THE BOUNDARY VOLUME OF A LATTICE POLYTOPE

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Abstract. For a d-dimensional convex lattice polytope P, a formula for the boundary volume $\mathrm{vol}(\partial P)$ is derived in terms of the number of boundary lattice points on the first $\lfloor d/2 \rfloor$ dilations of P. As an application we give a necessary and sufficient condition for a polytope to be reflexive, and derive formulae for the f-vector of a smooth polytope in dimensions 3, 4, and 5. We also give applications to reflexive order polytopes, and to the Birkhoff polytope.

1. Introduction

A *lattice polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{Z}^d$. We shall assume throughout that P is of maximum dimension, so that $\dim P = d$. Denote the boundary of P by ∂P. The boundary volume $\mathrm{vol}(\partial P)$ is the volume of each facet of P normalised with respect to the sublattice containing that facet, i.e.

$$\mathrm{vol}(\partial P) := \sum_{F \text{ facet of } P} \frac{\mathrm{vol}_{d-1}(F)}{\det(\operatorname{aff} F \cap \mathbb{Z}^d)},$$

where $\mathrm{vol}_{d-1}(F)$ denotes the $(d-1)$-dimensional volume, and $\det\big(\operatorname{aff} F \cap \mathbb{Z}^d\big)$ is the determinant of the sublattice contained in the affine hull of F.

In two dimensions, the number of lattice points on the boundary of P is equal to the boundary volume. In three dimensions there is a well-known relationship which can be derived directly from Euler's formula and Pick's Theorem (see, for example, [Kas06, Proposition 10.2.3]):

Proposition 1.1. *Let P be a three-dimensional convex lattice polytope. Then*

$$\mathrm{vol}(\partial P) = \left|\partial P \cap \mathbb{Z}^3\right| - 2.$$

We shall prove the following generalisation to arbitrary dimension:

Theorem 1.2. *Let P be a d-dimensional convex lattice polytope. Then*

$$\mathrm{vol}(\partial P) = \frac{\det(\mathcal{A})}{\det(\mathcal{D})} \tag{1.1}$$

$$= \frac{1}{(d-1)!} \sum_{m=0}^{n} (-1)^{n+m} \left(\binom{d-1}{n-m} + (-1)^{d-1} \binom{d-1}{n+m} \right) \left|\partial(mP) \cap \mathbb{Z}^d\right|, \tag{1.2}$$

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where $n := \lfloor d/2 \rfloor$, $\left|\partial(0P) \cap \mathbb{Z}^d\right| := 1$, *and* $\mathcal{A}$ *and* $\mathcal{D}$ *are invertible* $n \times n$ *matrices defined by*

$$\mathcal{A}_{ij} := \begin{cases} \left|\partial(iP) \cap \mathbb{Z}^d\right| - 2(d-2n), & \text{if } j = 1, \\ i^{d-2j+1}, & \text{otherwise;} \end{cases}$$
$$\mathcal{D}_{ij} := i^{d-2j+1}.$$

The boundary volume formula for each dimension $4 \leq d \leq 10$ are listed in Table 1.

2. A General Boundary Volume Formula

Let $L_P(m) := \left|mP \cap \mathbb{Z}^d\right|$ denote the number of lattice points in P dilated by a factor of $m \in \mathbb{Z}_{\geq 0}$. Similarly, let $L_{\partial P}(m) := \left|\partial(mP) \cap \mathbb{Z}^d\right|$ denote the number of lattice points on the boundary of mP. In two dimensions the relationship between L_P and $L_{\partial P}$ is given by Pick's Theorem [Pic99]. In three dimensions Reeve proved an analogous result:

Theorem 2.1 ([Ree57, Theorem 1])**.** *Let* P *be a three-dimensional convex lattice polytope. Then*

$$2(m-1)m(m+1)\mathrm{vol}(P) = 2(L_P(m) - m\left|P \cap \mathbb{Z}^3\right|) - (L_{\partial P}(m) - m\left|\partial P \cap \mathbb{Z}^3\right|),$$

and

$$L_{\partial P}(m) = 2(1-m^2) + m^2\left|\partial P \cap \mathbb{Z}^3\right|.$$

In general the function L_P is a polynomial of degree d, and is called the *Ehrhart polynomial*. Ehrhart showed that certain coefficients of L_P have natural interpretations in terms of P.

Theorem 2.2 ([Ehr67])**.** *Let* P *be a* d*-dimensional convex lattice polytope with Ehrhart polynomial* $L_P(m) = c_d m^d + \ldots + c_1 m + c_0$*. Then:*

(i) $c_d = \mathrm{vol}(P)$*;*
(ii) $c_{d-1} = (1/2)\mathrm{vol}(\partial P)$*;*
(iii) $c_0 = 1$.

The values of the remaining coefficients of L_P have been studied in, for example, [Pom93, DR97, BDLD$^+$05]. Particular attention has been paid to the connection with toric geometry; under some additional assumptions, the function $L_P(m)$ calculates $h^0(-mK)$.

Let P° denote the strict interior of P. Ehrhart conjectured, and Macdonald proved, a remarkable reciprocity formula connecting $L_P(m)$ and $L_{P^\circ}(m)$ (see [Dan78] for a proof exploiting Serre–Grothendieck duality).

Theorem 2.3 ([Mac71])**.** *Let* P *be a* d*-dimensional convex lattice polytope. Then*

$$L_P(-m) = (-1)^d L_{P^\circ}(m).$$

Since $L_P(m) = L_{\partial P}(m) + L_{P^\circ}(m)$ we have the following immediate corollary:

Corollary 2.4. *Let* P *be a* d*-dimensional convex lattice polytope. The coefficients* c_{d-1}*,* c_{d-3}*,* c_{d-5}*, ... of* L_P *satisfy the system of equations:*

$$\frac{1}{2}L_{\partial P}(m) = \sum_{i=1}^{\lceil d/2 \rceil} m^{d-2i+1} c_{d-2i+1}, \quad \text{for all } m \in \mathbb{Z}_{>0}.$$

d	$(d-1)!\operatorname{vol}(\partial P)$
4	$L_{\partial P}(2) - 2L_{\partial P}(1)$
5	$2\left(L_{\partial P}(2) - 4L_{\partial P}(1) + 6\right)$
6	$L_{\partial P}(3) - 4L_{\partial P}(2) + 5L_{\partial P}(1)$
7	$2\left(L_{\partial P}(3) - 6L_{\partial P}(2) + 15L_{\partial P}(1) - 20\right)$
8	$L_{\partial P}(4) - 6L_{\partial P}(3) + 14L_{\partial P}(2) - 14L_{\partial P}(1)$
9	$2\left(L_{\partial P}(4) - 8L_{\partial P}(3) + 28L_{\partial P}(2) - 56L_{\partial P}(1) + 70\right)$
10	$L_{\partial P}(5) - 8L_{\partial P}(4) + 27L_{\partial P}(3) - 48L_{\partial P}(2) + 42L_{\partial P}(1)$

TABLE 1. The relationship between the boundary volume and the number of boundary points, for each dimension $4 \leq d \leq 10$ (see Theorem 1.2).

