative competence is key: sharing the contextual background knowledge that allows a language to have semantic meaning is the basis for ‘understanding’. This is different from quickly and accurately manipulating the symbols in a language in a syntactic fashion: no *shared* understanding entails from such activity. Now it is possible that meaning could be attributed to rules or symbols by the person in the Chinese room, but, without the ability to test these against another person who has semantic understanding of the symbols, no interpersonal communication or shared understanding is achievable. The meaning so developed would be a private language that would not necessarily correspond to that of the interlocutor. Thus the person in the Chinese room might imagine that they were having a discussion about a family’s vacation outing when in fact the interlocutor interpreted the exchange of symbols as being a mathematical discourse on the solution to an algebraic problem (or vice versa).

CONCLUSION

These ideas about the nature of the relationship between syntax and semantics, procedure and concept, and instrumental and relational understanding do not undermine the importance of procedural fluency. Pimm (1995) addresses the issue in depth and identifies some of the important features of fluency in mathematics education:

> For me, fluency is about ease of production and mastery of generation – it is used also in relation to a complex system. ‘Fluent’ may be related to efficient, or just no wasted effort. It is often about working with the *form*. Finally, it can be about not having to pay conscious attention. (ibid. p. 174, original emphasis)

Thus fluency, including syntactical fluency, can serve as partial evidence of understanding in a communicational context. Mathematical fluency, as in nonmathematical communication, is a sign of communicative competence, which is a prerequisite for interpersonal understanding according to the hermeneutic/communicational tradition (Habermas, 1984; Sierpinska, 1994). Thus when we say that a human being does not follow rules in the same sense as a computer, we mean that the symbolic rule following (or algorithmic manipulation of syntax) is done in the context of mathematical communication, and thus has semantic framing.

Habermas' articulation of rational behaviour in discursive practices has been identified as productive for the analysis of shared cognition in mathematics education (Boero et al., 2010). In communicative action participants achieve shared goals by coordinating action (including speech action) through the development of a shared understanding. Thus, establishing shared goals and coordinating action around an appropriately designed mathematical task could serve as an interpretive basis for the researcher (or other virtual participant) to make a judgement about the understanding of the participants in collaborative learning of mathematics (Kent, 2013).

We suggest that consideration of Searle's (1984) critique of cognitive science allows for ongoing productive insight into what mathematical thinking is and its relation to education. An important problem faced by the mathematics education community is how we can use ideas of relational understanding and instrumental understanding in a sophisticated manner to promote the learning of mathematics. Learners of mathematics should gain genuine experience of real mathematical sense-making rather than engage in a charade of imitating what they think such behavior should look like. The increasing focus on fluency in policy in the UK (DfE, 2013) suggests the need for tools and practices to be developed which coordinate ideas of cognition, mathematical understanding and educational practices of teaching and assessment. Our consideration of Searle's Chinese Room argument has sought to highlight the nuance involved in these issues and the kinds of practices and theoretical frameworks that could be leveraged to address the problem of interpreting learners' mathematical understanding.

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