Research Article

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Modular iterated integrals associated with cusp forms

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Abstract: We construct an explicit family of modular iterated integrals which involves cusp forms. This leads to a new method of producing modular invariant functions based on iterated integrals of modular forms. The construction will be based on an extension of higher-order modular forms which, in contrast to the standard higher-order forms, applies to general Fuchsian groups of the first kind and, as such, is of independent interest.

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1 Introduction

This paper deals with two classes of functions that have not been previously studied together, namely *modular iterated integrals* and *higher-order modular forms*. We show that they are interrelated in a way that key features of one of them can be elucidated through constructions in the other.

The first class of objects, modular iterated integrals, were introduced recently [2–4] by Brown in the context of the theory of *real-analytic modular forms*. Those are real-analytic functions f on the upper half-plane $\mathfrak{H}$, characterized by the following (cf. Section 2.1 for the precise definition):

(i) A transformation law of the form

$$f(\gamma z) = (cz+d)^r (c\bar{z}+d)^s f(z) \quad \text{for all } z \in \mathfrak{H}$$

and for all $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ in a suitable group Γ.

(ii) An expansion of the form

$$f(z) = \sum_{|j| \le M} y^j \Big(\sum_{m,n \ge 0} a^{(j)}_{m,n} q^m \bar{q}^n \Big),$$

where $y = \mathrm{Im}(z)$ and $q = e^{2\pi i z}$.

The motivation for their introduction included their possible use towards arithmetic questions involving periods and evidence that the modular graph functions of String Theory are real-analytic modular forms.

A special subclass of the class of real-analytic modular forms consists of the spaces $\mathcal{MI}_\ell$ of *modular iterated integrals of length* ℓ. Their defining relation is

$$\partial \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + M[y] \times \mathcal{MI}_{\ell-1} \quad \text{and} \quad \bar{\partial} \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + \overline{M}[y] \times \mathcal{MI}_{\ell-1}, \tag{1.1}$$

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where $M[y]$ (resp. $\bar{M}[y]$) denotes polynomials in $y = \mathrm{Im}(z)$ with coefficients in the space of standard holomorphic (resp. anti-holomorphic) modular forms, and $\partial, \bar{\partial}$ are certain differential operators that will again be defined precisely in Section 2.1.

A reason for the special interest in this class is that the modular graph functions are expected to belong to it (cf. [2, Sections 1 and 9], where explicit evidence of this is provided). A second reason is that its structure seems to have arithmetic significance, as indicated by evidence provided by Brown (in [2, 4]) for the motivic nature of the space and by the association to it of classical number theoretic invariants, such as L-functions and period polynomials, by Drewitt and the author [10].

In this paper, we will address two questions that arise from the works mentioned above.

Question 1. The elements of $\mathcal{MI}_\ell$ almost exclusively studied in the papers above are those that satisfy a condition more specific than (1.1), namely

$$\partial \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + E[y] \times \mathcal{MI}_{\ell-1} \quad \text{and} \quad \bar{\partial} \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + \overline{E}[y] \times \mathcal{MI}_{\ell-1},$$

where E is the *subspace of M generated by Eisenstein series.*

What can be said about the remaining modular iterated integrals, i.e. the part originating in cusp forms? This is important not only because an answer describes more completely the structure of $\mathcal{MI}_\ell$, but, especially, because arithmetic information is normally expected to be encapsulated by forms that are cuspidal. This is particularly relevant in view of the evidence for the arithmetic significance of $\mathcal{MI}_\ell$ mentioned above.

We will provide an answer to this question by constructing (in Section 2.2.2) an explicit family of such functions originating in cusp forms. We denote those functions by $\phi^{\pm}_{h;r,s}$. To this end, we will first restate (in Section 2.2.1) the question in a concrete and precise form, a task of independent interest.

Question 2. A more explicit characterization of the space $\mathcal{MI}_\ell$ can be given in terms of Γ-*invariant* linear combinations of real and imaginary parts of iterated integrals of modular forms. This is proven in the case of elements of $\mathcal{MI}_\ell$ originating in Eisenstein series [4] and is conjectured to hold in general. Constructing such "modular invariant versions" of iterated integrals of modular forms is one of the important themes of [2], especially in relation to the applications to the theory of modular graph functions.

Here we discuss a new approach to this problem: We construct explicit "real-analytic iterated integrals", denoted by $\psi^{\pm}_{h;r,s}$, which naturally decompose into a modular invariant piece and a piece that can be thought of as known because it is expressed in terms of lower length modular iterated integrals. The modular invariant piece can immediately be read off of the formula for the function $\phi^{\pm}_{h;r,s}$ that was the basis for the answer to Question 1 above. At the same time, this modular invariant piece belongs to (an extension of) $\mathcal{MI}_\ell$. The construction is carried out, in the case of $\ell = 2$, in Section 3.2, where it is shown how $\psi^{\pm}_{h;r,s}$ yield polynomials whose coefficients are Γ-invariant elements of (an extension of) $\mathcal{MI}_\ell$.

The tool with which we achieve both of those two aims is based on *higher-order forms*, the second object we deal with in this work.

The characterizing feature of higher-order modular forms, in the special case of order 2 and weight k, for instance, is the transformation law

$$f|_k\gamma\delta - f|_k\gamma - f|_k\delta + f = 0 \quad \text{for all } \gamma, \delta \in \Gamma,$$

where the action of the group on the function is given by

$$g|_k\gamma(z) = g(\gamma z)(cz + d)^{-k}.$$

The precise definition will be given in Section 3, where the original notion of higher-order forms will be generalized. Higher-order modular forms have been studied from various perspectives (analytic, adelic, algebraic, spectral etc.) and led to applications to modular symbols, mathematical physics etc. In all these cases, the theory had to be developed on congruence subgroups of level higher than 1, because such forms were parametrized by cusp forms of weight 2, which are trivial in level 1. This was a unnatural constraint because it excluded integrals of higher weight forms from consideration and it also prevented availing oneself of simplifications occurring in $\mathrm{SL}_2(\mathbb{Z})$.

In this paper, we resolve this problem too, by proposing a very general framework within which to consider higher-order forms. The class obtained includes several known and new objects, including higher-order forms for all levels and iterated integrals.

It turns out that the solution to this problem allows us to realize the constructions behind our answers to Questions 1 and 2 above. Firstly, it allows us to produce a family of (extended) modular iterated integrals of length 2 originating in cusp forms (Question 1). These modular iterated integrals, in turn, by their very construction, are obtained from a class of second-order modular forms whose prototypes are exactly the iterated integrals of cusp forms (Question 2). At the same time, these second-order modular forms are not of an ad hoc nature. They form a basis of the class of second-order modular forms they belong to (Theorem 3.8). This suggests a deeper relation between the two objects that are the subject of this paper.

2 Subclasses of the space of real-analytic modular forms

2.1 Review of definitions and notation

We start by introducing some of the notation we will be using and by recalling the definitions of real-analytic modular forms and of modular iterated integrals.

2.1.1 Basic spaces and actions

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and set

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

If $\mathfrak{H}$ denotes the upper half-plane and $z = x + iy$, set

$$\begin{aligned} \mathcal{R} &:= \{\text{real analytic } f : \mathfrak{H} \to \mathbb{C} \mid f(z) = O(y^C) \text{ as } y \to \infty, \text{ uniformly in } x \text{ for some } C > 0\}, \\ \mathcal{R}_c &:= \{f \in \mathcal{R} \mid \text{for all } c > 0,\ f(z) = O(e^{-cy}) \text{ as } y \to \infty, \text{ uniformly in } x\}, \\ \mathcal{O} &:= \{\text{holomorphic } f \in \mathcal{R}\}, \\ \mathcal{O}_c &:= \{\text{holomorphic } f \in \mathcal{R}_c\}. \end{aligned}$$

For $r, s \in \mathbb{Z}$, $f \in \mathcal{R}$ and $\gamma \in \Gamma$, define a function $f \underset{r,s}{|} \gamma$ by

$$f \underset{r,s}{|} \gamma(z) = j(\gamma, z)^{-r} j(\gamma, \bar{z})^{-s} f(\gamma z) \quad \text{for all } z \in \mathfrak{H}.$$

Here

$$j(\gamma, z) = c_\gamma z + d_\gamma, \quad \text{where } \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

We extend the action to $\mathbb{C}[\Gamma]$ by linearity.

We can now give the definition of a real-analytic modular form.

We call an $f \in \mathcal{R}$ (resp. $f \in \mathcal{R}_c$) a *real-analytic modular* (resp. *cusp*) *form of weights* (r, s) *for* Γ if the following conditions hold:

(i) For all $\gamma \in \Gamma$ and $z \in \mathfrak{H}$, we have $f \underset{r,s}{|} \gamma = f$, i.e.

$$f(\gamma z) = j(\gamma, z)^r j(\gamma, \bar{z})^s f(z) \quad \text{for all } z \in \mathfrak{H}.$$

(ii) For some $M \in \mathbb{N}$ and $a^{(j)}_{m,n} \in \mathbb{C}$,

$$f(z) = \sum_{|j| \le M} y^j \Big(\sum_{m,n \ge 0} a^{(j)}_{m,n} q^m \bar{q}^n \Big), \quad q := \exp(2\pi i z). \tag{2.1}$$

We denote the space of real analytic modular (resp. cusp) forms of weights (r, s) for Γ by $\mathcal{M}_{r,s}$ (resp. $\mathcal{S}_{r,s}$). We set $\mathcal{M} = \bigoplus_{r,s} \mathcal{M}_{r,s}$ (resp. $\mathcal{S} = \bigoplus_{r,s} \mathcal{S}_{r,s}$.)