A formula for the volume of an even-dimensional convex lattice polytope was derived by Macdonald in [Mac63]:

$$\operatorname{vol}(P) = \frac{1}{d!}\left(\sum_{m=1}^{d/2}(-1)^{d/2-m}\binom{d}{d/2-m}\left(2\left|(mP)^\circ \cap \mathbb{Z}^d\right| + \left|\partial(mP)\cap\mathbb{Z}^d\right|\right) + (-1)^{d/2}\binom{d}{d/2}\right).$$

Kołodziejczyk was able to compute the odd-dimensional formula in [Koł00]:

$$\operatorname{vol}(P) = \frac{1}{(d+1)!}\sum_{m=1}^{(d+1)/2}(-1)^{(d+1)/2-m}\binom{d+1}{(d+1)/2-m}m\left(2\left|(mP)^\circ\cap\mathbb{Z}^d\right| + \left|\partial(mP)\cap\mathbb{Z}^d\right|\right).$$

It is worth noticing that, with a little rearranging, one can combine these results to give a general form remarkably similar to equation (1.2).

Theorem 2.5. *Let P be a d-dimensional lattice polytope. Then*

$$\operatorname{vol}(P) = \frac{1}{d!}\sum_{m=0}^{N}(-1)^{N+m}\left(\binom{d}{N-m} + (-1)^d\binom{d}{N+m}\right)\left(\left|mP\cap\mathbb{Z}^d\right| - \frac{1}{2}\left|\partial(mP)\cap\mathbb{Z}^d\right|\right),$$

where $N := \lceil d/2\rceil$ and $\left|\partial(0P)\cap\mathbb{Z}^d\right| := 1$.

Proof of Theorem 1.2. We wish to express the value of the penultimate coefficient c_{d-1} of L_P in terms of $L_{\partial P}$. A formula for $\operatorname{vol}(\partial P)$ follows from Theorem 2.2 (ii). We shall handle the even dimensional and odd dimensional cases separately. For brevity let us define

$$b_m := \frac{1}{2m}L_{\partial P}(m), \qquad \text{for all } m \in \mathbb{Z}_{>0}.$$

When $d = 2n$ is even, Corollary 2.4 tells us that the coefficients satisfy the linear system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2^2 & \dots & 2^{d-2} \\ \vdots & \vdots & & \vdots \\ 1 & n^2 & \dots & n^{d-2} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{d-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}. \tag{2.1}$$

Equation (1.1) follows from an application of Cramer's rule and some elementary matrix operations.

To obtain the explicit description (1.2), consider the square matrix on the left hand side of (2.1). This is a Vandermonde matrix; we can express its inverse in terms of the product $U \cdot L$ ([Tur66, equations (5) and (7)]), where U is an upper triangular matrix with 1s on the diagonal, and L is a lower triangular matrix given by

$$L_{ij} = \begin{cases} 0, & \text{if } i < j, \\ 1, & \text{if } i = j = 1, \\ \displaystyle\prod_{\substack{k=1 \\ k \neq j}}^{i} \frac{1}{j^2 - k^2}, & \text{otherwise.} \end{cases}$$

More explicitly,

$$\begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{d-1} \end{pmatrix} = \begin{pmatrix} 1 & & \star \\ & 1 & \\ 0 & \ddots & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & 0 \\ -\frac{1}{3} & \frac{1}{3} & & \\ \vdots & \vdots & \ddots & \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Since we need only know the bottom row of L in order to determine the coefficient c_{d-1}, we obtain

$$\begin{aligned} c_{d-1} &= \sum_{m=1}^{n} \Big(\prod_{\substack{k=1 \\ k \neq m}}^{n} \frac{1}{m^2 - k^2} \Big) b_m \\ &= 2 \sum_{m=1}^{n} \frac{(-1)^{n+m} m^2}{(n+m)!(n-m)!} b_m \\ &= \frac{1}{(2n)!} \sum_{m=1}^{n} (-1)^{n+m} \binom{2n}{n+m} m L_{\partial P}(m). \end{aligned}$$

Observing that

$$\frac{m}{n} \binom{2n}{n+m} = \binom{2n-1}{n-m} - \binom{2n-1}{n+m}$$

we obtain the result in the even-dimensional case:

$$c_{d-1} = \frac{1}{2 \cdot (2n-1)!} \sum_{m=0}^{n} (-1)^{n+m} \left(\binom{2n-1}{n-m} - \binom{2n-1}{n+m} \right) L_{\partial P}(m).$$

When $d = 2n + 1$ is odd we obtain the linear system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2^2 & \dots & 2^{d-3} \\ \vdots & \vdots & & \vdots \\ 1 & n^2 & \dots & n^{d-3} \end{pmatrix} \cdot \begin{pmatrix} c_2 \\ c_4 \\ \vdots \\ c_{d-1} \end{pmatrix} = \begin{pmatrix} b_1 - 1 \\ b_2/2 - 1/2^2 \\ \vdots \\ b_n/n - 1/n^2 \end{pmatrix}.$$

Once again, Cramer's rule yields (1.1).

Solving as in the even case, we have that

$$c_{d-1} = \frac{1}{(2n)!} \sum_{m=1}^{n} (-1)^{n+m} \binom{2n}{n+m} \left(L_{\partial P}(m) - 2\right).$$

From the identity

$$\sum_{m=0}^{2n} (-1)^m \binom{2n}{m} = 0$$

we deduce that

$$2 \sum_{m=1}^{n} (-1)^{n+m} \binom{2n}{n+m} = (-1)^{n+1} \binom{2n}{n}.$$

Hence:

$$\begin{aligned} c_{d-1} &= \frac{1}{(2n)!} \left((-1)^n \binom{2n}{n} + \sum_{m=1}^{n} (-1)^{n+m} \binom{2n}{n+m} L_{\partial P}(m) \right) \\ &= \frac{1}{2 \cdot (2n)!} \left(\sum_{m=0}^{n} (-1)^{n+m} \left(\binom{2n}{n-m} + \binom{2n}{n+m} \right) L_{\partial P}(m) \right). \end{aligned}$$

This gives us (1.2). □

3. Applications to Reflexive Polytopes

In [Sta80] Stanley proved that the generating function for L_P can be written as a rational function

$$\mathrm{Ehr}_P(t) := \sum_{m \geq 0} L_P(m) t^m = \frac{\delta_0 + \delta_1 t + \ldots + \delta_d t^d}{(1-t)^{d+1}},$$

where the coefficients δ_i are non-negative. The sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ is known as the *δ-vector* of P. For an elementary proof of this and other relevant results, see [BS07] and [BR07].

The following corollary is a consequence of Theorem 2.2.

Corollary 3.1. *Let P be a d-dimension convex lattice polytope with δ-vector $(\delta_0, \delta_1, \ldots, \delta_d)$. Then:*

(i) $\delta_0 = 1$;
(ii) $\delta_1 = \left|P \cap \mathbb{Z}^d\right| - d - 1$;
(iii) $\delta_d = \left|P^\circ \cap \mathbb{Z}^d\right|$;
(iv) $\delta_0 + \ldots + \delta_d = d! \,\mathrm{vol}(P)$.

