Dynamic Innovation and Pricing Decisions in a Supply-Chain

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Abstract

This paper studies dynamic innovation and pricing decisions in a two-echelon supply chain. We model a distribution channel where a seller sells a product to an independent buyer who ultimately sells it to the customers. We refer to innovation as efforts made on the product quality improvement, or on process improvement. Both the players can put innovation efforts over time which in turn may enhance the goodwill of the product in market. The product demand increases with goodwill and decreases with the retail price. The innovation efforts can also impact the unit processing cost of the product at the upstream firm's end positively or negatively. We model the problem as a Stackelberg differential game in which the seller first announces its wholesale price and innovation efforts over time and the buyer responds by deciding the retail price and its innovation efforts over time. We obtain feedback equilibrium strategies for a central decision maker in centralized channel, and for both the players in a decentralized channel. We also obtain several useful managerial insights using analytical as well as numerical means.

Key words: Product innovation; process innovation; quality improvement; Stackelberg differential game; feedback equilibrium; pricing.

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1 Introduction

Innovation and continuous improvement have long been regarded as powerful engine behind economic growth. Their role as drivers of competitiveness along with the accelerating pace of technological progress have made firms to concentrate on innovation activities. In today's changing global environment, technologies often go beyond the capability of an individual firm. Instead of looking at innovation activities from an internal perspective alone, an emerging viewpoint and approach is to take a holistic and a supply-chain view of product and process innovation. Facing intensified global competition, firms are increasingly engaged in collaborating with various partners, ranging from universities, research institutes and enterprises to even their suppliers and customers (Belderbos et al. 2004, Wu 2014) to constantly innovate and improve their products and processes. Collaboration usually has multiple compelling advantages in contemporary R&D activities. The collaboration in innovation helps partner firms to share costs and risks, gain new skills, crack business markets, and even control competitive forces (Tyler and Steensma (1995), Veugelers (1998)). Both empirical and theoretical studies consistently suggest the positive effect of R&D collaboration on improving firm performance, see, for e.g. Cassiman and Veugelers (2002), Hagedoorn (2002), Hagedoorn and Duysters (2002), Miotti and Sachwald (2003), and Becker and Dietz (2004). In general, innovation can be characterized as cost reducing (Amir et al. 2003) as well as demand enhancing (Gilbert and Cvsa 2003). Effective R&D collaboration should contribute to the firm via attributes of products such as quality, performance, and goodwill; which in turn are important factors that influence customers' demands (Karlsson et al. 2004, Ragatz et al. 2002).

Collaboration between two firms in innovation and R&D activities can be primarily divided into two categories in the literature, i.e., horizontal and vertical. Several empirical and theoretical studies have addressed key issues in the economics of horizontal R&D collaboration. D'Aspremont and Jacquemin (1988) developed a pioneering two-stage game model for R&D cooperation mixing with competition for a duopoly with information spillovers. Several papers have built on this framework and have modelled horizontal R&D collaboration. Some key examples include Kamien et al. (1992), Amir (2003), Suetens (2004), Ge and Hu (2008), Cellini and Lambertini (2009) etc. However, the focus of our work is on vertical interactions between two firms in innovation and improvement efforts.

Firms may utilize vertical collaboration programs to improve the quality of their products as well as their processes. Vertical R&D cooperation in this context, can be viewed as a cost and risk sharing, and/or a sales promoting mechanism for the firm to incentivize its suppliers or buyers. In recent times, there have been several examples of vertical collaboration in innovation activities between firms. Airbus for e.g., has been collaborating on development and production of carbon and fibreglass materials with its vendor SGL carbon, which is one of the world's leading manufacturers of carbon and graphite materials, carbon fibres, and composites (Tyrell (2019)). SGL had also been in a similar partnership with BMW for carbon fibre reinforced plastic parts and components used in the passenger compartment of BMW's i series (PricwaterhouseCoopers innovation survey report [1]). BMW and SGL formed a joint venture for this purpose, however, more recently in 2017, BMW withdrew from this joint-venture and sold its minority stake to SGL (Nica (2017)). In 2010 Tesla motors and its lithium-ion battery cells supplier Panasonic announced their collaboration for development of battery-cells for Tesla's electric vehicles (Tesla Inc., 2010). Their collaboration eventually lead to a new level with their joint investment in 2014 in the Gigafactory project for development and production of battery cells. However, more recently, their Gigafactory collaboration has hit roadblocks with reports of Tesla working on its own battery-cell R&D to reduce its dependence on Panasonic (Kolodny (2019)). In the case of bike-sharing economy in China, Zhou et al. (2019) cite an example of large bicycle companies involved in cooperation with firms in their industry chain, in particular, they mention the example of Phoenix bicycles (bicycle manufacturer) and ofo (Beijing based bicycle sharing company) signing an agreement to collaborate on bicycle development and improvement of user experience.

In this paper we model a two-echelon supply chain in which an upstream firm manufactures a product and sells it to a downstream firm who eventually sells it to the end consumers. Both the firms can put innovation efforts to improve the quality of the product and/or of the manufacturing process of the upstream firm. Consider, for example, a collaboration between a smart-phone manufacturer and supplier of its camera. The demand for such a product may depend on the improvements made in the design quality (better resolution, more options etc.), as well as on the conformance quality (fewer breakdowns) of the camera with the supplier. This may make this product (smart-phone) more appealing to the customers, thereby increasing its demand. The two firms can collaborate to improve the quality of the camera itself that is manufactured by the upstream firm, or to improve the production technology and process at the upstream firm. Both the firms can contribute by putting efforts in innovation activities over time and will incur cost of their respective efforts over time. In our context, effort could mean a commitment of resources such as for e.g., skilled labour-hours or infrastructure in innovation activities. In the smart-phone example, it could mean that the camera supplier as well as the phone manufacturer dedicate skilled R&D personnel to improve the quality of the camera or that of its manufacturing process.