For $s = 0$, upon restriction to holomorphic functions, we retrieve the space of standard holomorphic modular (resp. cusp) forms denoted by M_r (resp. S_r). We also set $M = \bigoplus_r M_r$ (resp. $S = \bigoplus_r S_r$.)

2.1.2 Lie structure

The Lie algebra $\mathfrak{sl}_2$ acts on $\mathcal{M}$ via the Maass operators $\partial_r : \mathcal{M}_{r,s} \to \mathcal{M}_{r+1,s-1}$ and $\bar{\partial}_s : \mathcal{M}_{r,s} \to \mathcal{M}_{r-1,s+1}$ given by

$$\partial_r = 2iy\frac{\partial}{\partial z} + r \quad \text{and} \quad \bar{\partial}_s = -2iy\frac{\partial}{\partial \bar{z}} + s$$

(for proofs and further details on this and the rest of this subsection, see [2, Section 2.2]). These operators induce bigraded derivations on $\mathcal{M}$ denoted by ∂ and $\bar{\partial}$, respectively. We set

$$\partial^{(m)} := \partial \circ \partial \circ \cdots \circ \partial : \mathcal{M}_{r,s} \to \mathcal{M}_{r+m,s-m},$$

with a similar definition for $\bar{\partial}^{(m)}$.

Two pairs of identities we will be using often, taken from [2, Lemma 2.5 and (2.13)], are

$$\partial_r(g \underset{r,s}{|} \gamma) = (\partial_r g) \underset{r+1,s-1}{|} \gamma \quad \text{and} \quad \bar{\partial}_r(g \underset{r,s}{|} \gamma) = (\bar{\partial}_r g) \underset{r-1,s+1}{|} \gamma, \tag{2.2}$$

$$\partial_r(y^k g) = y^k \partial_{r+k}(g) \quad \text{and} \quad \bar{\partial}_s(y^k g) = y^k \bar{\partial}_{s+k}(g). \tag{2.3}$$

To simplify notation, we will omit the index of ∂_r (resp. $\bar{\partial}_s$) when this is implied by the context.

2.1.3 Modular iterated integrals

We can now recall the definition of the space $\mathcal{MI}_\ell$ of *modular iterated integrals of length* ℓ *for* $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Set recursively:

- $\mathcal{MI}_{-1} = 0$.
- For each integer $\ell \geq 0$, we let $\mathcal{MI}_\ell$ be the largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}_{r,s}$ which satisfies

$$\partial \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + M[y] \times \mathcal{MI}_{\ell-1},$$
$$\bar{\partial} \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + \overline{M}[y] \times \mathcal{MI}_{\ell-1},$$

where $\overline{M}$ is the ring of anti-holomorphic modular forms.

In [2, Lemma 3.10] it is proved that $\mathcal{MI}_0 = \mathbb{C}[y^{-1}]$ and, in [2, Corollary 4.4], the following statement is shown.

Proposition 2.1. *Let $r, s \in \mathbb{Z}$ be of the same parity and such that $r + s > 2$. Let $\mathcal{E}_{r,s}$ be the real-analytic Eisenstein series given, for $z \in \mathfrak{H}$, by*

$$\mathcal{E}_{r,s}(z) = \sum_{\gamma \in B\backslash\Gamma} \frac{1}{j(\gamma, z)^r j(\gamma, \bar{z})^s}$$

(where $B = \{\pm T^n \mid n \in \mathbb{Z}\}$). Then $\mathcal{E}_{r,s} \in \mathcal{M}_{r,s}$ and

$$\mathcal{MI}_1 = \mathbb{C}[y^{-1}] \otimes \bigoplus_{r,s\geq 1, r+s\geq 4} \mathbb{C} y \mathcal{E}_{r,s}.$$

Note that our normalization of $\mathcal{E}_{r,s}$ is different from that of [2] and that the indices have been shifted relative to [2].

2.2 The space of extended modular iterated integrals

2.2.1 Motivating remarks and definition

With Proposition 2.1 and the definition of $\mathcal{MI}_\ell$, we see that the space $\mathcal{MI}_2$ is defined as the largest subspace of $\bigoplus_{r,s\geq 0}\mathcal{M}_{r,s}$ which satisfies

$$\partial\mathcal{MI}_2 \subset \mathcal{MI}_2 + \bigoplus_{\substack{j\in\mathbb{Z}\\ r,s\geq 1,\ r+s\geq 4}} y^j\mathcal{E}_{r,s}M, \tag{2.4}$$

$$\bar\partial\mathcal{MI}_2 \subset \mathcal{MI}_2 + \bigoplus_{\substack{j\in\mathbb{Z}\\ r,s\geq 1,\ r+s\geq 4}} y^j\mathcal{E}_{r,s}\bar{M}. \tag{2.5}$$

Now, to the best of our knowledge, all explicit examples of elements of $\mathcal{MI}_2$ considered in the literature correspond to elements of M in the right-hand side of (2.4) that are of a specific kind, namely *(holomorphic) Eisenstein series*. This, for instance, is the case for the important examples studied in [2, Section 9]. It is natural then to ask what is the nature of the functions corresponding to the remaining "cuspidal piece". Specifically, we would like to investigate the largest subspace $\mathcal{N}$ of $\mathcal{M}$ satisfying inclusions of the form

$$\partial\mathcal{N} \subset \mathcal{N} + \bigoplus y^j\mathcal{E}_{r,s}S \quad \text{and} \quad \bar\partial\mathcal{N} \subset \mathcal{N} + \bigoplus y^j\mathcal{E}_{r,s}\bar{S}$$

for a suitable choice of indices. The motivation for that, apart from the general aim of understanding more fully the space $\mathcal{MI}_2$, is to study modular iterated integrals whose L-functions are more likely to have classical arithmetic significance than those originating in Eisenstein series.

Before we state our definition, we note that, in contrast to the definition of $\mathcal{MI}_\ell$ (Section 2.1.3), it does not seem possible to define it so that the space is contained in the "first quadrant" $\bigoplus_{r,s\geq 0}\mathcal{M}_{r,s}$ only.

We provide a heuristic argument why this should not be possible. Assume, for contradiction, that the "cuspidal part" is indeed restricted to the "first quadrant" and that $f\in\mathcal{M}_{r,s}$ belongs to the space. Since $F := \partial^{(s)}f$ satisfies $\partial_{s+r}F\in\mathcal{M}_{r+s+1,-1}$, our assumption means that we should have

$$\partial_{s+r}F = \sum y^j\mathcal{E}_{m,l}g,$$

where the sum ranges over a finite number of $g\in S$, $j\in\mathbb{Z}$ and $m,l\geq 1$ with $m+l\geq 4$. With (2.3), this gives $\partial_0(y^{r+s}F) = \sum y^{r+s+j}\mathcal{E}_{m,l}g$, and hence, since g is cuspidal,

$$F = \frac{y^{-s-r}}{2}\sum y^j\int_{\infty}^{0}(y+t)^{r+s+j-1}\mathcal{E}_{m,l}(z+it)g(z+it)\,dt + y^{-r-s}\overline{h(z)}$$

for some holomorphic function $h(z)$. Since, again according to our assumption, F belongs to a space contained in the "first quadrant", $\bar\partial^{(r+1)}F$ should belong to $\bigoplus y^j\mathcal{E}_{r,s}\bar{S}$. On the other hand, the recursive relations of [2, Proposition 4.1], combined with (2.3), show that $\bar\partial^{(r+s+1)}F$ is a linear combination of elements of the form

$$y^j\int_{\infty}^{0}t^{j'}\mathcal{E}_{m,l}(z+it)g(z+it)\,dt \quad \text{and} \quad y^{-r-s}\overline{\partial^{(r+s+1)}h(z)},$$

where, for compactness of notation, we have taken $\mathcal{E}_{0,m}$ to stand for $y\mathbb{G}_{m+2}$ in accordance to [2, (4.1)]. Such linear combinations do not seem to belong to $\bigoplus y^j\mathcal{E}_{r,s}\bar{S}$, but we have not been able to show that rigorously. The above argument therefore remains heuristic and only serves to suggest that it may not be possible to avoid a key difference between $\mathcal{MI}_2$ and the following space (we will further comment on this difference after the statement of the definition).

Definition 2.2. Let the space $\mathcal{MI}'_2$ of *extended modular iterated integrals of length* 2 be the largest subspace of $\mathcal{M}$ which satisfies

$$\partial\mathcal{MI}'_2 \subset \mathcal{MI}'_2 + \bigoplus_{\substack{j\in\mathbb{Z}\\ r,s\in\mathbb{Z},\ r+s\geq 4}} y^j\mathcal{E}_{r,s}S \quad \text{and} \quad \bar\partial\mathcal{MI}'_2 \subset \mathcal{MI}'_2 + \bigoplus_{\substack{j\in\mathbb{Z}\\ r,s\in\mathbb{Z},\ r+s\geq 4}} y^j\mathcal{E}_{r,s}\bar{S}. \tag{2.6}$$

There are two differences between $\mathcal{MI}_2$ and $\mathcal{MI}_2'$: Firstly, compared with (2.4) and (2.5), the space M in the right-hand side has been replaced by S in (2.6). It is in this sense that the elements of the space $\mathcal{MI}_2'$ we will construct in the sequel are said to originate in cusp forms.

