

## The dynamics of coset dimensional reduction.

Josef L. P. Karthausser<sup>1§</sup>, P. M. Saffin<sup>1,2¶</sup>

<sup>1</sup>*Department of Physics and Astronomy, University of Sussex  
Falmer, Brighton, BN1 9QJ, UK.*

<sup>2</sup>*School of Physics and Astronomy, University of Nottingham  
University Park, Nottingham, NG7 2RD, UK.*

### Abstract

The evolution of multiple scalar fields in cosmology has been much studied, particularly when the potential is formed from a series of exponentials. For a certain subclass of such systems it is possible to get “assisted“ behaviour, where the presence of multiple terms in the potential effectively makes it shallower than the individual terms indicate. It is also known that when compactifying on coset spaces one can achieve a consistent truncation to an effective theory which contains many exponential terms, however, if there are too many exponentials then exact scaling solutions do not exist. In this paper we study the potentials arising from such compactifications of eleven dimensional supergravity and analyse the regions of parameter space which could lead to scaling behaviour.

---

<sup>§</sup>email: jlk23@sussex.ac.uk

<sup>¶</sup>email: paul.saffin@nottingham.ac.uk

# 1 Introduction

Extra dimensions are ubiquitous in unified models of gravity and particles, dating from the early work of Kaluza and of Klein to the modern ideas of string and M-theory. This raises the question of what to do with the extra dimensions? In a fully dynamical model we expect that at some time these extra dimensions should evolve, possibly having some impact on the Universe we observe today. In the context of dimensional reduction, where the extra dimensions take the form of a small, compact manifold, the dynamics of the internal space typically manifests itself through the dynamics of scalar fields in an effective theory. Such scalar fields are often termed moduli and their values describe the shape and size of the compact space. The evolution of scalar fields in cosmology is a well established area of study, finding applications in inflationary model building, quintessence and many others. Indeed, cosmologists seem able to solve most problems with the introduction of a new scalar field. A particularly attractive system of scalars is the one where the potential is formed from a series of exponential terms, such a system can be phrased in the form of an autonomous dynamical system whose critical points allow for a simplified description of the evolution [1, 2]. Fortunately such a nice description does not go to waste as exponential potentials are common in unified gravity models with exponentials coming from dimensional reduction, gaugino condensation and instanton corrections [3, 4, 5, 6].

The critical points of the autonomous system formed in cosmology using scalars with exponential potentials reveals that *tracking solutions* are possible, and indicate that *assisted behaviour* can occur. By tracking solution we mean that the scalars evolve in such a way that their energy density remains at a constant fraction of the total energy density of the Universe, with the rest of the energy density being composed typically of a barotropic fluid. One may also have *scaling* behaviour, where the energy density of the field evolves in proportion to  $H^2$  without reference to other matter. Assisted behaviour refers to the behaviour of multiple scalars, where their combined effect can be such that the Universe expands more rapidly than one may naively expect by looking at the individual terms in the potential.

In this paper we aim to study dimensional reduction on homogeneous manifolds, namely those formed as a coset of compact Lie groups  $G/H$ . In order to be concrete we shall take eleven dimensional supergravity to be our starting point, whose bosonic field content comprises a metric and a three-form potential. As we are aiming for a four dimensional effective theory we shall reduce on the seven dimensional cosets which have been classified in [7]. Such coset reductions are familiar in the supergravity literature, where the supersymmetric stationary points are well known. Here however we concern ourselves with the full effective potential and the dynamics that follow, paying particular attention to the regimes which lead to tracking or scaling behaviour.

We shall structure the paper by first introducing the bosonic action of eleven dimensional supergravity, along with the ansatz for dimensional reduction. Having done that we shall produce the effective potentials for all the 7D cosets classified in [7]. With these in place we are able, after a brief introduction to scaling solutions, to study the scaling properties of supergravity reduced on coset manifolds. Throughout the paper we shall be using technical results for dimensional reduction and for cosets, the details of which can be found in the appendices. We end the paper with our concluding remarks, and comments for future work.

## 2 The model

If the Universe is fundamentally described by a higher dimensional theory then there must be some mechanism whereby four spacetime dimensions are picked out at the expense of others. One such

mechanism is the famous Kaluza-Klein procedure, utilising a compact space whose size sets the mass scale for a tower of massive modes [8, 9]. These modes should be at a high enough scale that we have not yet observed them, leaving a 4D low energy theory effective for the moduli fields. The underlying theory allows these moduli fields to evolve and, as scalar fields, they are a natural source for the myriad scalars which cosmologists require to solve various problems; the difficulty lies in finding an internal space which has the requisite properties. The simplest type of internal manifold is a torus [10], however one may also consider the internal space to be a group manifold [11, 12], an Einstein manifold, a direct product of Einstein manifolds [13], or products of twisted manifolds [14, 15].

Here however we consider another case, namely reducing on homogeneous manifolds described as coset spaces. These are particularly attractive because they maintain the useful properties of group manifolds while not being as restrictive, and they do have a long history of use in Kaluza-Klein models, with their structure providing non-Abelian gauge groups via the Killing vectors living on the coset [7, 16]. As a way of connecting eleven dimensional supergravity to standard model physics in four dimensions it was pointed out in [17] that seven is the minimum number of dimensions for a homogeneous manifold invariant under the action of  $SU(3) \otimes SU(2) \otimes U(1)$ . Although there is a large body of work regarding cosets as internal manifolds, the existing literature is mostly concerned with finding stationary points of the effective potential, corresponding to Einstein metrics on the coset space, rather than studying the dynamics of evolving moduli. In this paper we shall seek to resolve this starting with a review of the framework of cosets and then applying it to the study of the evolution of coset spaces in cosmology, concentrating on eleven dimensional supergravity as the higher dimensional theory [18]. Using the conventions of [19] the bosonic action for the theory is

$$\hat{S} = \frac{1}{2\kappa_{11}^2} \int \left[ \star \hat{\mathcal{R}} - \frac{1}{2} \star F \wedge F - \frac{1}{6} C \wedge F \wedge F \right], \quad (2.1)$$

where  $\hat{\mathcal{R}}$  is the 11D Ricci scalar. The equations of motion following from this are

$$\hat{\mathcal{R}}_{\mu\nu} - \frac{1}{12} \left[ F_{\mu\rho_1\rho_2\rho_3} F_{\nu}{}^{\rho_1\rho_2\rho_3} - \frac{1}{12} g_{\mu\nu} F^2 \right] = 0, \quad (2.2)$$

$$d \star F + \frac{1}{2} F \wedge F = 0. \quad (2.3)$$

As we are interested in the case where the internal manifold is a coset, with squashing parameters that vary as a function of spacetime, we write the metric as

$$ds^2 = e^{2\psi(x)} ds_{(4)}^2 + g_{ij}(x) e^i \otimes e^j, \quad (2.4)$$

where the one-forms  $e^i$  span the cotangent space of the coset manifold (see appendix A). We also allow there to be a Freund-Rubin flux [20] of the form

$$F = f \eta_4, \quad (2.5)$$

where  $\eta_4$  is the volume form on spacetime and  $f$  is a function to be determined. In order to have a convenient four dimensional description we require an effective action in four dimensions which reproduces the equations of motion required to solve the full 11D system. This raises the issue of consistent truncation which is known to impose certain constraints on the internal manifold. In the case of group reductions, for example, one requires the group to be unimodular [21]; see [22] for a nice history and discussion of this issue and see [23, 24] for recent work. For coset reduction we shall find that the effective action for the fields from the gravity sector does actually correspond

$M_7$	$G$	$H$	ref
$S^7, J^7, V_{5,2}$	$SO(5)$	$SO(3)$	[7]
$M^{pqr}$	$SU(3) \times SU(2) \times U(1)$	$SU(2) \times U(1) \times U(1)$	[26]
$N^{pqr}$	$SU(3) \times U(1)$	$U(1) \times U(1)$	[27]
$Q^{pqr}$	$SU(2) \times SU(2) \times SU(2)$	$U(1) \times U(1)$	[28]

Table 1: the cosets of 11d supergravity.

to simply substituting the 11D metric ansatz into the 11D action, but the coset must again satisfy a particular constraint. This issue is not just related to the gravity sector, we also find it in the flux sector where in order to get the correct sign from the flux contribution one must appeal to the underlying equations of motion [25].

### 3 Choosing a coset

The eleven dimensional supergravity permits classical solutions where the space time is partitioned as  $M_{11} = M_4 \times M_7$  and the internal seven dimensional part is compact giving an effective theory in four dimensions. We are interested in the case where  $M_7$  takes the form of a coset manifold, fortunately such cosets have been classified in [7] and are shown in table 1. We have not explicitly included  $SO(8)/SO(7)$  as this is metrically equivalent to a particular case of the  $SO(5)/SO(3)$  coset, where the metric is proportional to the identity. Each of these describes a number of different cosets, depending upon the exact embedding of the subgroup  $H$  in  $G$ . For instance the group  $SO(5)$  has two orthogonal  $SO(3)$  subgroups, referred to as  $A$  and  $B$ . We can form a coset by dividing out by either of these, or by taking some combination of them. In this way we find that there are three cosets which are referred to as  $A$ ,  $A+B$  and  $MAX$ . In the same way the M, N and Q spaces have their subgroup embeddings parametrised by the integers  $p, q$  and  $r$ .

We now describe the effective theories derived from each type of coset. With all the effective actions in hand we shall then study their scaling behaviour.

#### 3.1 Equations for $SO(5)/SO(3)_A$

The general procedure for dimensional reduction on a coset is given in appendix B, here we produce the results for the various allowed cosets presented in table 1. Our first example is one of the  $SO(5)/SO(3)$  cosets,  $SO(5)/SO(3)_A$ , presented in appendix C, and we discover that there are seven moduli describing the coset metric. However, we need to make sure that the truncation is consistent, meaning that the 11D Ricci tensor must satisfy (2.2). If we look at the components of  $\hat{\mathcal{R}}_{\mu i}$  from (B.12) and use the structure constants relevant for this coset we find that in general one must restrict to a diagonal coset metric of the form

$$g_{ab} = \text{diag}(e^{2A}, e^{2B}, e^{2C}, e^{2D}, e^{2D}, e^{2D}, e^{2D}). \quad (3.6)$$

This is an important restriction, showing that the general metric which respects the coset symmetries does not form a consistent truncation. This does not however necessarily mean that there are no special configurations which contain off-diagonal terms for which  $\hat{\mathcal{R}}_{\mu i}$  happens to vanish.