Hibi proved [Hib94] the following lower bound on the δ_i, commonly referred to as the *Lower Bound Theorem*:

Theorem 3.2. *Let P be a d-dimensional convex lattice polytope with $\left|P^\circ \cap \mathbb{Z}^d\right| > 0$. Then $\delta_1 \leq \delta_i$ for every $2 \leq i \leq d-1$.*

As a consequence of the Lower Bound Theorem we have a bound on the number of lattice points in P in terms of the volume of P. Note that this bound is sharp: equality is given in each dimension by the d-simplex $\operatorname{conv}\{e_1, \ldots, e_d, -e_1 - \ldots - e_d\}$, where $e_1, \ldots, e_d$ is a basis of $\mathbb{Z}^d$.

Corollary 3.3. *Let P be a d-dimensional convex lattice polytope with $\left|P^\circ \cap \mathbb{Z}^d\right| > 0$. Then*

$$d! \operatorname{vol}(P) \geq (d-1)\left|P \cap \mathbb{Z}^d\right| - d^2 + 3.$$

We have equality if and only if the δ-vector of P equals

$$(1, \left|P \cap \mathbb{Z}^d\right| - d - 1, \left|P \cap \mathbb{Z}^d\right| - d - 1, \ldots, \left|P \cap \mathbb{Z}^d\right| - d - 1, 1).$$

Proof. This is a consequence of Corollary 3.1 parts (ii) and (iv), and Theorem 3.2. □

A convex lattice polytope P is called *Fano* if $P^\circ \cap \mathbb{Z}^d = \{0\}$; i.e. if the origin is the only interior lattice point of P. A convex lattice polytope P is called *reflexive* if the dual polyhedron

$$P^\vee := \{u \in \mathbb{R}^d \mid \langle u, v\rangle \leq 1 \text{ for all } v \in P\}$$

is also a lattice polytope. Clearly any reflexive polytope is Fano. Reflexive polytopes are of particular relevance to toric geometry: they correspond to Gorenstein toric Fano varieties (see [Bat94]). There are many interesting characterisations of reflexive polytopes (for example the list in [HM06]).

Theorem 3.4. *Let P be a d-dimensional Fano polytope. The following are equivalent:*

(i) *P is reflexive;*
(ii) *$L_P(m) = L_{\partial P}(m) + L_P(m-1)$ for all $m \in \mathbb{Z}_{>0}$;*
(iii) *$d \operatorname{vol}(P) = \operatorname{vol}(\partial P)$;*
(iv) *$\delta_i = \delta_{d-i}$ for all $0 \leq i \leq d$.*

Theorem 3.4 (iv) is commonly known as *Hibi's Palindromic Theorem* [Hib91] and can be generalised to duals of non-reflexive polytopes [FK08]. It is a consequence of a more general result of Stanley's [Sta78] concerning Gorenstein rings. Clearly any polytope giving equality in Corollary 3.3 must be reflexive.

Remark 3.5. Of course, as a consequence of equation (1.2) and Theorem 3.4 (iii), one can add the equivalent condition:

(v) $$\operatorname{vol}(P) = \frac{1}{d!}\sum_{m=0}^{n}(-1)^{n+m}\left(\binom{d-1}{n-m} + (-1)^{d-1}\binom{d-1}{n+m}\right)\left|\partial(mP) \cap \mathbb{Z}^d\right|.$$

We are now in a position to express the volume of a reflexive polytope in terms of the number of lattice points in the first n dilations.

Corollary 3.6. *Let P be a d-dimensional reflexive polytope. Then*

$$\mathrm{vol}(P) = \frac{1}{d!}\sum_{m=0}^{n}(-1)^{n+m}\left(\binom{d}{n-m} + (-1)^{d-1}\binom{d}{n+m+1}\right)\left|mP \cap \mathbb{Z}^d\right|, \tag{3.1}$$

where $n := \lfloor d/2 \rfloor$.

Proof. This follows from Theorem 3.4 (ii), Remark 3.5, and the recursive definition of the binomial coefficient. □

It is tempting to conjecture that the converse of Corollary 3.6 is true. However, suppose that P is a three-dimensional convex lattice polytope satisfying equation (3.1). By Theorem 2.1 we have that:

$$\begin{aligned} L_P(m) - L_{\partial P}(m) - L_P(m-1) =& (\left|P \cap \mathbb{Z}^3\right| - \left|\partial P \cap \mathbb{Z}^3\right| - 1)m^2 - \\ & (\left|P \cap \mathbb{Z}^3\right| - \left|\partial P \cap \mathbb{Z}^3\right| - 1)m + (\left|P \cap \mathbb{Z}^3\right| - \left|\partial P \cap \mathbb{Z}^3\right| - 1). \end{aligned}$$

Thus we require the additional assumption that $\left|P^\circ \cap \mathbb{Z}^3\right| = 1$; only then would it follow (by Theorem 3.4 (ii)) that P is reflexive.

More generally we can make use of Theorems 1.2 and 2.5 to write down a necessary and sufficient relation between the number of points in, and on the boundary of, the first N dilations of P.

Theorem 3.7. *Let P be d-dimensional Fano polytope. P is reflexive if and only if*

$$0 = \begin{cases} \displaystyle\sum_{m=0}^{N}(-1)^{N+m}\binom{2N}{N+m}\left(d\left|mP \cap \mathbb{Z}^d\right| - (N+m)\left|\partial(mP) \cap \mathbb{Z}^d\right|\right), & \text{if } d \text{ is even,} \\ \displaystyle\sum_{m=0}^{N}(-1)^{N+m}\binom{2N}{N+m}\left(md\left|mP \cap \mathbb{Z}^d\right| + \left(N^2 - m^2 - \frac{md}{2}\right)\left|\partial(mP) \cap \mathbb{Z}^d\right|\right), & \text{if } d \text{ is odd,} \end{cases}$$

where $N := \lceil d/2 \rceil$ and $\left|\partial(0P) \cap \mathbb{Z}^d\right| := 1$.

Proof. Suppose first that d is even, so that $N = n$. By Theorem 3.4 (iii), P is reflexive if and only if

$$\begin{aligned} \sum_{m=0}^{n}(-1)^{n+m}\left(\binom{d-1}{n-m} - \binom{d-1}{n+m}\right)\left|\partial(mP) \cap \mathbb{Z}^d\right| = \\ \sum_{m=0}^{n}(-1)^{n+m}\left(\binom{d}{n-m} + \binom{d}{n+m}\right)\left(\left|mP \cap \mathbb{Z}^d\right| - \frac{1}{2}\left|\partial(mP) \cap \mathbb{Z}^d\right|\right), \end{aligned}$$

where the left hand side follows from Theorem 1.2, and the right hand side from Theorem 2.5.

Using the binomial identity

$$\binom{d-1}{n-m} - \binom{d-1}{n+m} = \frac{2m}{d}\binom{d}{n+m},$$

we have that

$$\sum_{m=0}^{n}(-1)^{n+m}\binom{d}{n+m}m\left|\partial(mP)\cap\mathbb{Z}^d\right| = \\ d\sum_{m=0}^{n}(-1)^{n+m}\binom{d}{n+m}\left(\left|mP\cap\mathbb{Z}^d\right|-\frac{1}{2}\left|\partial(mP)\cap\mathbb{Z}^d\right|\right).$$

Noticing that $d/2 = n$, we obtain our result.