[1] https://www.pwc.com/gx/en/automotive/industry-publications-and-thought-leadership/assets/pwc-highway-to-growth-strategies-for-automotive-innovation.pdf

One relevant real world example that the authors can relate to through their personal experience is that of a major auto firm in India. This auto manufacturer committed its expert designers and engineers to assist one of its gear parts suppliers in a joint effort to address design and conformance quality issues in some critical gearbox components. Such innovation activities could have an impact on the unit production cost as well at the supplier's end.

With these motivating examples, some of the key research questions that we look to explore are as follows. First, we attempt to understand the incentives for both the firms to commit to such innovation efforts over time and answer a key question for both the firms, i.e., what should be their innovation effort policy over time. More specifically, how should each firm's effort depend upon an observable state (such as perceived quality of the phone or brand goodwill) that evolves with time. We have mentioned above the example of SGL Carbon and BMW's partnership where BMW entered into a collaboration and then later withdrew from the joint venture; and similarly that of Panasonic-Tesla collaboration where Tesla's recent moves indicate towards scaling down of R&D collaboration and eventually focussing more on R&D at its own end. These examples indicate that two firms in a supply chain can have different incentives for collaboration and those can evolve with time, and in our paper we attempt to understand these. In fact some of our results do indicate that the two firms' optimal efforts can have very different dependence on product goodwill, and consequently, can have very different trajectories over time. Second, to understand how various exogenous factors such as: the base unit cost of manufacturing, the value of the product, market factors such as demand sensitivity to price, etc. can have an impact on optimal strategies of two firms. One of the interesting aspects that we investigate is how the incentives for the two firms differ when the impact of innovation efforts are cost reducing (i.e., improve production efficiency) vs when they increase the unit production cost. Finally, we look to gain insights on whether the supply chain structure can also have an impact on the overall innovation efforts. Thus, for e.g., if the phone manufacturer starts manufacturing camera as well instead of purchasing it, what will be that single centralized firm's overall innovation efforts policy with time? Our paper attempts to answer these questions and attempts to bring useful insights for decision makers.

The rest of this paper is organized as follows. In the next section we discuss the background literature and highlight our contribution to the existing literature and knowledge. We describe the model in Section 3. In Section 4, we consider the model with prices as exogenous variables we and analyse it to obtain optimal innovation efforts strategies for a centralized as well as a decentralized channel in feedback form. We obtain several useful insights through analytical means. In Section 5, we solve the model with pricing decisions and obtain optimal feedback pricing and innovation efforts policies. We also perform numerical analysis to obtain several managerial insights. We finally conclude the paper in Section 6. For ease of reading, all the major proofs of our results as well as all the figures from numerical analysis pertaining to Section 5 are relegated to the Appendix.

2 Background Literature

More and more firms prefer to utilize cooperation with their upstream or downstream partner in R&D activities to maximize their payoffs (Harabi 2002). Geroski (1992) argues that compared to horizontal cooperation, which may lead to collusion, the vertical may perform better. Similar ideas are discussed in Riggs and Von Hippel (1994), and Harabi (1998). Arranza and Fdez de Arroyabe (2008) find that vertical cooperation happens more frequently than the horizontal in practice. Results from a study by Tsai and Hsieh (2009) indicate that innovation efforts involving downstream partner can help the upstream member in gaining accurate information on customer requirements. In comparison, Bendoly et al. (2012) suggest that R&D cooperation with the upstream partner makes firms benefit from high quality raw materials and cost reducing mechanisms. Tomlison and Fai (2013) conducted an extensive empirical study on SME's in UK and find that in SMEs, horizontal cooperation has no significant impact upon innovation. They argue that innovation activities benefit from good, close dyadic relations within the supply chain. While there have been a number of qualitative and empirical studies on the issue of vertical collaboration, the modelling and optimization of vertical interactions in innovation and improvement efforts in a supply chain, particularly in a dynamic environment, hasn't been sufficiently addressed in our opinion. Below, we discuss some models that are relevant to this work.

Ishii (2004) investigate the effects of cooperative R&D in a supply chain consisting of two upstream and two downstream members, with horizontal and vertical spillovers. Bhaskaran and Krishnan (2009) consider a collaborative model for joint development of products involving two firms with different development capabilities. Balachandran et al. (2013) consider make or buy decisions of a contractor in which the contractor invests on the innovation of a new product and decides on whether to subcontract the production to a subcontractor. The contractor can also manufacture in-house but at lower efficiency and the subcontractor can misappropriate innovation related key information. In their model the contractor determines investment on product innovation and the subcontractor makes investment on production process which can reduce the fixed cost of production. Ge et al. (2014) model different modes of vertical inter-firm R&D cooperation in a supply chain with knowledge spillovers, where two firms cooperate in R&D investments and decide the production quantity according to a wholesale price contract. In the case of Ishii (2004) and Ge et al. (2014), innovation is cost-reducing which does not impact product demand, whereas in Bhaskaran (2009) innovation is product focussed which has no impact on marginal cost. Ishii (2004) and Bhaskaran (2009) do not consider pricing decisions. More recently, Wang and Shin (2015) consider a supply chain where the upstream supplier invests in innovation and the downstream manufacturer sells the product to consumers. They consider different types of contracts. In their case, innovation is product focussed and they do not consider innovation effort by the downstream party. Similarly Song et al. (2017) consider a supply chain in which only the upstream firm invests in innovation and therefore they do not account for collaboration. Chen et al (2019) consider a model of innovation for sustainability in a supply chain where a manufacturer and a retailer can share the cost of innovation and the government can provide support for innovation for sustainability through subsidy or a tax credit. In their model it is only the manufacturer who puts effort in innovation and the retailer can share its cost. The manufacturer's per unit production cost can increase with the innovation efforts. Zhou et al. (2020) extend the problem in Ge et al. (2014) to consider uncertain technology efficiency where the cost reducing impact of innovation efforts is uncertain in nature. Yu et al. (2021) take motivation from electric vehicle market and model R&D collaboration for green technologies in a two echelon supply chain consisting of an upstream manufacturer (say for e.g. a battery manufacturer) and a downstream firm termed as a 'marketer' (for e.g. a car manufacturer). They study different contract settings, viz., vertical R&D collaboration in which only the manufacturer puts innovation efforts, and a co-development setting in which both firms can put R&D efforts. All the above models however, are static in nature and consider one time decisions by the decision makers.

