Class field theory, its three main generalisations, and applications

Ivan Fesenko

Abstract. This work presents branches of class field theory. Special and general approaches to class field theory, and their roles, are discussed. Three main generalisations of class field theory: higher class field theory, Langlands correspondences and anabelian geometry, and their further developments are discussed. Several directions of unification of generalisations of class field theory are proposed. New fundamental open problems are included.

Class Field Theory (CFT) is the main achievement of algebraic number theory of the 20th century. Its reach, beauty and power, stemming from the first steps in algebraic number theory by Gauß, have substantially influenced number theory. Shafarevich wrote [12, Foreword]:

> *Weil was undoubtedly right when he asserted, in the preface to the Russian edition of his book on number theory* [72]*, that since class field theory pertains to the foundation of mathematics, every mathematician should be as familiar with it as with Galois theory. Moreover, just like Galois theory before it, class field theory was reputed to be very complicated and accessible only to experts.* [...] *For class field theory, on the other hand, there is a wide range of essentially different expositions, so that it is not immediately obvious even whether the subject is the same.*

Weil's opinion has proved to be quixotic: these days even some number theorists are not familiar with the substance of CFT.

This text reviews the enduring process of discovering new branches of CFT and its generalisations. Many of such developments were complicated at their early stages and some were difficult or impossible to understand for their contemporaries. Three main generalisations of CFT and their further extensions will be presented and some of their key fundamental features will be discussed. This text proposes eight fundamental problems.

We start with Kummer theory, a purely algebraic exercise, whose highly nontrivial arithmetic analogues over arithmetic fields are supplied by CFT. Kummer theory is

an algebraic predecessor of CFT including its existence theorem. Then we discuss the fundamental *split* of (one-dimensional) CFT into *special CFT* (SCFT) and *general CFT* (GCFT). This split has enormously affected many developments in number theory. Section 3 delves into four fundamental parts of CFT including the reciprocity map, existence theorem, explicit formulas for the Hilbert symbol and its generalisations, and interaction with ramification theory. Section 4 briefly touches on higher Kummer theory using Milnor K-groups, i.e. the norm residue isomorphism property.

Three generalisations of CFT: *Langlands correspondences* (LC), *higher CFT*, and *anabelian geometry* are discussed in Section 5. We note that the split of CFT into SCFT and GCFT is currently somehow reproduced at the level of generalisations of CFT: LC over number fields does not yet have any development parallel to GCFT, while higher CFT is parallel to GCFT and it does not have substantial developments similar to SCFT.

In the last section we specialise to elliptic curves over global fields, as an illustration. There we consider two further developments: Mochizuki's inter-universal Teichmüller theory (IUT) which is pivoted on anabelian geometry and two-dimensional adelic analysis and geometry which uses structures of two-dimensional CFT. We also consider the fundamental role of zeta integrals which may unite different generalisations of CFT. Similarly to the situation with LC, the current studies of special values of zeta- and L-functions of elliptic curves over number fields, except two-dimensional adelic analysis and geometry, use special structures and are not of general type.

There is no attempt to mention all the main results in CFT and all of its generalisations or all of their parts, and the text does not include all of bibliographical references.

1. Kummer theory

For a field k denote its absolute Galois group by G_k and its maximal abelian quotient by

$$G_k^{\mathrm{ab}} = G_k/[G_k, G_k].$$

For a positive integer m the group of roots of unity of order m is denoted by μ_m.

For a field k whose characteristic does not divide integer $m > 1$, the exact sequence of G_k-modules

$$1 \longrightarrow \mu_m \longrightarrow \mathrm{k}^\times \longrightarrow \mathrm{k}^\times \longrightarrow 1,$$

where k is a separable closure of k, gives the Kummer map

$$k^\times/k^{\times m} \simeq H^1(G_k, \mu_m).$$

This is an H^1-theory.

If μ_m is in k, then the R.H.S. $= \mathrm{Hom}(G_k, \mathbb{Z}/m\mathbb{Z})$, and we get the Kummer pairing

$$k^\times/k^{\times m} \times G_k/([G_k, G_k]G_k^m) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

Kummer duality is easily deduced: finite abelian extensions of k of exponent m are in explicit one-to-one correspondence with subgroups B of $k^\times$ of exponent m: $B \longrightarrow k(\sqrt[m]{B})$. The analogue of this property in CFT is existence theorem of CFT; see Section 3.

This theory is not of much value for arithmetic fields, e.g. $\mathbb{Q}$ and $\mathbb{Q}_p$ and their finite extensions, since they contain too few roots, unlike function fields over algebraically closed fields. One can slightly extend Kummer theory, working with $\mathbb{Q}(\mu_m)$, for prime m, using the fact that $|\mathbb{Q}(\mu_m) : \mathbb{Q}|$ is prime to m, but this does not go too far.

First results in anabelian geometry show that every continuous automorphism of $G_{\mathbb{Q}}$ is inner. For nonarchimedean completions k_v of number fields not every continuous automorphism of the absolute Galois group G_{k_v} is inner. We do know an algebraic description of the absolute Galois groups of k_v (Demushkin, Yakovlev, Jannsen–Winberg), but relations of this description with higher ramification filtration are in need of further work. For number fields we still know little about G_k, unlike G_k^{ab} due to CFT. The Shafarevich conjecture: $[G_{\mathbb{Q}}, G_{\mathbb{Q}}]$ is free in the category of profinite groups, remains open.

Key Fundamental Problem. *How to reach to the maximal separable extension, maximal abelian extension, maximal nilpotent extension of fixed nilpotent class of arithmetic fields, and how to describe their finite subextensions in terms of objects associated to the ground field and in ways which allow a good range of applications? Which structures of arithmetic fields to use for such descriptions?*

This problem for abelian extensions is solved by various types of CFT and by higher CFT, for nilpotent extensions of local fields it is partially solved by arithmetic non-abelian local class field theory, whereas the Langlands correspondence provides some conjectural answers using representation theory, while anabelian geometry provides very different insights into the full structure of the absolute Galois group. Various developments that use appropriate features of CFT and its three generalisations may lead to a new powerful theory.

2. CFT of two types

CFT includes a construction of the reciprocity map, a homomorphism from an appropriate group associated to the ground field, to the Galois group of the maximal abelian extension of the field, and existence theorem which associates in 1-1 fashion, a finite abelian extension of the ground field to an appropriate open subgroup of the appropriate group: this open subgroup is the norm group of the abelian extension. Thus, the transfer/norm map plays the key role in CFT[1], and one of key features of general CFT (see below) is its functoriality with respect to finite separable extensions of the fields.

It is of fundamental importance to distinguish two different types of CFT:

CFT

SCFT   GCFT
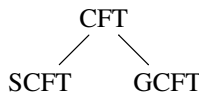
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[1]as well as in motivic cohomologies

Type I: *SCFT = special CFT*. These diverse theories use torsion/division points or values of appropriate functions at torsion points to generate finite extensions of the base fields under investigation and to describe the Galois action on them. Historically, this is the first approach. The global number fields versions of these theories work over certain small fields *only*, hence they are not functorial. The local (Lubin–Tate) and functional case theories work over any field.

The list of SCFTs includes (see [19] and other historical reviews of CFT for more information):

Cyclotomic (historically, the first): Kronecker, Weber, Hilbert.

Using elliptic curves with CM (historically, the second): Kronecker (including his Jugendtraum), Weber, a relevant portion of Takagi's work.

Using abelian varieties with CM: Shimura. These theories are not extendable to arbitrary number fields.

Positive characteristic: Hayes and Drinfeld (rank 1 Drinfeld modules). These theories work over all global fields of positive characteristic.

Local theory using Lubin–Tate formal groups. It works over any local field with finite residue field and does not work over local fields with infinite perfect residue field.