For this metric we have from (B.17) that

$$\frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} \quad (3.7)$$

$$\begin{aligned}
&= \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 + \frac{1}{4} \nabla_\mu g^{ij} \nabla^\mu g_{ij} - \frac{1}{8} g^{ij} \nabla_\mu g_{ij} g^{kl} \nabla_\mu g_{kl} + e^{2\psi} \mathcal{R}_{G/H} \right], \\
&= \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - (\nabla A)^2 - (\nabla B)^2 - (\nabla C)^2 - 4(\nabla D)^2 \right. \\
&\quad \left. - \frac{1}{2} [\nabla(A+B+C+4D)]^2 + e^{2\psi} \mathcal{R}_{G/H} \right],
\end{aligned}$$

where we have introduced  $V_{G/H}$  which is a *constant* volume of the unsquashed coset, defined by

$$V_{G/H} = \int_{G/H} e^1 \wedge e^2 \wedge e^3 \dots, \quad (3.8)$$

with the  $e^i$  being representatives of the coset cotangent space as in appendix A. This allows us to define a 4D gravitational coupling for the effective theory using

$$\frac{V_{G/H}}{2\kappa_{11}^2} = \frac{1}{2\kappa^2}. \quad (3.9)$$

We notice from (3.7) that the functions  $A, B, C, D$  have become scalar fields of the effective theory, but with non-canonical kinetic terms. In order to reduce this to standard form we diagonalise the gradient terms in the above expression using a Gram-Schmidt procedure and introduce

$$\begin{aligned}
A &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 - \frac{1}{\sqrt{2}} \varphi_2 - \frac{1}{\sqrt{6}} \varphi_3 - \frac{2}{\sqrt{21}} \varphi_4 \right), \\
B &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{1}{\sqrt{2}} \varphi_2 - \frac{1}{\sqrt{6}} \varphi_3 - \frac{2}{\sqrt{21}} \varphi_4 \right), \\
C &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \sqrt{\frac{2}{3}} \varphi_3 - \frac{2}{\sqrt{21}} \varphi_4 \right), \\
D &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{1}{2} \sqrt{\frac{3}{7}} \varphi_4 \right).
\end{aligned} \quad (3.10)$$

Having found the action coming from the Ricci scalar we now consider the dynamics coming from the Freund-Rubin flux. As we are presently only concerned with flux of the form (2.5) the  $F \wedge F$  term in (2.3) vanishes to leave

$$d \left( f e^{-4\psi} e^{A+B+C+4D} \right) = 0. \quad (3.11)$$

So, using (B.14), our flux parameter is given by

$$f = f_0 e^{6\psi}, \quad (3.12)$$

where  $f_0$  is an integration constant. We are now in a position to examine the equations of motion (2.2) finding that they can be derived from an effective action,

$$\begin{aligned}
S_4 &= \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right], \\
V(\varphi) &= -\frac{1}{2\kappa^2} e^{-\frac{3}{7}\sqrt{14}\kappa\varphi_1} \left[ 12e^{-\frac{1}{7}\sqrt{21}\kappa\varphi_4} - \mathcal{V}(-\kappa\varphi_2, -\kappa\varphi_3) e^{-\frac{10}{21}\sqrt{21}\kappa\varphi_4} \right. \\
&\quad \left. + \mathcal{V}(\kappa\varphi_2, \kappa\varphi_3) e^{\frac{4}{21}\sqrt{21}\kappa\varphi_4} - \frac{1}{2} \mathcal{V}(2\kappa\varphi_2, -2\kappa\varphi_3) e^{\frac{4}{21}\sqrt{21}\kappa\varphi_4} \right] + \frac{1}{2} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}.
\end{aligned} \quad (3.13)$$

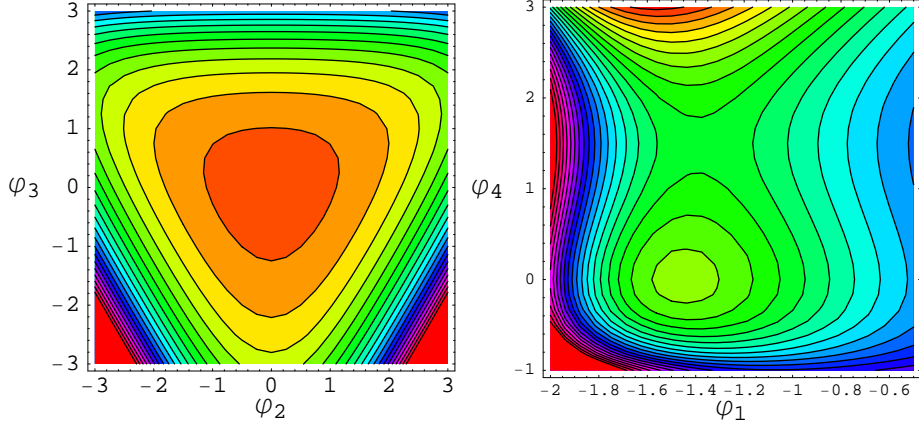


Figure 1: contour plots of the potential for the  $\text{SO}(5)/\text{SO}(3)_A$  coset.

We have introduced the function

$$\mathcal{V}(x, y) = e^{\sqrt{2}x + \sqrt{\frac{2}{3}}y} + e^{-\sqrt{2}x + \sqrt{\frac{2}{3}}y} + e^{-2\sqrt{\frac{2}{3}}y}, \quad (3.14)$$

as it aids our understanding of the dynamics in separate sectors. The function  $\mathcal{V}$  has a global minimum at  $\varphi_2 = 0 = \varphi_3$  which means that these fields will evolve to zero independently of what  $\varphi_1$  and  $\varphi_4$  are doing and we may consider how the potential depends on  $\varphi_1$  and  $\varphi_4$  alone. We see from figure 1 that there is indeed a minimum at  $\varphi_2 = \varphi_3 = 0$ . The figure also shows a local minimum in the  $\varphi_1 - \varphi_4$  plane along with a saddle point, these however depend on the value of  $\varphi_2$  and  $\varphi_3$ , with the plot using  $\varphi_2 = \varphi_3 = 0$ .

We can understand these extrema by studying the existing literature. In the early days of supergravity there was much interest in existence of Einstein metrics on coset manifolds as these were known to correspond to extrema of the effective potential. In [7] we find that for  $\text{SO}(5)/\text{SO}(3)_A$  there are two Einstein metrics on the coset, both with  $A = B = C$  but with one taking  $A = D$  and the other with  $A = D - \log \sqrt{5}$  (C.24); these correspond to a round seven sphere and a squashed seven sphere respectively and are given by the local minimum and saddle point of the potential. In terms of the canonical scalars the round metric takes the values  $\varphi_2 = \varphi_3 = \varphi_4 = 0$  and the squashed metric has  $\varphi_2 = \varphi_3 = 0$ ,  $\varphi_4 = \frac{2}{\kappa} \sqrt{\frac{3}{7}} \log \sqrt{5}$ . At both the round sphere and the squashed sphere one finds that the potential has an extremum at which point the potential is negative, allowing an  $\text{AdS}_4$  solution.

### 3.2 Equations for $\text{SO}(5)/\text{SO}(3)_{A+B}$

As mentioned in section 3.1 there is another coset of  $\text{SO}(5)$  where the  $\text{SO}(3)$  subgroup is taken to be the diagonal component of  $\text{SO}(3)_A \otimes \text{SO}(3)_B$ . In appendix D we show how such a coset is constructed and derive the general metric which is consistent with the coset symmetries. We restrict to a diagonal coset metric  $g_{ab} = \text{diag}(e^{2A}, e^{2A}, e^{2A}, e^{2B}, e^{2C}, e^{2C}, e^{2C})$ , which is needed for a consistent reduction. Following the same procedure as before we calculate from (B.17) the effective action coming from the Ricci scalar,

$$\frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} \quad (3.15)$$

$$= \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - 3(\nabla A)^2 - (\nabla B)^2 - 3(\nabla C)^2 - \frac{1}{2}[\nabla(3A + B + 3C)]^2 + e^{2\psi} \mathcal{R}_{G/H} \right].$$

Again we see that the parameters describing the size and shape of the internal manifold have become scalar fields with non-canonical kinetic terms, and in order to have standard kinetic terms we diagonalise the gradient terms as we did before. We find that the Gram-Schmidt procedure provides us with the following field redefinition,

$$\begin{aligned} A &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 - \sqrt{\frac{1}{12}} \varphi_2 - \frac{1}{2} \sqrt{\frac{3}{7}} \varphi_3 \right), \\ B &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{\sqrt{3}}{2} \varphi_2 - \frac{1}{2} \sqrt{\frac{3}{7}} \varphi_3 \right), \\ C &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{2}{\sqrt{21}} \varphi_3 \right). \end{aligned} \quad (3.16)$$

Turning our attention to the equations for the three-form potential (2.3), we find that the Freund-Rubin flux must satisfy

$$d \left( f e^{-4\psi} e^{3A+B+3C} \right) = 0, \quad (3.17)$$

and so using (B.14) our flux parameter is given by the same expression as before, (3.12). From this one may find the equations of motion and we discover that they can be derived from the following effective action,

$$S_4 = \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right], \quad (3.18)$$

$$\begin{aligned} V(\varphi) &= -\frac{1}{2\kappa^2} e^{-3\sqrt{\frac{2}{7}}\kappa\varphi_1} \left[ 9e^{-\frac{4}{\sqrt{21}}\kappa\varphi_3} + 9e^{\frac{1}{\sqrt{3}}\kappa\varphi_2 + \sqrt{\frac{3}{7}}\kappa\varphi_3} + 3e^{-\sqrt{3}\kappa\varphi_2 + \sqrt{\frac{3}{7}}\kappa\varphi_3} \right. \\ &\quad \left. - \frac{3}{2} e^{-\frac{4}{\sqrt{3}}\kappa\varphi_2 - \frac{4}{\sqrt{21}}\kappa\varphi_3} - \frac{3}{2} e^{\frac{4}{\sqrt{3}}\kappa\varphi_2 - \frac{4}{\sqrt{21}}\kappa\varphi_3} - \frac{3}{2} e^{-\frac{2}{\sqrt{3}}\kappa\varphi_2 + \frac{10}{\sqrt{21}}\kappa\varphi_3} \right] + \frac{1}{2} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}. \end{aligned} \quad (3.19)$$

On examining the contour plot of this potential in the  $\varphi_2 - \varphi_3$  plane (figure 3.2) we observe that there is an extremum, as is to be expected, which corresponds to the location where the metric is Einstein [7]; this point is independent of the scalar  $\varphi_1$ . The Einstein metric is derived in appendix D and is given by

$$\begin{aligned} \varphi_2 &= \frac{\sqrt{3}}{4\kappa} \log\left(\frac{3}{2}\right), \\ \varphi_3 &= -\frac{1}{\sqrt{7}} \varphi_2, \end{aligned} \quad (3.20)$$

at which point the potential becomes

$$V(\varphi) = -\frac{1}{2\kappa^2} \frac{63}{2} \left(\frac{3}{2}\right)^{1/7} e^{-3\sqrt{\frac{2}{7}}\kappa\varphi_1} + \frac{1}{2} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}, \quad (3.21)$$

which allows for an AdS<sub>4</sub> solution in the minimum.

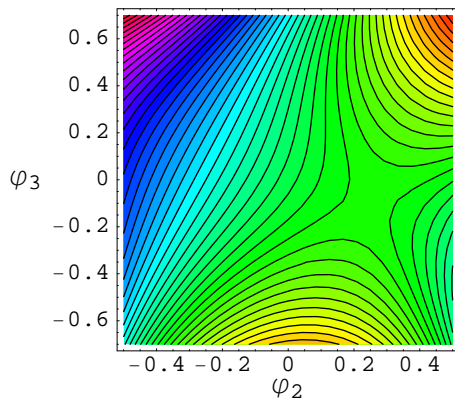


Figure 2: contour plots of the potential for the  $\text{SO}(5)/\text{SO}(3)_{A+B}$  coset.