Secondly, the restriction of the space belonging to the “first quadrant” present in the definition of $\mathcal{MI}_2$ is no longer required. A consequence of this is that arguments such as those leading to the full description of $\mathcal{MI}_1$ in [2] are no longer possible because they rely on the finiteness of chains of functions generated by repeated applications of the operators ∂ and $\bar{\partial}$. In the case of $\mathcal{MI}_1$, the finiteness is guaranteed by the “first quadrant” condition. However, as suggested by the heuristics above, it seems unlikely that there exist a space with such a condition that, at the same time, satisfies the properties we want to encode in $\mathcal{MI}_2'$.

Finally, it should be stressed that we do not claim that $\mathcal{MI}_2$ is the direct sum of $\mathcal{MI}_2'$ and the space of the functions corresponding to Eisenstein series previously studied in the literature.

2.2.2 An explicit sub-class of $\mathcal{MI}_2'$

We will now define an explicit family of elements of $\mathcal{MI}_2'$ that will give an answer to Question 1. In later sections, we will show that it originates in second-order modular forms. We will first introduce some preparatory constructions and results.

Let P_{k-2} denote the space of polynomials in $\mathbb{C}[X]$ of degree less than or equal to $k-2$, acted upon by $\underset{2-k,0}{|}$. Denote the tensor product of the representations

$$(\mathcal{R}, \underset{r,s}{|}) \quad \text{and} \quad (P_{k-2}, \underset{2-k,0}{|})$$

by $\underset{r,s,2-k}{|}$. The group Γ then acts on $\mathcal{R} \otimes P_{k-2}$ as

$$(f \underset{r,s,2-k}{|} \gamma)(z, X) = f(\gamma z, \gamma X) j(\gamma, z)^{-r} j(\gamma, \bar{z})^{-s} j(\gamma, X)^{k-2}.$$

We use the same notation for the sub-representations corresponding to $\mathcal{R}_c$, $\mathcal{O}$ and $\mathcal{O}_c$.

Let now

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a cusp form of weight k for Γ and consider its Eichler integrals

$$F_f^+(z, X) = \int_{i\infty}^{z} f(w)(w-X)^{k-2}\, dw \quad \text{and} \quad F_f^-(z, X) = \overline{\int_{i\infty}^{z} f(w)(w-X)^{k-2}\, dw},$$

where the bar means complex conjugation (acting trivially on X).

Let r, s be integers of the same parity and such that $r+s>k$. We set

$$r_f(\gamma; X) := r_f^+(\gamma; X) = \int_{\gamma^{-1} i\infty}^{i\infty} f(w)(w-X)^{k-2}\, dw, \quad r_f^-(\gamma; X) = \overline{\int_{\gamma^{-1} i\infty}^{i\infty} f(w)(w-X)^{k-2}\, dw}$$

and

$$\phi_{r,s}^{\pm}(f; z, X) := \sum_{\gamma \in B\backslash\Gamma} F_f^{\pm} \underset{r,s,2-k}{|} \gamma = \sum_{\gamma \in B\backslash\Gamma} \frac{F_f^{\pm}(\gamma z, \gamma X)}{j(\gamma, z)^r j(\gamma, \bar{z})^s} j(\gamma, X)^{k-2}. \tag{2.7}$$

We have the following proposition.

Proposition 2.3. *Suppose that $r, s \in \mathbb{Z}$ have the same parity and satisfy $r+s>k$. Then, for each $z \in \mathfrak{H}$, the series $\phi_{r,s}^{\pm}(f; z, X)$ converges absolutely and it is invariant under the action of $\underset{r,s,2-k}{|}$ of Γ. Viewed as a polynomial in X, its coefficients are functions of (at most) polynomial growth at infinity.*

Proof. We show it for ϕ^+, the proof for ϕ^- being deduced upon conjugating ϕ^+.

By the first equality of (2.7), $\phi^+_{r,s}$ is invariant under the action $\underset{r,s,2-k}{|}$ of Γ, for those r, s for which it converges.

To prove the statement about absolute convergence, we first note that the change of variables $w \to \gamma w$, the transformation law of f and the identity

$$(\gamma z - \gamma X) j(\gamma, z) j(\gamma, X) = z - X \tag{2.8}$$

imply that $F^+_f(\gamma z, \gamma X) j(\gamma, X)^{k-2}$ equals

$$\int_{\gamma^{-1} i\infty}^{z} f(w)(w - X)^{k-2}\, dw = r_f(\gamma; X) + F^+(z, X) \tag{2.9}$$

$$= \sum_{j=0}^{k-2} (-1)^j \binom{k-2}{j} \int_{\gamma^{-1} i\infty}^{i\infty} f(w)(w - \gamma^{-1}\infty)^j\, dw \cdot (X - \gamma^{-1}\infty)^{k-2-j} + F^+(z, X). \tag{2.10}$$

By applying this decomposition to the defining series for $\phi^+_{r,s}$, we get a sum of two terms.

To analyze the part corresponding to the first term of (2.10), we note that each of the integrals appearing in the sum is (up to a power of i) the value at $s = l + 1$ of the "completed" L-function with additive twists. Specifically,

$$\Lambda_f\Big(s, \frac{p}{q}\Big) := \int_0^\infty f\Big(\frac{p}{q} + ix\Big) x^{s-1}\, dx = \Gamma(s)(2\pi)^{-s} \sum_{n=1}^{\infty} \frac{a(n) e^{2\pi i n p/q}}{n^s}. \tag{2.11}$$

It is well-known that $\Lambda_f(s, p/q)$ has a functional equation (see [11] for a general version of the functional equation) and, with convexity, this implies that, for each $j = 0, \ldots, k-2$,

$$q^{j+1} \Lambda_f(j+1, p/q) \ll q^{\frac{k+1}{2} + \epsilon}. \tag{2.12}$$

Also,

$$X - \gamma^{-1} i\infty = (X - z) + (z - g^{-1} i\infty) = (X - z) + j(\gamma, z)/c_\gamma. \tag{2.13}$$

Upon applying the binomial formula to (2.13) and substituting into the polynomial

$$\sum_{\gamma \in B\backslash\Gamma} \frac{\int_{\gamma^{-1} i\infty}^{i\infty} f(w)(w - \gamma^{-1}\infty)^j\, dw \cdot (X - \gamma^{-1}\infty)^{k-2-j}}{j(\gamma, z)^r j(\gamma, \bar z)^s},$$

we get an expansion in $(X - z)^{k-2-j-m}$ $(0 \le m \le k-2-j)$. In this expansion, the coefficient of $(X - z)^{k-2-j-m}$ equals

$$\binom{k-2-j}{m} i^{j+1} \sum_{\gamma \in B\backslash\Gamma} \frac{\Lambda_f(j+1, \gamma^{-1}\infty)}{c_\gamma^m j(\gamma, z)^{r-m} j(\gamma, \bar z)^s} \ll \sum_{\gamma \in B\backslash\Gamma} \frac{c_\gamma^{\frac{k-1}{2} + \epsilon - j - m}}{|j(\gamma, z)|^{r+s-m}}.$$

For the last estimate we used (2.12). The elementary inequality $|c_\gamma| \le |j(\gamma, z)|\, \mathrm{Im}(z)^{-1}$ implies that the sum is

$$\le y^{\frac{1-k}{2} - \epsilon + j + m} \sum_{\gamma \in B\backslash\Gamma} |j(\gamma, z)|^{\frac{k-1}{2} - r - s - j + \epsilon}, \tag{2.14}$$

which converges uniformly for z in a compact set since $-\frac{k-1}{2} + r + s + j - \epsilon > 2$.

The second term of (2.10) gives the polynomial

$$\mathcal{E}_{r,s}(z) \int_{i\infty}^{z} f(w)(w - X)^{k-2}\, dw,$$

which, by Proposition 2.1, has real-analytic functions as coefficients since $r + s > 2$.

Therefore, both pieces of $\phi^+_{r,s}$ induced by (2.10) will converge to a polynomial in X with coefficients that are real-analytic functions if $-\frac{k-1}{2} + r + s + j - \epsilon > 2$ for all $j = 0, \ldots, k-2$ and $r + s > 2$. This is indeed the case if $r + s > k$.

The bound (2.14) and the polynomial growth of $E_{r,s}(z)$ show that the coefficients of $(X - z)^j$ (and of X^j) in $\phi^+_{r,s}$ are of, at most, polynomial growth as $y \to \infty$. □

The series $\phi^{\pm}_{r,s}(f; z, X)$ can be decomposed in terms of elements of $\mathcal{M}$. Specifically, let

$$\phi^{\pm}_{r,s}(f; i, z), \quad i = 0, \dots, k-2,$$

be functions such that

$$\phi^{\pm}_{r,s}(f; z, X) = \sum_{i=0}^{k-2} \phi^{\pm}_{r,s}(f; i, z)(X-z)^i (X-\bar{z})^{k-2-i}. \tag{2.15}$$

From [2, Proposition 7.1], we know that $\phi^{\pm}_{r,s}(f; i, z)$ is $|_{r+i,s+k-2-i}$-invariant. To show that it actually belongs to

$$\mathcal{M}_{r+i,s+k-2-i},$$

we need to show that it has an expansion of the form (2.1). This is part of the content of the next proposition.

