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On the robustness of entanglement in analogue gravity systems

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Abstract. We investigate the possibility of generating quantum-correlated quasi-particles utilizing analogue gravity systems. The quantumness of these correlations is a key aspect of analogue gravity effects and their presence allows for a clear separation between classical and quantum analogue gravity effects. However, experiments in analogue systems, such as Bose–Einstein condensates (BECs) and shallow water waves, are always conducted at non-ideal conditions, in particular, one is dealing with dispersive media at non-zero temperatures. We analyse the influence of the initial temperature on the entanglement generation in analogue gravity phenomena. We lay out all the necessary steps to calculate the entanglement generated between quasi-particle modes and we analytically derive an upper bound on the maximal temperature at which given modes can still be entangled. We further investigate a mechanism to enhance the quantum correlations. As a particular example, we analyse the robustness of the

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entanglement creation against thermal noise in a sudden quench of an ideally homogeneous BEC, taking into account the super-sonic dispersion relations.

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1. Introduction

Can quantum effects in curved spacetimes be simulated in compact, laboratory-based experimental setups? Following the formal analogy between quantum field theory on curved spacetimes and classical fluid systems, which was established by Unruh in [1], this question has captivated researchers for decades (see e.g. [2] for a recent review). In particular, the prospect of accessible experimental setups to test the quantum effects of varying spacetime backgrounds has motivated scientists, who have subsequently directed their ingenuity and effort towards the study of analogue gravity systems. The range of physical systems in which such simulations can be performed is vast, reaching from actual shallow water waves [3–6] and Bose–Einstein condensates (BECs) [7–9] to laser pulse filaments [10, 11], to name but a few.

A central aim in such studies is the observation of radiation that can be associated with quantum pair creation processes, for instance, to the Hawking-, Unruh- and the dynamical Casimir effect. All of these effects rely on similar mechanisms in quantum field theory, i.e. particle creation due to time-dependent gravitational fields and boundary conditions, or the presence of horizons. It is only natural then to ask what are the criteria for associating the effects of quantum field theory with the analogue systems. How can these criteria *distinguish quantum from classical scattering processes*? In particular, which systems actually exhibit quantum effects is a matter of ongoing debate, while others, e.g. water waves, are not expected to produce genuine quantum effects at all. One aspect that has already been studied theoretically [12], and has been rigorously tested in experiments, is the spectrum of the radiation and its relation to the effective gravitational field (see e.g. [4]). However, it might be argued that the ingredient that is missing so far is the verification of the quantumness of the observed radiation via a direct detection of entanglement, which cannot be inferred from the spectrum alone without additional information. Efforts have been directed towards addressing this issue by studying non-classical behaviour as captured by sub-Poissonian statistics and the connected violation of Cauchy–Schwarz inequalities [13, 14]. However, the most paradigmatic quantum mechanical feature—*entanglement*—has not yet been verified in analogue gravity systems, despite the fact that typical pair creation processes in curved spacetime scenarios [15] involve the creation of mode entanglement. For instance, entanglement is created between modes of quantum fields in an expanding spacetime (see e.g. [16]), a system that has been previously studied in the context of analogue gravity (see e.g. [17–19]). However, the presence of quantum correlations depends

not only on the presence of entangling processes, but also on the initial state of the system. The temperature in the system needs to be low enough to allow entanglement to be generated.