### 3.3 Equations for $\text{SO}(5)/\text{SO}(3)_{MAX}$

The last coset one can construct out of  $\text{SO}(3)$  subgroups of  $\text{SO}(5)$  is where the  $\text{SO}(3)$  subgroup is maximal. We construct this subgroup in appendix E finding only a single modulus, which describes just a breathing mode for the internal space and therefore there are no shape moduli. We are only allowed the single parameter diagonal coset metric which we parametrise as  $g_{ab} = \text{diag}(e^{2A}, e^{2A}, e^{2A}, e^{2A}, e^{2A}, e^{2A}, e^{2A})$  and for which we calculate from (B.17)

$$\frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} = \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - \frac{63}{2} (\nabla A)^2 + e^{2\psi} \mathcal{R}_{G/H} \right]. \quad (3.22)$$

These are the terms which will give us the kinetic terms for the effective action. In order to have canonical kinetic terms we rescale  $A$  as follows,

$$A = \frac{1}{3} \sqrt{\frac{2}{7}} \kappa \varphi_1. \quad (3.23)$$

The effect of the Freund-Rubin flux is calculated as before, with (2.3) giving

$$d \left( f e^{-4\psi} e^{7A} \right) = 0, \quad (3.24)$$

and once again our flux parameter is given by (3.12). We find that the evolution of the moduli coming from 11D equations of motion can then be described by the following effective action,

$$S_4 = \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right], \quad (3.25)$$

$$V(\varphi) = -\frac{1}{2\kappa^2} \frac{189}{10} e^{-3\sqrt{\frac{2}{7}} \kappa \varphi_1} + \frac{1}{2} f_0^2 e^{-\sqrt{14} \kappa \varphi_1},$$

which has an  $\text{AdS}_4$  extremum at  $\varphi_1 = \frac{1}{4\kappa} \sqrt{\frac{7}{2}} \log\left(\frac{10f_0^2 \kappa^2}{81}\right)$ .



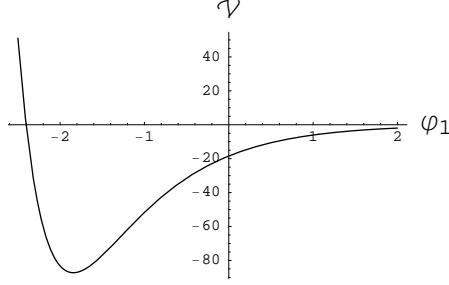


Figure 3: Plot of the potential for the  $SO(5)/SO(3)_{MAX}$  coset.

### 3.4 Equations for $M^{pqr} = SU(3) \times SU(2) \times U(1)/SU(2) \times U(1) \times U(1)$

We now consider another class of cosets,  $SU(3) \times SU(2) \times U(1) / SU(2) \times U(1) \times U(1)$ , characterised by integers  $p, q, r$ , the construction of which we show in appendix F. A full discussion of these spaces can be found in [7, 17, 29], in particular the curvature depends only on the ratio  $p/q$ , and the space  $M^{pqr}$  is the covering space of  $M^{pqr}$ .

We parametrise the metric for this class of coset (F.37) in maximum generality as the diagonal metric  $g_{ab} = \text{diag}(e^{2A}, e^{2A}, e^{2A}, e^{2A}, e^{2B}, e^{2B}, e^{2C})$ , and following the same procedure as before we calculate from (B.17) the effective action coming from the Ricci scalar,

$$\begin{aligned} & \frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} \\ &= \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - 4(\nabla A)^2 - 2(\nabla B)^2 - (\nabla C)^2 - 2[\nabla(2A + B + \frac{1}{2}C)]^2 + e^{2\psi} \mathcal{R}_{G/H} \right]. \end{aligned} \quad (3.26)$$

Again we see that the parameters describing the size and shape of the internal manifold have become scalar fields with non-canonical kinetic terms, and so in order to have standard kinetic terms we diagonalise the gradient terms as we did before. We find that the Gram-Schmidt procedure provides us with the following field redefinition,

$$\begin{aligned} A &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 - \frac{1}{\sqrt{6}} \varphi_2 - \frac{1}{\sqrt{42}} \varphi_3 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\ B &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{1}{\sqrt{3}} \varphi_2 - \frac{1}{\sqrt{42}} \varphi_3 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\ C &= \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \sqrt{\frac{6}{7}} \varphi_3 \right) - \frac{8}{9} \log\left(\frac{q}{\zeta}\right), \end{aligned} \quad (3.27)$$

with  $\zeta = \sqrt{3p^2 + q^2 + 2r^2}$ . In these expressions we have also shifted the fields by a constant factor to simplify the potential.

We find that the Freund-Rubin flux must satisfy

$$d \left( f e^{-4\psi} e^{4A+2B+C} \right) = 0, \quad (3.28)$$

and so using (B.14), our flux parameter is given by the same expression as before, (3.12). From this one may find the equations of motion and we discover that they can be derived from the following

effective action,

$$S_4 = \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right], \quad (3.29)$$

$$V(\varphi) = -\frac{1}{2\kappa^2} e^{-3\sqrt{\frac{2}{7}}\kappa\varphi_1} \left[ -\frac{9p^2}{4q^2} e^{\frac{2}{\sqrt{3}}\kappa\varphi_2 + 8\sqrt{\frac{2}{21}}\kappa\varphi_3} + 2e^{-\frac{2}{\sqrt{3}}\kappa\varphi_2 + \sqrt{\frac{2}{21}}\kappa\varphi_3} \right. \\ \left. + 6e^{\frac{1}{\sqrt{3}}\kappa\varphi_2 + \sqrt{\frac{2}{21}}\kappa\varphi_3} - \frac{1}{2} e^{-\frac{4}{\sqrt{3}}\kappa\varphi_2 + 8\sqrt{\frac{2}{21}}\kappa\varphi_3} \right] + \frac{1}{2} \left( \frac{q^2}{3p^2 + q^2 + 2r^2} \right)^{1/3} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}. \quad (3.30)$$

### 3.5 Equations for $N^{pqr} = \text{SU}(3) \times \text{U}(1)/\text{U}(1) \times \text{U}(1)$

Another class of cosets to consider are  $\text{SU}(3) \times \text{U}(1) / \text{U}(1) \times \text{U}(1)$ , which are also characterised by integers  $p, q, r$ ; the construction is shown in appendix G, but further details may be found in [30, 27]. For this coset one finds again that the curvature is independent of  $r$ , with  $N^{pq0}$  being the covering space of  $N^{pqr}$ .

The general metric for this class of coset is again diagonal (G.42), which by taking  $g_{ab} = \text{diag}(e^{2A}, e^{2A}, e^{2B}, e^{2B}, e^{2C}, e^{2C}, e^{2D})$  and calculating as before from (B.17) we obtain the effective action coming from the Ricci scalar,

$$\frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} \quad (3.31) \\ = \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - 2(\nabla A)^2 - 2(\nabla B)^2 - 2(\nabla C)^2 - 2[\nabla(A + B + C + \frac{1}{2}D)]^2 + e^{2\psi} \mathcal{R}_{G/H} \right].$$

Introducing a constant shift in the field redefinition following from the Gram-Schmidt process we find the following gives canonical kinetic terms for the scalars  $\varphi_i$ ,

$$A = \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 - \frac{1}{2} \varphi_2 - \frac{1}{\sqrt{6}} \varphi_3 - \frac{1}{\sqrt{42}} \varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \quad (3.32) \\ B = \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{1}{2} \varphi_2 - \frac{1}{\sqrt{6}} \varphi_3 - \frac{1}{\sqrt{42}} \varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\ C = \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \frac{1}{\sqrt{3}} \varphi_3 - \frac{1}{\sqrt{42}} \varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\ D = \kappa \left( \frac{1}{3} \sqrt{\frac{2}{7}} \varphi_1 + \sqrt{\frac{6}{7}} \varphi_4 \right) - \frac{8}{9} \log\left(\frac{q}{\zeta}\right),$$

with  $\zeta = \sqrt{3p^2 + q^2 + 2r^2}$ .

The flux equations of motion require the Freund-Rubin flux to satisfy

$$d \left( f e^{-4\psi} e^{2A+2B+2C+D} \right) = 0, \quad (3.33)$$

and we discover that the equations of motion can be derived from the effective action,

$$S_4 = \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right] \quad (3.34)$$

$$\begin{aligned}
V(\varphi) = & -\frac{1}{2\kappa^2}e^{-3\sqrt{\frac{2}{7}}\kappa\varphi_1} \left[ -\frac{(3p-q)^2}{8q^2}e^{-\frac{4}{\sqrt{3}}\kappa\varphi_3+8\sqrt{\frac{2}{21}}\kappa\varphi_4} + 3e^{-\frac{2}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} \right. \\
& -\frac{1}{2}e^{-2\kappa\varphi_2-\frac{2}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} - \frac{1}{2}e^{2\kappa\varphi_2-\frac{2}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} + 3e^{-\kappa\varphi_2+\frac{1}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} \\
& + 3e^{\kappa\varphi_2+\frac{1}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} - \frac{1}{2}e^{\frac{4}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} - \frac{1}{2}e^{2\kappa\varphi_2+\frac{2}{\sqrt{3}}\kappa\varphi_3+8\sqrt{\frac{2}{21}}\kappa\varphi_4} \\
& \left. -\frac{(3p+q)^2}{8q^2}e^{-2\kappa\varphi_2+\frac{2}{\sqrt{3}}\kappa\varphi_3+8\sqrt{\frac{2}{21}}\kappa\varphi_4} \right] + \frac{1}{2} \left( \frac{q^2}{3p^2+q^2+2r^2} \right)^{1/3} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}. \tag{3.35}
\end{aligned}$$

### 3.6 Equations for $Q^{pqr} = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)/\text{U}(1) \times \text{U}(1)$

The final class of cosets in the classification are the  $Q^{pqr}$  spaces, given by the quotient  $\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) / \text{U}(1) \times \text{U}(1)$  where the integers  $p, q, r$  characterise the different embeddings of the subgroup. This coset is constructed in appendix H, but further details on its structure may be found in [31].