Some of the SCFT are not easy. Weber published several erroneous proofs of the complex multiplication CFT, due to mistakes with the 2-primary part; see [64]. In comparison to simplified explicit versions of GCFT, SCFT perhaps stays too close to detailed structures and is not able to see more general easier to use structures. Hilbert Problem 12 was about extensions of SCFT to number fields, the best was achieved by Shimura.

Type II: *GCFT = general CFT*. These theories follow very different conceptual patterns than SCFT. They work over any global or local field. In particular, there are fundamental conceptual features of GCFT which one does not find in SCFT. Working at the level of small number fields does not uncover various general structures which make GCFT work over any finite separable extension of the ground field.

The list of GCFTs for *arithmetic fields* includes:

Takagi, this seems to be the first work in GCFT with his general *existence theorem* and its applications;

Artin *reciprocity map*, whose full construction uses Chebotarev's theorem;

Hasse, the use of *the Brauer group* in CFT, the first local CFT, local-to-global aspects;

Chevalley's invention of *ideles*, local-to-global, the global reciprocity map as the product of the local reciprocity maps, whose kernel contains the diagonal image of global elements.

Classical approaches to CFT are presented, among many sources, in Hasse's Klassenkörperbericht, and in Weil's and Lang's books.

Cohomological approaches: Artin–Tate's book [2] is a classical cohomological presentation, and there are many other presentations of varying degree of understanding of what is

actually going on. All these class field theories are not especially easy to embrace. Finding explicit formulas for the Hilbert pairing and its generalisations (see Section 3 below) was one of the ways to get more explicit information about the reciprocity map.

Positive characteristic general CFTs: there are various approaches, including some parallel to the number fields case. The GSFT Kawada–Satake theory [36] that uses Witt duality for the most difficult p-primary part is as easy as the SCFT theory of Hayes–Drinfeld.

Explicit, post-cohomological and cohomologically-free theories, local CFT: Tate–Dwork (Dwork's Lemma proved by Tate), Hazewinkel, Neukirch, Fesenko, global CFT: Neukirch. These theories clarified and made explicit some of the key structures of CFT. These are less dependent on torsion and they do not use the Brauer group. It is amazing that the fact that the Brauer group computation is not needed for CFT is still not well known. These explicit CFT are easy and they really explain CFT. One can say that explicit GCFT, working over any global and local field, is easier than SCFT working over small number fields only. Note the very different nature of explicitness in explicit GCFT in comparison to that in SCFT. The explicit GCFT was highly useful in developing aspects of higher CFT.

3. Four fundamental parts of CFT

Part I. The *functorial reciprocity map*

$$\Psi_k : C_k \longrightarrow G_k^{\mathrm{ab}}$$

is almost a topological isomorphism, from the topological abelian group C_k (the multiplicative group of a local field or the idele class group of a global field) associated to the ground field k. For a finite Galois extension l/k it induces the isomorphism

$$C_k / N_{l/k} C_l \simeq G(l/k)^{\mathrm{ab}}.$$

Part II. The *existence theorem*, describing finite abelian extensions of the ground field k in terms of open subgroups of finite index of C_k, is important in many applications. Existence theorem is the origin of the name 'class field': open subgroups of finite index in C_k are in one-to-one correspondence with finite abelian extensions l/k:[2,3]

$$l/k \longrightarrow N_{l/k} C_l.$$

The existence theorem allows to construct and study abelian extensions by operating with open subgroups of finite index.

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[2]In various generalisations of the classical class field theory such as class field theory for local fields with quasi-finite or perfect residue field, [5], the topology associated with norm subgroups is strictly weaker than the discrete valuation topology on the multiplicative group.

[3]See also [13] which connects aspects of existence theorem with topological reflexivity with respect to a related explicit pairing.

Part III. *Explicit formulas* for the (wild) Hilbert pairing, already asked for in Hilbert's Problem 9. Important for numerous applications and computations. The nature of explicitness here differs from that in explicit CFT.

Part IV. *Compatibility with ramification theory* is an additional arithmetic flavour of CFT. The local reciprocity map (a) makes compatible the filtration on groups of units by higher subgroups of principal units with the higher ramification groups filtration on the abelian part of the absolute Galois group. The latter filtration, on any Galois group of a local field, is due to an application of (b) Herbrand's theorem on the compatibility of the upper ramification filtration with taking quotients of Galois groups, and it satisfies (c) the Hasse–Arf theorem: higher ramification jumps of abelian extensions are integers.

Now we discuss various features of Part I and Part III of CFT.

Cohomological approaches: there are several, all are relatively well known. Cohomological CFT is an H^2-theory. One of the approaches uses the Kummer map, locally

$$\text{Hom}(G_k, \mathbb{Z}/m\mathbb{Z}) \times k^\times/k^{\times m} \longrightarrow H^1(G_k, \mathbb{Z}/m\mathbb{Z}) \times H^1(G_k, \mu_m)$$
$$\longrightarrow H^2(G_k, \mu_m) = {}_m\text{Br}(k) \simeq \mathbb{Z}/m\mathbb{Z},$$

which gives the map

$$k^\times \longrightarrow \varprojlim k^\times/k^{\times m} \longrightarrow G_k^{\text{ab}}.$$

The map $H^1 \times H^1 \to H^2$ is defined using the cup product. The cup product is non-explicit, which implies the non-explicit nature of cohomological class field theory.

Remark. Already Hasse used information about the Brauer group to deduce the reciprocity map. However, the information about the Brauer group is of less use for existence theorem and explicit formulas. CFT is not reduced to the computation of the Brauer group and the Brauer group is not needed in explicit CFT, see above.

Fifty years ago Hasse wrote [19]:

the sharply profiled lines and individual features of this magnificent edifice seem to me to have lost somewhat of their original splendour and plasticity by the penetration of class field theory with cohomological concepts and methods.

The *class formations* approach was a long search for clarification of CFT in the following sense: deduce as much as possible in CFT from as few axioms as possible. Typically, the axioms are about H^0 (index of the norm group) and H^1 (Hilbert Theorem 90) for cyclic extensions. Many papers are devoted to the *CFT mechanism*, which is about how to deduce all the main theorems of CFT from a small number of assumptions (CFT axioms) in a purely topological group theoretical way, hence *without using ring structures*. Thus, from the point of view of anabelian geometry, the CFT mechanism can play a special role.[4] At the same time, proving CFT axioms in all known types of CFT involves the ring structure.

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[4]See more on this in 'Reciprocity and IUT', https://www.maths.nottingham.ac.uk/plp/pmzibf/jl.pdf.

Explicit CFT provides much more information about the image in $C_k/N_{l/k}C_l$ of a Galois automorphism σ of a finite Galois extension l/k, [60,61]. *The Neukirch method* has a very short description, and, due to its fundamental importance and its potential value for future developments, we briefly present it now. One of its key points is to work with appropriate infinite extensions to get information about finite extensions. In explicit CFT one works with appropriate free abelian pro-extensions of rank 1: the maximal unramified extension for local fields, the extension fk/k for algebraic number fields k where $f/\mathbb{Q}$ is the only $\widehat{\mathbb{Z}}$-subextension of $\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}$, and the maximal constant extension of a global field of positive characteristic.

Start with an abelian (discrete topological) group A endowed with a continuous action by a profinite group G. For simplicity, let G be the absolute Galois group G_k of a field k. For an open subgroup G_K of G_k denote by A_K the G_K-fixed elements of A. For an open subgroup G_L of an open subgroup G_K of G_k denote by $N_{L/K}: A_L \to A_K$ the product of the action of right representatives of G_L in G_K, this does not depend on their choice.