Here we study how robust possible entanglement generation phenomena in analogue gravity systems, e.g. in BEC simulations of an expanding universe [16, 20, 21] and the dynamical Casimir effect [9], are against such thermal noise. For this purpose, we connect the techniques for quantum information processing with continuous variables (see e.g. [22]) and relativistic quantum information (RQI). During the last decade, the RQI community has developed techniques to quantify entanglement in quantum field theory (for a review see e.g. [23]). While previous work, see e.g. [24, 25], has already provided model specific discussions, in this paper we apply the framework recently developed within RQI [26, 27] to establish the first general description of entanglement generation in analogue gravity systems, including such effects as initial temperatures and nonlinear dispersion relations. Connecting RQI and analogue gravity promises to be a fruitful endeavour. We address the central question: *is it in principle possible to observe quantum correlations in analogue gravity systems?* Naively, the answer is: yes. But, the effects are highly sensitive to the levels of thermal noise. We provide closed analytical expressions for the required maximal background temperatures. Our framework applies directly to all analogue gravity systems where only two degrees of freedom are mixed. For example, all homogeneous analogue gravity systems and such inhomogeneous systems where only two phononic modes become entangled. In cases where more degrees of freedom are mixed, for example in the presence of inhomogeneities, our results can be considered as a limiting case that supplies an upper bound for the allowed temperature by virtue of the monogamy of entanglement [28]. Our criteria thus represent critical benchmarks that need to be taken into account by the next generation of analogue gravity experiments. In addition, we consider a scheme—entanglement resonances—to enhance the generation of quantum correlations to overcome temperature restrictions. We illustrate our results for a particular example, an ideally homogeneous BEC undergoing a single sudden quench, i.e. a change of the speed of sound, or a series of such transformations, for which we explicitly include the effects of nonlinear dispersion and we assume that the dispersive effects are negligible.

2. Entanglement in analogue gravity systems

Let us start with a brief overview of the description and quantification of entanglement. For an introductory review, see e.g. [29]. A quantum state is called *separable* with respect to the bipartition into subsystems A and B if the corresponding density operator ρ_{AB} can be written as

$$\rho_{AB} = \sum_i p_i |\psi_i^A\rangle\langle\psi_i^A| \otimes |\psi_i^B\rangle\langle\psi_i^B|, \quad (1)$$

where $\sum_i p_i = 1$, $p_i \geq 0$ and $|\psi_i^{A/B}\rangle$ are pure states of the subsystems A or B respectively. Such states can contain classical correlations, i.e. correlations that depend on the local (in the sense of the subsystems A and B) choice of basis. Quantum states that cannot be decomposed in the form of ρ_{AB} above are called *entangled*. A formal way to quantify the entanglement of a given state ρ_{AB} is the *entanglement of formation* E_{of} , defined as [30]

$$E_{\text{of}}(\rho_{AB}) = \inf_{\{p_i, \psi_i^{AB}\}} \sum_i p_i \mathcal{E}(|\psi_i^{AB}\rangle). \quad (2)$$

Here $\mathcal{E}(|\psi^{AB}\rangle)$ is the entropy of entanglement, given by the von Neumann entropy of the reduced state $\rho_A = \text{tr}_B |\psi^{AB}\rangle\langle\psi^{AB}|$. The minimum in equation (2) is taken over all pure state decompositions $\{p_i, \psi_i^{AB}\}$ that realize the state ρ_{AB} , i.e. such that $\rho_{AB} = \sum_i p_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$. For mixed states of arbitrary dimension, this measure is not computable but for some special cases, including those we discuss in this paper, the minimization in equation (2) can be carried out and E_{of} can be computed analytically.

In the analogue gravity context that we want to consider here, the subsystems A and B will be two (out of an ensemble of possibly infinitely many) bosonic modes, e.g. associated to phonons in a BEC [21]. The corresponding annihilation and creation operators, a_i and a_i^\dagger satisfy the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ and $[a_i, a_j] = 0$. The operators a_i and a_i^\dagger may be combined into the quadrature operators $q_j := \frac{1}{\sqrt{2}}(a_j + a_j^\dagger)$ and $p_j := \frac{-i}{\sqrt{2}}(a_j - a_j^\dagger)$, which, in turn, can be collected into the vector

$$\mathbb{X} := (q_1, p_1, q_2, p_2, \dots)^T. \quad (3)$$

For the particularly important class of *Gaussian states*, which includes, e.g. the vacuum and thermal states, all the information about the state ρ is encoded in the first moments $\langle\mathbb{X}_i\rangle_\rho$ and the second moments

$$\Gamma_{ij} := \langle\mathbb{X}_i\mathbb{X}_j + \mathbb{X}_j\mathbb{X}_i\rangle_\rho - 2\langle\mathbb{X}_i\rangle_\rho\langle\mathbb{X}_j\rangle_\rho, \quad (4)$$

where $\langle\mathbb{X}_i\rangle_\rho$ denotes the expectation value of the operator \mathbb{X}_i in the state ρ (see [22]). Furthermore, the *covariance matrix* Γ contains all the relevant information about the entanglement between the modes.