Proposition 2.4. *Suppose r, s, k are as in Proposition 2.3. Then, for each j = 0, ..., k − 2, we have*

$$\phi^{+}_{r,s}(f; j, z) = (-1)^j \binom{k-2}{j} y^{2-k} \Big(\int_{i\infty}^{z} f(w)(w-\bar{z})^j (w-z)^{k-2-j}\, dw \Big) \mathcal{E}_{r,s} + \sum_{m=0}^{j} \sum_{n=0}^{k-2-j} \alpha_{m,n} y^{2-k} \sum_{1 \neq \gamma \in B\backslash\Gamma} \frac{\Lambda_f(m+n+1, \gamma^{-1}(\infty)) c_\gamma^{m+n-k+2}}{j(\gamma, z)^{r+j+n+2-k} j(\gamma, \bar{z})^{s+m-j}} \tag{2.16}$$

and

$$\phi^{-}_{r,s}(f; j, z) = (-1)^j \binom{k-2}{j} y^{2-k} \Big(\overline{\int_{i\infty}^{z} f(w)(w-\bar{z})^{k-2-j} (w-z)^{j}\, dw} \Big) \mathcal{E}_{r,s} + \sum_{m=0}^{j} \sum_{n=0}^{k-2-j} \alpha_{m,n} y^{2-k} \sum_{1 \neq \gamma \in B\backslash\Gamma} \frac{\overline{\Lambda_f(m+n+1, \gamma^{-1}(\infty)) c_\gamma^{m+n-k+2}}}{j(\gamma, z)^{2-k+r+m+j} j(\gamma, \bar{z})^{s-j+n}}, \tag{2.17}$$

where

$$\alpha_{m,n} := i^{1-2j-m-n} \binom{k-2}{j} \binom{j}{m} \binom{k-2-j}{n}.$$

Further, each $\phi^{\pm}_{r,s}(f; j, z)$ is in $\mathcal{M}$.

Proof. Replacing $w - X$ in $F_f^+(z, X)$ according to the identity

$$w - X = ((w-z)(X-\bar{z}) + (\bar{z}-w)(X-z)) / \operatorname{Im} z$$

and expanding with the binomial theorem, we see that the coefficients in the right-hand side of (2.15) can be written as

$$\phi^{+}_{r,s}(f; j, z) = (-1)^j \binom{k-2}{j} \sum_{\gamma \in B\backslash\Gamma} \Big((\operatorname{Im} z)^{2-k} \int_{i\infty}^{z} f(w)(w-\bar{z})^j (w-z)^{k-2-j}\, dw \Big) \Big|_{r+j,k-2+s-j,0} \gamma,$$

respectively

$$\phi^{-}_{r,s}(f; j, z) = (-1)^j \binom{k-2}{j} \sum_{\gamma \in B\backslash\Gamma} \Big((\operatorname{Im} z)^{2-k} \overline{\int_{i\infty}^{z} f(w)(w-\bar{z})^{k-2-j} (w-z)^{j}\, dw} \Big) \Big|_{r+j,k-2+s-j,0} \gamma.$$

Upon unraveling the definition of the action $|$, the sum in $\phi^{+}_{r,s}(f; j, z)$ equals

$$\sum_{\gamma \in B\backslash\Gamma} \frac{y^{2-k} \int_{i\infty}^{\gamma z} f(w)(w-\gamma\bar{z})^j j(\gamma, \bar{z})^j (w-\gamma z)^{k-2-j} j(\gamma, z)^{k-2-j}\, dw}{j(\gamma, z)^r j(\gamma, \bar{z})^s}.$$

The change of variables $w \to \gamma w$ and the transformation law of f imply that the integral equals

$$\int_{\gamma^{-1} i\infty}^{z} f(w)(w-\bar{z})^j (w-z)^{k-2-j}\, dw = \Big(\int_{\gamma^{-1} i\infty}^{i\infty} + \int_{i\infty}^{z} \Big) f(w)(w-\bar{z})^j (w-z)^{k-2-j}\, dw. \tag{2.18}$$

The first integral in the right-hand side is 0 for $\gamma = 1$. For $\gamma \neq 1$, the binomial theorem leads to

$$\sum_{m=0}^{j}\sum_{n=0}^{k-2-j}\binom{j}{m}\binom{k-2-j}{n}\frac{j(\gamma,\bar z)^{j-m}j(\gamma,z)^{k-2-j-n}}{(-c_\gamma)^{k-2-m-n}}\int_{\gamma^{-1}i\infty}^{i\infty} f(w)(w-\gamma^{-1}i\infty)^{m+n}\,dw.$$

Equation (2.16) follows from this and (2.11) combined with (2.18). Equation (2.17) can be deduced from equation (2.16) upon a conjugation.

To show that $\phi^{+}_{r,s}(f;j,z)$ has an expansion of the form (2.1), we apply the usual double coset decomposition to the series

$$\sum_{1\neq\gamma\in B\backslash\Gamma}\frac{\Lambda_f(m,\gamma^{-1}(\infty))c_\gamma^n}{j(\gamma,z)^p j(\gamma,\bar z)^t},\quad \text{where } m,n,p,t \text{ are integers.}$$

Then this becomes

$$\sum_{c>0}\sum_{d \bmod c}\sum_{l\in\mathbb{Z}}\frac{\Lambda_f(m,-\frac{d}{c})c^n}{(c(z+l)+d)^p(c(\bar z+l)+d)^t}$$
$$=\sum_{c>0}c^{n-p-t}\sum_{d \bmod c}\Lambda_f\Big(m,-\frac{d}{c}\Big)\sum_{l\in\mathbb{Z}}e^{2\pi i l(x+\frac{d}{c})}\int_{\mathbb{R}}\frac{e^{-2\pi i l t_1}\,dt_1}{(t_1+iy)^p(t_1-iy)^t},\tag{2.19}$$

where we used the Poisson formula followed by a change of variables. With [12, Section 3.2, equation (12)], combined with [14, equation 13.14.9], the integral equals $P_l(y)e^{-2\pi|l|y}$ for some polynomial $P_l(y)$ of degree less than or equal to $|p-2|$ in $y^{\pm}$. This implies that (2.19) can be written as

$$\sum_{l\geq 0}q^l P_l(y)\sum_{c>0}\sum_{d \bmod c}\frac{\Lambda_f(m,-\frac{d}{c})e^{2\pi i l\frac{d}{c}}}{c^{p+t-n}}+\sum_{l<0}\bar q^{-l}P_l(y)\sum_{c>0}\sum_{d \bmod c}\frac{\Lambda_f(m,-\frac{d}{c})e^{2\pi i l\frac{d}{c}}}{c^{p+t-n}}.$$

We can apply this with m, n, p, t replaced by $m+n+1, m+n-k+2, , r+j+n+2-k, s+m-j$, respectively, because, by (2.12), the inner series converge with $r+s>k$. We deduce that $\phi^{+}_{r,s}(f;j,z)$ has an expansion of the form (2.1). The analogous assertion for $\phi^{-}_{r,s}(f;j,z)$ can be deduced from this after a conjugation. □

We can now verify an identity for $\phi^{\pm}_{r,s}(f;z,X)$ that will allow us to show that $\phi^{\pm}_{r,s}(f;j,z)\in\mathcal{MI}'_2$.

Proposition 2.5. *Let r, s be integers of the same parity and such that $r+s>k$. We have*

$$\partial_r(\phi^+_{r,s}) = r\phi^+_{r+1,s-1}+2iyf(z)(X-z)^{k-2}\mathcal{E}_{r,s}\quad \text{and}\quad \bar\partial_s(\phi^+_{r,s}) = s\phi^+_{r-1,s+1},$$
$$\bar\partial_s(\phi^-_{r,s}) = s\phi^-_{r-1,s+1}-2iy\overline{f(z)}(X-\bar z)^{k-2}\mathcal{E}_{r,s}\quad \text{and}\quad \partial_r(\phi^-_{r,s}) = r\phi^-_{r+1,s-1}.$$

Proof. With (2.2), we have

$$\partial_r(\phi^{\pm}_{r,s}) = \sum_{\gamma\in B\backslash\Gamma}\partial_r(F^{\pm}_f)\,\Big|_{r+1,s-1,2-k}\gamma.\tag{2.20}$$

The definition of ∂_r and the identity $\mathrm{Im}(\gamma z)=\mathrm{Im}(z)/(j(\gamma,z)j(\gamma,\bar z))$ imply that, in the plus-case, this equals

$$r\phi^+_{r+1,s-1}+2iy\sum_{\gamma\in B\backslash\Gamma}\frac{f(\gamma z)(\gamma z-\gamma X)^{k-2}j(\gamma,X)^{k-2}}{j(\gamma,z)^r j(\gamma,\bar z)^s}.$$

With the transformation law for $f(z)$ and (2.8), this implies the statement in this case.

In the minus-case, (2.20) equals $r\phi^-_{r+1,s-1}$ as required. The identities involving $\bar\partial$ are proved upon conjugating those we just proved. □

This proposition will enable us to define a subset of $\mathcal{MI}'_2$ which, in turn, will be the basis for the solution of one of our motivating problems. In fact, the equations proved in Proposition 2.5 and their difference from the equations of the analogous proposition [2, Proposition 4.1] help explain one of the differences between the definitions of $\mathcal{MI}_2$ and $\mathcal{MI}'_2$. Specifically, in [2, Proposition 4.1], the application of ∂ to functions with indices $(r,s)=(k-2,0)$ gives (up to a factor of y) only a classical Eisenstein series. By contrast, in our case, the outcome involves other $\phi^+_{-,-}$ as well. Therefore, the “chain” obtained in this way cannot be finite.