A relevant question for background literature pertaining to our study is to investigate how product and/or process quality improvement is modelled in dynamic settings, and how such decisions are made in conjunction with other factors such as pricing? One of the few early works in this area include Kouvelis and Mukhopadhyay (1995), who formulate an optimal control model with design quality level and price as control variables to maximize total discounted profit over the life cycle of the product. More recent examples include Vörös (2006, 2013), Fruchter (2009), Lambertini and Mantovani (2009), Chenavaz (2012), Liu et al. (2015), Duarte et al. (2016), and Pan and Li (2016). However, all these models solve a single firm's problem and do not consider a supply-chain perspective. Nair and Narsimhan (2006) develop a Nash differential game to obtain open-loop policies for optimal quality and advertising investment of two competing firms. Some papers which do study product/process quality improvement in a supply chain perspective are mentioned below. El Ouardighi et al. (2008) formulate a differential game model and study optimal operational and marketing strategies of the members of a supply chain for both decentralized and centralized management in a dynamic setting. De Giovanni (2011) consider a single manufacturer - single buyer channel in which the retailer controls the advertising efforts while the manufacturer controls the quality improvements and wholesale price. Both advertising and quality contribute to the build-up of goodwill and the demand depends on price and goodwill. Gurnani and Erkoc (2008) consider a channel in which the manufacturer puts a quality improvements effort and the retailer puts a demand enhancing effort, and the demand depends on both. In their model however, the retailer does not collaborate in quality improvement and the quality effort does not impact the unit production cost. Furthermore, their model in static in nature.

We model our problem as a Stackelberg differential game with the seller as the leader and the buyer as the follower. We first focus only on innovation effort decisions by considering prices as exogenous decision variables, and later also include dynamic pricing decisions for the two firms. We would like to emphasize that in our context innovation refers to the efforts made on the quality of the product, and/or on the quality of the manufacturing/service process. Such efforts may lead to one or more of the following desirable outcomes: better design quality of the product; better conformance quality; more streamlined/efficient/in control processes that may generate fewer failures; etc. Ultimately, these efforts may lead to an increased customer satisfaction, higher perception of quality, greater brand goodwill, and increased demand. We use the Nerlove-Arrow (1962) framework to model the dynamics of demand. We assume that the innovation efforts and the retail pricing strategies impact the goodwill of the product. The product demand is dynamic in nature and depends on the goodwill and the retail price at any time. The demand increases with goodwill and decreases with the retail price. To capture the marginally diminishing returns of investments in innovation efforts, we model the immediate cost of innovation efforts as square of a firm's effort. In addition to the immediate cost of such efforts (R&D staff, infrastructure etc.), such efforts may also have an impact on the unit production cost of the product. It may lead to an increase in unit cost due to a superior design and additional features (for e.g. a sharper lens in the camera leading to a higher cost of phone); or even a decrease in the unit cost due to an improved process quality (for e.g. reduction in failure costs and reworks). We incorporate the long term dynamic innovation and the pricing strategies of the two firms as decision variables, and thereby, attempt to cover the strategic and operational, as well as the marketing aspects of a supply chain. In supply chain literature, there is a strong emphasis on pricing decisions and integration of pricing with other operational and marketing decisions. See for e.g., Dolgui and Proth (2010) on dynamic pricing and relevant literature in the supply chain. While there may be circumstances where the firm may face price as an exogenous variable given market conditions or may have to take strategic and operational decisions independent of pricing, a complete holistic approach will involve integrating pricing policy along with innovation policy decisions. In marketing and supply chain literature it is well known that price can play an important role in customers' perception of product quality or goodwill. Moreover, since innovation efforts incur cost and may impact unit production cost as well, a firm would like to use pricing as an important lever along with innovation decisions to not only manage its margins but also to influence consumer demand with an objective of maximizing its profits. Finally, a combined view of pricing along with innovation decisions can help a firm in strategic positioning of its product in the market, for e.g. a high-innovation high-price product vs a low-innovation low-price product. All these factors highlight the importance of making innovation decisions in conjunction with pricing decisions.