Make two assumptions. First, assume that there is a surjective homomorphism of profinite groups

$$\deg = \deg_k: G_k \to \widehat{\mathbb{Z}}.$$

Denote its kernel $G_{\widetilde{k}}$. For an open subgroup G_K of G_k, deg induces a surjective homomorphism

$$\deg_K = |G_k : G_K G_{\widetilde{k}}|^{-1} \deg_k: G_K \to \widehat{\mathbb{Z}}.$$

Call any element of G_K which is sent by $\deg_K$ to $1 \in \widehat{\mathbb{Z}}$ a frobenian with respect to $\deg_K$. Extensions of K in $K\widetilde{k}$ will be called unramified. Second, assume that there is a homomorphism $v: A_k \to \widehat{\mathbb{Z}}$ whose image is $\mathbb{Z}$ or $\widehat{\mathbb{Z}}$ and such that the equality

$$v(N_{L/K}A_L) = |G_K G_{\widetilde{k}} : G_L G_{\widetilde{k}}| \, v(A_K)$$

holds for any open subgroup G_L of any open subgroup G_K of G_k.

For the pair $(\deg, v)$ we get a reciprocity map (not well defined, unless some further conditions are imposed) in the following way. For a finite extension K of k and a finite Galois extension L/K and σ in its Galois group, find any $\widetilde{\sigma} \in G(L\widetilde{k}/K)$ such that $\deg(\widetilde{\sigma})$ is an integer $\geqslant 1$ (arithmetic orientation) and $\widetilde{\sigma}|_L = \sigma$. Denote by Σ the fixed field of $\widetilde{\sigma}$. Then $L\widetilde{k} = \Sigma\widetilde{k}$ and $L\widetilde{k}/\Sigma$ is an unramified extension and $\widetilde{\sigma}$ is a frobenian of G_Σ. Thus, every element σ in the Galois group of L/K can be lifted to a frobenian, after taking a composite with some appropriate finite extension Σ/K depending on σ. Call $\pi_K \in A_K$ such that $v(N_{K/k}\pi_K) = |\widehat{\mathbb{Z}} : \deg(G_K)|$ a prime element of A_K. The element π_K remains prime in all finite (unramified) subextensions of K in $K\widetilde{k}$.

Frobenians are mapped to prime elements with respect to the reciprocity map for unramified extensions, so then it is natural to define the (inverse to the) reciprocity map by the simple formula

$$\Upsilon_{L/K}: \sigma \mapsto N_{\Sigma/K}\pi_\Sigma \quad \mathrm{mod}\ N_{L/K}A_L,$$

where π_Σ is any prime element of A_Σ. This involves two *indeterminacies* with respect to the choice of $\widetilde{\sigma}$ and of π_Σ.

Now we need two axioms for the G-module A (axioms of CFT). The first is that the kernel of the norm map $N_{L/K}$ for cyclic extensions of prime degree is

$$\{(1-\sigma)A_L : \sigma \in G(L/K)\}.$$

The second axiom is a numerical (hence, *a priori*, not geometrical) property for finite abelian extensions L/K: the index of $N_{L/K}A_L$ in A_K equals the degree of L/K. Then one rather easily proves that $\Upsilon_{L/K}$ is a well defined homomorphism, and it induces an isomorphism

$$G(L/K)^{\mathrm{ab}} \to A_K/N_{L/K}A_L$$

satisfying all the standard functorial properties.

For a local field of mixed characteristic k one can use the maximal unramified (in the usual sense) extension generated by roots of order prime to the residue characteristic as $\widetilde{k}$. One can also use other (ramified in the usual sense) $\widehat{\mathbb{Z}}$-extensions as $\widetilde{k}$. Usually one uses the canonical (arithmetic) Frobenius automorphism to fix deg and takes v as the usual surjective discrete valuation, then one gets the canonical local reciprocity map. We use the unique $\widehat{\mathbb{Z}}$-subextension of $\mathbb{Q}$ inside its maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ as $\widetilde{\mathbb{Q}}$. For a number field k we use the composite of k and $\widetilde{\mathbb{Q}}$ as $\widetilde{k}$. The map v is first defined on ideles as the composite of the product of the inverse to the local reciprocity homomorphisms (with the canonical local Frobenius automorphisms), which is then restricted to $G(\widetilde{k}/k)$ and composed with deg. The reciprocity law for cyclotomic extensions of $\mathbb{Q}$ implies that v factorises through idele classes. There is no canonical choice of a generator of $G(\widetilde{\mathbb{Q}}/\mathbb{Q})$, hence no canonical choice of deg, but the global reciprocity map does not depend on this choice, since using a deg with $a \in \widehat{\mathbb{Z}}^{\times}$ changes v to av and the pairs (deg, v) and (a deg, av) define the same reciprocity map. In characteristic p one uses the maximal constant field extensions; one can also use Artin–Schreier–Witt theory to explicitly get the p-primary part of CFT.

Explicit formulas for the (local) Hilbert pairing. Let k contain m roots of unity of order m, μ_m. The Hilbert pairing

$$(\, , \,)_m : k^{\times}/k^{\times m} \times k^{\times}/k^{\times m} \longrightarrow \mu_m$$

is defined as $(a,b)_m = \sigma(c)/c$, where $c^m = b$ and σ is the restriction on $k(\sqrt[m]{b})/k$ of the image of a with respect to the local reciprocity map. If one uses the Kummer map, then

$$(\, , \,)_m : k^{\times}/k^{\times m} \times k^{\times}/k^{\times m} \longrightarrow H^1(G_k, \mu_m) \times H^1(G_k, \mu_m)$$
$$\longrightarrow H^2(G_k, \mathbb{Z}/m\mathbb{Z}) = {}_m\mathrm{Br}(k) \simeq \mu_m$$

after appropriate choices are made.

If m is prime to the residue characteristic p, then the Hilbert pairing is a power of the tame symbol with its explicit linear algebra formula. When $m = p^r$, explicit formulas for the wild Hilbert symbol and its generalisations are far from linear algebra. Such formulas are at the heart of arithmetic.

Two types of explicit formulas (for the wild Hilbert pairing and its generalisation to formal groups):

Partial/special explicit formulas: Artin–Hasse, Iwasawa, Coates–Wiles, Fontaine, Perrin-Riou, Kato–Kurihara–Tsuji, Benois and others.

Full explicit formulas (for full values of the argument): Shafarevich, Vostokov, Brückner, Henniart, Fesenko, Kato. Because of its importance, let's include the Vostokov formula. Let k be a finite extension of $\mathbb{Q}_p$ and let it contain a primitive p^nth root ζ_{p^n} of unity, $p > 2, n \geqslant 1$. Choose a prime element π of k. Let O_0 be the ring of integers of the inertia subfield of k, $\mathrm{Tr} = \mathrm{Tr}_{O_0/\mathbb{Z}_p}$ and φ be the Frobenius automorphism acting on O_0. Then for $\alpha, \beta \in k^\times$,

$$(\alpha, \beta)_{p^n} = \zeta_{p^n}^{\mathrm{Tr\,res}\,\Phi(A,B)\,(1/S+1/2)}, \quad \Phi(A, B) = l(B)\,dA/A - l(A)\frac{1}{p}dB^\triangle/B^\triangle,$$

where $A, B \in O_0((X))^\times$ are any series such that

$$A(\pi) = \alpha, \quad B(\pi) = \beta,$$

$S = S_1^{p^n} - 1$, $S_1 \in 1 + XO_0[\![X]\!]$ is any series such that

$$S_1(\pi) = \zeta_{p^n},$$

$$l(A) = \log\big(A^p/A^\triangle\big)/p, \quad \Big(\sum a_i X^i\Big)^\triangle = \sum \varphi(a_i)X^{pi};$$

see Chapter VII of [12]. Thus, this formula for the Hilbert pairing involves *indeterminacies* in relation to the choice of π, A, B, S_1. The right-hand side of the previous displayed formula is defined independently of CFT, it is called the *Vostokov symbol* and it can be used to establish part of CFT for Kummer extensions of k.