We can now define our sub-class of $\mathcal{MI}_2'$ as the vector space $\mathcal{A}$ generated over $\mathbb{C}$ by $\phi^{\pm}_{r,s}(f; i, -)$ for all $f \in S_k$ ($k \geq 12$), all integers r, s such that $r+s$ is even and greater than k, and $0 \leq i \leq k-2$. With this definition, we can state our answer to Question 1 as follows.

Theorem 2.6. *The space $\mathcal{A}$ is a subspace of the space $\mathcal{MI}_2'$ of extended modular iterated integrals of length* 2.

Proof. An elementary computation implies that for all real-analytic f_j we have

$$\partial_m\Big(\sum_{j=0}^{k-2} f_j(z)(X-z)^j(X-\bar{z})^{k-2-j}\Big) = \sum_{j=0}^{k-2}(\partial_{m+j}f_j(z) - (j+1)f_{j+1}(z))(X-z)^j(X-\bar{z})^{k-2-j}, \tag{2.21}$$

where, for convenience, f_{k-1} is set to equal 0.

Proposition 2.5, combined with (2.21), implies that each $\partial_r\phi^{\pm}_{r,s}(f; i, -)$ is a linear combination of $\phi^{\pm}_{r,s}(f; i, -)$ (for varying r, s, i) and an element of $S[y] \otimes \bigoplus_{r,s} \mathcal{E}_{r,s}$. Therefore, $\mathcal{A}$ satisfies the first of the inclusions (2.6).

In the same way, we verify the analogous statement for $\bar{\partial}_s\phi^{\pm}_{r,s}(f; i, -)$ with $\bar{S}$ in place of S.

Finally, by Proposition 2.4, $\mathcal{A}$ is a subspace of $\mathcal{M}$. Since $\mathcal{MI}_2'$ is, by definition, the largest subspace of $\mathcal{M}$ satisfying the inclusions (2.6), $\mathcal{A}$ is contained in $\mathcal{MI}_2'$. □

3 The space of iterated invariants

To define our extended higher-order modular forms, it will be necessary to describe a general framework involving a family of representations; see [8, 9] for two alternative general definitions of higher-order objects, which are built on only one representation and which use the formalism of the augmentation ideal.

Exceptionally, we will give the next definition for general Fuchsian groups Γ of the first kind acting on $\mathfrak{H}$ with non-compact quotient $\Gamma \backslash \mathfrak{H}$. The reason is that we want to compare it with the standard higher-order modular forms, as defined in [6], which are trivial in level 1.

Let $V = (\rho_i, V_i)_{i\geq 0}$ be a sequence of representations of Γ, where the *right*-action on each V_i is denoted by ".". Assume further that the $\mathbb{C}$-vector spaces V_i are finite-dimensional when $i \geq 1$. For each $n \in \mathbb{N}$, we consider the tensor representation $\bigotimes_{i=0}^{n-1} V_i$. To ease notation, we will generally denote the action on it also by ".". It will generally be clear which representation it refers to in each case, but, in cases of potential ambiguity, it will be explained separately.

In the following definition, if V is a Γ-module, we view $H^0(\Gamma, V)$ as a subset of V.

Definition 3.1. Set $M^{(0)} := \{0\}$ and define, inductively, $M^{(n)} = M^{(n)}(V)$ to be the subspace of $\bigotimes_{i=0}^{n-1} V_i$ given by

$$M^{(n)} = \mathrm{pr}^{-1}\, H^0\Big(\Gamma, \Big(\bigotimes_{i=0}^{n-1} V_i\Big)/(M^{(n-1)} \otimes V_{n-1})\Big),$$

where the implied action is induced by that of Γ on $\bigotimes_{i=0}^{n-1} V_i$ and pr is the canonical projection of $\bigotimes_{i=0}^{n-1} V_i$ onto $\bigotimes_{i=0}^{n-1} V_i/(M^{(n-1)} \otimes V_{n-1})$. We then set

$$M_c^{(n)} = M^{(n)} \cap \bigcap_{\text{parabolic } \pi} H^0\Big(\langle\pi\rangle, \bigotimes_{i=0}^{n-1} V_i\Big),$$

where $\langle\pi\rangle$ is the subgroup generated by π.

We call the elements of $M^{(n)}(V)$ *iterated invariants of order n*.

In the next proposition, we show that this definition is well-founded and we give an equivalent formulation of it.

Proposition 3.2. (i) *For each $n \in \mathbb{N}$, $M^{(n)}$ and $M^{(n-1)} \otimes V_{n-1}$ are closed under the action of* Γ.

(ii) *We have*

$$M^{(n-1)} \otimes V_{n-1} \subset M^{(n)}.$$

Hence, composing with a map $M^{(n-1)} \hookrightarrow M^{(n-1)} \otimes V_{n-1}$, *induced by* $v \to v \otimes v_0$ *(for some* $v_0 \neq 0$*), we have* $M^{(n-1)} \hookrightarrow M^{(n)}$ *and* $M_c^{(n-1)} \hookrightarrow M_c^{(n)}$.

(iii) *The space* $M_c^{(n)}$ *is isomorphic to the space of* $f \in \otimes V_i$ *such that, for each* $\gamma \in \Gamma$,

$$f.(\gamma - 1) \in M^{(n-1)} \otimes V_{n-1}$$

and, for each parabolic $\pi \in \Gamma$,

$$f.(\pi - 1) = 0.$$

Proof. (i) We show this by induction over n. Let f be an element of $M^{(n)}$ $(n \geq 1)$. Then, by definition, $f.(\varepsilon - 1) \in M^{(n-1)} \otimes V_{n-1}$ for each $\varepsilon \in \Gamma$. Let $\gamma \in \Gamma$. Then, for each $\delta \in \Gamma$,

$$(f.\gamma).(\delta - 1) = g.\gamma \quad \text{for } g := f.(\gamma\delta\gamma^{-1} - 1) \in M^{(n-1)} \otimes V_{n-1}.$$

Suppose that $g = \sum_{j=0}^{k_{n-1}} f_j \otimes v_j$ for some $f_j \in M^{(n-1)}$, where $\{v_i\}_{j=0}^{k_{n-1}}$ is a basis of V_{n-1}. Then

$$(f.\gamma).(\delta - 1) = g.\gamma = \sum_{j=0}^{k_{n-1}} f_j.\gamma \otimes v_j.\gamma, \tag{3.1}$$

which, by induction hypothesis, belongs to $M^{(n-1)} \otimes V_{n-1}$. Therefore, $f.\gamma$ belongs to $M^{(n)}$.

Because (3.1) holds for all $g \in M^{(n-1)} \otimes V_{n-1}$, the Γ-invariance of $M^{(n-1)}$ implies the Γ-invariance of $M^{(n-1)} \otimes V_{n-1}$.

(ii) We have $M^{(n-1)} \otimes V_{n-1} = \mathrm{pr}^{-1}(\{0\}) \subset M^{(n)}$.

(iii) This is seen by unraveling the definition, which, as shown in (i), is well-founded. □

A first result on the structure of the space of iterated invariants is provided by the following lemma. To state it, we introduce some additional notation for each Γ-module M:

$$\begin{aligned}
C^1(\Gamma, M) &= \{\alpha : \Gamma \to M\}, & C_c^1(\Gamma, M) &= \{\alpha \in C^1(\Gamma, M) : \alpha(\pi) = 0 \text{ for all parabolic } \pi \in \Gamma\},\\
Z^1(\Gamma, M) &= \{\text{1-cocycles of } \Gamma \text{ in } M\}, & Z_c^1(\Gamma, M) &= Z^1(\Gamma, M) \cap C_c^1(\Gamma, M),\\
B^1(\Gamma, M) &= \{\text{1-coboundaries of } \Gamma \text{ in } M\}, & B_c^1(\Gamma, M) &= B^1(\Gamma, M) \cap C_c^1(\Gamma, M),\\
H^1(\Gamma, M) &= Z^1(\Gamma, M)/B^1(\Gamma, M), & H_c^1(\Gamma, M) &= Z_c^1(\Gamma, M)/B_c^1(\Gamma, M).
\end{aligned}$$

With this notation, we have the following lemma.

Lemma 3.3. *Let* $n \in \mathbb{N}$. *There is a map* ψ *such that the following sequence is exact:*

$$0 \to H^0\Big(\Gamma, \bigotimes_{i=0}^{n-1} V_i\Big) \xrightarrow{\iota} M_c^{(n)} \xrightarrow{\psi} M^{(n-1)} \otimes C_c^1(\Gamma, V_{n-1}).$$

In particular, for $n = 2$ *we have the exact sequence*

$$0 \to H^0(\Gamma, V_0 \otimes V_1) \xrightarrow{\iota} M_c^{(2)} \xrightarrow{\psi} H^0(\Gamma, V_0) \otimes Z_c^1(\Gamma, V_1).$$

Proof. Fix a basis $\{u_i\}$ of $M^{(n-1)}$. Then, for every $f \in M_c^{(n)}$ and every $\gamma \in \Gamma$, we have

$$f.(\gamma - 1) = \sum \psi_i^f(\gamma) \otimes u_i$$

for some $\psi_i^f(\gamma) \in V_{n-1}$. By definition, each map $\gamma \to \psi_i^f(\gamma)$ gives an element of $C_c^1(\Gamma, V_{n-1})$. Therefore, the assignment $f \to \sum \psi_i^f \otimes u_i$ induces the map ψ of the proposition.