Now suppose that d is odd. In particular, $N = n+1$. In this case we have that P is reflexive if and only if

$$\sum_{m=0}^{n}(-1)^{n+m}2\binom{d-1}{n+m}\left|\partial(mP)\cap\mathbb{Z}^d\right| = \\ \sum_{m=0}^{n+1}(-1)^{n+m}\left(\binom{d}{n+m+1}-\binom{d}{n+m}\right)\left(\left|mP\cap\mathbb{Z}^d\right|-\frac{1}{2}\left|\partial(mP)\cap\mathbb{Z}^d\right|\right).$$

By standard binomial identities, we have that

$$\begin{aligned}\binom{d-1}{n+m} &= \frac{n+m+1}{d}\binom{d}{n+m+1}\\ &= \frac{n+m+1}{d}\binom{d}{n-m}\\ &= \frac{(n+m+1)(n-m+1)}{d(d+1)}\binom{d+1}{n+m+1},\end{aligned}$$

and that

$$\binom{d}{n+m+1}-\binom{d}{n-m+1} = -\frac{2m}{d+1}\binom{d+1}{n+m+1}.$$

Observing that $n-m+1$ vanishes when $m = n+1$, we obtain the equality

$$\sum_{m=0}^{n+1}(-1)^{n+m}(n+m+1)(n-m+1)\binom{d+1}{n+m+1}\left|\partial(mP)\cap\mathbb{Z}^d\right| = \\ \sum_{m=0}^{n+1}(-1)^{n+m+1}md\binom{d+1}{n+m+1}\left(\left|mP\cap\mathbb{Z}^d\right|-\frac{1}{2}\left|\partial(mP)\cap\mathbb{Z}^d\right|\right),$$

which is equivalent to

$$\sum_{m=0}^{N}(-1)^{N+m}\binom{d+1}{N+m}\left(md\left|mP\cap\mathbb{Z}^d\right|+\left((N+m)(N-m)-\frac{md}{2}\right)\left|\partial(mP)\cap\mathbb{Z}^d\right|\right) = 0.$$

□

The conditions given by Theorem 3.7 are summarised in Table 2 for low dimensions.

d	$f(P)$
3	$2L_P(2) - L_{\partial P}(2) - 4L_P(1) - 2L_{\partial P}(1) + 8$
4	$L_P(2) - L_{\partial P}(2) - 4L_P(1) + 3L_{\partial P}(1) + 3$
5	$2L_P(3) - L_{\partial P}(3) - 8L_P(2) + 10L_P(1) + 11L_{\partial P}(1) - 24$
6	$L_P(3) - L_{\partial P}(3) - 6L_P(2) + 5L_{\partial P}(2) + 15L_P(1) - 10L_{\partial P}(1) - 10$
7	$2L_P(4) - L_{\partial P}(4) - 12L_P(3) + 2L_{\partial P}(3) + 28L_P(2) + 10L_{\partial P}(2) - 28L_P(1) - 46L_{\partial P}(1) + 80$
8	$L_P(4) - L_{\partial P}(4) - 8L_P(3) + 7L_{\partial P}(3) + 28L_P(2) - 21L_{\partial P}(2) - 56L_P(1) + 35L_{\partial P}(1) + 35$

TABLE 2. A d-dimensional Fano polytope P is reflexive if and only if the equation $f(P)$ in the second column vanishes (see Theorem 3.7).

Notice that if P is a reflexive polytope and d is even then, by Theorem 3.4 (ii), Theorem 3.7 reduces to

$$\begin{aligned}0 &= \sum_{m=0}^{n}(-1)^{n+m}\binom{d}{n+m}\left(d\left|mP\cap\mathbb{Z}^d\right| - (n+m)\left(\left|mP\cap\mathbb{Z}^d\right| - \left|(m-1)P\cap\mathbb{Z}^d\right|\right)\right)\\ &= \sum_{m=0}^{n-1}(-1)^{n+m}\binom{2n}{n-m}(n-m)\left|mP\cap\mathbb{Z}^d\right| - \\ &\qquad\qquad \sum_{m=0}^{n-1}(-1)^{n+m}\binom{2n}{n+m+1}(n+m+1)\left|mP\cap\mathbb{Z}^d\right|.\end{aligned}$$

Clearly the right hand side vanishes, so we learn nothing new. The odd-dimensional case is different; the relation is given in Theorem 3.8 and calculated for small dimensions in Table 3.

Theorem 3.8. *Let P be a reflexive d-dimensional polytope, where d is odd. Then*

$$\sum_{m=0}^{N}(-1)^{N+m}\binom{d+2}{N-m}\left|mP\cap\mathbb{Z}^d\right| = 0,$$

where $N := \lceil d/2\rceil$.

Proof. From Theorem 3.4 (ii) and Theorem 3.7 we have that

$$\begin{aligned}0 &= (-1)^N\binom{2N}{N}N^2 + \sum_{m=1}^{N}(-1)^{N+m}\binom{2N}{N+m}\left(md\left|mP\cap\mathbb{Z}^d\right| + \right.\\ &\qquad\left.\left(N^2-m^2-\frac{md}{2}\right)\left(\left|mP\cap\mathbb{Z}^d\right| - \left|(m-1)P\cap\mathbb{Z}^d\right|\right)\right)\\ &= \sum_{m=0}^{N}(-1)^{N+m}\binom{2N}{N+m}\left(\frac{md}{2}+N^2-m^2\right)\left|mP\cap\mathbb{Z}^d\right| - \\ &\qquad\qquad \sum_{m=0}^{N}(-1)^{N+m}\binom{2N}{N+m+1}\left(\frac{(m+1)d}{2}-N^2+(m+1)^2\right)\left|mP\cap\mathbb{Z}^d\right|.\end{aligned}$$

d	$g(P)$
3	$L_P(2) - 5L_P(1) + 10$
5	$L_P(3) - 7L_P(2) + 21L_P(1) - 35$
7	$L_P(4) - 9L_P(3) + 36L_P(2) - 84L_P(1) + 126$
9	$L_P(5) - 11L_P(4) + 55L_P(3) - 165L_P(2) + 330L_P(1) - 462$

TABLE 3. If P is a d-dimensional reflexive polytope then the equation $g(P)$ in the second column will vanish (see Theorem 3.8).

Now:

$$\begin{aligned}\binom{2N}{N+m}\left(\frac{md}{2}+N^2-m^2\right)-\binom{2N}{N+M+1}\left(\frac{(m+1)d}{2}-N^2+(m+1)^2\right)\\ =\left(\binom{2N}{N+m}-\binom{2N}{N+m+1}\right)\frac{md}{2}+\\ \left(\binom{2N}{N+m}+\binom{2N}{N+m+1}\right)(N^2-m^2)-\binom{2N}{N+m+1}\left(2m+1+\frac{d}{2}\right),\end{aligned}$$

which, by standard results on the binomial coefficient, reduces to

$$\begin{aligned}\binom{2N+1}{N-m}\frac{md(2m+1)}{2(2N+1)}+\binom{2N+1}{N-m}(N^2-m^2)-\binom{2N+1}{N-m}\left(2m+1+\frac{d}{2}\right)\frac{N-m}{2N+1}\\ =\binom{2N+1}{N-m}\frac{1}{2N+1}\left(\frac{md}{2}(2m+1)+(N-m)(2N^2+N+2mN-m-\frac{d}{2}-1)\right).\end{aligned}$$