See [70] for a general review of explicit formulas, more information and references.

Remark. CFT is more than an abelian theory, it includes some information about abelian by finite extensions, the Shafarevich–Weil theorem: for an arithmetic field k and finite $m/l/k$, m/l abelian, $m/k, l/k$ Galois, the class of the group extension in $H^2\big(G(l/k), C_l/N_{m/l}C_m\big)$ of the exact sequence

$$1 \longrightarrow C_l/N_{m/l}C_m \longrightarrow G(m/k) \longrightarrow G(l/k) \longrightarrow 1$$

equals the image of the canonical class in $H^2\big(G(l/k), C_l\big)$. Koch–de Shalit's metabelian local class field theory, which is of SCFT type, may be viewed as one of the developments related to this theorem. A related development is general *arithmetic non-abelian CFT*, which is of GCFT type, it includes local theory for arithmetically profinite extensions, with its existence theorem and compatibility with ramification theory (Fesenko [7], Ikeda–Serbest [23–26]), and some global theory (Ikeda [22]).

Remark. For a discussion of some contrasting aspects of CFT and generalised Kummer theory see Section 4.2 of [55].

4. From CFT to higher Kummer theory involving Milnor K-groups

The Hilbert pairing satisfies the norm property: $(a, b)_m = 1$ iff $b \in N_{k(\sqrt[m]{a})/k} k(\sqrt[m]{a})^\times$. This implies the Steinberg property $(a, 1 - a)_m = 1$ for $a \neq 1$. So the Hilbert pairing is a symbol map and induces a map from Milnor $K_2(k)$ to μ_m. If m is the cardinality of roots of unity in k, the mth Hilbert pairing is a universal continuous map from $K_2(k)$ to finite abelian groups, see e.g. Chapter IX of [12].

For any field k of characteristic not dividing m, using the Kummer map and the cup-product, one gets the 2-Kummer map:

$$k^\times / k^{\times m} \times k^\times / k^{\times m} \longrightarrow H^1(G_k, \mu_m) \times H^1(G_k, \mu_m) \longrightarrow H^2(G_k, \mu_m \otimes \mu_m).$$

If k is local, this is closely related to the Hilbert pairing. The 2-Kummer map satisfies the Steinberg property, hence induces the norm residue symbol

$$K_2(k)/m K_2(k) \longrightarrow H^2(G_k, \mu_m \otimes \mu_m).$$

Key results about the norm residue symbol include: the norm residue symbol is an isomorphism for local and global fields (Tate, using CFT); the norm residue symbol is an isomorphism for all fields (Merkuriev–Suslin); the generalisation

$$K_r(k)/m K_r(k) \longrightarrow H^r(G_k, \mu_m^{\otimes r})$$

of the norm residue symbol to Milnor K_r is an isomorphism for all r (Voevodsky, previous results of Suslin, Morel and Rost play an important role, for the first complete presentation of the proof see [17]).

Milnor K-groups of higher fields play a fundamental role in higher CFT.

A classical result of Matsumoto tells that for an infinite field k its Milnor $K_2(k)$ is isomorphic to the fundamental group $\pi_1(G)$ for any split, simply connected, semi-simple, almost simple, algebraic group G over k, not of symplectic type. Due to the work of Morel (see [57]) and Voevodsky, and a more recent work by Morel and Sawant, the sheaf of Milnor K_2 is canonically isomorphic to the motivic sheaf $\pi_1^{\mathbb{A}^1}(G)$ with G the same as above.

5. Three generalisations of CFT

Each of the following three generalisations was started approximately half a century ago:

LC   higher CFT   anabelian geometry

CFT
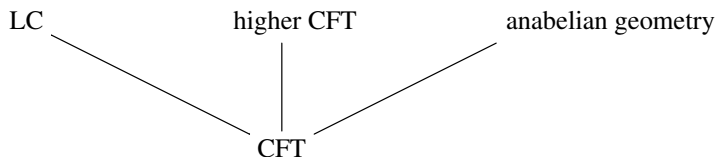
These generalisations use fundamental groups: the étale fundamental group in anabelian geometry, representations of the étale fundamental group (thus, forgetting something very essential about the full fundamental group) in Langlands correspondences and the (abelian) motivic $\mathbb{A}^1$ fundamental group (i.e. Milnor K_2) in two-dimensional (2d) higher class field theory.

5.1. Langlands correspondences (LC)

These theories are well known, they all use representation theory. Unlike CFT, whose modern expositions are not in need to involve any analytic results, the main arithmetic version of LC substantially involves analytic considerations. Its central analytic objects are two types of L-functions associated to Galois representations and to automorphic representations.

There are no known full analogues of Parts II, III of CFT in LC. For some very partial analogue of Part III of CFT in LC see e.g. Sections 1–2 of [1]. There is the higher ramification groups order formula for the Artin conductor, but it is rather not an LC analogue of Part IV of CFT.

Problem 1. *Develop fuller LC analogues of Parts II–IV of CFT.*

One way to characterise the correspondence in LC is to say that LC conjecturally classifies (irreducible) linear continuous representations of the Galois group (or related more complicated objects, such as the Weil or Weil–Deligne groups), using Artin L-functions and their generalisations, in terms of certain automorphic representations of local or adelic algebraic groups, using automorphic L-functions, in a way compatible with the classification of one-dimensional representations supplied by CFT. The L-functions, analytic objects, play fundamentally important role in the current form of LC for global fields and their completions. Recall that one can rewrite some of Part I of CFT for number fields as the property that for the L-function associated to a character of a finite abelian extension of a number field there is a unique primitive Hecke character of the number field with the same Hecke L-function. However, modern expositions of CFT do not use L-functions.

To some extent LC is a linear theory over abelian CFT. Since it is a representation theory, LC inevitably misses various important features of the full absolute Galois group that are not of linear representation type. Tannakian duality theoretically allows to recover a group from the tannakian category of its linear representations but this is difficult to do practically and, furthermore, not the full tannakian structure is taken care of by LC. For example, anabelian geometry uses the following two group theoretical properties of the absolute Galois group of a number field or of its nonarchimedean completion: each of its open subgroups is centre-free, each nontrivial normal closed subgroup H of any open subgroup, with the property that H is topologically finitely generated as a group, is open.[5] How can one use these properties at the level of representations of these Galois groups?

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[5]Profinite groups with the property that each nontrivial normal closed subgroup of any open subgroup is open are called hereditarily just-infinite, they form an important class in the classification of profinite groups, one representative is the Nottingham group and some of its hereditarily just-infinite subgroups can be realised as Galois groups of arithmetically profinite extensions of local fields, [6].

L. Lafforgue wrote[6]:

In the case of the absolute Galois group of a field, set-theoretic actions of this group correspond to separable extensions of the field. The world of its linear representations is so different that it is usually extremely difficult and deep to derive concrete consequences for separable extensions of results obtained for linear representations of this group

and

From the point of view of classifying toposes, going to linear representations is just a base change: instead of studying the classifying topos of a profinite group relatively to the absolute base, the topos of sets, you study it after base change with the classifying topos of the theory of vector spaces over some coefficient field. The fact that such a base change makes such a difference is extremely surprising.

The main conjectures of LC are stated over all global and local fields. There are important classes of representations of Galois groups naturally arising from torsion or division points of arithmetic objects or from appropriate cohomology groups. Over number fields and their completions, there has been progress in several directions, but we are still quite far from solutions of the key fundamental problems. Langlands wrote [46, p. 467]:

It appears that accidental, conditional phenomena are being used to establish general principles, a philosophically disagreeable circumstance.