Let us now consider a typical transformation occurring in analogue gravity systems, for example, the generation of phonons in a BEC that is undergoing a sudden change the speed of sound. Such transformations are described by *Bogoliubov transformations*, i.e. linear transformations

$$\tilde{a}_m = \sum_n (\alpha_{mn}^* a_n - \beta_{mn}^* a_n^\dagger) \quad (5)$$

between two sets of annihilation and creation operators, $\{(a_i, a_i^\dagger)\}$ and $\{(\tilde{a}_i, \tilde{a}_i^\dagger)\}$, ($i = 1, 2, \dots$), which leave the commutation relations invariant. The Bogoliubov transformation induces a unitary transformation on the Hilbert space of states. In phase space, on the other hand, these unitaries are realized as *symplectic* transformations S that satisfy $S\Omega S^T = \Omega$, where the symplectic form Ω is defined by the relation $i\Omega_{mn} = [\mathbb{X}_m, \mathbb{X}_n]$. The transformation S can be expressed in terms of the coefficients α_{mn} and β_{mn} , which allows us to quantify the entanglement that is being generated between the modes for a wide variety of scenarios (see [26]).

At this point, it is crucial to note that in the most favourable scenarios in analogue gravity (e.g. homogeneous systems), the structure of the Bogoliubov transformations can be particularly simple. In these cases, the transformations mix pairs of degrees of freedom p, p' and the only non-vanishing Bogoliubov coefficients are $\alpha_{pp}, \alpha_{p'p'}$ and $\beta_{pp'}$. For example, in the case of homogeneous systems, the absence of boundaries that couple counter-propagating modes along with momentum conservation implies that only coefficients α_{kk} and $\beta_{k(-k)} (\forall k)$ are non-zero, where k labels the momentum. This occurs even if the dispersion relation allows for more than two modes to correspond to the same frequency. In other words, the transformation cannot shift the momenta of individual excitations, but it allows for the creation of (quasi-)particle pairs with equal but opposite momenta. Furthermore, in systems that are inherently inhomogeneous,

such as analogue black hole setups, the coupling can be limited to pairs of frequencies due to the system's stationarity. In order to illustrate our techniques, we specialize to systems where the mixing occurs between modes of opposite momenta. This simple structure of the Bogoliubov coefficients permits us to consider the covariance matrix for any pairs of modes k and $-k$ independently of any other modes of the continuum. In the following, we consider this simple structure to be an approximation for inhomogeneous systems with minor density fluctuations. In realistic setups, entanglement is generated between more than two degrees of freedom. For example, in the setups we address here, the effect of the inhomogeneity is to distribute the entanglement that is generated also across modes with different momenta [31], introducing additional noise in the reduced state of the modes k and $-k$. Due to monogamy restraints [28], this will decrease the amount of entanglement produced between these modes, such that the results we obtain can be considered as upper bounds on the entanglement generation.

Dispersive effects on the other hand, e.g. in effective Friedmann–Robertson–Walker spacetimes, are easily taken into account in terms of modified Bogoliubov coefficients. These can be obtained by solving the corresponding equation of motion for the field modes, involving fourth-order derivatives in space for sub- and super-luminal dispersion relations. For the two chosen modes, the symplectic transformation S can be written as

$$S = \begin{pmatrix} \mathcal{M}_{kk} & \mathcal{M}_{k(-k)} \\ \mathcal{M}_{(-k)k} & \mathcal{M}_{(-k)(-k)} \end{pmatrix}, \quad (6)$$

where the 2×2 blocks are given by

$$\mathcal{M}_{nn} = \begin{pmatrix} \text{Re}(\alpha_{nn}) & \text{Im}(\alpha_{nn}) \\ -\text{Im}(\alpha_{nn}) & \text{Re}(\alpha_{nn}) \end{pmatrix}, \quad (7a)$$

$$\mathcal{M}_{n(-n)} = \begin{pmatrix} -\text{Re}(\beta_{n(-n)}) & \text{Im}(\beta_{n(-n)}) \\ \text{Im}(\beta_{n(-n)}) & \text{Re}(\beta_{n(-n)}) \end{pmatrix} \quad (7b)$$