Since $d = 2N - 1$ we can simplify the term in brackets:

$$\begin{aligned}\frac{md}{2}(2m+1)+(N-m)(2N^2+N+2mN-m-\frac{d}{2}-1)\\ =\frac{md}{2}(2m+1)+(N-m)(2N^2+N+md-\frac{d}{2}-1)\\ =\frac{md}{2}(2N+1)+(N-m)(2N^2-\frac{1}{2})\\ =\frac{N(2N-1)(2N+1)}{2}.\end{aligned}$$

Thus we have that

$$\frac{N(2N-1)}{2}\sum_{m=0}^{N}(-1)^{N+m}\binom{2N+1}{N-m}\left|mP\cap\mathbb{Z}^d\right|=0.$$

Finally, since we are free to divide through by a non-zero constant, we obtain our result. □

By exploiting Hibi's Palindromic Theorem one can express the δ_i in terms of $L_P(m)$, for $1 \leq m \leq \lfloor d/2 \rfloor$. When $d = 4$ we obtain the δ-vector

$$(1, \left|P\cap\mathbb{Z}^4\right| - 5, \left|2P\cap\mathbb{Z}^4\right| - 5\left|P\cap\mathbb{Z}^4\right| + 10, \left|P\cap\mathbb{Z}^4\right| - 5, 1), \tag{3.2}$$

and when $d = 5$ we have

$$(1, \left|P\cap\mathbb{Z}^5\right| - 6, \left|2P\cap\mathbb{Z}^5\right| - 6\left|P\cap\mathbb{Z}^5\right| + 15, \left|2P\cap\mathbb{Z}^5\right| - 6\left|P\cap\mathbb{Z}^5\right| + 15, \left|P\cap\mathbb{Z}^5\right| - 6, 1). \tag{3.3}$$

Corollary 3.9. *If P is a 4-dimensional reflexive polytope then the following bound is sharp:*

$$6\left|P\cap\mathbb{Z}^4\right| \leq \left|2P\cap\mathbb{Z}^4\right| + 15.$$

If P is a 5-dimensional reflexive polytope then the following bounds are sharp:

$$\left|P\cap\mathbb{Z}^5\right| \leq \frac{1}{7}\left|2P\cap\mathbb{Z}^5\right| + 3,$$
$$\left|2P\cap\mathbb{Z}^5\right| \leq \frac{1}{4}\left|3P\cap\mathbb{Z}^5\right| + 7.$$

Proof. By Theorem 3.2 we have that $\delta_1 \leq \delta_2$. Applying this to (3.2) gives

$$6\left|P\cap\mathbb{Z}^4\right| \leq \left|2P\cap\mathbb{Z}^4\right| + 15, \text{ when } d = 4,$$
$$\left|P\cap\mathbb{Z}^5\right| \leq \frac{1}{7}\left|2P\cap\mathbb{Z}^5\right| + 3, \text{ when } d = 5.$$

In the case when $d = 5$ we apply Theorem 3.8 to (3.3), obtaining the second bound.

If P is a 4-dimensional reflexive polytope such that $6\left|P\cap\mathbb{Z}^4\right| = \left|2P\cap\mathbb{Z}^4\right| + 15$ then it has δ-vector

$$(1, \left|P\cap\mathbb{Z}^4\right| - 5, \left|P\cap\mathbb{Z}^4\right| - 5, \left|P\cap\mathbb{Z}^4\right| - 5, 1)$$

and $4!\,\mathrm{vol}(P) = 3\left|P\cap\mathbb{Z}^4\right| - 13$. These conditions are satisfied by the simplex associated with $\mathbb{P}^4$ (see the remark preceding Corollary 3.3).

Suppose that P is a 5-dimensional reflexive polytope attaining both of the bounds above. Then

$$\left|2P\cap\mathbb{Z}^5\right| = 7\left|P\cap\mathbb{Z}^5\right| - 21,$$
$$\text{and } \left|3P\cap\mathbb{Z}^5\right| = 28\left|P\cap\mathbb{Z}^5\right| - 112.$$

In particular, the δ-vector is given by

$$(1, \left|P\cap\mathbb{Z}^5\right| - 6, \left|P\cap\mathbb{Z}^5\right| - 6, \left|P\cap\mathbb{Z}^5\right| - 6, \left|P\cap\mathbb{Z}^5\right| - 6, 1),$$

and $5!\,\mathrm{vol}(P) = 4\left|P\cap\mathbb{Z}^5\right| - 22$. An example satisfying these conditions is the simplex associated with $\mathbb{P}^5$. □

The examples given in Corollary 3.9 are not unique. A search through Øbro's classification of the smooth polytopes in dimensions 4 and 5 (which form a subset of the reflexive polytopes) gives many more examples[1]. These are recorded in Table 4.

[1] http://grdb.lboro.ac.uk/search/toricsmooth?id_cmp=in&id=24,25,127,128,138,139,144,145,147
http://grdb.lboro.ac.uk/search/toricsmooth?id_cmp=in&id=148,149,950,954,955,989,990,1008,1009,1010,1013

d	ID
4	24, 25, 127, 128, 138, 139, 144, 145, 147
5	148, 149, 950, 954, 955, 989, 990, 1008, 1009, 1010, 1013

TABLE 4. The smooth polytopes attaining the bounds in Corollary 3.9. The ID refers to the ID of the polytope in the online Graded Ring Database; the data was calculated using [Øbr07].

4. Applications to Smooth Polytopes

Let the number of i-dimensional faces of a polytope P be denoted by f_i. The vector $(f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of P. By convention $f_{-1} = f_d = 1$, representing the empty face $\emptyset$ and the entire polytope P. The f-vector satisfies *Euler's relation*

$$\sum_{i=-1}^{d} (-1)^i f_i = 0. \tag{4.1}$$

When P is simplicial (i.e. the facets of P are $(d-1)$-simplicies) the *Dehn-Sommerville equations* give some additional relations amongst the f_i. Conjectured by Dehn and first proved by Sommerville, these equations did not become widely known until they were rediscovered by Klee.

Theorem 4.1 ([Kle64])**.** *Let P be a d-dimensional simplicial lattice polytope with f-vector $(f_0, f_1, \ldots, f_{d-1})$. Then*

$$f_i = \sum_{j=i}^{d-1} (-1)^{d-1-j} \binom{j+1}{i+1} f_j, \qquad \text{for } 1 \leq i \leq d-2.$$

A d-dimensional convex lattice polytope P is called *smooth* if the vertices of any facet of P form a $\mathbb{Z}$-basis of the ambient lattice $\mathbb{Z}^d$. Any such P is simplicial and reflexive. Smooth polytopes are in bijective correspondence with smooth toric Fano varieties, and as such have been the subject of much study (see, for example, [Bat91, Øbr07]).

In [Par03] Park investigated the f-vector of smooth polytopes of dimension $3 \leq d \leq 5$ and established weak bounds on the f_i. We shall make use of Theorem 1.2 to give an explicit description of the f-vector in those dimensions.