We therefore contribute to the existing literature in the following aspects. While most of the studies on R&D collaboration focus on horizontal interactions, relatively few model vertical relationships, and even fewer integrate it with pricing decisions. Vertical R&D collaboration models in the literature (for e.g. Ishi (2004), Bhaskaran and Krishnan (2009), Ge et al.(2014), Wang and Shin (2015)) are static in nature. As Dawid et al. (2013) also argue, in order to capture key implications of R&D interactions in industry settings like the ones discussed above, a dynamic model is needed. Furthermore, as highlighted previously, there are additional differences between these paper and our study in terms of aspects such as: inclusion of pricing decisions, accomodation of both product and process innovation, and participation of multiple members of value chain in innovation activities. Among the stream of dynamic models which study product/process innovation (for e.g. Vörös (2006, 2013), Fruchter (2009), Chenavaz (2012), Liu et al. (2015), Pan and Li (2016) etc.), these decisions are made from the point of view of a single firm. Thus, this paper can also be looked as a supply-chain extension of papers such as Fruchter (2009) and Chenavaz (2012). Two papers which study dynamic innovation in a supply chain perspective are El Ouardighi et al. (2008) and De Giovanni (2011). However there are some key differences between these two papers and our model. In both these papers, it is only the manufacturer who puts a quality improvement effort, whereas in our model, both the players can put efforts aimed at improving product/process quality at upstream firm's end. In their case, the impact of quality improvement effort is a quadratic cost on the manufacturer's end only. However, in our case, apart from a quadratic cost of innovation for both firms, we allow for a scenario where the efforts of two firms may also impact the unit production cost positively or negatively. Thus in our model, the innovation effort may not be limited to product quality, but it may also aim at process quality. We allow pricing as a decision variable for both the parties, whereas in El Ouardighi et al. (2008) only the downstream firm makes the retail pricing decisions and in De Giovanni (2011) only the upstream firm makes wholesale price decision with the retail price decided based on a fixed endogenous margin. Therefore, in that respect, to the best of our knowledge, ours is the first paper looking at pricing and innovation decisions in a dynamic supply chain, where: (i) both the product as well as process improvement efforts are allowed for, and; (ii) both the members of the chain can put efforts on the product/process. Numerical analysis of our model also yields some very interesting insights which may impact firms' strategies on the brand positioning in terms of quality. These results suggest differences in the level of innovation and quality positioning in a centralized vs a decentralized supply chain. In our opinion, such insights add a new dimension to the call for supply chain collaboration in innovation by researchers such as Tomlison and Fai (2013). And finally, despite the computational complexity of our model, we are able to obtain a feedback Stackelberg solution. It is widely understood in game theory literature that feedback strategies are generally more difficult to obtain and are subgame-perfect as opposed to open-loop strategies (see for e.g. Cachon and Netessine (2004) and He et al. (2007)). We therefore, also hope to contribute to the wider theoretical literature on differential games in supply chain management and management science, by presenting an example of explicit feedback strategies.

Our findings add useful insights to the existing knowledge, particularly regarding the modelling aspects which we believe have not been sufficiently addressed in the literature. One of our key contributions is that we obtain several insights on the incentives of both a supplier as well as a buyer in a supply chain on investing in product and process innovation at the supplier's end over time. As highlighted earlier, except Ishii (2004), Ge et al. (2014), and Yu et al. (2021), all papers (static or dynamic) consider either a single firm's problem, or a supply chain scenario where only one firm (mostly upstream) invests in product/process improvement. The focus of Ishii (2004), Ge et al. (2014), and Yu et al. (2021) however is on comparing different modes of cooperation and contracts in innovation (such as for e.g., vertical R&D cartels vs horizontal, or R&D joint ventures) in a static setting, and not so much on studying similar or contrasting incentives for the two firms over time. We find that the two firms may have different, in fact sometimes diverging, incentives. This is reflected in how their optimal efforts evolve w.r.t. product goodwill and hence w.r.t. time. We also note that the innovation efforts policies for the two players could be different depending on whether pricing decisions are also part of decision making or not. The discussion below serves an example of this. In terms of the modelling and solution approach, one paper that is somewhat similar to us is De Giovanni (2011), which models a dynamic supply chain where only the upstream firm invests in improvement efforts. They find that the upstream player's effort are increasing in goodwill. In our paper, we find that the supplier's effort are increasing in goodwill when the innovation efforts lead to savings in unit costs, but can be decreasing in goodwill when they lead to increment in unit costs. As far as the buyer's efforts are concerned, in the case when prices are also decision variables, our numerical analysis indicates that they are relatively insensitive to changes in goodwill. When the prices are exogenous, we find that the buyer's efforts are constant and independent of goodwill if the innovation efforts are either cost increasing or do not lead to a significant decrease in unit costs. However, when the reduction in unit costs due to innovation efforts is high enough, the buyer's efforts are decreasing in goodwill under exogenous prices. We thus find cases where the efforts by two players evolve very differently (sometimes opposite) with goodwill, and hence with time. More such interesting findings are discussed in appropriate places. We discuss the impacts of some modelling parameters which we believe have not been discussed in the literature so far, such as for e.g., the impact of changes in the base cost of manufacturing, and the sensitivity of innovation efforts on unit costs.

3 The Model

We consider a dynamic supply chain consisting of a single seller (referred to with the subscript s), and a single buyer (referred to with the subscript b). The seller sells a product to the buyer who ultimately sells it to the final consumers. Both of them need to decide on their own pricing strategies and their respective innovation efforts strategy over time t. Specifically, player s decides on the wholesale price $w(t)$ as well as its innovation effort $I_s(t)$ at time $t \geq 0$, and player b decides on the retail price $p(t)$ as well as its innovation effort $I_b(t)$ at time $t \geq 0$. Their innovation efforts are aimed at product/process improvement at the seller's end. Before presenting our model in further detail, we introduce some key notations as follows:

t	Time $t, t \in [0, \infty)$;
i	Index of players, $i = s$ (for seller), b (for buyer);
$I_i(t) \geq 0$	Innovation effort by player i at time t;
$I(t) \geq 0$	Firm's innovation effort at time t in a centralized channel;
$r(t) \geq 0$	Goodwill of the product at time t;
$w(t) \geq 0$	Unit wholesale price at time t;
$p(t) \geq 0$	Unit retail price at time t;
$c(t)$	Unit production cost at time t;
C_0	Base unit production cost;
$D(t) \geq 0$	Product demand at time t;

$\delta > 0$	Innovation effectiveness parameter;
$\gamma_p > 0$	Pricing effectiveness parameter on goodwill;
$\gamma_r > 0$	Decay parameter of goodwill;
$\alpha > 0$	Base market size constant;
$\beta_p, \beta_r \in (0, \infty)$	Price and goodwill effectiveness parameters on product demand, respectively;
μ	Impact parameter of innovation efforts on unit production cost;
$\phi > 0$	Discount rate;
V^i	Value functions for player i;
$Sign\,[X]$	Indicates the sign of X, i.e., -ve or +ve.