with $n = k, -k$. The unitarity of the transformation implies

$$|\alpha_{kk}|^2 - |\beta_{k(-k)}|^2 = 1, \quad (8)$$

and, in general for $\Theta \in \mathbb{R}$, we have

$$\alpha_{kk} = e^{i\Theta} \alpha_{(-k)(-k)}, \quad (9a)$$

$$\beta_{k(-k)} = e^{i\Theta} \beta_{(-k)k}, \quad (9b)$$

where the phase $\Theta \in \mathbb{R}$ is left undetermined by the unitarity of the transformation. Let us now consider the effect of the Bogoliubov transformation on the entanglement between the modes k and $-k$, including dispersive and finite temperature effects. Ideally, these modes are initially in the ground state. However, in every analogue gravity setup, the background temperature is non-zero (see e.g. [9]). We are therefore applying the transformation S to the covariance matrix $\Gamma_{\text{th}}(T)$ of a thermal state at temperature T . Since in our case the modes k and $-k$ have the same initial frequency $\omega_{\text{in}} = \omega_{\text{in}}(|k|)$, their thermal covariance matrix is proportional to the identity and given by (see [32])

$$\Gamma_{\text{th}}(T) = \coth\left(\frac{\hbar \omega_{\text{in}}}{2k_{\text{B}} T}\right) \mathbb{1}, \quad (10)$$

such that the average particle number is distributed according to Bose–Einstein statistics. The transformed state has the form

$$\tilde{\Gamma} = S \Gamma_{\text{th}}(T) S^T = \begin{pmatrix} \tilde{\Gamma}_k & C \\ C^T & \tilde{\Gamma}_{-k} \end{pmatrix}, \quad (11)$$

where C is composed of the 2×2 matrices of equation (7) and the reduced state covariance matrices of the individual modes, $\tilde{\Gamma}_k$ and $\tilde{\Gamma}_{-k}$, are identical thermal states

$$\tilde{\Gamma}_k = \tilde{\Gamma}_{-k} = \coth\left(\frac{\hbar \omega_{\text{in}}}{2 k_B T}\right) (2|\beta_{k(-k)}|^2 + 1) \mathbb{1} \quad (12)$$

with non-zero temperature even when the initial temperature T is vanishing. We shall use this fact to define a characteristic temperature T_E of the individual modes via the relation

$$\coth\left(\frac{\hbar \omega_{\text{out}}}{2 k_B T_E}\right) = 2|\beta_{k(-k)}|^2 + 1, \quad (13)$$

where we have taken into account a possible change in frequency, $\omega_{\text{in}} \rightarrow \omega_{\text{out}}$, for fixed k , due to nonlinear dispersion. This *entanglement temperature* T_E , which corresponds to the Hawking temperature for a black hole evaporation process, can be attributed purely to the entanglement that is generated from the initial vacuum in a homogeneous system, in complete analogy to the mixedness that is quantified by the entropy of entanglement in equation (2).

If the first moments $\langle \mathbb{X}_i \rangle_\rho$ of the initial state vanish, the average particle number after the transformation can be computed from

$$\tilde{N}_k = \langle a_k^\dagger a_k \rangle = \frac{1}{4}(\tilde{\Gamma}_{11} + \tilde{\Gamma}_{22} - 2), \quad (14)$$

where the $\tilde{\Gamma}_{ij}$ are the elements of the 4×4 covariance matrix $\tilde{\Gamma}$ (see equation (4)), and we obtain

$$\tilde{N}_k = \frac{1}{2} \left(\coth\left(\frac{\hbar \omega_{\text{out}}}{2 k_B T_E}\right) \coth\left(\frac{\hbar \omega_{\text{in}}}{2 k_B T}\right) - 1 \right), \quad (15)$$