For the case $n = 2$, we note, with Proposition 3.2 (iii), that $M^{(1)} = M_c^{(1)} = H^0(\Gamma, V_0)$. Therefore, the 1-cocycle condition satisfied by $\gamma \to f.(\gamma - 1)$ is inherited by each $\psi_i^f \in C_c^1(\Gamma, V_1)$. □

Corollary 3.4. *Let* $\bar{\psi}$ *be induced by* ψ *and the natural projection* $Z_c^1(\Gamma, V_1) \to H_c^1(\Gamma, V_1)$. *Then we have the following exact sequence:*

$$0 \to H^0(\Gamma, V_0 \otimes V_1)/(H^0(\Gamma, V_0) \otimes H^0(\Gamma, V_1)) \xrightarrow{\bar{\iota}} M_c^{(2)}/(M^{(1)} \otimes V_1^c) \xrightarrow{\bar{\psi}} H^0(\Gamma, V_0) \otimes H_c^1(\Gamma, V_1),$$

where $\bar{\iota}$ *is induced by* ι *and* V_1^c *consists of* $v \in V_1$ *invariant under all parabolic* $\pi \in \Gamma$.

Proof. This is deduced directly from Lemma 3.3. One can also use the long exact sequence associated with

$$0 \to M^{(1)} \otimes V_1 \to M^{(2)} \to M^{(2)}/(M^{(1)} \otimes V_1) \to 0$$

to deduce the corollary. □

Since the maps ψ and $\bar{\psi}$ constructed in Lemma 3.3 and Corollary 3.4 will play an important role in the sequel, we restate their definition separately: Let $n \in \mathbb{N}$ and let $\{u_i\}$ be a basis of $M^{(n-1)}$. Then the map

$$\psi : M_c^{(n)} \to M^{(n-1)} \otimes C_c^1(\Gamma, V_{n-1})$$

is given, for each $f \in M_c^{(n)}$, by

$$\psi(f) = \sum \psi_i^f \otimes u_i,$$

where the $\psi_i^f \in C_c^1(\Gamma, V_{n-1})$ are such that

$$f.(\gamma - 1) = \sum \psi_i^f(\gamma) \otimes u_i \quad \text{for all } \gamma \in \Gamma.$$

For $n = 2$, let π denote the natural projection

$$M^{(1)} \otimes Z_c^1(\Gamma, V_1) \to M^{(1)} \otimes H_c^1(\Gamma, V_1).$$

Then

$$\bar{\psi} := \pi \circ \psi.$$

3.1 Extended higher-order modular forms

For $k_0 \in \mathbb{Z}$ and positive even integers $k_1, k_2, \ldots$, let

$$V = \mathfrak{O} = (\,|_{2-k_i,0}, V_i)_{i\geq 0},$$

where $V_0 = \mathcal{O}$ and $V_i = P_{k_i-2}[X_i]$ ($i \geq 1$) is the space of polynomials in X_i of degree less than or equal to $k_i - 2$. We call the elements of $M_c^{(n)}(\mathfrak{O})$ *extended modular forms of order n*. With Proposition 3.2 (iii), we see that this is the space of $f(z; X_1, \ldots, X_{n-1}) \in \mathcal{O}[X_1, \ldots, X_{n-1}]$ such that

$$f.(\gamma - 1) \in \begin{cases} \{0\} & (n = 1), \\ M^{(n-1)} \otimes P_{k_{n-1}-2}[X_{n-1}] & (n \geq 2), \end{cases}$$

and, for all parabolic $\pi \in \Gamma$,

$$f.\pi = f,$$

where the action of Γ is induced by

$$(f.\gamma)(z; X_1, \ldots, X_{n-1}) := f(\gamma z; \gamma X_1, \ldots \gamma X_{n-1}) j(\gamma, z)^{k_0-2} j(\gamma, X_1)^{k_1-2} \ldots j(\gamma, z)^{k_{n-1}-2}. \tag{3.2}$$

In particular,

$$M_c^{(1)}(\mathfrak{O}) = M_{2-k_0}(\Gamma) = \{\text{weight } 2 - k_0 \text{ holomorphic modular forms for } \Gamma\}. \tag{3.3}$$

Let

$$V = \mathfrak{O}_c = (\,|_{2-k_i,0}, V_i)_{i\geq 0},$$

where $V_0 = \mathcal{O}_c$ and $V_i = P_{k_i-2}[X_i]$. Then we obtain the space $M_c^{(n)}(\mathfrak{O}_c)$ of *extended cusp forms of order n*.

Remark. The adjective "extended" in the previous examples aims to distinguish them from the class of (standard) higher-order modular forms (see, e.g., [6]). We can retrieve the standard higher-order modular forms by setting $k_1 = \cdots = k_n = 2$. Then "." is simply

$$|_{2-k_0,0} \quad \text{for all } n$$

and the space $M_c^{(n)}(\mathfrak{O})$ consists of all $f \in \mathcal{O}$ such that, for all $\gamma \in \Gamma_0(N)$ and for all parabolic $\pi \in \Gamma_0(N)$,

$$f \underset{2-k_0,0}{|} (\gamma - 1) \in M_c^{(n-1)}(\mathfrak{O}) \quad \text{and} \quad f \underset{2-k_0,0}{|} \pi = f.$$

The standard higher-order modular forms become trivial in $\Gamma_0(1)$ because, as was shown in [6], they are parametrized by weight 2 cusp forms which are trivial in $\mathrm{SL}_2(\mathbb{Z})$. Finally, note that, in contrast to general iterated invariants, $M_c^{(n-1)}(\mathfrak{O})$ can be legitimately used instead of $M^{(n-1)}(\mathfrak{O})$ in the last displayed equation because of the identity $(\gamma - 1)(\pi - 1) = (\gamma\pi\gamma^{-1} - 1)\gamma - (\pi - 1))$.

3.1.1 Iterated Eichler integrals

Important examples and, indeed, some of the prototypes, of the standard higher-order forms mentioned in the closing remark of the last section are the antiderivatives of weight 2 cusp forms and their higher iterated analogues:

$$\int_{i\infty}^{z} f(w)\,dw, \qquad \int_{i\infty}^{z} f(w_1) \int_{i\infty}^{w_1} g(w_2) \ldots dw_2\,dw_1 \qquad \text{for weight 2 cusp forms } f, g, \ldots .$$

In Lemma 3.5 we show that, more generally, Eichler integrals and their iterated counterparts belong to $M_c^{(n)}(\mathfrak{O})$. Its content is not essentially new (e.g., it is implied by the more general arguments of [1, Section 5.3]), but we revisit it from the perspective of higher-order forms in order to motivate the term "real-analytic iterated integrals" introduced in the next subsection.

Lemma 3.5. *Let $k_0 = 2$ and $k_1, \ldots, k_{n-1} \in 2\mathbb{N}$. Suppose that, for $i = 1, \ldots, n-1$, f_i is a weight k_i cusp form for $\mathrm{SL}_2(\mathbb{Z})$. Let $F_n \in \mathcal{O}[X_1, \ldots, X_{n-1}]$ be defined by $F_1 = 1$ and, for $n \geq 2$,*

$$F_n(w; X_1, \ldots, X_{n-1}) := \int_{i\infty}^{w} f_1(w_1)(w_1 - X_1)^{k_1-2} \int_{i\infty}^{w_1} f_2(w_2)(w_2 - X_2)^{k_2-2} \ldots dw_{n-1} \ldots dw_1.$$

Then $F_n \in M_c^{(n)}(\mathfrak{O})$.

Remark. Though not "visible" in the statement, k_0 is necessary for the definition of $M_c^{(n)}(\mathfrak{O})$. It reflects the fact that, in terms of the variable w, Γ acts like a regular representation.

Proof. We first show the assertion for $n = 2$.

The action "." is given explicitly by (3.2). We then see that $F_2.\gamma$ is

$$\int_{i\infty}^{\gamma w} f_1(w_1)(w_1 - \gamma X_1)^{k_1-2} j(\gamma, X_1)^{k_1-2}\,dw_1 = \int_{\gamma^{-1} i\infty}^{w} f_1(w_1)(w_1 - X_1)^{k_1-2}\,dw_1,$$

where the last integral is obtained by a change of variables and (2.8). Therefore, with (3.3),

$$F_2.(\gamma - 1) = \int_{\gamma^{-1} i\infty}^{i\infty} f_1(w_1)(w_1 - X_1)^{k_1-2}\,dw_1 \in \mathbb{C} \otimes P_{k_1-2}[X_1] \subset M^{(1)}(\mathfrak{O}) \otimes P_{k_1-2}[X_1].$$

The same identity shows that $F_2.T = F_2$, and hence $F_2 \in M_c^{(2)}(\mathfrak{O})$.