Theorem 4.2. *If P is a 3-dimensional smooth polytope then its f-vector is given by*

$$\left(\left|\partial P \cap \mathbb{Z}^3\right|, 3\left|\partial P \cap \mathbb{Z}^3\right| - 6, 2\left|\partial P \cap \mathbb{Z}^3\right| - 4 \right).$$

If P is a 4-dimensional smooth polytope then its f-vector is given by

$$\left(\left|\partial P \cap \mathbb{Z}^4\right|, \left|\partial(2P) \cap \mathbb{Z}^4\right| - \left|\partial P \cap \mathbb{Z}^4\right|, 2\left|\partial(2P) \cap \mathbb{Z}^4\right| - 4\left|\partial P \cap \mathbb{Z}^4\right|, \right.$$
$$\left. \left|\partial(2P) \cap \mathbb{Z}^4\right| - 2\left|\partial P \cap \mathbb{Z}^4\right| \right).$$

If P is a 5-dimensional smooth polytope then its f-vector is given by

$$\big(\left|\partial P\cap\mathbb{Z}^5\right|, \left|\partial(2P)\cap\mathbb{Z}^5\right| - \left|\partial P\cap\mathbb{Z}^5\right|, 4\left|\partial(2P)\cap\mathbb{Z}^5\right| - 14\left|\partial P\cap\mathbb{Z}^5\right| + 20,$$
$$5\left|\partial(2P)\cap\mathbb{Z}^5\right| - 20\left|\partial P\cap\mathbb{Z}^5\right| + 30, 2\left|\partial(2P)\cap\mathbb{Z}^5\right| - 8\left|\partial P\cap\mathbb{Z}^5\right| + 12\big).$$

Proof. Let P be a d-dimensional smooth polytope. By definition each facet F of P is a simplex whose vertices generate the underlying lattice $\mathbb{Z}^d$. Hence $\operatorname{vol}(F) = 1/(d-1)!$, so

$$(d-1)!\operatorname{vol}(\partial P) = f_{d-1}.$$

Furthermore, $|\partial P\cap\mathbb{Z}^n| = f_0$.

$d=3$: Theorem 4.1 gives $2f_1 = 3f_2$, and Theorem 1.2 yields $f_2 = 2f_0 - 4$. Thus the f-vector is uniquely defined in terms of f_0.

$d=4$: In this case Theorem 4.1 gives $f_2 = 2f_3$. Applying (4.1) we obtain $f_1 = f_0 + f_3$. Finally, Theorem 1.2 tells us that $f_3 = \left|\partial(2P)\cap\mathbb{Z}^4\right| - 2f_0$. The result follows.

$d=5$: In dimension five Theorem 4.1 and equation (4.1) give three relations:

$$2f_1 = 3f_2 - 5f_4,$$
$$2f_3 = 5f_4,$$
$$2f_0 - f_2 + 2f_4 = 4.$$

From Theorem 1.2 we know that $f_4 = 2\left|\partial(2P)\cap\mathbb{Z}^5\right| - 8f_0 + 12$. Substituting, we see that the f-vector is uniquely defined in terms of $\left|\partial(2P)\cap\mathbb{Z}^5\right|$ and $\left|\partial P\cap\mathbb{Z}^5\right|$.

□

It is worth noting that Casagrande [Cas06] proves a sharp bound for $\left|\partial P\cap\mathbb{Z}^d\right|$ in terms of the dimension, and Batyrev [Bat99, Theorem 2.3.7] gives us a bound on f_{d-3} in terms of f_{d-2}. Bremner and Klee [BK99] tell us a lower bound on f_1 in terms of f_0 and d. These results are collected in the following theorem.

Theorem 4.3. *Let P be d-dimensional smooth polytope. Then the following inequalities hold:*

(i) $\left|\partial P\cap\mathbb{Z}^d\right| \leq \begin{cases} 3d, & \text{if } d \text{ is even;} \\ 3d-1, & \text{if } d \text{ is odd.} \end{cases}$

(ii) $12f_{d-3} \geq (3d-4)f_{d-2}$.

(iii) $df_0 \leq f_1 + \binom{d+1}{2}$.

Thus we obtain upper and lower bounds on $\left|\partial(2P)\cap\mathbb{Z}^d\right|$ when $d = 4$ or 5.

Corollary 4.4. *If P is a 4-dimensional smooth polytope then*

$$5\left|\partial P\cap\mathbb{Z}^4\right| - 10 \leq \left|\partial(2P)\cap\mathbb{Z}^4\right| \leq 5\left|\partial P\cap\mathbb{Z}^4\right|.$$

If P is a 5-dimensional smooth polytope then

$$42\left|\partial P\cap\mathbb{Z}^5\right| - 105 \leq 7\left|\partial(2P)\cap\mathbb{Z}^5\right| \leq 52\left|\partial P\cap\mathbb{Z}^5\right| - 90.$$

Proof. Apply Theorem 4.3 (ii) and (iii) to Theorem 4.2. □

d	Equality	ID
4	Lower	$24, 25, 127, 128, 138, 139, 144, 145, 147$
	Upper	$63, 100$
5	Lower	$148, 149, 950, 954, 955, 989, 990, 1008, 1009, 1010, 1013$
	Upper	None

TABLE 5. The smooth polytopes attaining one of the bounds in Corollary 4.4. The ID refers to the ID of the polytope in the online Graded Ring Database; the data was calculated using [Øbr07].

Corollary 4.5. *If P is a 4-dimensional smooth polytope then*

$$4! \operatorname{vol}(P) \leq 3f_0.$$

If P is a 5-dimensional smooth polytope then

$$5! \operatorname{vol}(P) \leq \frac{48f_0 - 96}{7}.$$

Proof. Recall that since P is smooth, $d! \operatorname{vol}(P) = (d-1)! \operatorname{vol}(\partial P) = f_{d-1}$. In each case Theorem 4.2 tells us the value for f_{d-1}. Applying Corollary 4.5 immediately gives the result in dimension four.

In dimension five we see that

$$\begin{aligned} 7 \cdot 5! \operatorname{vol}(P) &= 7 \cdot 4! \operatorname{vol}(\partial P) \\ &= 2(7 \left|\partial(2P) \cap \mathbb{Z}^5\right| - 28 \left|\partial P \cap \mathbb{Z}^5\right| + 42) \\ &\leq 2(24f_0 - 48), \end{aligned}$$

where the final inequality is an application of Corollary 4.5. □

The smooth polytopes attain either the lower or the upper limit in Corollary 4.4 are listed[2] in Table 5. The upper bound in dimension five is not sharp.

5. Reflexive order polytopes

Throughout let Q be a finite poset with $d := |Q|$ elements. Let $\Omega(Q, k)$ denote the number of order–preserving maps $f : Q \to C_k$, where C_k is the chain with $k \in \mathbb{Z}_{>0}$ elements; i.e. if $x \leq y$ in Q, then $f(x) \leq f(y)$. Then $\Omega(Q, k)$ is a polynomial in k of degree d, called the *order polynomial* of Q.

Let $\bar{\Omega}(Q, k)$ denote the number of strictly order–preserving maps $f : Q \to C_k$; i.e. if $x < y$ in Q, then $f(x) < f(y)$. Once again $\bar{\Omega}(Q, k)$ is a polynomial in k of degree d; it is called the *strict order polynomial* of Q.