The metric for this class of coset is the diagonal metric  $g_{ab} = \text{diag}(e^{2A}, e^{2A}, e^{2B}, e^{2B}, e^{2C}, e^{2C}, e^{2D})$  for which we calculate from (B.17) that the effective action coming from the Ricci scalar is,

$$\begin{aligned}
& \frac{1}{2\kappa_{11}^2} \int \sqrt{-\hat{g}_{11}} d^{11}x \hat{\mathcal{R}} \\
= & \frac{V_{G/H}}{2\kappa_{11}^2} \int \sqrt{-g_4} d^4x \left[ \mathcal{R}_4 - 2(\nabla A)^2 - 2(\nabla B)^2 - (\nabla C)^2 - (\nabla D)^2 \right. \\
& \left. - 2[\nabla(A+B+C+\frac{1}{2}D)]^2 + e^{2\psi} \mathcal{R}_{G/H} \right]. \tag{3.36}
\end{aligned}$$

To get back to canonical kinetic terms we introduce field redefinitions,

$$\begin{aligned}
A &= \kappa \left( \frac{1}{3}\sqrt{\frac{2}{7}}\varphi_1 - \frac{1}{2}\varphi_2 - \frac{1}{\sqrt{6}}\varphi_3 - \frac{1}{\sqrt{42}}\varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\
B &= \kappa \left( \frac{1}{3}\sqrt{\frac{2}{7}}\varphi_1 + \frac{1}{2}\varphi_2 - \frac{1}{\sqrt{6}}\varphi_3 - \frac{1}{\sqrt{42}}\varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\
C &= \kappa \left( \frac{1}{3}\sqrt{\frac{2}{7}}\varphi_1 + \frac{1}{\sqrt{3}}\varphi_3 - \frac{1}{\sqrt{42}}\varphi_4 \right) + \frac{1}{9} \log\left(\frac{q}{\zeta}\right), \\
D &= \kappa \left( \frac{1}{3}\sqrt{\frac{2}{7}}\varphi_1 + \sqrt{\frac{6}{7}}\varphi_4 \right) - \frac{8}{9} \log\left(\frac{q}{\zeta}\right), \tag{3.37}
\end{aligned}$$

with  $\zeta = \sqrt{p^2+q^2+r^2}$ , and the flux equation gives us

$$d\left(fe^{-4\psi}e^{2A+2B+2C+D}\right) = 0. \tag{3.38}$$

From this one may find the equations of motion and we discover that they can be derived from the following effective action,

$$S_4 = \int \sqrt{-g_4} d^4x \left[ \frac{1}{2\kappa^2} \mathcal{R}_4 - \frac{1}{2} \sum_i \nabla_\mu \varphi_i \nabla^\mu \varphi_i - V(\varphi) \right] \tag{3.39}$$

$$V(\varphi) = -\frac{1}{2\kappa^2}e^{-3\sqrt{\frac{2}{7}}\kappa\varphi_1} \left[ -\frac{p^2}{2q^2}e^{2\kappa\varphi_2+\frac{2}{\sqrt{3}}\kappa\varphi_3+8\sqrt{\frac{2}{21}}\kappa\varphi_4} + 2e^{-\frac{2}{\sqrt{3}}\kappa\varphi_3+\sqrt{\frac{2}{21}}\kappa\varphi_4} \right] \tag{3.40}$$

$$\begin{aligned}
& + 2e^{-\kappa\varphi_2 + \frac{1}{\sqrt{3}}\kappa\varphi_3 + \sqrt{\frac{2}{21}}\kappa\varphi_4} + 2e^{\kappa\varphi_2 + \frac{1}{\sqrt{3}}\kappa\varphi_3 + \sqrt{\frac{2}{21}}\kappa\varphi_4} - \frac{1}{2}e^{-2\kappa\varphi_2 + \frac{2}{\sqrt{3}}\kappa\varphi_3 + 8\sqrt{\frac{2}{21}}\kappa\varphi_4} \\
& - \frac{r^2}{2q^2}e^{-\frac{4}{\sqrt{3}}\kappa\varphi_3 + 8\sqrt{\frac{2}{21}}\kappa\varphi_4} \Big] + \frac{1}{2} \left( \frac{q^2}{p^2 + q^2 + r^2} \right)^{1/3} f_0^2 e^{-\sqrt{14}\kappa\varphi_1}.
\end{aligned}$$

## 4 Scaling solutions

Now that we have our set of effective actions for the full set of cosets we move on to examine their dynamics. While studying the different possible types of evolution a scalar field may have it becomes clear that scaling solutions coming from exponential potentials hold a rather prominent position [1, 2, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. This is because systems of exponential potentials can have attractor solutions, when viewed from a dynamical systems perspective [1, 2], where the scalar field constitutes a fixed fraction of the energy density in the Universe [42]. We have seen in the above sections that coset spaces naturally give rise to exponential potentials, a situation which is common in dimensionally reduced theories. Another nice property of having many scalars is that while each term in the potential may be too steep to support an accelerating scale factor, together they may combine in such a way as to allow inflationary behaviour [43]. We can understand this by noting that in some cases it is possible to perform a field redefinition such that only one of the new fields is involved in the evolution, or some smaller subset than was being used in the initial formulation [2]. In this section we shall investigate how the scalars evolve for the coset reductions discussed above, we start with a general discussion of how scaling solutions appear. In doing this we shall follow [36, 39] by adding a fluid component which accounts for other forms of matter/radiation that could be present. We describe this fluid by the following equation of state,

$$P_\gamma = (\gamma - 1)\rho_\gamma. \quad (4.41)$$

Taking the cosmological ansatz of homogeneous fields,  $\phi(x^\mu) = \phi(t)$  and the FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\underline{x}^2, \quad (4.42)$$

for flat spatial sections we find the usual equations

$$H^2 = \frac{\kappa^2}{3} \left[ \sum_i \frac{1}{2} \dot{\phi}_i^2 + V + \rho_\gamma \right], \quad (4.43)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ \sum_i \dot{\phi}_i^2 + \gamma\rho_\gamma \right], \quad (4.44)$$

$$\dot{\rho}_\gamma = -3\gamma H\rho_\gamma \quad (4.45)$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_{,i} = 0. \quad (4.46)$$

A central part of scaling solutions is that all terms in a given equation have the same behaviour. Given that, it is easy to see the above equations require

$$H^2 \propto \dot{\phi}_i^2 \propto V \propto \rho_\gamma \propto \dot{H}, \quad (4.47)$$

which can be consistent only for exponential potentials, and  $H \propto 1/t$ . A crucial point in searching for these scaling solutions is that one cannot have more independent terms in the potential than scalar fields, otherwise the system is overdetermined and has no scaling solution. We see straight away that for our system we always have more terms than scalars and so there cannot be any true

scaling solution. That does not rule out there being regions where the evolution is well approximated by scaling solutions, and as pointed out in [39] this can in fact be a useful tool, with the evolution moving from scaling solution to scaling solution. Given that there is no true scaling solution we shall now consider regions of parameter space where we expect approximate scaling behaviour.

As an example, consider the case of a scalar field with potential

$$\mathcal{V} = A \exp(-b\kappa\phi), \quad (4.48)$$

$A > 0$ , scale factor with  $a(t) \sim t^p$  and fluid density  $\rho = \tilde{\rho}/(\kappa^2 t^2)$ . We find that

$$p = 2/(3\gamma), \quad \tilde{\rho} = 3p(p - 2/b^2) \quad (4.49)$$

with a fluid and

$$p = 2/b^2 \quad (4.50)$$

without the fluid. There are also the consistency relations

$$b^2 > 3\gamma \quad (4.51)$$

with a fluid and

$$b^2 < 6 \quad (4.52)$$

without. These come from demanding positivity of the fluid energy density and potential  $\mathcal{V}$ . If we were to allow  $A < 0$  then the scaling solution, without fluid, would have the opposite bound for  $b$ ,  $b^2 > 6$ . However, in this case the solution corresponds to the scalar field rolling up the potential due to a large initial velocity, and is an unstable evolution. Because of this we shall only be trying to construct scaling solutions from positive terms in the potential.

## 5 Scaling regions

Taking a general potential of the form[2],

$$V = \sum_{i=1}^N V_i \quad (5.53)$$

$$V_i = \Lambda_i \exp(\underline{\alpha}_i \cdot \underline{\varphi}) = \Lambda_i \exp\left(\sum_{I=1}^n \alpha_{iI} \varphi_I\right), \quad (5.54)$$

with  $n$  scalar fields and  $N$  terms then scaling behaviour can be expected only for potentials with  $N \leq n$ , otherwise we find an over-determined system of equations as described in the last section. In our system we have more terms than fields, therefore we look for directions in field space which can support scaling solutions owing to a subset of terms being subdominant. For term  $V_d$  to dominate over term  $V_s$  in direction  $\underline{\Phi}$  we need

$$\underline{\alpha}_s \cdot \underline{\Phi} < \underline{\alpha}_d \cdot \underline{\Phi}. \quad (5.55)$$

In order for scaling to be achieved without going to very large field values we shall require the subdominant terms ( $V_s$ ) to die off exponentially, while the dominant terms ( $V_d$ ) increase exponentially

$$\underline{\alpha}_d \cdot \underline{\Phi} > 0, \quad \underline{\alpha}_s \cdot \underline{\Phi} < 0. \quad (5.56)$$

Furthermore, we guarantee that the scaling solution rolls down the potential, rather than up, by enforcing the terms with negative  $\Lambda_i$  to be subdominant terms. We call all these requirements the conditions for *strong scaling*.

## 5.1 SO(5)/SO(3)<sub>A</sub>

In order to make the procedure clearer we shall start with a special, restricted, example by choosing the coset SO(5)/SO(3)<sub>A</sub>, taking the restriction  $\varphi_2 = 0 = \varphi_3$  (3.13). Then we find the terms in the potential are given by the following  $\underline{\alpha}$  vectors (3.13)(5.54), with those contributing to negative terms ( $\Lambda_i < 0$ ) indicated with  $\bullet$  marks,

$$\begin{aligned}
\bullet \underline{\alpha}_1 &= \frac{\sqrt{7}}{21} \kappa(-9\sqrt{2}, 0, 0, -3\sqrt{3}), \\
\underline{\alpha}_2 &= \frac{\sqrt{7}}{21} \kappa(-9\sqrt{2}, 0, 0, -10\sqrt{3}), \\
\bullet \underline{\alpha}_3 &= \frac{\sqrt{7}}{21} \kappa(-9\sqrt{2}, 0, 0, 4\sqrt{3}), \\
\underline{\alpha}_4 &= \frac{\sqrt{7}}{21} \kappa(-21\sqrt{2}, 0, 0, 0).
\end{aligned} \tag{5.57}$$

The procedure now is to find a direction in field space,  $\underline{\Phi}$ , which satisfies

$$\underline{\alpha}_1 \cdot \underline{\Phi} = -\eta^2, \tag{5.58}$$

$$\underline{\alpha}_3 \cdot \underline{\Phi} = -\zeta^2, \tag{5.59}$$

where  $\eta$  and  $\zeta$  are real constants, thereby ensuring that the negative terms,  $V_1$  and  $V_3$ , are exponentially small in the direction  $\underline{\Phi}$ . With these conditions we now parametrise  $\underline{\Phi}$  as

$$\underline{\Phi} = \frac{21}{\sqrt{7}\kappa}(\sqrt{2}a, 0, 0, \sqrt{3}b), \tag{5.60}$$

where the numerical factors are simply for convenience, and we then derive that

$$\underline{\alpha}_2 \cdot \underline{\Phi} = -2\eta^2 + \zeta^2, \tag{5.61}$$

$$\underline{\alpha}_4 \cdot \underline{\Phi} = -\frac{4}{3}\eta^2 - \zeta^2. \tag{5.62}$$

We therefore conclude that if the negative terms in the potential,  $V_1$  and  $V_3$ , are exponentially small then so is  $V_4$  as  $\underline{\alpha}_4 \cdot \underline{\Phi} < 0$ . However, there are regions where  $V_2$  can be large as  $\underline{\alpha}_2 \cdot \underline{\Phi}$  will be positive if  $\zeta^2 > 2\eta^2$ . In this example we find that if

$$a + 2b > 0, \quad 3a + 5b < 0 \tag{5.63}$$

then the term  $V_2$  will be exponentially large with  $V_1, V_3, V_4$  being exponentially small. Thus, this system meets the requirements of *strong scaling* as given earlier.