Currently, the main arithmetic achievements in LC are of special type only.[7] In the case of $\mathbb{Q}$, one of them is the work of Wiles and R. Taylor on modularity of L-functions of semistable elliptic curves over $\mathbb{Q}$ and its extensions to some small number fields. However, even the general Taniyama–Shimura conjecture over all number fields still stands unsolved, and its future general solution, as well as solutions of general conjectures of LC will be very different from the special methods used so far.

The $GL_2(\mathbb{Q})$ case is still open, 'We still don't know how to associate an Artin representation to a Galois Maass form' and 'the problem of Galois Maass forms is just the tip of an iceberg'[8].

There are Harris–Taylor and Henniart's results about the local case for general linear groups using global methods, and still no fully local proof of local LC for all GL_n is known.

Together with the correspondence issue (not an entirely analytic statement), the second most important issue in LC is functoriality (an analytic statement). Langlands wrote [46, p. 458]:

progress in functoriality, as a part of the theory of automorphic forms, has been largely analytic, exploiting very few arithmetic *arguments* [. . .] *Relations between the analytic theory and the arithmetical have often been uneasy* [. . .]

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[6]From L. Lafforgue's email message, Feb 21, 2018.
[7]Compare with p. 467 of [46].
[8]From R. Taylor's email messages of Mar 4 and Mar 13, 2018.

The proof of the fundamental lemma, a local statement, by Ngô, using Hitchin systems (which played an important role in geometric LC), was another highlight in all characteristics. Using it, Arthur established his trace formulas and various arithmetic applications.

The approach to functoriality through appropriate Fourier transforms and Poisson formulas was proposed for the first time in 2000 by Braverman and Kazhdan. L. Lafforgue, [41, 42], proposed a more precise formulation, both in characteristic zero and in positive characteristic, and proved the equivalence between functoriality and the existence of such *non-additive* Fourier transforms verifying a Poisson formula. In particular, these new Fourier transforms are defined and the associated Poisson formulas are proved in the function fields case. This reformulation asks for a definition of the Fourier transform on functional spaces for a general reductive algebraic group where one cannot use the obvious relation of GL_n to matrices, without using the spectral decomposition results.

Fifty years after its start, most fundamental problems in arithmetic LC remain open. Is this a testament to the difficulty of the problems or to them not yet have been stated in the 'right' form? Work on this text revealed that researchers in LC often have a restricted knowledge of CFT, and in particular do not have a good knowledge of GCFT. Could that be one of key reasons why the most fundamental problems in arithmetic LC have not yet been solved? It is quite clear that some new fundamental arithmetic insights about LC are missing. Linear algebraic and geometric methods used in LC cannot fully cover or substitute various profound arithmetic issues, in particular those revealed by CFT and its other generalisations.

LC is non-abelian but not utmost non-abelian in comparison to anabelian geometry.

Question. *Can the conjectures in arithmetic LC be fully established remaining solely inside the use of representation theory for adelic objects and Galois groups and class field theory? Or should one use more information about the absolute Galois group of global and local fields, which cannot be reached via representation theory, e.g. such as in anabelian geometry?*

The third important prediction of LC is that the L-factors of zeta functions of regular proper arithmetic schemes coincide with appropriate automorphic L-functions, [46]. The best result in this direction is the theorem of Wiles and its extension for all elliptic curves of $\mathbb{Q}$. See Section 6 below for a 2dAAG approach to the zeta functions.

In positive characteristic there has been much more progress using geometrical tools which are not available in the standard approaches to LC over number fields. The local GL_r case was established by Laumon, Stuhler and Rapoport for $r \geqslant 2$. The global GL_r case was established by Drinfeld for $r = 2$, and by L. Lafforgue for all r (see [40]) using Drinfeld modules and FH-sheaves[9]. Further important work on functoriality by V. Lafforgue, using in particular ideas from the geometric theory, described how to go from

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[9]Named so by G. Harder and D. Kazhdan in [18], F for Frobenius, H for Hecke, instead of the original name shtuki (pl.). One of the main translations of shtuka (sing.) is whatsit, thingummy, gizmo.

the automorphic side to the Galois side in positive characteristic for arbitrary reductive algebraic groups, [43]. To establish functoriality one currently has to go through Galois representations.

L. Lafforgue wrote[10]:

the methods used to construct LC for function fields or pieces of this correspondence (for number fields) are highly non-explicit, as they consist in considering and studying l-adic cohomology spaces of sophisticated algebraic varieties. Galois representations constructed as components of cohomology spaces are not at all objects one can describe.

Remark. Inaba's work on matrix Artin–Schreier theory [27,28] describes all finite Galois extensions of degree a power of p of fields of characteristic p. It may lead to a simplified p-primary part of LC in characteristic p.

The *geometric Langlands correspondence*[11], further away from arithmetical aspects of CFT, was initiated by Deligne, Drinfeld and Laumon and developed by Beilinson–Drinfeld, Gaitsgory, Frenkel, and others. Hitchin systems proved to be very useful for this theory. There are various links between geometric Langlands correspondence and physics such as the paper of Kapustin–Witten. A categorical geometric Langlands correspondence was proposed by Gaitsgory in [16]. While one of the key features of CFT is its functoriality with respect to finite separable base change, both LC and geometric LC currently experience difficulties with its analogues. Basic problems remain unsolved or even non-stated, e.g. functoriality; partial progress uses the theta sheaf constructed by Lysenko, [47–49]. It is still not clear, even conjecturally, what happens when one passes from a curve to its finite étale cover, and the ramified case remains substantially open too. Frenkel [15] discusses possible analytic approached to automorphic fundings on complex algebraic curves; see also [3].

Remark. Apparently, no significant development within LC is parallel to (or can be viewed as a generalisation of or parallel to) GCFT. Thus, *in the arithmetic Langlands correspondence, 50 years after its start, we are still in the 'pre-Takagi' stage, similar to where we were 100 years ago in CFT*.

Problem 2. *Find a version of enhanced arithmetic LC parallel to GCFT.*

Problem 3. *Find a version of enhanced LC parallel to SCFT but which works over all number fields.*

Problem 4. *Find a version of enhanced LC parallel to some of post-cohomological CFT, thus circumventing the problem of using non-abelian cohomology.*

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[10]From L. Lafforgue's email message, Feb 21, 2018.
[11]For the most recent review see [14].

Note that since LC is a linear theory whereas CFT is not, the asked enhancements of LC have to be non-linear. Hopefully, better knowledge and use of GCFT can be helpful for further progress in LC. Langlands emphasised the importance of Hasse's Klassenkörperbericht.[12]

Concerning Problem 3, such a mixture is a property of Mochizuki's IUT theory, a further development of anabelian geometry. This theory is of special type and works over all number fields.

A very different approach to the description of certain non-commutative Galois extensions of function fields of curves over finite fields was started by Ihara, [21]. For example, this theory describes the maximal unramified cover of the projective line over $\mathbb{F}_{p^2}$ minus three points which is at most tamely ramified at the three points and in which some special finite set of points decomposes completely, in terms of subgroups of finite index of the quotient of the subgroup of $SL_2(\mathbb{Z}[1/p])$ congruent to the identity mod 2 by its cyclic subgroup of order 2.

5.2. 2d (and higher) CFTs

Classical Lang's geometric CFT for function fields of varieties over finite fields used commutative algebraic groups over finite fields and their isogenies, [44, 45]. Even though one can deduce some of its main results from higher CFT, see [34], the point of view of geometric CFT on obtaining abelian étale coverings as pullbacks from isogenies of commutative group schemes has a larger geometric content.