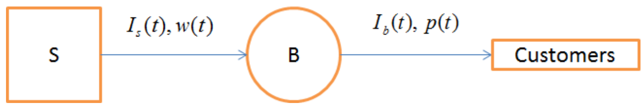

Figure 1: Model structure

Fig 1 depicts the present model structure. The sequence of events is as follows. The seller acts as a leader, first announces its wholesale price and innovation efforts over time, i.e, $w(t)$, and $I_s(t)$, respectively. Given the seller's policies, the buyer, in response decides its retail price $p(t \mid I_s(t), w(t))$ and innovation effort $I_b(t \mid I_s(t), w(t))$ over time. The buyer then sells the product to the customers, with the demand $D(t)$ being dynamic in nature and as described below. We assume that there is no information delay between the buyer and the seller and no inefficiency due to the information asymmetry in a supply chain. For ease of notation and presentation, we also refer to the buyer's decisions as simply $p(t)$ and $I_b(t)$.

We formulate our model as optimal control problems for both the players. To model the dynamics of the problem and the consumer demand, we derive motivation from the classical Nerlove-Arrow framework (1962). The state of the system $r(t)$ is defined as the goodwill of the product in the market at time t. We use the following dynamics as our state equation to define the evolution of state over time:

$$\dot{r}(t) = \frac{dr(t)}{dt} = \gamma_p p(t) - \gamma_r r(t) + \delta(I_s(t) + I_b(t)), \quad r(0) = r_0 \geq 0. \tag{1}$$

The first term $\gamma_p p(t)$ on the right side represents the direct impact of retail price on the goodwill, and the second term reflects the rate at which the goodwill decays over time, represented by $\gamma_r r(t)$. The last term $\delta(I_s(t)+I_b(t))$ reflects the effect of innovation efforts of the seller and the buyer on the goodwill of the product. One way to interpret the goodwill in our context is the perceived quality of the product, i.e., the quality of the product as perceived by the customers. In this regard, the customer's perception of quality increases as more efforts into innovation are made. It is also sometimes common for the consumers to associate a higher price tag with a higher perceived quality, and hence the first term in our dynamics. In marketing there are several studies which point out that customer perception of quality is often a function of its price (see for e.g. Zeithaml (1988), and more recently Yang et al. (2019)). With the perceived quality as a form of goodwill, we can closely relate our dynamics to Fruchter (2009). The demand is expressed as a linear function of price and goodwill. We assume that the demand decreases linearly in price and increases with the goodwill. Such formulation of demand as a function of price and goodwill (perceived quality) is very common in literature with the Nerlove-Arrow framework, see, for e.g., Fruchter (2009), De Giovanni (2011), and Pan and Li (2016). We write the dynamic demand as:

$$D(t) = \alpha - \beta_p p(t) + \beta_r r(t). \tag{2}$$

This formulation is derived from the classic linear demand function $D(t) = \alpha - \beta_p p(t)$, where demand is a function of retail price only. The constant $\alpha > 0$ is initial market potential size. In (2), the parameters β_p, and β_r, represent the direct effects of price, and goodwill, respectively, on the demand.

We assume that unit production cost of manufacturing at the seller's end is C_0 in the absence of any innovation efforts. The innovation efforts may impact the unit cost of production and we assume that this effect is linear in the innovation efforts of the two parties. As discussed in some existing works in literature such as Amir et al. (2003) and Ge et al. (2014), a firm's innovation effort could be an effective production cost reducing mechanism. It is possible that improvements in the product or even the process of manufacturing may make the production more efficient and may reduce the cost of production. On the other hand, if innovation results in new features of the product, it is possible that incorporating those features may lead to a higher cost of production. This increment in unit cost could be due to different raw materials needed to provide new design features and/or due to changes in the manufacturing process to incorporate these. In this paper, we consider the scenario when the two firms decide to collaborate in R&D where each firm commits to a specific amount of effort (i.e., I_b and I_s). We assume that the overall impact of the innovation and improvement activities on the unit cost (as well as on the demand) is a function of total overall effort, i.e., $I_b + I_s$. An example of such a scenario could be one where the two firms form a joint R&D venture or a joint improvement team and may independently commit to resources such as manpower over a period of time. This is somewhat similar to an R&D joint venture discussed in Ge et al. (2014) where they also assume that overall impact of innovation is a function of the sum of individual efforts. Thus, we write the unit cost of production $c(t)$ at the seller's end as:

$$c(t) = C_0 - \mu(I_s(t) + I_b(t)), \tag{3}$$

where the parameter μ is a constant and could take positive or negative values. Following a common assumption of marginally diminishing returns for R&D expenditures (or quality improvement expenditure) in the literature (see for e.g., D'Aspremont and Jacquemin (1988), De Giovanni (2011), Ge et al. (2014)), we assume that the costs of innovation is quadratic in the innovation effort. Thus, the seller's and buyer's innovation expenditures at time $t \geq 0$ are given by $I_s^2(t)$ and $I_b^2(t)$, respectively.