As an example we consider an initial condition commensurate with (5.63) by choosing as a direction for  $\underline{\Phi}$ ,

$$\begin{aligned}
\underline{\Phi} &= \frac{1}{\sqrt{5}}(\sqrt{2}\varphi_1 - \sqrt{3}\varphi_4), \\
\underline{\chi} &= \frac{1}{\sqrt{5}}(\sqrt{3}\varphi_1 + \sqrt{2}\varphi_4),
\end{aligned} \tag{5.64}$$

where we have introduced the orthogonal partner,  $\underline{\chi}$ , to  $\underline{\Phi}$ . Note that as this is an SO(2) rotation the kinetic terms remain canonical. Substituting this into the potential (3.13) we find that the large  $\underline{\Phi}$  limit is described by

$$V \sim \frac{3}{2\kappa^2} \exp\left(\frac{4\sqrt{7}}{7\sqrt{5}}\kappa\underline{\Phi} - \frac{19\sqrt{42}}{21\sqrt{5}}\kappa\underline{\chi}\right). \tag{5.65}$$

Performing one more SO(2) field redefinition,

$$\psi_1 = \sqrt{3/22} \left( \frac{4\sqrt{7}}{7\sqrt{5}}\Phi - \frac{19\sqrt{42}}{21\sqrt{5}}\chi \right), \quad \psi_2 = \sqrt{3/22} \left( \frac{4\sqrt{7}}{7\sqrt{5}}\chi + \frac{19\sqrt{42}}{21\sqrt{5}}\Phi \right) \quad (5.66)$$

the potential becomes

$$V \sim \frac{3}{2\kappa^2} \exp(\sqrt{22/3}\kappa\psi_1). \quad (5.67)$$

We now see that (4.48)(4.51)(4.52) imply that a scaling solution is possible, but only in the presence of a background fluid. Note that in this example whenever  $V_2$  dominates over the other terms it is always possible to perform a field redefinition so that the effective potential will take exactly the same form as (5.67), showing that tracking solutions are allowed in the region defined by (5.63).

If we now consider the more general case with  $\varphi_2$  and  $\varphi_3$  non-vanishing then a similar analysis shows that there can be up to four positive terms dominant in the potential, as there are four scalars this allows there to be scaling solutions and the full system can meet the requirements of *strong scaling*. The terms which can dominate in these more general regions are given in table 2. This table lists the terms which are dominant in some particular directions, for example: there are directions in the SO(5)/SO(3)<sub>A</sub> case where only the  $V_2$  increases exponentially; there are also directions where both  $V_2$  and  $V_3$  increase with the rest decreasing; there are, however, no directions where  $V_2$  and  $V_8$  increase with the rest decreasing. We shall defer the study of these more complicated directions for future work.

## 5.2 SO(5)/SO(3)<sub>A+B</sub>

We can perform a similar analysis for the potential (3.19), for which there are three scalar fields so we are allowed at most three dominant terms in the potential for a scaling solution. The exponents in the potential are described by the following seven vectors,

$$\begin{aligned} \bullet \underline{\alpha}_1 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, 0, -4), \\ \bullet \underline{\alpha}_2 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, \sqrt{7}, 3), \\ \bullet \underline{\alpha}_3 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, -3\sqrt{7}, 3), \\ \underline{\alpha}_4 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, -4\sqrt{7}, -4), \\ \underline{\alpha}_5 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, 4\sqrt{7}, -4), \\ \underline{\alpha}_6 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, -2\sqrt{7}, 10), \\ \underline{\alpha}_7 &= \frac{1}{\sqrt{21}}\kappa(-7\sqrt{6}, 0, 0). \end{aligned} \quad (5.68)$$

In this case we find that a different structure appears. Whereas the SO(5)/SO(3)<sub>A</sub> case had directions in field space using the maximum number of terms in the potential allowed by scaling, here we find that there can be at most one single term. By requiring that the negative terms  $V_1$ ,  $V_2$  and  $V_3$  all decrease we find there are regions where either  $V_4$ ,  $V_5$  or  $V_6$  dominate and each such region gives an exponent of  $b = \sqrt{26/3}$  in (4.48). Thus SO(5)/SO(3)<sub>A+B</sub> can satisfy *strong scaling*.

### 5.3 M(p,q,r)

The coset  $SU(3) \times SU(2) \times U(1) / SU(2) \times U(1) \times U(1)$  leads to the potential given in (3.30), which we can describe using the following vectors,

$$\begin{aligned}
\underline{\alpha}_1 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 2\sqrt{7}, 8\sqrt{2}), \\
\bullet \underline{\alpha}_2 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, -2\sqrt{7}, \sqrt{2}), \\
\bullet \underline{\alpha}_3 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, \sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_4 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, -4\sqrt{7}, 8\sqrt{2}), \\
\underline{\alpha}_5 &= \frac{1}{\sqrt{21}} \kappa(-7\sqrt{6}, 0, 0).
\end{aligned} \tag{5.69}$$

When looking for *strong scaling* regions we discover that although there are four fields, at most there are two that will be relevant. The terms that can dominate individually are  $V_1$ ,  $V_4$ ,  $V_5$  for which we find the effective exponents  $\sqrt{10}$ ,  $\sqrt{14}$  and  $\sqrt{14}$  respectively. There is also a set of directions where both  $V_1$  and  $V_4$  dominate, the study of these directions will be included in forthcoming work.

### 5.4 N(p,q,r)

The directions in the potential for  $N^{pqr}$  are given by the following vectors,

$$\begin{aligned}
\underline{\alpha}_1 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 0, -4\sqrt{7}, 8\sqrt{2}), \\
\bullet \underline{\alpha}_2 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 0, -2\sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_3 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, -2\sqrt{21}, -2\sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_4 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 2\sqrt{21}, -2\sqrt{7}, \sqrt{2}), \\
\bullet \underline{\alpha}_5 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, -\sqrt{21}, \sqrt{7}, \sqrt{2}), \\
\bullet \underline{\alpha}_6 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, \sqrt{21}, \sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_7 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 0, 4\sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_8 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, 2\sqrt{21}, 2\sqrt{7}, 8\sqrt{2}), \\
\underline{\alpha}_9 &= \frac{1}{\sqrt{21}} \kappa(-3\sqrt{6}, -2\sqrt{21}, 2\sqrt{7}, 8\sqrt{2}), \\
\underline{\alpha}_{10} &= \frac{1}{\sqrt{21}} \kappa(-7\sqrt{6}, 0, 0, 0).
\end{aligned} \tag{5.70}$$

In this case each of the positive terms,  $V_1$ ,  $V_3$ ,  $V_4$ ,  $V_7$ ,  $V_8$ ,  $V_9$ ,  $V_{10}$ , in the potential can satisfy *strong scaling* individually. In these cases the effective exponent  $b$  in (4.48) is  $\sqrt{14}$ ,  $\sqrt{8}$ ,  $\sqrt{8}$ ,  $\sqrt{8}$ ,  $\sqrt{14}$ ,  $\sqrt{14}$ ,  $\sqrt{14}$ , respectively. We also find that there can be up to three dominant terms while still maintaining the *strong scaling* criteria; such terms are given in table 2.



coset	terms allowed
SO(5)/SO(3) <sub>A</sub>	{2},{3},{4},{8},{9},{10}, {2,3},{2,4},{2,9},{3,4},{3,8},{4,10}, {2,3,4},{2,3,8},{2,3,9},{2,4,9},{2,4,10},{3,4,8},{3,4,10}, {2,3,4,8},{2,3,4,9},{2,3,4,10}
SO(5)/SO(3) <sub>A+B</sub>	{4},{5},{6}
SO(5)/SO(3) <sub>max</sub>	{2}
M <sup>pqr</sup>	{1},{4},{5},{1,4}
N <sup>pqr</sup>	{1},{3},{4},{7},{8},{9},{10}, {1,3},{1,4},{1,8},{1,9},{3,9},{3,10},{4,8},{4,10},{7,8},{7,9},{7,10},{8,9}, {1,3,9},{1,4,8},{1,8,9},{7,8,9},{1,3,8,9},{1,4,8,9},{1,7,8,9}
Q <sup>pqr</sup>	{1},{5},{6},{7},{1,5},{1,6},{5,6},{1,5,6}.

Table 2: Terms allowed in *strong scaling* directions for the cosets of 11d supergravity.

## 5.5 Q(p,q,r)

The directions in the Q<sup>pqr</sup> potential are described by the following vectors,

$$\begin{aligned}
\underline{\alpha}_1 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, 2\sqrt{21}, 2\sqrt{7}, 8\sqrt{2}), \\
\bullet \underline{\alpha}_2 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, 0, -2\sqrt{7}, \sqrt{2}), \\
\bullet \underline{\alpha}_3 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, -\sqrt{21}, \sqrt{7}, \sqrt{2}), \\
\bullet \underline{\alpha}_4 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, \sqrt{21}, \sqrt{7}, \sqrt{2}), \\
\underline{\alpha}_5 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, -2\sqrt{21}, 2\sqrt{7}, 8\sqrt{2}), \\
\underline{\alpha}_6 &= \frac{1}{\sqrt{21}}\kappa(-3\sqrt{6}, 0, -4\sqrt{7}, 8\sqrt{2}), \\
\underline{\alpha}_7 &= \frac{1}{\sqrt{21}}\kappa(-7\sqrt{6}, 0, 0, 0).
\end{aligned} \tag{5.71}$$

As in the N<sup>pqr</sup> case we find that each of the positive terms in the potential,  $V_1, V_5, V_6, V_7$ , can satisfy *strong scaling* on their own, in each case we find that the effective exponent is  $\sqrt{14}$ . We also find that there can be up to three terms in the potential which can dominate together over the rest, these are shown in table 2.

## 6 Concluding remarks

In this paper we have given an explicit construction of the consistent dimensional reduction of eleven dimension supergravity over the homogeneous spaces classified in [7]. By including a Freund-Rubin four-form flux we have seen that this leads to an effective theory in four dimensions that consists of Einstein gravity coupled to a series of moduli fields, and that these fields experience a potential which is composed of a series of exponentials. Motivated by the large body of material which studies gravity+scalars with exponential potentials we studied the possibility of scaling solutions

in this setup. However, for exact scaling solutions to exist there can be at most the same number of terms in the potential as there are fields; if this is violated then one finds an over constrained algebraic system for which there is no solution. In all the cosets of the classification we find that there are more terms in the potential than there are moduli fields. While this rules out the existence of exact scaling solutions, one still has the possibility that there are regions in field space where only a subset of terms are relevant, thereby allowing approximate scaling solutions. Having given a set of criteria for such regions to exist we showed that it is possible to get approximate scaling in each of the coset examples. The cases which we analysed explicitly were only the ones where there is a single dominant term, and we explicitly gave the exponent of the effective potential. We also noted that there are examples where more than one term could dominate, giving the possibility of assisted behaviour in these directions; we shall leave the detailed study of such examples for future work.

Aside from the Freund-Rubin flux that we included, there is also the possibility that one may include internal flux. This internal flux may take the form of a combination of axions coming from three-form potentials as  $C_3 \sim \phi(x)c_3(y)$ , and non-exact flux, i.e. a four form flux that cannot be written globally as  $dc$ . In such a scheme, the consistent truncation would require us to take fluxes that matched the symmetries of the coset involved and would lead to a finite, manageable set of scalar fields. This would then lead to a four dimensional effective theory containing axions and geometrical moduli. Both of these types fields are to be expected in such dimensional reduction schemes and it remains to be seen how the presence of such axions in coset models affects the dynamics.