CFT for n-dimensional schemes take into account CFT at the $(n-1)$-residue level. Almost all higher CFTs typically use Milnor K-groups. They are of GCFT type. A large number of main properties in higher CFT has been established.

Cohomological higher CFT was developed by Kato (higher local fields), [32, 33], while local-global and global theories were developed by Kato, Sh. Saito, Kato–Sh. Saito, [34, 35], Bloch, Spieß, Jannsen–Sh. Saito. They generalise some of the cohomological approaches to CFT. The local part of Kato's theory works with $H^3(G_F, \mu_m \otimes \mu_m)$ instead of the m-torsion part of the Brauer group. The cohomological 2dCFT is an H^3-theory. Other 1d cohomological approaches cannot be (easily) generalised to 2d. In particular, for a finite Galois extension L/F the homomorphism $K_2(F) \longrightarrow K_2(L)^{G(L/F)}$ is neither injective nor surjective in general, so the 1d class formations approach does not generalise directly.

An explicit higher local CFT in any characteristic, including the most difficult case of mixed characteristic fields, was constructed in [4], it generalised the Neukirch method and worked with several topologies on Milnor K-groups of higher fields and their quotient K^t, making use of a higher local fields version of the Vostokov symbol, [70], and of the Artin–Schreier–Witt pairing. These two pairings are higher analogues of explicit formulas of Part III of CFT and exist independently of CFT.

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[12]In two email messages of Feb 9, 2016 and Feb 24, 2018.

Remark. For the p-primary part of CFT in characteristic p one cannot use Kummer theory. However, the Kawada–Satake method and its generalisations [13], using Witt theory, makes the p-primary part the easiest part of CFT in characteristic p. The nature of existence theorem in positive characteristic is clarified in [13]: it corresponds to topological reflexivity of (generally non-locally compact) groups with respect to a generalised explicit pairing. An explicit approach to higher CFT in positive characteristic by Parshin, [62, 63], is a higher local fields extension of the Kawada–Satake method, [36], several errors and gaps in [62, 63] were fixed in [13].

Perhaps, the shortest way to state the reciprocity map of 2dCFT is probably this: the 2d reciprocity map for a 2d global field K is a continuous homomorphism

$$\Psi_K \colon K_2^t(\mathbf{A}) / \big(K_2^t(\mathbf{B}) + K_2^t(\mathbf{C}) \big) \longrightarrow G_K^{\mathrm{ab}},$$

where $\mathbf{A}$ is 2d geometric adeles associated to a proper regular model of K, $\mathbf{B}$ is its subspace corresponding to divisors and $\mathbf{C}$ corresponds to closed points,

$$K_2^t(\mathbf{A}) = K_2(\mathbf{A}) / \big(\cap_{l \geqslant 1} l K_2(\mathbf{A}) \big)$$

and $K_2^t(B)$, $K_2^t(C)$ are images in $K_2^t(A)$ of $K_2(B)$ and $K_2(C)$, [8, Part II].

All these 2dCFT are GCFT, they do have analogues of Parts II, III of CFT, but not of Part IV due to lack of a satisfactory 2d ramification theory satisfying analogues of each of (a), (b), (c) at the end of Section 3. There are many different approaches to 2d local ramification theory, including those by Kato, Hyodo, Zhukov, Fesenko, Kato–T. Saito, Borger, Abbes–T. Saito, each having merits and disadvantages, and none capable to serve as a comprehensive 2d ramification theory.

Problem 5. *Develop a general ramification theory for surfaces compatible with 2dCFT and taking into careful account ramification theory at the one-dimensional residue level.*

An approach to parts of higher global CFT in any characteristic, which uses the étale fundamental group and does not use Milnor K-groups, was developed by Wiesend, [73, 74]. It analyses which finite compatible covers of closed points and curves on a regular arithmetic scheme correspond to a finite Galois cover of the scheme. This leads to a simplified approach to parts of global higher CFT. Unusually for CFT, this approach does not (yet) have local and local-to-global parts.[13]

Remark. Unlike the classical one-dimensional case, 2dCFT is somehow separated from geometry. The distance between full 2d geometric adeles $\mathbf{A}$ and the function field K is two-step, intermediate local-global adelic objects are $\mathbf{B}$ and $\mathbf{C}$. The Picard group of a proper regular model of K is isomorphic to $\mathbf{B}^\times / (K^\times \cdot \text{units})$.

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[13]Some corrections, extensions and applications of Wiesend's approach were later produced by Schmidt and Kerz, [37, 38].

Remark. No significant development within higher CFT is parallel to (or can be viewed as a generalisation of) SCFT.

Problem 6. *Develop a special higher CFT which uses torsion structures, to provide new insights into 2dCFT.*

We observe some analogy between the role of special and general approaches to CFT and to its generalisations such as LC and higher CFT. In particular, the main achievements of LC in the number field case are of special type.

5.3. Anabelian geometry

The contributors include Neukirch, Iwasawa, Ikeda, Uchida (for 1d fields, it uses the computation of the Brauer group of arithmetic fields); Pop (birational anabelian geometry for finitely generated fields); Belyi; Nakamura, Tamagawa, Mochizuki (hyperbolic curves over finite fields and subfields of local number fields); Stix (positive characteristic).

A very different birational geometry for function fields of varieties of dimension > 2 over $\mathbb{C}$, which uses Milnor K-groups, was pioneered by Bogomolov and developed by Bogomolov–Tschinkel, and contributed to later by Pop and Topaz.

There is no analogue of Parts II, III of CFT in anabelian geometry, but there might be some analogue of Part IV of CFT.

Anabelian geometry is a sort of utmost non-abelian and non-linear theory, working with full topological groups such as the absolute Galois groups and fundamental groups of hyperbolic curves (smooth projective geometrically connected curve whose Euler characteristic is negative). Rigidity of certain Galois and fundamental groups is a key feature of anabelian geometry.

Tamagawa's theorem states that for two non-proper hyperbolic curves C_1, C_2 over a finitely generated field k over $\mathbb{Q}$, the natural morphism from k-isomorphisms of k-schemes C_1 to C_2 to continuous G_k-isomorphisms of their étale fundamental groups modulo inner automorphisms of the étale fundamental group of $C_2 \times_k k^{\mathrm{alg}}$ is bijective, [67]. One of Mochizuki's theorem extends this property to all hyperbolic curves, [51]. Another stronger theorem of Mochizuki states that the natural morphism from dominant k-morphisms of hyperbolic curves C_1, C_2 over a subfield k of a field finitely generated over p-adic numbers to open continuous G_k-homomorphisms of their étale fundamental groups, considered up to composition with an inner automorphism of the étale fundamental group of $C_2 \times_k k^{\mathrm{alg}}$, is bijective, [52, 53].

Mono-anabelian geometry developed by Mochizuki further extends anabelian geometry. It includes strong results on algorithmical reconstruction of an arithmetic object from certain fundamental groups. One of the main theorems of mono-anabelian geometry is Mochizuki's algorithmic reconstruction of k and the function field of a hyperbolic curve of strictly Belyi type over k, with k either a number field or its nonarchimedean completion, from the étale fundamental group of the curve, [54, Part III]. Recall that in general one cannot reconstruct a finite extension of $\mathbb{Q}_p$ from its absolute Galois group, thus the

previous theorem demonstrates that using the next dimension by involving certain hyperbolic curves over the field fundamentally improves the situation.