We now define the dynamic optimization problems of the seller and the buyer. Both the channel members obtain their optimal pricing and innovation decisions over time by solving an optimization problem to maximize the present value of their respective profit stream over the infinite horizon. The seller's optimal control problem can be written as

$$V^s(r) = \max_{I_s(t)\geq 0,\, w(t)\geq 0,\, t\geq 0} \left\{ \int_0^\infty e^{-\phi t} \left[(w(t) - c(t))D(t) - I_s^2(t) \right] dt \right\}, \tag{4}$$

subject to the state dynamics in equation (1), where $\phi > 0$ is the discount rate. Thus, for the seller, $I_s(t)$ and $w(t)$ are control variables and $r(t)$ is the state variable. Similarly, given the seller's decisions, the buyer's optimal control problem can be written as

$$V^b(r) = \max_{I_b(t)\geq 0,\, p(t)\geq 0,\, t\geq 0} \left\{ \int_0^\infty e^{-\phi t} \left[(p(t) - w(t))D(t) - I_b^2(t) \right] dt \right\}, \tag{5}$$

subject to (1). Due to the high analytical complexity of our model, to ensure the analytical tractability and the feasibility of a closed form solution, we make certain assumptions on the model parameters. These are discussed at respective appropriate places in the main text or in the Appendix. Finally, as discussed earlier, we obtain optimal policies in feedback form, i.e., the optimal values of decision variables are obtained as functions of the state variable, i.e., goodwill (r). This implies that when prices are also decision variables, the supplier's equilibrium innovation effort and the wholesale price at time t, $t \geq 0$ are $I_s(r(t))$, and $w(r(t))$, respectively. Similarly, in response to these decisions by the seller, the buyer's optimal innovation efforts and retail price at time t are $I_b(r(t) \mid I_s(r(t)), w(r(t)))$, and $I_b(r(t) \mid I_s(r(t)), w(r(t)))$, respectively. We will refer to these optimal decisions simply as $I_s(r), w(r), I_b(r)$, and $p(r)$. In the next section, however, we focus on the case when the prices are exogenous in nature and focus only on the optimal innovation efforts policies of the two firms.

4 Model without pricing decisions

We first consider a model where the wholesale and retail prices are exogenous in nature and thus the two players determine their optimal innovation efforts over time in feedback form. We first solve a centralized decision maker's problem and then that of a decentralized channel.

4.1 Centralized channel

In this sub-section, we consider the problem of a single integrated firm which manufactures the product and then sells to the end customers. It's objective is to determine its optimal strategy of innovation efforts over time. The optimal control problem for the centralized decision maker is

$$V(r) = \max_{I(t) \geq 0} \left\{ \int_0^\infty e^{-\phi t} \left[(p(t) - c(t))D(t) - I^2(t) \right] dt \right\},$$

subject to the state dynamics

$$\dot{r}(t) = \gamma_p p(t) - \gamma_r r(t) + \delta I(t), \qquad r(0) = r_0 \geq 0, \tag{6}$$

where $V(r)$ is the so-called value function of the firm. We modify equation (3) and write the dependence of unit cost of manufcturing on innovation efforts as $c(t) = C_0 - \mu I(t)$.

The Hamilton-Jacobi-Bellman (HJB) equation for the centralized decision maker's value function can now be written as

$$\phi V(r) = \max_I \left[(p + \mu I - C_0)(\alpha - p\beta_p + r\beta_r) - I^2 + V_r(p\gamma_p - r\gamma_r + I\delta) \right], \tag{7}$$

where $V_r = \frac{\partial V(r)}{\partial r}$, subject to the boundary condition on the value function

$$\lim_{t \to \infty} e^{-\phi t} V(r(t)) = 0. \tag{8}$$

To obtain the optimal innovation effort policy, we apply the first-order condition (f.o.c.) w.r.t. $I(r)$ in (7), and get

$$I(r) = \frac{\mu(\alpha - p\beta_p + r\beta_r) + V_r\delta}{2}. \tag{9}$$

It can be seen that the second order condition (s.o.c.) is also satisfied as we get $\frac{\partial^2 V(r)}{\partial I_i^2} = -\frac{2}{\phi} < 0$. We then substitute (9) in (7), and re-write (7) as

$$\begin{aligned} 4\phi V(r) = 4p\alpha - 4p^2\beta_p + 4pr\beta_r - 4C_0(\alpha - p\beta_p + r\beta_r) + 4pV_r\gamma_p - 4rV_r\gamma_r + V_r^2\delta^2 \\ + 2V_r(\alpha - p\beta_p + r\beta_r)\delta\mu + (\alpha - p\beta_p + r\beta_r)^2\mu^2. \end{aligned} \tag{10}$$

Equation (10) above is a partial differential equation (p.d.e.), the solution to which gives us the value function of the firm $V(r)$, and consequently the optimal policy of the firm using (9). After some observation, one can see that a value function that is quadratic in state will satisfy the p.d.e. in (10), and therefore, we conjecture a value function of the following form.

$$V(r) = a_2r^2 + a_1r + a_0, \tag{11}$$

where a_2, a_1, and a_0 are constants whose values depend on model parameters. With the above form of value function in (11), we clearly have

$$V_r = 2a_2r + a_1. \tag{12}$$

Using (12), the optimal policy in (9) can be rewritten as

$$I(r) = \left[a_2\delta + \frac{\beta_r\mu}{2}\right] * r + \frac{1}{2}(a_1\delta + (\alpha - \beta_p p)\mu). \tag{13}$$

We then use (11) to rewrite the left-hand-side (LHS) of (10), and use (12) to rewrite the right-hand-side (RHS) of (10); and then compare the coefficients of r^2, r, and the constant term on the LHS and the RHS of equation (10). As a result, we get the following non-liner algebraic equations in the coefficients a_2, a_1, and a_0.

$$a_2 = \frac{-8a_2\gamma_r + (\mu\beta_r + 2a_2\delta)^2}{4\phi} \tag{14}$$

$$a_1 = \frac{-2C_0\beta_r - 2a_1\gamma_r + (\mu\alpha + a_1\delta)(\mu\beta_r + 2a_2\delta) + p((2 - \mu^2\beta_p)\beta_r + 4a_2\gamma_p - 2a_2\mu\beta_p\delta)}{2\phi} \tag{15}$$

$$a_0 = \frac{p^2\beta_p(\mu^2\beta_p - 4) - 4C_0(\alpha - p\beta_p) + (\mu\alpha + a_1\delta)^2 - 2p(\alpha(\mu^2\beta_p - 2) - 2a_1\gamma_p + a_1\mu\beta_p\delta) + 4a_2\sigma^2}{4\phi}. \tag{16}$$