which reduces to the usual Bose–Einstein statistics for $\beta_{k(-k)} = 0$, while it takes the familiar form [15] $\tilde{N}_k(T=0) = |\beta_{k(-k)}|^2$ starting from the initial vacuum. Initially, i.e. for $T_E = 0$, the distribution (15) is thermal but after the transformation, this is not necessarily the case since $T_E = T_E(k)$. From equation (12) we can further infer that $\tilde{\Gamma}$ is a *symmetric* state, i.e. $\det \Gamma_k = \det \Gamma_{-k}$, for which the entanglement of formation E_{oF} can be computed explicitly (see [22]). It is simply given as a function of the parameter $\nu_- \geq 0$, the smallest eigenvalue of $|\text{i}\Omega P_k \tilde{\Gamma} P_k|$, where $P_k = \text{diag}\{1, -1, 1, 1\}$ represents partial transposition of mode k . If $0 \leq \nu_- < 1$, the transformed state $\tilde{\Gamma}$ is entangled and E_{oF} is a monotonously decreasing function of ν_- , given by

$$E_{\text{oF}} = \begin{cases} h(\nu_-) & \text{if } 0 \leq \nu_- < 1, \\ 0 & \text{if } \nu_- \geq 1, \end{cases} \quad (16)$$

where

$$h(x) = \frac{(1+x)^2}{4x} \ln \frac{(1+x)^2}{4x} - \frac{(1-x)^2}{4x} \ln \frac{(1-x)^2}{4x}. \quad (17)$$

For the state $\tilde{\Gamma}$ of equation (11), we find

$$\nu_-(T) = \coth\left(\frac{\hbar \omega_{\text{in}}}{2 k_B T}\right) (|\alpha_{kk}| - |\beta_{k(-k)}|)^2. \quad (18)$$

While equations (16)–(18) completely quantify the entanglement that is generated by the Bogoliubov transformation in the initial thermal state of temperature T , it is equation (18) alone that is needed to determine whether or not any entanglement is created at all. In particular, we can identify the *sudden death temperature* T_{SD} , i.e. the initial temperature at which a given Bogoliubov transformation no longer generates any entanglement between the fixed modes k and $-k$. It is determined by the condition $\nu_-(T_{SD}) = 1$. By combining the entanglement temperature and equation (18), we can express this condition as

$$\coth\left(\frac{\hbar\omega_{\text{out}}}{2k_B T_E}\right) = \coth\left(\frac{\hbar\omega_{\text{in}}}{k_B T_{SD}}\right), \quad (19)$$

which, in turn, implies

$$T_{SD} = 2 \frac{\omega_{\text{in}}}{\omega_{\text{out}}} T_E. \quad (20)$$

This simple relation provides us with a powerful tool to determine whether a particular analogue gravity setup can, in principle, be expected to produce entanglement, for instance in a BEC simulating an expanding universe [16]. However, we can also formulate a simple procedure to identify transformations whose repetitions—should they be implementable—will resonantly enhance the entanglement produced between particular modes. In the following, we will study the symplectic representation S of such a repeatable transformation and identify the conditions for enhancing the ability of detecting entanglement.

3. Entanglement resonances

Any symplectic transformation of two modes can be decomposed into a passive transformation S_P , with $S_P^T S_P = \mathbb{1}$, representing rotations and beam splitting, and an active transformation $S_A = S_A^T$, consisting of single- and two-mode squeezing, i.e. $S = S_P S_A$ (see [33]). From the reduced states in equation (12), we can easily see that the transformation described by equation (6) contains no single-mode squeezing. Consequently, S_A is a pure two-mode squeezing operation, $S_A = S_{TMS}(r)$, the paradigm Gaussian entangling operation, where $r \in \mathbb{R}$ is the squeezing parameter. For the initial states of two modes that are proportional to the identity, as in our case, we can consider a *resonance condition* as discussed in [27], i.e.

$$[S, S^T] = 0. \quad (21)$$

Since the typical Bogoliubov transformations in analogue gravity systems do not contain any single-mode squeezing, i.e. $S = S_A S_{TMS}$, the condition of equation (21) has a very intuitive interpretation. It suggests that the state $\Gamma_{TMS} = S_{TMS} S_{TMS}^T$ is invariant under the passive transformation S_P , $S_P \Gamma_{TMS} S_P^T = \Gamma_{TMS}$. Since two-mode squeezing operations form a one-parameter subgroup of the symplectic transformations, $S_{TMS}(r_1) S_{TMS}(r_2) = S_{TMS}(r_1 + r_2)$, one can easily see that a transformation which satisfies the resonance condition (21) will accumulate entanglement when repeated. In particular, the entanglement of formation of $\Gamma_{TMS}(r)$ is given by $h(e^{-2|r|})$, see equations (16) and (17). The increase of entanglement for *particular modes* is then a matter of tuning the transformation at hand to fulfill the resonance condition (21).