The proof for general n is an application of the shuffle product formula for iterated integrals, but we give a direct proof by induction. As before, by the definition of the action "." on $\mathcal{O}[X_1, \ldots, X_{n-1}]$, the changes of variables $w_i \to \gamma w_i$ and (2.8), we deduce that, for $n > 2$, $F_n.\gamma$ equals

$$\int_{\gamma^{-1} i\infty}^{w} f_1(w_1)(w_1 - X_1)^{k_1-2} \Big(\int_{\gamma^{-1} i\infty}^{w_1} f_2(w_2)(w_2 - X_2)^{k_2-2} \ldots dw_2 \Big)\,dw_1$$

$$= \Big(\int_{i\infty}^{w} + \int_{\gamma^{-1} i\infty}^{i\infty} \Big) f_1(w_1)(w_1 - X_1)^{k_1-2} \Big(\Big(\int_{i\infty}^{w_1} + \int_{\gamma^{-1} i\infty}^{i\infty} \Big) f_2(w_2)(w_2 - X_2)^{k_2-2} \ldots dw_2 \Big)\,dw_1,$$

where the sum of integral signs indicates that they are both applied to the integrand following them. Therefore, $F_n.(\gamma-1)$ is a sum of iterated integrals such that each iterated integral includes at least one constituent integral with limits $i\infty$ and $\gamma^{-1}i\infty$, e.g.,

$$\int_{i\infty}^{w} f_1(w_1)(w_1-X_1)^{k_1-2} \int_{\gamma^{-1}i\infty}^{i\infty} f_2(w_2)(w_2-X_2)^{k_2-2} \int_{i\infty}^{w_2} f_3(w_3)(w_3-X_3)^{k_3-2} \dots dw_3\, dw_2\, dw_1.$$

This, on the one hand, implies that $F_n.(T-1)=0$ and, on the other, that $F_n.(\gamma-1)$ is a sum of products of the form $F_{i-1}\cdot P_{i-1}(X_{i-1},\dots,X_{n-1})$ $(i=2,\dots n)$, where the polynomials P_{i-1} are of degree less than or equal to k_j-2 in X_j $(j=i-1,\dots,n-1)$ and independent of w. By induction, each product

$$F_{i-1}\cdot P_{i-1}(X_{i-1},\dots,X_{n-1})$$

belongs to

$$M^{(i-1)}(\mathfrak{O})\otimes P_{k_i-1}[X_i]\otimes\dots P_{k_{n-1}-2}[X_{n-1}]\subset M^{(n-1)}(\mathfrak{O})\otimes P_{k_{n-1}-2}[X_{n-1}].$$

This completes the proof of the statement. □

Upon tensoring with a space of cusp forms, we obtain the following corollary.

Corollary 3.6. *Let $k_0,\dots,k_{n-1}\in 2\mathbb{Z}$. Suppose that, for $i=1,\dots,n-1$, f_i is a weight k_i cusp form for $\mathrm{SL}_2(\mathbb{Z})$ and that f_0 is a cusp form of weight $2-k_0$. Let $F_n\in\mathcal{O}[X_1,\dots,X_{n-1}]$ be defined by*

$$f_0(w)\int_{i\infty}^{w} f_1(w_1)(w_1-X_1)^{k_1-2}\int_{i\infty}^{w_1} f_2(w_2)(w_2-X_2)^{k_2-2}\dots dw_{n-1}\dots dw_1.$$

Then $F_n\in M_c^{(n)}(\mathfrak{O}_c)$.

3.2 The space of real-analytic iterated integrals

As mentioned in Section 1, modular iterated integrals are conjectured to be given in terms of Γ-*invariant* linear combinations of iterated integrals of modular forms (see [7] for a somewhat analogous phenomenon, whereby linear combinations of multiple Hecke L-values are expressed in terms of usual Hecke L-series). In view of this conjectured connection, we discuss here a way to obtain invariant objects from iterated integrals associated with modular forms. To this end, we first associate iterated invariants to real analytic modular forms.

Let $V=(V_i,\rho_i)_{i\geq 0}$, where $V_0=\mathcal{R}$ (resp. $V_0=\mathcal{R}_c$) with $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ acting through $|_{r,s}$, and $V_i=P_{k_i-2}[X_i]$ $(i\geq 1)$ with the usual action $|_{2-k_i,0}$ on polynomials. Then we have that, for $i>0$,

$$V_i^c=H^0(\Gamma,V_i)=H^0(\Gamma,P_{k_i-2})=0 \quad \text{if } k_i>2 \text{ (by translation invariance)}, \tag{3.4}$$

$$H_c^1(\Gamma,V_i)\cong S_{k_i}\oplus\bar{S}_{k_i} \quad \text{(by Eichler–Shimura combined with [13, Lemma 1 of VI, Section 5])}. \tag{3.5}$$

Notice that, although $H^0(\Gamma,\mathcal{R})$ is very similar to $\mathcal{M}_{r,s}$, they are not the same, because the former includes functions that do not have an expansion of the form (2.1).

We also consider the one-dimensional subspace $\tilde{\mathcal{M}}_{r,s}$ of $\mathcal{M}_{r,s}\subset H^0(\Gamma,\mathcal{R})$ generated by $\mathcal{E}_{r,s}$. With the map $\bar{\psi}$ defined in Corollary 3.4, we set

$$\tilde{M}_c^{(2)}(\mathcal{R}):=\bar{\psi}^{-1}(\tilde{\mathcal{M}}_{r,s}\otimes H_c^1(\Gamma,V_1)).$$

Explicit examples of elements of $\tilde{M}_c^{(2)}(\mathcal{R})$ are certain real-analytic analogues of iterated Eichler integrals in the case of $\mathcal{E}_{r,s}$. We will make this more specific in the case $n=2$ with the following corollary of Lemma 3.5.

Corollary 3.7. *Suppose that $k_1\in 2\mathbb{N}$ and that f_1 is a weight k_1 cusp form for $\mathrm{SL}_2(\mathbb{Z})$. Let F_2 be defined by*

$$F_2(w,X_1):=\mathcal{E}_{r,s}(w)\int_{i\infty}^{w} f_1(w_1)(w_1-X_1)^{k_1-2}\,dw_1.$$

Then $F_2\in\tilde{M}_c^{(2)}(\mathcal{R})$.

Proof. This follows immediately from Lemma 3.5 combined with the identity $F_2.(\gamma-1)=\mathcal{E}_{r,s}r_f(\gamma;X_1)$ and the definition of $\bar{\psi}$. □

The functions of this corollary have been the prototypes for the elements of $\tilde{M}_c^{(2)}(\mathcal{R})$. At the same time, these functions are, up to multiplication with real-analytic $\mathcal{E}_{r,s}$, special cases of the iterated integrals of Lemma 3.5. Therefore, we refer to $\tilde{M}_c^{(2)}(\mathcal{R})$ as the space of *real-analytic iterated integrals*, even though its elements are not necessarily representable by iterated integrals in the strict sense.

Now, with the definition of $\tilde{M}_c^{(2)}(\mathcal{R})$ and with (3.4) and (3.5), Corollary 3.4 becomes

$$0\to H^0(\Gamma,\mathcal{R}\otimes P_{k_1-2}[X_1])\xrightarrow{\bar{\iota}}\tilde{M}_c^{(2)}(\mathcal{R})\xrightarrow{\bar{\psi}}\tilde{\mathcal{M}}_{r,s}\otimes(S_{k_1}\oplus\bar{S}_{k_1}).$$

We will show that this can be completed to a right exact sequence.

Theorem 3.8. *Suppose that $r+s>k_1$. The sequence of maps*

$$0\to H^0(\Gamma,\mathcal{R}\otimes P_{k_1-2}[X_1])\xrightarrow{\bar{\iota}}\tilde{M}_c^{(2)}(\mathcal{R})\xrightarrow{\bar{\psi}}\tilde{\mathcal{M}}_{r,s}\otimes(S_{k_1}\oplus\bar{S}_{k_1})\to 0$$

is exact.

Proof. The only part remaining to be proved is the surjectivity of $\bar{\psi}$. Let $\mathcal{E}_{r,s}\otimes(f,\bar{g})$ be an arbitrary basis element of $\tilde{\mathcal{M}}_{r,s}\otimes(S_{k_1}\oplus\bar{S}_{k_1})$. With the notation of Section 2.2.2, assign to each $h\in S_{k_1}$ a function $\psi^{\pm}_{h;r,s}$ given by

$$\psi^{\pm}_{h;r,s}(z,X):=\phi^{\pm}_{r,s}(h;z,X)-F^{\pm}_h(z,X)\mathcal{E}_{r,s}=\sum_{\gamma\in B\backslash\Gamma}\frac{r^{\pm}_h(\gamma;X)}{j(\gamma,z)^r j(\gamma,\bar{z})^s}.$$

By Proposition 2.3, this is absolutely convergent and its coefficients are of polynomial growth at infinity, and thus they belong to $\mathcal{R}$.

The image of $\psi^{+}_{f;r,s}+\psi^{-}_{g;r,s}$ under ψ is induced by the mapping

$$\begin{aligned}\gamma\to(\psi^{+}_{f;r,s}+\psi^{-}_{g;r,s})\underset{r,s,2-k_1}{|}(\gamma-1)&=-(F^{+}_f+F^{-}_g)\underset{0,0,2-k_1}{|}(\gamma-1)\mathcal{E}_{r,s}(z)\\&=-(r_f(\gamma;X)+\overline{r_g(\gamma;X)})\mathcal{E}_{r,s}(z).\end{aligned}$$

For the two equalities, we have used Proposition 2.3 and (2.9). By the explicit formula for the Eichler–Shimura map, we deduce that

$$\bar{\psi}(-\psi^{+}_{f;r,s}-\psi^{-}_{g;r,s})=\mathcal{E}_{r,s}\otimes(f,\bar{g}).$$

This shows that $-\psi^{+}_{f;r,s}-\psi^{-}_{g;r,s}\in\tilde{M}_c^{(2)}(\mathcal{R})$ and that its image is the element $\mathcal{E}_{r,s}\otimes(f,\bar{g})$. □

Remark. The theorem could be stated in more general form so that the real-analytic analogues of both Eisenstein and Poincaré series are captured. That would have the advantage of accounting for the full space $M_c^{(2)}$ instead of $\tilde{M}_c^{(2)}$, but we would need to enlarge our investigations to objects that do not satisfy (2.1). This is because the “real-analytic Poincaré series” do not satisfy (2.1). However, they are clearly interesting objects, worthwhile studying, and they are the subject of work in progress with F. Strömberg.