Equations (14)-(16) essentially characterize the optimal feedback innovation policy of the firm. It means that a solution of (14)-(16) in the coefficients a_2, a_1, and a_0 will give us the optimal decision using (13), and the value function of the firm using (11). It is to be noted that solving (14)-(16) gives two solutions for the coefficients. We eliminate a solution that does not satisfy the boundary condition in (8) and then obtain a unique solution as shown in (17)-(18).[2] Further details on this can be found in the Appendix.

$$a_2 = \frac{2\gamma_r - \mu\beta_r\delta + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta^2} \tag{17}$$

$$\begin{aligned} a_1 = {} & \frac{\delta(\alpha\mu(2\gamma_r + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)}) - 2C_0\beta_r\delta)}{\delta^2(\phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)})} \\ & + \frac{p(2\gamma_p(2\gamma_r - \beta_r\delta\mu + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)}) - \delta(2\beta_r\delta + \beta_p\mu(-2\gamma_r - \phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)})))}{\delta^2(\phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)})}, \end{aligned} \tag{18}$$

We can now present the following result.

Proposition 1: *The optimal feedback innovation policy of the centralized decision maker is linear in the state r, and is given by*

$$I(r) = r * \left[\frac{2\gamma_r + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta}\right] + \frac{\delta(\mu\alpha(\gamma_r + \phi) - C_0\beta_r) - p[\delta(\mu\beta_p(\gamma_r + \phi) - \beta_r\delta) + \gamma_p(\sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)} - (2\gamma_r - \mu\beta_r\delta + \phi))]}{\delta(\phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)})}. \tag{19}$$

The value function of the centralized decision maker can be obtained by (11), where a_2 and a_1 are given by (17) and (18), respectively, and a_0 can be obtained by using (17)-(18) in (16).

We can further analyse the result in Proposition 1 to obtain a number of corollaries, each of which brings about some useful managerial insights, as presented below. Each corollary is followed by a discussion on key insights from it.

Corollary 1.1: *Optimal innovation effort's dependence on product goodwill.*
It can be seen that $\frac{\partial I(r)}{\partial r} = \frac{2\gamma_r + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta}$, *and after a few steps of algebra it can*

[2] To ensure a real solution, we assume that $(2\gamma_r - 2\beta_r\delta\mu + \phi) \geq 0$, i.e., $\mu \leq \frac{2\gamma_r + \phi}{2\beta_r\delta}$

be shown that $Sign[\frac{\partial I(r)}{\partial r}] = Sign[\mu]$.

Corollary 1.1 gives a very interesting insight that how the optimal innovation effort evolves w.r.t. goodwill is very much dependent on whether the innovation effort leads to a reduction in unit cost or an increment in it. More specifically, It implies that the feedback innovation policy is increasing in the state variable r when innovation leads to a reduction in the production cost and decreasing in r when innovation leads to an increase in the production cost. Thus when $\mu > 0$ we see a cyclic effect that as goodwill increases (decreases) the innovation efforts increases (decreases), which then given the state dynamics, further increases (decreases) the goodwill. Thus a clear distinction is more likely to happen where we have either a high innovation high goodwill scenario or a low innovation low goodwill scenario. On the other hand, if $\mu < 0$, we are likely to see a more balancing situation where a higher (lower) goodwill leads to lower (higher) innovation efforts, which then given the state dynamics impacts the goodwill negatively (positively). This could be explained by the fact that when $\mu < 0$, innovation efforts increase unit cost of manufacturing, and this could then balance off the benefits due to higher goodwill.

Corollary 1.2: *Special case when* $\mu = 0$.
In the special case when innovation efforts have no impact on the unit production cost, the optimal innovation effort is constant, i.e., independent of state, and is equal to

$$I(r(t)) = \frac{p\beta_r\delta}{2(\gamma_r + \phi)},$$

and the value function is linear in the state variable and given by

$$V(r) = r * \frac{p\beta_r}{(\gamma_r + \phi)} + \frac{p(4\alpha(\gamma_r + \phi)^2 + p(\beta_r^2\delta^2 + 4\beta_r\gamma_p(\gamma_r + \phi) - 4\beta_p(\gamma_r + \phi)^2))}{4\phi(\gamma_r + \phi)^2}.$$

It is interesting to see that when the innovation efforts have no impact on unit manufacturing cost and hence its impact is only demand-enhancement, the firm's optimal innovation effort does not change with time. In a more general scenario however, as we have noted, the innovation effort evolves with goodwill (and hence with time) depending on whether innovation increases or decreases the unit cost.

We now study sensitivity of the optimal feedback innovation effort policy $I(r)$ w.r.t. diferent exogenous model parameters. We simply find the first partial derivative of the optimal policy in Proposition 1 w.r.t. these parameters and investigate these partial derivatives, in particular their signs, i.e., positive or negative. In some cases, we are able to obtain insights through analytical means and we discuss these in the results that we obtain and present below. In other cases we had to rely on numerical analysis to obtain such insights, which are discussed later in Section 4.3.