A fundamental related development is the study of the (profinite) Grothendieck–Teichmüller group (GT) in relation to its subgroup $G_{\mathbb{Q}}$. The previous role of Grothendieck's theory of *dessins d'enfants* and the contributions of Drinfeld, Lochak and Schneps and others are well known. A recent paper of Mochizuki–Hoshi–Minamide [20] shows that GT is, up to S_{n+3}, just the group of outer automorphisms of the étale fundamental group of the n-dimensional configuration space, $n > 1$, associated to a hyperbolic curve of genus 0 with 3 punctures over an algebraically closed field of characteristic zero. This result suggests $G_{\mathbb{Q}}$ as a proper subgroup of GT. One of corollaries, by Minamide and Nakamura, is the description of the profinite Grothendieck–Teichmüller group as the outer automorphism group of the mapping class group of a topological torus with two marked points, [50]. Another recent result by Tsujimura establishes a surjective homomorphism from the (largest) p-adic Grothendieck–Teichmüller group to $G_{\mathbb{Q}_p}$ whose restriction on $G_{\mathbb{Q}_p}$ is the identity map, [69].

Remark. The anabelian geometry viewpoint provides a new understanding of the role of the class formations in axiomatic approaches to CFT. Checking the validity of axioms of CFT requires the use of the ring structure, while the CFT mechanisms, i.e. deducing the reciprocity map and other properties from the axioms of CFT, is purely group theoretical. It is interesting to note that Neukirch's explicit approach to CFT (Section 3), was partially motivated by his experience in anabelian geometry for number fields.

6. Three generalisations of CFT in arithmetic of elliptic curves

In this last section we look at the particular example of an elliptic curve E over a number field k. Choose a proper regular model $\mathcal{E}$ flat over the ring of integers of k. Denote $K = k(E) = k(\mathcal{E})$.

LC: here the main achievement is of Wiles–Taylor et al. for elliptic curves over $\mathbb{Q}$. It is of special type. The associated Galois and l-adic representations of G_k use torsion points of E over a separable closure of k. Current approaches to LC for elliptic curves over number fields work with the generic fibre of $\mathcal{E}$ or with special fibres of $\mathcal{E}$, but rather not with the full arithmetic surface $\mathcal{E}$ as a two-dimensional geometric object.

2dCFT: The 2d reciprocity map

$$\Psi_K \colon K_2^t(\mathbf{A})/(K_2^t(\mathbf{B}) + K_2^t(\mathbf{C})) \longrightarrow G_K^{\mathrm{ab}}$$

is the outcome of two global reciprocity laws with respect to $K_2^t(\mathbf{B})$ and $K_2^t(\mathbf{C})$, associated to curves and to points of $\mathcal{E}$, of 2d adelic product of 2d local reciprocity maps

$$\Psi_F \colon K_2^t(F) \longrightarrow G_F^{\mathrm{ab}},$$

where F is a 2d local field associated to a point on a curve on $\mathcal{E}$. The local reciprocity map sends K_2^t-prime elements such as $\{t_1, t_2\}$, where t_2 is a local parameter of F and t_1 is a lift

to F of a local parameter of its residue field, to an automorphism of the maximal abelian extension of F which when restricted to the maximal prime-to-p-cyclotomic extension of F is equal to its Frobenius automorphism over F. A generalisation of the 1d Neukirch method works for K_2^t-objects, [4].

Anabelian geometry: one can work with various hyperbolic curves associated to E, e.g. $X = E \setminus \{0\}$ over k, and with the homomorphism $\pi_1(X) \to G_k$. From the homomorphism $\pi_1(X) \to G_k$ one can algorithmically recover $k(X)$ and the fields k and completions k_v of k, [54]. It is well known that from G_{k_v} one cannot in general recover k_v.

Anabelian geometry is intensively used in Mochizuki's *IUT = arithmetic deformation theory* and its applications to some of the abc inequalities, and the Szpiro and Vojta conjectures, [55, 56]. It is interesting to observe that similarly to the Neukirch explicit CFT and the Vostokov symbol in explicit formulas for the Hilbert pairing, IUT involves several *indeterminacies* at its crucial stage of multi-radial representation. IUT uses generalised Kummer theory and the computation of the local Brauer group, it does not use anything else from CFT. It works with values of certain nonarchimedean functions (étale theta functions) at torsion points, in this respect it is nearer to SCFT; on the other hand, it works over any number field and in this respect it is nearer to GCFT.[14]

Informally speaking, IUT deals with Galois groups as tangent bundles, see the beginning of Sections 2.6 and 4.3 (ii) of [55]. To a certain degree, global class field theory does kind of the same with abelian Galois groups: abelian Galois groups over a global field correspond to idele classes, while adeles are dual to generalised differential forms.

2dAAG = 2d adelic analysis and geometry studies properties of the zeta function of $\mathcal{E}$ by involving 2d analytic adeles and 2d zeta integrals.[15] Recall that the (Hasse) zeta function of $\mathcal{E}$ is

$$\zeta_{\mathcal{E}}(s) = \prod_x \left(1 - |k(x)|^{-s}\right)^{-1},$$

x runs through all closed points of $\mathcal{E}$. Its conjectural functional equation is

$$c_{\mathcal{E}}^{2-s} \zeta_{\mathcal{E}}(2-s)^2 = c_{\mathcal{E}}^s \zeta_{\mathcal{E}}(s)^2$$

with some positive rational $c_{\mathcal{E}}$ not depending on the archimedean data; note the absence of the Γ-factors. This functional equation is known to hold if $k = \mathbb{Q}$ by the work of Wiles and Taylor and others. Dependent on the generic fibre E of $\mathcal{E}$ only, the (Hasse–Weil) zeta function $\zeta_E(s)$ of E satisfies the equation $\zeta_{\mathcal{E}}(s) = n_{\mathcal{E}}(s)\,\zeta_E(s)$ where the factor $n_{\mathcal{E}}(s)$ is the product of finitely many zeta functions of affine lines over finite fields, corresponding to what happens at singular fibres of $\mathcal{E}$. If $\mathcal{E}$ is the global minimal Weierstraß model of E, then $n_{\mathcal{E}}(s) = 1$. The L-factor (i.e. the Hasse–Weil L-function of E) is the denominator of ζ_E:

$$\zeta_E(s) = \zeta_{\mathbb{P}^1(k)}(s)/L_E(s).$$

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[14]See also Remark 2.3.3 of [56, Part IV] and Section 4.2 of [55].

[15]See Sections I, J, K of https://www.maths.nottingham.ac.uk/personal/ibf/mp.html.

Thus, the conjectural functional equation of L_E is more complicated than that of $\zeta_{\mathcal{E}}$, and its Gamma-factor has its origin due to the 1d Gamma-factor in the functional equation of $\zeta_k(s)\,\zeta_k(s-1)$.

The L-factors of zeta-functions of proper regular arithmetic schemes are like micro/quantum objects, whose non-commutative study requires the use of representation theory, once developed for the needs of quantum mechanics and then finding applications in various parts of mathematics including number theory. Thus, in some sense they are more complicated objects than the zeta functions of proper regular arithmetic schemes which are like macro objects and can be studied without using representation theory, using commutative (but higher dimensional) methods.

Now we consider the role of zeta integrals in the classical setting of the Iwasawa–Tate theory, and in 2dAAG and LC.

Zeta integrals. Classically, i.e. in the Iwasawa–Tate theory, [29–31, 68], a completed zeta function is written as a zeta integral

$$\int_{\mathbb{A}_k^\times} f(\alpha)\,|\alpha|^s\,d\mu_{\mathbb{A}_k^\times}(\alpha)$$

over ideles $\mathbb{A}_k^\times$ of the product of a Bruhat–Schwartz function f and the module function raised to complex power s, against a nontrivial translation invariant measure on the locally compact group of ideles. More generally, one can replace $|\alpha|^s$ with an arbitrary quasi-character χ of $\mathbb{A}_k^\times$ vanishing on $k^\times$. To deduce the functional equation and meromorphic continuation of the zeta integral one uses dualities of additive adeles and objects from appropriate 1d CFT, such as $\mathbb{A}_k^\times$, $\mathbb{A}_k^\times/k^\times$, but not 1d CFT itself.