[2] http://grdb.lboro.ac.uk/search/toricsmooth?id_cmp=in&id=24,25,127,128,138,139,144,145,147
http://grdb.lboro.ac.uk/search/toricsmooth?id_cmp=in&id=148,149,950,954,955,989,990,1008,1009,1010,1013

Definition 5.1. A poset Q is said to be *graded* if there exists an order–preserving function f such that whenever y covers x, $f(y) = f(x) + 1$. Equivalently, all maximal chains of Q have the same length r. Following Stanley [Sta97, Chapter 3.1] we shall call r the *rank* of Q. In particular one can adjoin a unique minimum element $\hat{0}$ and unique maximum element $\hat{1}$ to Q to obtain a bounded, graded poset $\hat{Q}$ of rank $r + 2$.

Definition 5.2. A bijective order–preserving map is called a *linear extension* of Q. The number of linear extensions is denoted by $e(Q)$.

Let $e_s(Q)$ denote the number of surjective order–preserving maps $f : Q \to C_s$.

Example 5.3. If Q is the antichain with $|Q| = d$ then $\bar{\Omega}(Q, k) = \Omega(Q, k) = k^d$ and $e(Q) = d!$. If Q is the chain C_d then $\Omega(Q, k) = \binom{d+k-1}{d}$, $\bar{\Omega}(Q, k) = \binom{k}{d}$, and $e(Q) = 1$.

Theorem 5.4 ([Sta70])**.** *Let Q be a finite poset with $|Q| = d$ and order polynomial $\Omega(Q, k) = a_d k^d + \ldots + a_1 k + a_0$. Then:*

(i) $\bar{\Omega}(Q, k) = (-1)^d \Omega(Q, -k)$ *for all* $k \in \mathbb{Z}$;
(ii) *If Q is graded of rank r, then* $a_{d-1} = \frac{re(Q)}{2(d-1)!}$;
(iii) *If Q is graded of rank r, then* $\Omega(Q, -r - k) = (-1)^d \Omega(Q, k)$ *for each* $k \in \mathbb{Z}$;
(iv) $a_d = \frac{e(Q)}{d!}$.
(v) $\Omega(Q, k) = \sum_{s=1}^{d} e_s(Q) \binom{k}{s}$.
(vi) *If Q is graded of rank r, then* $2e_{d-1}(Q) = (d - r + 1)e(Q)$.

Theorem 5.4 (i) is commonly referred to as the *Reciprocity Theorem for the Order Polynomial*.

Definition 5.5. The *order polytope* $\mathcal{O}(Q)$ of a poset Q is the set of order-preserving maps from Q to the interval $[0, 1]$, i.e. the set of all functions f satisfying

$$\begin{aligned} 0 \leq f(x) \leq 1, &\quad \text{for all } x \in Q; \\ f(x) \leq f(y), &\quad \text{if } y \text{ covers } x \text{ in Q.} \end{aligned}$$

Given the bounded poset $\hat{Q}$ one can define $\hat{\mathcal{O}}(Q)$ as the set of all functions g such that

$$\begin{aligned} g(\hat{0}) &= 0, \\ g(\hat{1}) &= 1, \\ \text{and} \quad g(x) &\leq g(y), \text{ if } y \text{ covers } x \text{ in } \hat{Q}. \end{aligned}$$

Then the bijective linear map $\rho : \hat{\mathcal{O}}(Q) \to \mathcal{O}(Q)$ given by restriction to Q defines a combinatorial equivalence of polytopes. Stanley was able to derive the entire facial structure of $\hat{\mathcal{O}}(Q)$ ([Sta86, §1]). In particular, the number of facets of $\mathcal{O}(Q)$ is equal to the number of cover relations in $\hat{Q}$, and the number of vertices of $\mathcal{O}(Q)$ is given by:

$$\left| \{ I \subset Q \mid \text{ if } x \in I \text{ and } y \geq x \text{ then } y \in I \} \right|.$$

Theorem 5.6 ([Sta86, §4])**.** *Let Q be a finite poset with $|Q| = d$. Then:*

(i) $L_{\mathcal{O}(Q)}(k) = \left| k\mathcal{O}(Q) \cap \mathbb{Z}^d \right| = \Omega(Q, k + 1)$ *for each* $k \in \mathbb{Z}$;
(ii) $\mathrm{vol}(\mathcal{O}(Q)) = \frac{e(Q)}{d!}$.

Corollary 5.7. *Let Q be a finite poset with order polytope $P := \mathcal{O}(Q)$. Then*

$$\left|(kP)^\circ \cap \mathbb{Z}^d\right| = \bar{\Omega}(Q, k-1), \qquad \text{for all } k \in \mathbb{Z}_{>0}.$$

Proof. This is immediate from Theorems 2.3, 5.4 (i), and 5.6 (i):

$$\begin{aligned} L_{P^\circ}(k) &= (-1)^d L_P(-k) \\ &= (-1)^d \Omega(Q, 1-k) \\ &= \bar{\Omega}(Q, k-1), \qquad \text{for any } k \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

□

Remark 5.8. Suppose that Q is graded of rank r. Stanley showed [Sta97, Corollary 4.5.17] that

$$\Omega(Q, -1) = \Omega(Q, -2) = \ldots = \Omega(Q, -r) = 0,$$

and that

$$\Omega(Q, -r-1) = (-1)^d.$$

From Corollary 5.7 we see that $(r+2)\mathcal{O}(Q)$ is the smallest integral dilation of $\mathcal{O}(Q)$ with an interior lattice point; in fact $(r+2)\mathcal{O}(Q)$ contains a *unique* interior lattice point.

Proposition 5.9. *Let Q be a poset with $|Q| = d$. Let $P := \mathcal{O}(Q)$ be the order polytope of Q. Then the boundary volume of P is*

$$\operatorname{vol}(\partial P) = \frac{(3-d)e(Q) + 2e_{d-1}(Q)}{(d-1)!}.$$

If in addition Q is a graded poset of rank r then the boundary volume of P is

$$\operatorname{vol}(\partial P) = \frac{(r+2)e(Q)}{(d-1)!}.$$

Proof. Since $L_P(k) = \Omega(Q, k+1)$ for each $k \in \mathbb{Z}$ by Theorem 5.6 (i), hence if we express $\Omega(Q, n)$ as

$$\Omega(Q, n) = \sum_{i=0}^{d} a_i n^i,$$

then

$$L_P(n) = \Omega(Q, n+1) = a_d(n+1)^d + a_{d-1}(n+1)^{d-1} + \sum_{i=0}^{d-2} a_i (n+1)^i.$$

If we express $L_P(n)$ in the form

$$L_P(n) = \sum_{i=0}^{d} c_i n^i$$

then $c_{d-1} = a_{d-1} + d a_d$.