**Acknowledgements** The authors would like to thank Beatriz de Carlos, Mark Hindmarsh, Thomas House and Eran Palti for useful discussions. Both JLPK and PMS are supported by PPARC.

## Appendices

### A Review of invariant objects on a coset

Decompose the Lie algebra as  $\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$ , where  $\mathcal{H} \subset \mathcal{G}$  is a subgroup of  $\mathcal{G}$ . We consider only compact groups, so the coset space  $G/H$  is reductive [16] i.e.

$$\begin{aligned} [\mathcal{H}_a, \mathcal{H}_b] &= \mathcal{H}_c f_{ab}^c, \\ [\mathcal{H}_a, \mathcal{K}_i] &= \mathcal{K}_j f_{ai}^j, \\ [\mathcal{K}_i, \mathcal{K}_j] &= \mathcal{H}_a f_{ij}^a + \mathcal{K}_k f_{ij}^k. \end{aligned} \tag{A.1}$$

We use  $i, j, \dots$  indices to label elements of  $\mathcal{K}$  and  $a, b, \dots$  to label elements of  $\mathcal{H}$ . Now consider the left-invariant form on  $G/H$ ,  $\Theta$ , which we expand by introducing the one-forms  $e^a$  and  $e^i$ .

$$\Theta = L^{-1}dL = \mathcal{H}_a e^a + \mathcal{K}_i e^i. \tag{A.2}$$

To find the algebra of these one-forms consider

$$d\Theta = dL^{-1} \wedge dL = -L^{-1}dL \wedge L^{-1}dL = -\Theta \wedge \Theta \tag{A.3}$$

which then gives

$$\begin{aligned} de^a &= -\frac{1}{2}f^a_{bc}e^b \wedge e^c - \frac{1}{2}f^a_{ij}e^i \wedge e^j \\ de^i &= -\frac{1}{2}f^i_{jk}e^j \wedge e^k - f^i_{aj}e^a \wedge e^j \end{aligned} \quad (\text{A.4})$$

We may use these left-invariant one-forms to construct a homogeneous, G-invariant metric on  $G/H$ ,

$$ds^2_{G/H} = g_{ij}e^i \otimes e^j. \quad (\text{A.5})$$

Homogeneity requires the parameters  $g_{ij}$  to be independent of the co-ordinates on  $G/H$  and G-invariance requires [44]

$$g_{kj}f^k_{ia} + g_{ik}f^k_{ja} = 0. \quad (\text{A.6})$$

This relation gets generalised for objects with more indices, such as form fields

$$T_{ki_2i_3\dots}f^k_{i_1a} + T_{i_1ki_3\dots}f^k_{i_2a} + T_{i_1i_2k\dots}f^k_{i_3a} + \dots = 0. \quad (\text{A.7})$$

Note that for  $G$  to be unimodular then  $f^I_{IJ} = 0$  and for  $H$  to be unimodular then  $f^a_{ab} = 0$  and for  $G/H$  being reductive then  $f^a_{bi} = 0$ , all of which show that  $f^i_{ij} = 0$  and  $f^i_{ia} = 0$ .

## B Reducing the Ricci scalar

We choose a higher dimensional metric to consist of a spacetime part and an internal coset part according to

$$\begin{aligned} ds^2 &= e^{2\psi(x)} ds^2_{(1,d-1)} + g_{ij}(x)e^i \otimes e^j, \\ &= e^{2\psi(x)} \eta_{\mu\nu}e^\mu \otimes e^\nu + g_{ij}(x)e^i \otimes e^j \\ &= \hat{g}_{\hat{\mu}\hat{\nu}}e^{\hat{\mu}} \otimes e^{\hat{\nu}}, \end{aligned} \quad (\text{B.8})$$

with the co-ordinates on spacetime being represented by  $x$  and those on the coset by  $y$ ,  $\psi(x)$  represents a freedom to choose the spacetime co-ordinates. In the following we shall analyse this space using the frame  $e^{\hat{\mu}} = (e^\mu, e^i)$ , note that this is not an orthonormal frame. In order to find the connection one-forms,  $\omega^{\hat{\mu}}_{\hat{\nu}}$ , we need to solve

$$\begin{aligned} d\hat{g}_{\hat{\mu}\hat{\nu}} - \omega^{\hat{\rho}}_{\hat{\mu}}g_{\hat{\rho}\hat{\nu}} - \omega^{\hat{\rho}}_{\hat{\nu}}g_{\hat{\mu}\hat{\rho}} &= 0 \\ de^{\hat{\mu}} + \omega^{\hat{\mu}}_{\hat{\nu}} \wedge e^{\hat{\nu}} &= 0, \end{aligned} \quad (\text{B.9})$$

and the curvature two-forms follow from

$$\hat{R}^{\hat{\mu}}_{\hat{\nu}} = d\omega^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}}_{\hat{\rho}} \wedge \omega^{\hat{\rho}}_{\hat{\nu}}. \quad (\text{B.10})$$

We find that the Ricci tensor is given by

$$\begin{aligned} \hat{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{(d)\mu\nu} - (d-2)\nabla_\mu\nabla_\nu\psi - \eta_{\mu\nu}\nabla_\rho\nabla^\rho\psi - (d-2)\eta_{\mu\nu}\nabla_\rho\psi\nabla^\rho\psi + (d-2)\nabla_\mu\psi\nabla_\nu\psi \\ &\quad - \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} - \frac{1}{2}g^{ij}\nabla_\mu\nabla_\nu g_{ij} + \frac{1}{2}g^{ij}(\nabla_\mu g_{ij}\nabla_\nu\psi + \nabla_\nu g_{ij}\nabla_\mu\psi) - \frac{1}{2}\eta_{\mu\nu}g^{ij}\nabla_\rho g_{ij}\nabla^\rho\psi \end{aligned} \quad (\text{B.11})$$

$$\hat{\mathcal{R}}_{\mu j} = -\frac{1}{2}g^{kl}\nabla_\mu g_{km}f^m_{lj} \quad (\text{B.12})$$

$$\hat{\mathcal{R}}_{ij} = \tilde{\mathcal{R}}_{ij} + e^{-2\psi} \left( \frac{1}{2}g^{kl}\nabla_\mu g_{ik}\nabla^\mu g_{jl} - \frac{1}{2}\nabla_\mu\nabla^\mu g_{ij} + \frac{1}{4}g_{kl}\nabla_\mu g^{kl}\nabla^\mu g_{ij} - \frac{1}{2}(d-2)\nabla_\mu\psi\nabla^\mu g_{ij} \right).$$

In deriving this we have used the fact that compact Lie groups are unimodular, giving  $f_{IJ}^I = 0$  [21, 22, 45].  $\tilde{\mathcal{R}}_{ij}$  denotes the curvature of the coset space, treating the  $g_{ij}$  as constant and the covariant derivatives,  $\nabla_\mu$  are for the metric  $ds_{(1,d-1)}^2$  with their indices raised by  $\eta^{\mu\nu}$ . Given the Ricci curvatures above we can see one of the issues related to the consistency of truncation, namely that there is nothing to source  $\mathcal{R}_{\mu j}$  and so it must vanish by the 11D equations of motion. For the cases we consider, we find that this term does vanish.

We may now trace the above to find the following Ricci scalar

$$\begin{aligned} \hat{\mathcal{R}} &= \mathcal{R}_{G/H} + e^{-2\psi} [\mathcal{R}_{(d)} - 2(d-1)\nabla^2\psi - (d-1)(d-2)\nabla_\mu\psi\nabla^\mu\psi - g^{ij}\nabla^2g_{ij} \\ &\quad - \frac{3}{4}\nabla_\mu g^{ij}\nabla_\mu g_{ij} - (d-2)g^{ij}\nabla_\mu g^{ij}\nabla^\mu\psi - \frac{1}{4}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla^\mu g_{kl}]. \end{aligned} \quad (\text{B.13})$$

Making use of the gauge freedom we choose

$$e^{(d-2)\psi}\sqrt{g_{ij}} = 1 \quad (\text{B.14})$$

showing that the physical volume of the internal space is given by

$$V_{phys} = V_{G/H}e^{(2-d)\psi}. \quad (\text{B.15})$$

This gauge choice enables us to write

$$\begin{aligned} \hat{\mathcal{R}}_{\mu\nu} &= \mathcal{R}_{(d)\mu\nu} + \frac{1}{2(d-2)}g^{ij}\nabla_\sigma\nabla^\sigma g_{ij}\eta_{\mu\nu} + \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} + \frac{1}{2(d-2)}\eta_{\mu\nu}\nabla_\sigma g^{ij}\nabla^\sigma g_{ij} \\ &\quad - \frac{1}{4(d-2)}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla_\nu g_{kl} \\ &= \mathcal{R}_{(d)\mu\nu} + \frac{1}{4}\nabla_\mu g^{ij}\nabla_\nu g_{ij} - \eta_{\mu\nu}\nabla^2\psi - (d-2)\nabla_\mu\psi\nabla_\nu\psi \\ \hat{\mathcal{R}}_{ij} &= \tilde{\mathcal{R}}_{ij} + \frac{1}{2}e^{-2\psi} [g^{kl}\nabla_\mu g_{ik}\nabla^\mu g_{jl} - \nabla_\mu\nabla^\mu g_{ij}] \\ \hat{\mathcal{R}} &= e^{-2\psi} [\mathcal{R}_{(d)} - 2(d-1)\nabla^2\psi - g^{ij}\nabla^2g_{ij} - \frac{3}{4}\nabla_\mu g^{ij}\nabla_\mu g_{ij} - \frac{1}{4(d-2)}g^{ij}\nabla_\mu g_{ij}g^{kl}\nabla^\mu g_{kl}] \\ &\quad + \mathcal{R}_{G/H}. \end{aligned} \quad (\text{B.17})$$

## C Coset $\text{SO}(5)/\text{SO}(3)_A$

### C.1 The invariant metric

The Lie algebra of  $\text{SO}(n)$  may be written in terms of one-forms,  $\Lambda_{IJ}$  as

$$d\Lambda_{IJ} = -2\Lambda_{IK} \wedge \Lambda_{KJ} \quad (\text{C.18})$$

where  $\Lambda_{IJ} = -\Lambda_{JI}$  and  $I, J, \dots = 0, 1, \dots, n-1$ . Taking  $N = 5$  we can introduce a different basis,  $\lambda_i$ ,  $\rho_i$  and  $\Pi_a$

$$\begin{aligned} \lambda_i &= -2 \left( \Lambda_{0i} + \frac{1}{2}\epsilon_{ijk}\Lambda_{jk} \right) \\ \rho_i &= 2 \left( \Lambda_{0i} - \frac{1}{2}\epsilon_{ijk}\Lambda_{jk} \right) \\ \Pi_p &= 2\Lambda_{p4}. \end{aligned} \quad (\text{C.19})$$

In this basis we find that the  $\lambda_i$  and  $\rho_i$  form commuting  $\mathfrak{so}(3)$  sub-algebras, labelled  $\mathfrak{so}(5)_A$  and  $\mathfrak{so}(5)_B$  respectively. Given these sub-algebras we may construct the coset  $\text{SO}(5)/\text{SO}(3)_A$ . In order to find a  $G$  invariant metric we apply the techniques of appendix A to  $g_{ab}$  using (A.7) and discover that the most general metric on the coset, consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & \phi & \theta & 0 & 0 & 0 & 0 \\ \phi & \beta & \psi & 0 & 0 & 0 & 0 \\ \theta & \psi & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \quad (\text{C.20})$$

so there are seven allowed parameters.