Recall that some of Part I of CFT for global fields is equivalent to the following property: the L-function of a global field associated to a character of its absolute Galois group, after being appropriately completed with Gamma-factors is equal to

$$\int_{\mathbb{A}_k^\times} f(\alpha)\,\chi(\alpha)\,d\mu_{\mathbb{A}^\times}(\alpha)$$

for an appropriate quasi-character χ of $\mathbb{A}_k^\times$ vanishing on $k^\times$. For one-dimensional objects of number theory, the three zeta/L-functions coincide: scheme theoretic, abelian Galois representation theoretic and abelian automorphic representation theoretic.

The role of appropriately generalised zeta integrals in 2dAAG and LC is fundamental. 2dAAG includes a 2d zeta integral presentation

$$\int_{\mathbb{A}^\times \times \mathbb{A}^\times} f(\alpha)\,\|\alpha\|^s\,d\mu_{\mathbb{A}^\times \times \mathbb{A}^\times}(\alpha)$$

of the square of $\zeta_{\mathcal{E}}(s)$ times appropriately scaled squares of 1d zeta integrals of finitely many horizontal curves on $\mathcal{E}$, where f is a 2d Bruhat–Schwartz function and $\|\ \|^s$ is a twisted module homomorphism, against a 2d translation invariant measure on the multiplicative group of 2d analytic adeles $\mathbb{A}$ (which are fundamentally different from the

2d geometric adeles $\mathbf{A}$ of $\mathcal{E}$ and from the 1d adeles $\mathbb{A}_k$), [8, Part II]. This study has applications to the meromorphic continuation and functional equality of $\zeta_{\mathcal{E}}(s)$ via a 2d theta-formula and resulting boundary term and mean-periodicity conjecture, [8, 66]; to the GRH for $\zeta_{\mathcal{E}}(s)$ via the Laplace–Carleman spectrum of the boundary term, [8, 65]; and to the BSD conjecture via the boundary term and an interaction between the multiplicative groups of 2d geometric and analytic adeles, [8, Part III]. In particular, an interaction between the multiplicative groups of the two adelic structures

$$K_1(\mathbf{A}) \times K_1(\mathbb{A}) \to K_2(\mathbf{A}),$$

originating from explicit *existence theorem* in 2dCFT, an analogue of Part II of CFT, leads to a relation between the analytic and geometric adelic structures that can be viewed as an adelic lift of the well known conjecture about the two ranks of E, [8, Part III]. This seems to be the only general (i.e. working over all global fields) approach to the study of special values of the zeta function $\zeta_{\mathcal{E}}(s)$. All this information about $\zeta_{\mathcal{E}}(s)$ implies the corresponding information about $L_E(s)$. Thus, similar to the property of mono-anabelian geometry mentioned in Section 5.3, the use of 2d objects in 2dAAG allows one to see deeper properties of the zeta-function and L-function of an elliptic curve over a global field. Without using the hard part of the theory of [8, Part II] the mean-periodic conjecture was extended by Fesenko, Ricotta and Suzuki to the zeta functions of all proper regular arithmetic schemes in [11]. It directly tells, without looking at automorphicity of the L-factors, what is an equivalent condition for the zeta function to have a meromorphic extension and functional equation.

For the L-function of an irreducible GL_n-representation of the absolute Galois group G_k of a global field k, its conjectural automorphicity, due to the converse theorems, is closely related to the following conjectural property: its completed L-function and its twists by appropriate characters, after multiplying with appropriate Gamma-factors, is equal to an appropriate zeta integral, for an appropriate $M(n, \mathbb{A}_k)$-Bruhat–Schwartz function f,

$$\int_{GL_n(\mathbb{A}_k)} f(\alpha)\, c(\alpha)\, |\det(\alpha)|^s \, d\mu_{GL_n(\mathbb{A}_k)}(\alpha).$$

The additional factor

$$c(\alpha) = \int_{GL_n(\mathbb{A}_k)^1/GL_n(k)} g_1(\gamma\alpha)\, g_2(\gamma)\, d\mu(\gamma) \quad \text{for } n > 1$$

involves two cuspidal functions g_i. Here, $GL_n(\mathbb{A}_k)^1$ is the preimage of the unit circle with respect to the determinant, and a complex valued cuspidal function g is a smooth (in the adelic sense) function such that the α-variable integral of $g(\alpha\beta)$ over $U_{r,n}(\mathbb{A}_k)/U_{r,n}(k)$ is zero for all $\beta \in GL_n(\mathbb{A}_k)$, where $U_{r,n}$ is the group of block upper-triangle matrices with two diagonal blocks being the identity matrices of order r and $n-r$, $1 \leqslant r \leqslant n-1$. For $n \geqslant 4$ one may need a similar property of zeta integral presentation to hold for the twists of the L-function by appropriate L-functions of irreducible GL_m-representations for all $2 \leqslant m \leqslant n-2$, to use the converse theorems.

It is quite possible that such complex functions should include the Kurokawa–Selberg zeta function of the fundamental group of hyperbolic curves. This zeta function is defined in Section 1 of [39] which proposes a conjectural relation of three types of zeta/L-functions with the Kurokawa–Selberg zeta function.

Here are some relations between the three generalisations of CFT and their further developments:
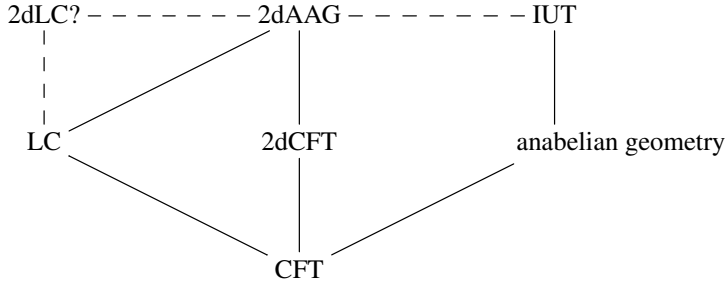
2dLC should exist but not much is known about it. An object, generalising the quotient $GL_2(\mathbb{A})/GL_2(k)$ in 1d case, certain functions on which may serve as 2d automorphic functions of $\mathcal{E}$ is described in the last section of [9]. It is expected that local 2dLC will use higher translation invariant measure and integration on algebraic groups over 2d local fields in [9, 58, 59, 71].

Relations between 2dAAG and IUT include, in addition to those mentioned in [10], the following analogy. IUT uses (after fixing a prime number l) two fundamental symmetries: additive geometric and multiplicative arithmetic. 2dAAG uses adelic structures on arithmetic surfaces: geometric adeles (their additive structure is related to Zariski cohomology and the intersection pairing, their K_2-structure is important for 2dCFT) and multiplicative analytic adeles $\mathbb{A}^\times$ (used to study the 2d zeta integral, zero cycles).

As for potential relations between IUT and LC, one of the key activities in anabelian geometry is the restoration of ring structure, i.e. the second operation of addition, when the multiplicative structure is already known. It is interesting to compare with [41] which reformulates the functoriality in LC as a problem to find for an arbitrary reductive algebraic group an analogue of the relation between the group GL_n and the ring of square matrices of order n.

Problem 7. *Find more direct relations between the generalisations of CFT. Use them to produce a single unified generalisation of CFT.*[16]

Such relations, when found, may lead to new approaches to LC which are parallel to general CFT, or, at least have some mixture of features of special CFT but work over all number fields, as they do in IUT. They may even lead to a unified powerful generalisation of CFT which specialises to its three generalisations.

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[16]Stated as the result of e-communication with V. Shokurov.

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Ivan Fesenko
School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK;
ivan.b.fesenko@gmail.com