Using Theorem 5.4 (iv) and (v), we get that

$$c_{d-1} = d\frac{e(Q)}{d!} + \frac{e_{d-1}(Q)}{(d-1)!} - \frac{\binom{d}{2} e(Q)}{d!} = \frac{(e(Q)(3-d) + 2e_{d-1}(Q)}{2(d-1)!}.$$

But it follows from Theorem 2.2 (ii), that

$$(1/2)\mathrm{vol}(\partial P) = c_{d-1} = \frac{(e(Q)(3-d) + 2e_{d-1}(Q)}{2(d-1)!}.$$

Combining these results gives

$$\mathrm{vol}(\partial P) = \frac{(3-d)e(Q) + 2e_{d-1}(Q)}{(d-1)!}.$$

When Q is a graded poset, applying Theorem 5.4 (vi) to the previous formula gives

$$\mathrm{vol}(\partial P) = \frac{(r+2)e(Q)}{(d-1)!}.$$

□

Lemma 5.10. *Let Q be a graded poset of rank r with $|Q| = d$. Then $(r+2)\mathcal{O}(Q)$ is a translate of a reflexive polytope.*

Proof. Let $P := (r+2)\mathcal{O}(Q)$ be the $(r+2)$-th dilate of the order polytope $\mathcal{O}(Q)$ of Q. It is enough to prove that $d\,\mathrm{vol}(P) = \mathrm{vol}(\partial P)$. But

$$\begin{aligned} d\,\mathrm{vol}(P) &= d\,\mathrm{vol}((r+2)\mathcal{O}(Q)) \\ &= d(r+2)^d \mathrm{vol}(\mathcal{O}(Q)) \\ &= d(r+2)^d \frac{e(Q)}{d!} \\ &= (r+2)^{d-1} \frac{(r+2)e(Q)}{(d-1)!} \\ &= (r+2)^{d-1} \mathrm{vol}(\partial \mathcal{O}(Q)) \\ &= \mathrm{vol}(\partial P)\,. \end{aligned}$$

□

Since $(r+2)\mathcal{O}(Q)$ is a (translate of a) reflexive polytope, we can reinterpret our results from Section 3 in terms of the order polytope:

Corollary 5.11 (c.f. Corollary 3.6)**.** *Let Q be a finite graded poset of rank r with $|Q| = d$. Let $e(Q)$ denote the number of linear extensions of Q. Then*

$$(r+2)^d e(Q) = \sum_{m=0}^{n} (-1)^{n+m} \left(\binom{d}{n-m} + (-1)^{d-1} \binom{d}{n+m+1} \right) \Omega(Q, m(r+2)+1),$$

where $n := \lfloor d/2 \rfloor$.

Theorem 5.12 (c.f. Theorem 3.8)**.** *Let Q be a finite graded poset of rank r with $|Q| = d$. Suppose that d is odd. Then*

$$\sum_{m=0}^{N} (-1)^{N+m} \binom{d+2}{N-m} \Omega(Q, m(r+2)+1) = 0,$$

where $N := \lceil d/2 \rceil$.

6. The Birkhoff polytope

Let $B(d)$ denote the *Birkhoff polytope* (or *transportation polytope*) of $d \times d$ doubly stochastic matrices in $\mathbb{R}^{d^2}$. That is, $B(d)$ is defined by

$$x_{i,j} \geq 0, \quad \sum_{i=1}^{d} x_{i,j} = 1, \quad \sum_{j=1}^{d} x_{i,j} = 1, \quad \text{for all } 1 \leq i, j \leq d.$$

Because of its rich combinatorial properties, the Birkhoff polytope has been intensively studied. In particular, methods for estimating and computing the volume and Ehrhart polynomial are of considerable interest (see [Pak00, BP03, CM09]). The following theorem summarises some of the key facts about $B(d)$:

Theorem 6.1. *Let $B(d)$ denote the polytope of $d \times d$ doubly stochastic matrices in $\mathbb{R}^{d^2}$. Let $H_n(r)$ denote the number of $n \times n$ magic squares with linear sums equal to r. Let $P_n(r)$ denote the number of $n \times n$ positive magic squares with linear sums equal to r, where positive means that all entries of the matrix are positive. Then:*

(i) $\dim B(d) = (d-1)^2$;
(ii) $L_{B(d)}(m) = H_d(m)$ *for all* $d \in \mathbb{Z}_{>0}$ *and* $m \in \mathbb{Z}_{\geq 0}$;
(iii) $L_{B(d)}(-d-t) = (-1)^{(d-1)^2} L_{B(d)}(t)$ *for all* $t \in \mathbb{Z}$;
(iv) *the vertices of $B(d)$ are the permutation matrices;*
(v) $L_{B(d)^\circ}(m) = P_d(m)$ *for all* $d \in \mathbb{Z}_{>0}$ *and* $m \in \mathbb{Z}_{>0}$.

In fact – as the following two lemmas show – it is easy to see that the d-th dilation of the Birkhoff polytope contains precisely one interior lattice point, and that this dilation is a translate of a reflexive polytope.

Lemma 6.2. *Let $B(d)$ denote the polytope of $d \times d$ doubly stochastic matrices in $\mathbb{R}^{d^2}$. Then $\left|dB(d)^\circ \cap \mathbb{Z}^{d^2}\right| = 1$.*

Proof. Using Theorem 6.1 (v),

$$\left|dB(d)^\circ \cap \mathbb{Z}^{d^2}\right| = L_{B(d)^\circ}(d) = P_d(d).$$

But if Q is a $d \times d$ positive magic square whose lines sum to d, then Q must be the matrix with all entries equal to one. Hence $P_d(d) = 1$. □

Lemma 6.3. *Let $P := dB(d) - Q$ denote the translation of the d-th dilate of the Birkhoff polytope by Q, where Q is the matrix with all entries equal to one. Then P is a reflexive polytope.*

Proof. From Theorem 3.4 (i) and (ii) it is enough to show that

$$(-1)^{(d-1)^2} L_{dB(d)}(-m) = L_{dB(d)}(m-1)$$

for all $m \in \mathbb{Z}_{>0}$. But setting $t = d(m-1)$ in Theorem 6.1 (iii) gives:

$$L_{dB(d)}(m) = L_{B(d)}(dm) = (-1)^{(d-1)^2} L_{B(d)}(d(m-1)) = (-1)^{(d-1)^2} L_{dB(d)}(m-1).$$

□

We can now reinterpret our results in Section 3 in terms of the Birkhoff polytope. In particular an explicit formula for the volume of the Birkhoff polytope is given in terms of the the first $\lfloor (d-1)^2/2 \rfloor$ dilations.

Corollary 6.4 (c.f. Corollary 3.3)**.** *Let $B(d)$ denote the polytope of $d \times d$ doubly stochastic matrices in $\mathbb{R}^{d^2}$. Then*

$$((d-1)^2)!\, d^{(d-1)^2} \mathrm{vol}(B(d)) \geq (d-1)^2 H_d(d) - (d-1)^2 + 3$$

Corollary 6.5 (c.f. Corollary 3.6)**.** *Let $B(d)$ denote the polytope of $d \times d$ doubly stochastic matrices in $\mathbb{R}^{d^2}$. Then*

(6.1)

$$\mathrm{vol}(B(d)) = \frac{1}{((d-1)^2)! d^{(d-1)^2}} \sum_{m=0}^{n} (-1)^{n+m} \left(\binom{(d-1)^2}{n-m} + (-1)^d \binom{(d-1)^2}{n+m+1} \right) H_d(md),$$

where $n := \lfloor (d-1)^2/2 \rfloor$.

Theorem 6.6 (c.f. Theorem 3.8)**.** *Suppose that d is even. Then*

$$\sum_{m=0}^{N} (-1)^{N+m} \binom{d^2 - 2d + 3}{N-m} H_d(dm) = 0, \tag{6.2}$$

where $N := \lceil (d-1)^2/2 \rceil$.

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