## C.2 The Ricci tensor

In order to simplify the analysis of the dynamics we restrict the most general metric to the diagonal case,  $g_{ab} = \text{diag}(\alpha, \beta, \gamma, \delta, \delta, \delta, \delta)$  and find that the Ricci tensor is given by

$$\begin{aligned} \mathcal{R}_{11} &= \frac{\alpha^2}{2\beta\gamma} - \frac{\beta}{2\gamma} - \frac{\gamma}{2\beta} + \frac{\alpha^2}{\delta^2} + 1, \\ \mathcal{R}_{22} &= \frac{\beta^2}{2\alpha\gamma} - \frac{\alpha}{2\gamma} - \frac{\gamma}{2\alpha} + \frac{\beta^2}{\delta^2} + 1, \\ \mathcal{R}_{33} &= \frac{\gamma^2}{2\alpha\beta} - \frac{\alpha}{2\beta} - \frac{\beta}{2\alpha} + \frac{\gamma}{\delta^2} + 1, \\ \mathcal{R}_{44} = \mathcal{R}_{55} = \mathcal{R}_{66} = \mathcal{R}_{77} &= -\frac{\alpha}{2\delta} - \frac{\beta}{2\delta} - \frac{\gamma}{2\delta} + 3. \end{aligned} \quad (\text{C.21})$$

When considering the dynamics one discovers that there are special points in the moduli space given by Einstein metrics, corresponding to stationary points of the effective potential. This condition states that the Ricci curvature is proportional to the metric, we find two such points given by [7]

$$\alpha = \beta = \gamma = \delta \quad (\text{C.22})$$

$$\mathcal{R}_{ab} = \frac{3}{2\alpha} g_{ab} \quad (\text{C.23})$$

and

$$\alpha = \beta = \gamma = \delta/5 \quad (\text{C.24})$$

$$\mathcal{R}_{ab} = \frac{27}{50\alpha} g_{ab}. \quad (\text{C.25})$$

## D Coset $\text{SO}(5)/\text{SO}(3)_{A+B}$

### D.1 The invariant metric

One may construct another  $\text{SO}(3)$  subgroup of  $\text{SO}(5)$  based on the diagonal part of  $\text{SO}(3)_A \times \text{SO}(3)_B$ ,  $\text{SO}(3)_{A+B}$ . To see this we introduce a new basis of one-forms,

$$\sigma_i = -\epsilon_{ijk} \Lambda_{jk} = \frac{1}{2}(\lambda_i + \rho_i), \quad (\text{D.26})$$

$$\begin{aligned}
\tau_i &= 2\Lambda_{0i} = -\frac{1}{2}(\lambda_i - \rho_i), \\
\tau_4 &= 2\Lambda_{04} = \Pi_0, \\
\tau_5 &= 2\Lambda_{14} = \Pi_1, \\
\tau_6 &= 2\Lambda_{24} = \Pi_2, \\
\tau_7 &= 2\Lambda_{34} = \Pi_3.
\end{aligned}$$

In this basis we find that the  $\sigma_i$  form a commuting  $\mathfrak{so}(3)$  sub-algebra, labelled  $\mathfrak{so}(5)_{A+B}$ . In order to find a  $G$  invariant metric we apply (A.7) to  $g_{ab}$  and discover that the most general metric on the coset consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 & \gamma \end{pmatrix}, \quad (\text{D.27})$$

and so there are four allowed parameters.

## D.2 The Ricci scalar

Again we restrict to a diagonal metric given by  $g_{ab} = \text{diag}(\alpha, \alpha, \alpha, \beta, \gamma, \gamma, \gamma)$  and find that the Ricci tensor has the following components

$$\begin{aligned}
\mathcal{R}_{11} = \mathcal{R}_{22} = \mathcal{R}_{33} &= \frac{\alpha^2}{2\beta\gamma} - \frac{\beta}{2\gamma} - \frac{\gamma}{2\beta} + 3, \\
\mathcal{R}_{44} &= -\frac{3\alpha}{2\gamma} + \frac{3\beta^2}{2\alpha\gamma} - \frac{3\gamma}{2\alpha} + 3, \\
\mathcal{R}_{55} = \mathcal{R}_{66} = \mathcal{R}_{77} &= -\frac{\alpha}{2\beta} - \frac{\beta}{2\alpha} + \frac{\gamma}{2\alpha\beta} + 3.
\end{aligned} \quad (\text{D.28})$$

We may also find the points in moduli space for which such metrics satisfy the Einstein condition, we find [7]

$$\alpha = \gamma = \frac{2}{3}\beta \quad (\text{D.29})$$

$$\mathcal{R}_{ab} = \frac{9}{4\alpha}g_{ab}. \quad (\text{D.30})$$

## E Coset $\text{SO}(5)/\text{SO}(3)_{MAX}$

### E.1 The invariant metric

The final  $\text{SO}(3)$  subgroup of  $\text{SO}(5)$  is the maximal subgroup, which we can see by introducing the following basis of one-forms

$$\sigma_1 = \frac{1}{\sqrt{5}}(\lambda_1 + \sqrt{3}\Pi_2), \quad (\text{E.31})$$

$$\begin{aligned}
\sigma_2 &= \frac{1}{\sqrt{5}}(\lambda_2 + \sqrt{3}\Pi_1), \\
\sigma_3 &= \frac{1}{2\sqrt{5}}(\lambda_3 - 3\rho_3), \\
\tau_1 &= -\frac{1}{4\sqrt{5}}(3\lambda_1 - 2\sqrt{3}\Pi_2 - 5\rho_1), \\
\tau_2 &= -\frac{1}{4\sqrt{5}}(3\lambda_2 - 2\sqrt{3}\Pi_1 - 5\rho_2), \\
\tau_3 &= \frac{1}{2\sqrt{5}}(3\lambda_3 + \rho_3), \\
\tau_4 &= -\frac{1}{4}(\sqrt{3}\lambda_1 - 2\Pi_2 + \sqrt{3}\rho_1), \\
\tau_5 &= \frac{1}{4}(\sqrt{3}\lambda_2 - 2\Pi_1 + \sqrt{3}\rho_2), \\
\tau_6 &= \Pi_0, \\
\tau_7 &= \Pi_3.
\end{aligned}$$

In this basis we find that the  $\sigma_i$  form a commuting  $\mathfrak{so}(3)$  sub-algebra, labelled  $\mathfrak{so}(5)_{MAX}$ , we shall now consider the coset  $\text{SO}(5)/\text{SO}(3)_{MAX}$ . In order to find a  $G$  invariant metric we apply (A.7) to  $g_{ab}$  and discover that the most general metric on the coset consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \quad (\text{E.32})$$

so there is only one allowed parameter and the metric is already diagonal. One finds that this metric is already an Einstein metric, with the Ricci curvature being given by

$$\mathcal{R}_{ab} = \frac{27}{10\alpha}g_{ab}. \quad (\text{E.33})$$

## F Coset $M^{pqr} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) / \text{U}(1) \times \text{U}(1)$

### F.1 The invariant metric

The generators of the Lie algebra of  $M^{pqr}$  may be written as,

$$\begin{aligned}
T^1 &= -\frac{1}{2}i\lambda_4 & T^2 &= -\frac{1}{2}i\lambda_5 & T^3 &= -\frac{1}{2}i\lambda_6 \\
T^4 &= -\frac{1}{2}i\lambda_7 & T^5 &= -\frac{1}{2}i\sigma_1 & T^6 &= -\frac{1}{2}i\sigma_2 \\
T^7 &= -\frac{i}{2\zeta} \left( 2ry + \sqrt{3}p\lambda_8 + q\sigma_3 \right) \\
T^8 &= -\frac{1}{2}i\lambda_1 & T^9 &= -\frac{1}{2}i\lambda_2 & T^{10} &= -\frac{1}{2}i\lambda_3 \\
T^{11} &= \frac{i}{\sqrt{2}\zeta\eta} \left( (3p^2 + q^2)y - \sqrt{3}pr\lambda_8 - qr\sigma_3 \right)
\end{aligned} \quad (\text{F.34})$$

$$T^{12} = -\frac{i}{2\eta} \left( \sqrt{3}p\sigma_3 - q\lambda_8 \right)$$

where  $\sigma_i$  with  $i \in \{1, 2, 3\}$  are the Pauli matrices with  $[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$ , and the  $\lambda_i$  with  $i \in \{1..8\}$  are the Gell-Mann  $SU(3)$  matrices normalised such that  $[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$  with the totally antisymmetric  $\epsilon$  and  $f$  given by,

$$\begin{aligned} \epsilon_{123} &= 1 \\ f_{123} &= 1 \\ f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ f_{458} &= f_{678} = \frac{\sqrt{3}}{2}, \end{aligned} \tag{F.35}$$

and  $y$  is the generator of the separate  $U(1)$  factor. The three primes  $p$ ,  $q$  and  $r$  characterise the embedding of the  $U(1) \times U(1)$  subgroup. For convenience we define the following quantities,

$$\zeta = \sqrt{3p^2 + q^2 + 2r^2} \quad \eta = \sqrt{3p^2 + q^2}. \tag{F.36}$$

In order to find a  $G$  invariant metric we apply the techniques of appendix A to  $g_{ab}$  using (A.7) and discover that the most general metric on the coset, consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \tag{F.37}$$

so there are three allowed parameters.

## F.2 The Ricci tensor

Taking the metric to be,  $g_{ab} = \text{diag}(\alpha, \alpha, \alpha, \alpha, \beta, \beta, \gamma)$  and find that the Ricci tensor is given by

$$\begin{aligned} \mathcal{R}_{11} = \mathcal{R}_{22} = \mathcal{R}_{33} = \mathcal{R}_{44} &= \frac{3}{2} - \frac{9}{8} \frac{p^2}{\zeta^2} \frac{\gamma}{\alpha}, \\ \mathcal{R}_{55} = \mathcal{R}_{66} &= 1 - \frac{1}{2} \frac{q^2}{\zeta^2} \frac{\gamma}{\beta}, \\ \mathcal{R}_{77} &= \frac{\gamma^2}{2\zeta^2} \left( \frac{9}{2} \frac{p^2}{\alpha^2} + \frac{q^2}{\beta^2} \right). \end{aligned} \tag{F.38}$$

## G Coset $N^{pqr} = SU(3) \times U(1) / U(1) \times U(1)$

### G.1 The invariant metric

The generators of the Lie algebra of  $N^{pqr}$  may be written as,

$$T^1 = -\frac{1}{2}i\lambda_1 \quad T^2 = -\frac{1}{2}i\lambda_2 \quad T^3 = -\frac{1}{2}i\lambda_4 \tag{G.39}$$



$$\begin{aligned}
T^4 &= -\frac{1}{2}i\lambda_5 & T^5 &= -\frac{1}{2}i\lambda_6 & T^6 &= -\frac{1}{2}i\lambda_7 \\
T^7 &= -\frac{i}{2\zeta} \left( 2ry + q\lambda_3 + \sqrt{3}p\lambda_8 \right) \\
T^8 &= \frac{i}{\sqrt{2}\zeta\eta} \left( (3p^2 + q^2)y - qr\lambda_3 - \sqrt{3}pr\lambda_8 \right) \\
T^9 &= -\frac{i}{2\eta} \left( \sqrt{3}p\lambda_3 - q\lambda_8 \right)
\end{aligned}$$

where  $\lambda_i$  with  $i \in \{1..8\}$  are the Gell-Mann  $SU(3)$  matrices normalised such that  $[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$  with the totally antisymmetric  $f$  given by,

$$\begin{aligned}
f_{123} &= 1 & (G.40) \\
f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\
f_{458} &= f_{678} = \frac{\sqrt{3}}{2},
\end{aligned}$$

and  $y$  is the generator of the separate  $U(1)$  factor. The three primes  $p$ ,  $q$  and  $r$  characterise the embedding of the  $U(1) \times U(1)$  subgroup. For convenience we define the following quantities,

$$\zeta = \sqrt{3p^2 + q^2 + 2r^2} \quad \eta = \sqrt{3p^2 + q^2}. \quad (G.41)$$

In order to find a  $G$  invariant metric we apply the techniques of appendix A to  $g_{ab}$  using (A.7) and discover that the most general metric on the coset, consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \quad (G.42)$$

so there are four allowed parameters.