The family $\{\psi^{\pm}_{h;r,s}\}$ constructed in Theorem 3.8 allows us to describe our approach to Question 2 of Section 1. Specifically, for $h\in S_{k_1}$, set

$$\psi^{\pm}_{h;r,s}(z,X):=\sum_{\gamma\in B\backslash\Gamma}\frac{r^{\pm}_h(\gamma;X)}{j(\gamma,z)^r j(\gamma,\bar{z})^s}.$$

The family addresses Question 2, inasmuch as it satisfies the following three properties:
- Firstly, by Theorem 3.8, the $\psi^{\pm}_{h;r,s}$ belong to the space $\tilde{M}_c^{(2)}(\mathcal{R})$ of real-analytic iterated integrals.
- Secondly, this family is “canonical” in the sense that it induces a generating set for $\tilde{\mathcal{M}}_{r,s}\otimes(S_{k_1}\oplus\bar{S}_{k_1})$.
- Thirdly, it is possible to obtain, by a simple process, explicit Γ-equivariant versions of the real-analytic iterated integrals $\psi^{\pm}_{h;r,s}$.

This process is given in the following proposition, which also formalizes a link between the two main themes of this note, namely second-order modular forms and iterated integrals.

Proposition 3.9. *Let $r + s > k_1$. There is a well-defined linear map from the subspace of $\mathcal{M}_c^{(2)}(\mathcal{R})$ generated by the family $\{\psi^{\pm}_{h;r,s}\}$ to $\bigoplus_{i=0}^{k_1-2} \mathcal{MI}'_2$.*

Proof. For each $\psi^{\pm}_{h;r,s}(z, X)$, consider

$$\phi^{\pm}_{r,s}(h; z, X) = \psi^{\pm}_{h;r,s}(z, X) + F^{\pm}_h(z, X)\mathcal{E}_{r,s}.$$

By Theorem 2.6, the coefficients $\phi^{\pm}_{r,s}(h; i, z, X)$ of $(X - z)^i(X - \bar{z})^{k_1-2-i}$ in $\phi^{\pm}_{r,s}(h; z, X)$ belong to $\mathcal{MI}_2$. Therefore, the assignment

$$\psi^{\pm}_{h;r,s}(z, X) \to (\phi^{\pm}_{r,s}(h; 0, z, X), \dots, \phi^{\pm}_{r,s}(h; k - 2, z, X))$$

defines the sought map. □

3.2.1 A classification of holomorphic second-order forms

We close with another implication of Corollary 3.4 to *holomorphic* extended second-order modular forms. The resulting theorem is the analogue of [6, Theorem 2.3] to the class of extended second-order modular forms.

Let ρ be a complex representation of Γ of dimension $k_1 - 1$ induced by

$$\rho(-I_2) = I_{k_1-1}, \quad \rho(S) = ((-1)^i \delta_{j,k_1-2-i})_{i,j=0}^{k_1-2}, \quad \rho(T) = \left((-1)^{i+j}\binom{j}{i}\right)_{i,j=0}^{k_1-2}.$$

For an even $k > 0$, we denote by $M_k(\rho)$ the space of vector-valued modular forms for the representation ρ.

Proposition 3.10. *Suppose that $k > k_1 > 2$. The sequence of maps*

$$0 \to M_k(\rho) \to M_c^{(2)}(\mathfrak{O}) \xrightarrow{\bar{\psi}} M_k \otimes (S_{k_1} \oplus \bar{S}_{k_1}) \to 0$$

is exact. In particular,

$$\dim M_c^{(2)}(\mathfrak{O}) = 2\dim(M_k)\dim(S_{k_1}) + \frac{5 + k}{12}(k_1 - 1) + \frac{i^{k+k_1-2}}{4} - \frac{1}{3}\Big(\frac{k_1 - 1}{3}\Big)\Big(\frac{k - 1}{3}\Big).$$

Proof. We apply Corollary 3.4 to $V = \mathfrak{O}$ with $k_0 := 2 - k$ to deduce, in the first instance,

$$0 \to H^0(\Gamma, \mathcal{O} \otimes P_{k_1-2}) \xrightarrow{\bar{\imath}} M_c^{(2)}(\mathfrak{O}) \xrightarrow{\bar{\psi}} M_k \otimes (S_{k_1} \oplus S_{k_1}).$$

Here we used (3.4) and (3.5).

The isomorphism $H^0(\Gamma, \mathcal{O} \otimes P_{k_1-2}) \cong M_k(\rho)$ is induced by the mapping

$$g = \sum_{j=0}^{k_1-2} f_j(z)X^j \to (f_0, f_1, \dots, f_{k_1-2})^T.$$

By a direct computation, we see that the invariance of g under S and T is equivalent to the invariance of the associated vector under S and T in terms of the representation ρ.

To prove the surjectivity, let

$$f \otimes (g, \bar{h}) \in M_k \otimes (S_{k_1} \oplus \bar{S}_{k_1}).$$

Assume that $f = \sum_{n\geq 0} \lambda_n P_n$, where P_n is the classical Poincaré series of weight k given by

$$P_n(z) = \sum_{B\backslash\Gamma} \frac{e^{2\pi i n\gamma z}}{j(\gamma, z)^k}$$

(here $\lambda_n = 0$ for all but finitely many integer $n \geq 0$). For $n \geq 0$ and $h \in S_{k_1}$, set

$$G^{\pm}_{n,h}(z, X) := \sum_{\gamma\in B\backslash\Gamma} (F^{\pm}_f \cdot e^{2\pi i n -}) \mid_{k,0,2-k_1} \gamma - F^{\pm}_h(z, X)P_n = \sum_{\gamma\in B\backslash\Gamma} \frac{r^{\pm}_h(\gamma; X)e^{2\pi i n\gamma z}}{j(\gamma, z)^k}.$$

Since $|e^{2\pi i n\gamma z}| \le 1$, the absolute convergence of this series, for $k > k_1$, follows from the absolute convergence of $\phi^{\pm}_{k,0}$ proved in Proposition 2.3. The same proposition implies the polynomial growth of the polynomial coefficients of $G^{\pm}_{n,h}$.

We then have, for all $\gamma \in \Gamma$,

$$-\sum_{n\ge 0} \lambda_n (G^+_{n,g} + G^-_{n,h}) \big|_{k,0,2-k_1} (\gamma - 1) = \sum_{n\ge 0} \lambda_n (r^+_g(\gamma; X) + r^-_h(\gamma; X)) P_n = (r^+_g(\gamma; X) + r^-_h(\gamma; X)) f.$$

By the definition of the map $\bar{\psi}$, we deduce that the image of $\sum \lambda_n (G^+_{n,g} + G^-_{n,h})$ is $f \otimes (g, \bar{h})$.

To deduce the dimension formula, we use [5, Theorem 6.3 and Remark 6.4] to compute the dimension of $M_k(\rho)$. Indeed, ρ is an even representation, $\dim \rho = k_1 - 1$ and a direct computation gives

$$\mathrm{Tr}(\rho(ST)^2) = \mathrm{Tr}(\rho(ST)) = \mathrm{Tr}(\rho(S)\rho(T)) = \sum_{i=0}^{k_1-2} (-1)^i \binom{i}{k_1 - 2 - i} \left(\frac{k_1 - 1}{3}\right),$$

where $(\frac{\cdot}{\cdot})$ stands for the Legendre symbol. This is seen by noting that the sequence given by

$$a_n = \sum_{i,j\in\mathbb{Z},\, i+j=n} (-1)^i \binom{i}{j}, \quad \text{where } \binom{i}{j} = 0, \text{ unless } i \ge j \ge 0,$$

satisfies the recurrence relation $a_n + a_{n-1} + a_{n-2} = 0$, and therefore, by induction, $a_n = \frac{n+1}{3}$

We can also see, by induction, that, for $\xi = e^{\pi i/3}$,

$$\frac{\xi^k}{1 - \xi^2} + \frac{x^{2k}}{1 - \xi^{-2}} = -\left(\frac{k-1}{3}\right).$$

Since the only eigenvalue of $\rho(T)$ is 1 and the corresponding eigenspace has dimension 1, there is one Jordan block with 1 in the diagonal. By [5, Theorem 3.4], we deduce that the trace for a standard choice of exponents for $\rho(T)$ is 0. This, together with [5, Remark 6.3], implies that, since $k > k_1$, the dimension of $M_k(\rho)$ is given by the formula of [5, Theorem 6.3], which, by the preceding remarks, is

$$\frac{5 + k}{12}(k_1 - 1) + \frac{i^{k+k_1-2}}{4} - \frac{1}{3}\left(\frac{k_1 - 1}{3}\right)\left(\frac{k-1}{3}\right). \qquad \square$$

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