4. Sudden quench of a Bose–Einstein condensate

Let us now discuss a specific application of the general principles we have mentioned so far. We shall consider the Bogoliubov transformation that describes a single, sudden change in the

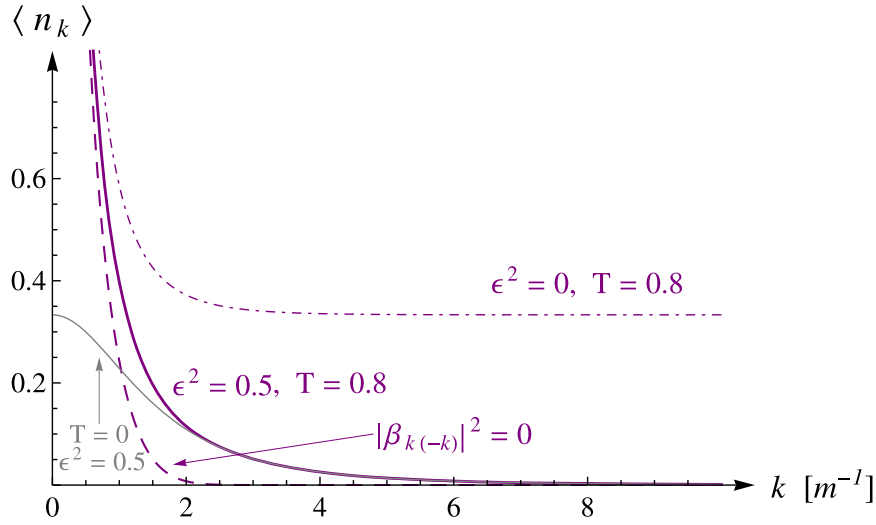


Figure 1. Average particle number across the spectrum: the average particle number $\langle n_k \rangle$ after the sudden quench is plotted as a function of the wave number k for a nonlinear dispersion relation. All curves are plotted for $c_{\text{in}} = 1 \text{ ms}^{-1}$ and $c_{\text{out}} = 3 \text{ ms}^{-1}$. Values for the parameter ϵ are shown in units of $\text{m}^2 \text{ s}^{-1}$, while temperatures T are displayed in units of $(\hbar/k_B) \text{ K}$. For a thermal state at temperature $T = 0.5$ with nonlinear dispersion ($\epsilon^2 = 0.5$), the initial average particle number (dashed purple curve) increases due to the sudden quench (solid purple curve). The increase is strongly suppressed for higher-mode numbers with respect to the case of linear dispersion (dotted-dashed purple curve).

speed of sound, a ‘quench’, of a BEC at some initial temperature T . This can be achieved via a Feshbach resonance, i.e. a sudden change of the interaction strength, such that the density and phase of the BEC remain continuous; see [21] for details. As before, we are going to make the approximation that the system is homogeneous throughout the process and that any effects of the inhomogeneity, in particular any caused by the quench, will enter as noise that reduces the entanglement. The non-adiabatic adjustment of the speed of sound, $c_{\text{in}} \rightarrow c_{\text{out}}$, causes the frequencies of the phononic modes to be altered, $\omega_{\text{in}} \rightarrow \omega_{\text{out}}$, while the momenta k remain the same. The Bogoliubov coefficients thus have exactly the previously discussed structure for $\Theta = 0$. More specifically, we have [21]

$$\alpha_{kk} = \frac{1}{2} \left(\sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} + \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \right) e^{i(\omega_{\text{out}} - \omega_{\text{in}})t_0}, \quad (22a)$$

$$\beta_{k(-k)} = \frac{1}{2} \left(\sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} - \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \right) e^{-i(\omega_{\text{out}} + \omega_{\text{in}})t_0}, \quad (22b)$$

where t_0 is the time of the transition which becomes relevant for consecutive quenches. The frequencies are given by the *nonlinear dispersion relation*

$$\omega^2 = c^2 k^2 \pm \epsilon^2 k^4 \quad (23)$$

with $\epsilon = \hbar/(2m)$, where \hbar is Planck’s constant and m is the mass of the atoms of the BEC. For a BEC, only the positive sign—*super-sonic* dispersion—occurs in equation (23), but it