## G.2 The Ricci tensor

Taking the metric to be,  $g_{ab} = \text{diag}(\alpha, \alpha, \beta, \beta, \gamma, \gamma, \delta)$  and find that the Ricci tensor is given by

$$\begin{aligned}
\mathcal{R}_{11} = \mathcal{R}_{22} &= \frac{3}{2} - \frac{q^2}{2\zeta^2} \frac{\delta}{\alpha} + \frac{1}{4\beta\gamma} (\alpha^2 - \beta^2 - \gamma^2), & (G.43) \\
\mathcal{R}_{33} = \mathcal{R}_{44} &= \frac{3}{2} - \frac{(3p+q)^2}{8\zeta^2} \frac{\delta}{\beta} - \frac{1}{4\alpha\gamma} (\alpha^2 - \beta^2 + \gamma^2), \\
\mathcal{R}_{55} = \mathcal{R}_{66} &= \frac{3}{2} - \frac{(3p-q)^2}{8\zeta^2} \frac{\delta}{\gamma} - \frac{1}{4\alpha\beta} (\alpha^2 + \beta^2 - \gamma^2), \\
\mathcal{R}_{77} &= \frac{\delta^2}{2\zeta^2} \left( \frac{q^2}{\alpha^2} + \frac{(3p+q)^2}{4\beta^2} + \frac{(3p-q)^2}{4\gamma^2} \right).
\end{aligned}$$

## H Coset $Q^{pqr} = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) / \text{U}(1) \times \text{U}(1)$

### H.1 The invariant metric

The generators of the Lie algebra of  $Q^{pqr}$  may be written as,

$$\begin{aligned}
 T^1 &= -\frac{i}{1} \sigma_1, & T^2 &= -\frac{i}{2} \sigma_2, & T^3 &= -\frac{i}{2} \tilde{\sigma}_1, \\
 T^4 &= -\frac{i}{2} \tilde{\sigma}_2, & T^5 &= -\frac{i}{2} \tilde{\sigma}_1, & T^6 &= -\frac{i}{2} \tilde{\sigma}_2, \\
 T^7 &= -\frac{i}{2\zeta} (p \sigma_3 + q \tilde{\sigma}_3 + r \hat{\sigma}_3), \\
 T^8 &= -\frac{i}{2\zeta \eta} (p r \sigma_3 + q r \tilde{\sigma}_3 - (p^2 + q^2) \hat{\sigma}_3), \\
 T^9 &= \frac{i}{2\eta} (p \tilde{\sigma}_3 - q \sigma_3)
 \end{aligned} \tag{H.44}$$

where  $\sigma_i, \tilde{\sigma}_i, \hat{\sigma}_i$  with  $i \in \{1, 2, 3\}$  are three sets of mutually commuting Pauli matrices. The three primes p, q and r characterise the embedding of the  $\text{U}(1) \times \text{U}(1)$  subgroup. For convenience we define the following quantities,

$$\zeta = \sqrt{p^2 + q^2 + r^2} \quad \eta = \sqrt{p^2 + q^2}. \tag{H.45}$$

In order to find a  $G$  invariant metric we apply the techniques of appendix A to  $g_{ab}$  using (A.7) and discover that the most general metric on the coset, consistent with its symmetries is

$$g_{ab} = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix}, \tag{H.46}$$

so there are four allowed parameters.

### H.2 The Ricci tensor

Taking the metric to be  $g_{ab} = \text{diag}(\alpha, \alpha, \beta, \beta, \gamma, \gamma, \delta)$  we find that the Ricci tensor is given by

$$\begin{aligned}
 \mathcal{R}_{11} = \mathcal{R}_{22} &= 1 - \frac{1}{2} \frac{p^2}{\zeta^2} \frac{\delta}{\alpha}, \\
 \mathcal{R}_{33} = \mathcal{R}_{44} &= 1 - \frac{1}{2} \frac{q^2}{\zeta^2} \frac{\delta}{\beta}, \\
 \mathcal{R}_{55} = \mathcal{R}_{66} &= 1 - \frac{1}{2} \frac{r^2}{\zeta^2} \frac{\delta}{\gamma}, \\
 \mathcal{R}_{77} &= \frac{\delta^2}{2\zeta^2} \left( \frac{p^2}{\alpha^2} + \frac{q^2}{\beta^2} + \frac{r^2}{\gamma^2} \right).
 \end{aligned} \tag{H.47}$$

## References

- [1] E. J. Copeland, A. R. Liddle and D. Wands, *Phys. Rev. D* **57**, 4686 (1998) [arXiv:gr-qc/9711068].
- [2] A. Collinucci, M. Nielsen and T. Van Riet, *Class. Quant. Grav.* **22**, 1269 (2005) [arXiv:hep-th/0407047].
- [3] J. P. Derendinger, L. E. Ibanez and H. P. Nilles, *Phys. Lett. B* **155**, 65 (1985).
- [4] M. Dine, R. Rohm, N. Seiberg and E. Witten, *Phys. Lett. B* **156** (1985) 55.
- [5] E. Witten, *Nucl. Phys. B* **474**, 343 (1996) [arXiv:hep-th/9604030].
- [6] G. W. Moore, G. Peradze and N. Saulina, *Nucl. Phys. B* **607**, 117 (2001) [arXiv:hep-th/0012104].
- [7] L. Castellani, L. J. Romans and N. P. Warner, *Nucl. Phys. B* **241**, 429 (1984).
- [8] P. Forgacs and Z. Horvath, *Gen. Rel. Grav.* **11** (1979) 205.
- [9] P. Forgacs and Z. Horvath, *Gen. Rel. Grav.* **10** (1979) 931.
- [10] J. E. Lidsey, D. Wands and E. J. Copeland, *Phys. Rept.* **337**, 343 (2000) [arXiv:hep-th/9909061].
- [11] B. S. DeWit, “Relativity, Groups And Topology.” Les Houches (1963) Gordon and Breach 1964.
- [12] T. Biswas and P. Jaikumar, *JHEP* **0408**, 053 (2004) [arXiv:hep-th/0407063].
- [13] C. M. Chen, P. M. Ho, I. P. Neupane, N. Ohta and J. E. Wang, *JHEP* **0310**, 058 (2003) [arXiv:hep-th/0306291].
- [14] I. P. Neupane and D. L. Wiltshire, *Phys. Rev. D* **72**, 083509 (2005) [arXiv:hep-th/0504135].
- [15] I. P. Neupane and D. L. Wiltshire, *Phys. Lett. B* **619** (2005) 201 [arXiv:hep-th/0502003].
- [16] M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Phys. Rept.* **130**, 1 (1986).
- [17] E. Witten, *Nucl. Phys. B* **186**, 412 (1981).
- [18] E. Cremmer, B. Julia and J. Scherk, *Phys. Lett. B* **76**, 409 (1978).
- [19] J. P. Gauntlett and S. Pakis, *JHEP* **0304**, 039 (2003) [arXiv:hep-th/0212008].
- [20] P. G. O. Freund and M. A. Rubin, *Phys. Lett. B* **97**, 233 (1980).
- [21] J. Scherk and J. H. Schwarz, *Nucl. Phys. B* **153**, 61 (1979).
- [22] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, *Class. Quant. Grav.* **20**, 5161 (2003) [arXiv:hep-th/0306043].
- [23] J. M. Pons and P. Talavera, *Nucl. Phys. B* **678** (2004) 427 [arXiv:hep-th/0309079].
- [24] J. M. Pons and P. Talavera, *Nucl. Phys. B* **703** (2004) 537 [arXiv:hep-th/0401162].

- [25] M. J. Duff, Phys. Lett. B **226**, 36 (1989).
- [26] L. Castellani, R. D'Auria and P. Fre, Nucl. Phys. B **239** (1984) 610.
- [27] L. Castellani and L. J. Romans, Nucl. Phys. B **238** (1984) 683.
- [28] R. D'Auria, P. Fre and P. van Nieuwenhuizen, Phys. Lett. B **136** (1984) 347.
- [29] D. N. Page and C. N. Pope, Phys. Lett. B **145**, 337 (1984).
- [30] D. N. Page and C. N. Pope, Phys. Lett. B **147**, 55 (1984).
- [31] D. N. Page and C. N. Pope, Phys. Lett. B **144**, 346 (1984).
- [32] C. Wetterich, Nucl. Phys. B **302**, 668 (1988).
- [33] A. B. Burd and J. D. Barrow, Nucl. Phys. B **308** (1988) 929.
- [34] N. Kaloper and K. A. Olive, Astropart. Phys. **1**, 185 (1993).
- [35] R. Brustein and P. J. Steinhardt, Phys. Lett. B **302**, 196 (1993) [arXiv:hep-th/9212049].
- [36] T. Barreiro, B. de Carlos and E. J. Copeland, Phys. Rev. D **58**, 083513 (1998) [arXiv:hep-th/9805005].
- [37] T. Barreiro, B. de Carlos and N. J. Nunes, Phys. Lett. B **497**, 136 (2001) [arXiv:hep-ph/0010102].
- [38] R. Brustein, S. P. de Alwis and P. Martens, Phys. Rev. D **70**, 126012 (2004) [arXiv:hep-th/0408160].
- [39] T. Barreiro, B. de Carlos, E. Copeland and N. J. Nunes, arXiv:hep-ph/0506045.
- [40] A. A. Coley and R. J. van den Hoogen, Phys. Rev. D **62**, 023517 (2000) [arXiv:gr-qc/9911075].
- [41] D. Blais and D. Polarski, Phys. Rev. D **70**, 084008 (2004) [arXiv:astro-ph/0404043].
- [42] S. A. Kim, A. R. Liddle and S. Tsujikawa, Phys. Rev. D **72**, 043506 (2005) [arXiv:astro-ph/0506076].
- [43] A. R. Liddle, A. Mazumdar and F. E. Schunck, Phys. Rev. D **58** (1998) 061301 [arXiv:astro-ph/9804177].
- [44] F. Mueller-Hoissen and R. Stuckl, Class. Quant. Grav. **5**, 27 (1988).
- [45] Y. M. Cho and P. G. O. Freund, Phys. Rev. D **12**, 1711 (1975).