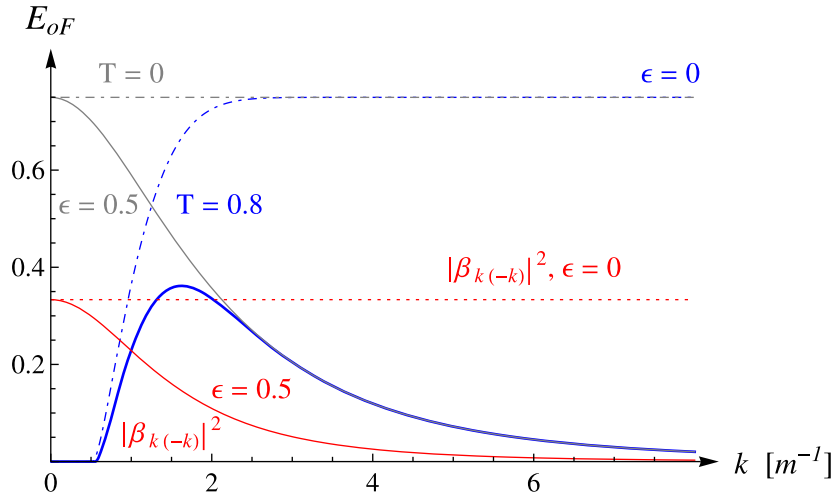


Figure 2. Entanglement generation across the spectrum: the entanglement of formation E_{oF} that is generated by a sudden quench is plotted as a function of the wave number k with a nonlinear dispersion relation and non-zero temperature (solid thick blue line). For a linear dispersion relation ($\epsilon = 0$) and zero temperature, $T = 0$, the generated entanglement (horizontal blue dashed line) and the particle production as measured by $|\beta_{k(-k)}|^2$ (horizontal red dotted line) are independent of k . The nonlinearity of the dispersion relation, e.g. as shown here for $\epsilon = 0.5$ (in units of m^2s^{-1}), is damping the generation of particles (solid red curve (bottom)) and of the entanglement (solid grey line) for higher energy modes, i.e. large k . Non-zero temperature, on the other hand, suppresses the entanglement generation for small energies (dotted-dashed blue line). All curves are plotted for $c_{\text{in}} = 1 \text{ ms}^{-1}$ and $c_{\text{out}} = 3 \text{ ms}^{-1}$. Temperatures are displayed in units of $(\hbar/k_B) \text{ K}$.

is straightforward to consider the sub-sonic case for other systems. The nonlinear dispersion relation now enters the problem of entanglement creation due to the Bogoliubov transformation in two places. First, the nonlinear effects influence the k -dependence of the initial temperature distribution, i.e. the average number of phonons is still a thermal distribution but it is strongly suppressed for higher-mode numbers, as illustrated in figure 1. Additionally, the nonlinearity enters directly in the Bogoliubov coefficients (22), which decreases the number of quasi-particles that are produced by the quench. Together, the effects of the nonlinearity and the initial temperature compete to determine the entanglement generation in equation (18). It thus becomes evident that, given a specific transformation and dispersion relation, the entanglement generation is optimal for particular regions in k -space. We have illustrated this behaviour in figure 2 for convenient, but not necessarily experimentally accessible, values of the parameters T , ϵ , c_{in} and c_{out} .

Finally, we can attempt to construct a resonant transformation from the single quench to enhance the entanglement production. For this transformation to be repeatable, it has to take $\omega_{\text{in}} \rightarrow \omega_{\text{in}}$. We can achieve a non-trivial transformation of this type by combining a first quench with $\omega_{\text{in}} \rightarrow \omega_{\text{out}}$ at time t_1 with a second quench that takes $\omega_{\text{out}} \rightarrow \omega_{\text{in}}$ at time t_2 . We then evaluate the resonance condition (21) for the total transformation, for which we find the two necessary

conditions

$$\cos(\omega_{\text{in}} t_+) \sin(\omega_{\text{out}} t_-) f(\omega_{\text{in}}, \omega_{\text{out}}, t_1, t_2) = 0, \quad (24a)$$

$$\sin(\omega_{\text{in}} t_+) \sin(\omega_{\text{out}} t_-) f(\omega_{\text{in}}, \omega_{\text{out}}, t_1, t_2) = 0, \quad (24b)$$

where $t_{\pm} = (t_1 \pm t_2)$, and

$$f = \omega_+^2 \sin(\omega_- t_-) - \omega_-^2 \sin(\omega_+ t_-) \quad (25)$$

with $\omega_{\pm} = (\omega_{\text{in}} \pm \omega_{\text{out}})$. Equation (24) is trivially satisfied if $|t_-| = (n\pi/\omega_{\text{out}})$ for any $n \in \mathbb{N}$. In these cases, the combined transformation reduces to local rotations that do not change the initial state. However, for $|t_-| \neq (n\pi/\omega_{\text{out}})$, the remaining condition $f(\omega_{\text{in}}, \omega_{\text{out}}, t_1, t_2) = 0$ can still suffice. Then the resonance condition reduces to the transcendent equation

$$\sin(\omega_- t_-)/\omega_-^2 = \sin(\omega_+ t_-)/\omega_+^2, \quad (26)$$

which can be solved numerically. However, for some special values an analytical solution lies close at hand. For instance, the transformation can be picked such that the ratio of ω_+ and ω_- is rational, i.e. $m\omega_- = n\omega_+$, $m, n \in \mathbb{Z}$. Inserting this into (26), the transcendent equation can be easily solved for $t_- = n\pi/\omega_-$. Since we are excluding the trivial transformations for which $t_- = (l\pi/\omega_{\text{out}})$, $l \in \mathbb{Z}$, we obtain the resonant solutions of equation (24) by additionally requiring that m and n have odd separation, i.e. $(m - n) \neq 2l$.

While such a series of sudden transformations might be only a rough estimation of the sinusoidal modification of the speed of sound applied in [9], our framework allows us to estimate whether a given system is above or below the sudden death temperature by measuring the average particle number for a given frequency and the initial temperature. With this information, equation (15) can be used to provide T_{SD} and determine whether entanglement is present even if the explicit Bogoliubov coefficients are not known. Since tests of entanglement in analogue systems can be rather involved, this check is vital to ensure the viability of such experiments. As mentioned above, our results are, in principle, applicable to both super- and sub-luminal types of nonlinear dispersion, for instance surface waves.

5. Conclusions

While our analysis assumes that the transformations of interest mix only couples of degrees of freedom (i.e. within homogeneous systems), the sudden death temperature we provide represents an upper bound on the allowed temperature for the inhomogeneous case as well. In particular, in the case of resonant enhancement, e.g. by driving the transformation at a fixed frequency [9], the inhomogeneity leads to a smearing of the sharp peaks and the entanglement is distributed over several adjacent modes.

In conclusion, we have conducted an analysis of the entanglement generation in analogue gravity systems at finite initial temperature. We find that the entanglement generation is fully determined by the Bogoliubov transformations describing the simulated gravitational or relativistic effects. For every pair of quasi-particle modes of the system, the problem can be phrased in terms of an effective entanglement temperature T_E . If the initial temperature is above the benchmark of $2(\omega_{\text{in}}/\omega_{\text{out}})T_E$, then no entanglement is produced between the particular modes corresponding to T_E , regardless of the homogeneity of the system. The detection of entanglement in analogue gravity systems is a major ambition of future setups, for instance, to test Bell inequalities [34] in similar settings as in [9]. Our results provide clear-cut criteria

for the feasibility of such endeavours that are applicable to a broad range of current analogue gravity experiments. In particular, we want to direct attention to [35], which appeared shortly after submission of our work and analyses closely related questions.

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