Corollary 1.3: Sensitivity of innovation effort w.r.t. base cost, retail price, and demand sensitivity to price.

i. $\frac{\partial I}{\partial C_0} < 0$

ii. $Sign\left[\frac{\partial I}{\partial p}\right] = Sign\left[\delta\left(\beta_r\delta - \beta_p\mu(\gamma_r+\phi)\right) + \gamma_p\left(2\gamma_r - \beta_r\delta\mu + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\beta_r\delta\mu+\phi)}\right)\right].$
Furthermore, $\frac{\partial}{\partial\gamma_p}\left(\frac{\partial I}{\partial p}\right) \geq 0$, *and* $Sign\left[\frac{\partial}{\partial\beta_p}\left(\frac{\partial I}{\partial p}\right)\right] = Sign\left[-\mu\right]$

iii. $Sign\left[\frac{\partial I}{\partial\beta_p}\right] = Sign\left[-\mu\right].$

i. Impact of base cost C_0 : We find that if the retail price p and all other parameters are kept constant, as the base unit cost of manufacturing C_0 increases, the innovation effort decreases. Thus a central decision maker is likely to put less effort for high base cost products. As base cost increases but if the price is kept constant, the 'base margin' for the firm, i.e., $p - C_0$ decreases, thereby possibly resulting in a lower incentive for the firm to enhance demand by investing in innovation. Interestingly, this is regardless of whether innovation increases or decreases the unit manufacturing cost.
ii. Impact of retail price p : The impact of retail price depends on several other model parameters as well. We can divide the insights in two cases. i) When innovation efforts increase the unit manufacturing cost, i.e., $\mu < 0$, we can show that $\frac{\partial I}{\partial p} > 0$. This implies that with all other factors unchanged, innovation efforts are greater for higher priced products. ii) When innovation reduce the unit manufacturing cost, i.e., $\mu > 0$, while the partial derivative $\frac{\partial I}{\partial p}$ depends on various other model parameters as well, we can however make the following observation after some investigation. Innovation efforts are increasing (decreasing) in retail price when demand sensitivity to price (β_p) is low (high) and impact of price on goodwill (γ_p) is high (low).
iii. Impact of demand sensitivity to price β_p : When innovation effort is cost reducing ($\mu > 0$), it decreases as the demand becomes more sensitive to price; whereas innovation efforts increase with price sensitivity of demand if the innovation effort increases the unit cost ($\mu < 0$).

Partial derivative of I w.r.t. various other model take much more complex expressions and their signs depend on the values of various other model parameters.

The result in Corollary 1.3 i. is valid when among all the other parameters, the retail price is also kept constant. If there are small changes in the base manufacturing cost the firm might want to keep its retail price constant given the market constraints, and so these insights will hold valid. However, firm might chose to change the retail price as well as the base cost changes, in order to maintain steady margins. In such a scenario it is useful to study the changes in innovation effort as the base cost changes and the firm also changes the retail price accordingly to earn a constant base margin. For this purpose, we considered two cases, one in which the firm aims to earn a fixed margin in terms of dollar value, i.e., constant $(p-C_0)$; and second in which the firm aims to earn a fixed mark-up in % above the base cost, i.e., constant p/C_0. Note that here the retail price is still exogenous, but changes as one changes the base cost in the model. In this context, for presentation in the rest of the paper, we will refer to a product with a higher base cost C_0 as a higher "value" product. Corollary 1.4 presents insights in such cases.

Corollary 1.4: Sensitivity of innovation effort when the firm keeps a constant base margin.

i. *When* $p = C_0 + M$, *where* M *is constant margin earned by the firm over the base cost* C_0, *we have* $Sign\left[\frac{\partial I}{\partial C_0}\right] = Sign\Big[\gamma_p\big(2\gamma_r - \beta_r\delta\mu + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\beta_r\delta\mu+\phi)}\big) - \delta\beta_p\mu(\gamma_r+\phi)\Big]$.

ii. *When* $p = C_0(1+m)$, *where* m *indicates a constant % margin earned by the firm over the base cost* C_0, *we have* $Sign\left[\frac{\partial I}{\partial C_0}\right] = Sign\Big[(1+m)\gamma_p\big(2\gamma_r - \beta_r\delta\mu + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\beta_r\delta\mu+\phi)}\big) + \delta\big(m\beta_r\delta - (1+m)\beta_p\mu(\gamma_r+\phi)\big)\Big]$.

iii. In both the cases above, i.e., i., and ii., $\frac{\partial}{\partial\gamma_p}\left(\frac{\partial I}{\partial C_0}\right) \geq 0$, and $Sign\left[\frac{\partial}{\partial\beta_p}\left(\frac{\partial I}{\partial C_0}\right)\right] = Sign\left[-\mu\right]$.

We can clearly contrast the result in Corollary 1.4 to that in Corollary 1.3. As opposed to when retail price does not change with the base cost, in this case, the sensitivity of optimal innovation effort policy w.r.t. base cost is more complex relationship. We can categorize the insights in two cases depending on whether the innovation efforts are cost-reducing or cost-enhancing. i) When $\mu < 0$, i.e., when innovation effort leads to increase in unit manufacturing cost, innovation efforts are are increasing in base cost. ii) However, when $\mu > 0$, i.e., when innovation effort leads to reduction in base cost, $\frac{\partial I}{\partial C_0}$ depends on various model parameters. In particular, when $\mu > 0$, innovation effort is increasing in base cost when price has a high impact on goodwill (γ_p), and low impact on demand (β_p). The opposite holds when we have low values of γ_p and high values of β_p.

4.1.1 A steady state solution

We also consider the steady state condition which we obtain by first computing the steady state value of goodwill ($\bar{r}$) and then using this value in feedback innovation effort policy in (19). To obtain $\bar{r}$, we rewrite the state dynamics in (6) using (19), and then solve $\dot{r} = 0$. We get the following result

$$\bar{r} = \frac{\delta(-C_0\beta_r\delta + \alpha\mu(\gamma_r + \phi)) + p(\gamma_p(2\gamma_r - \beta_r\delta\mu + 2\phi) + \delta(\beta_r\delta - \beta_p\mu(\gamma_r + \phi)))}{2\gamma_r^2 - \beta_r\delta\mu\phi + 2\gamma_r(-\beta_r\delta\mu + \phi)}$$

$$\bar{I} = I^*(\bar{r}) = \frac{-C_0\beta_r\gamma_r\delta + \alpha\gamma_r\mu(\gamma_r + \phi) + p(\mu(\gamma_r + \phi)(\beta_r\gamma_p - \beta_p\gamma_r) + \beta_r\gamma_r\delta)}{2\gamma_r^2 - \beta_r\delta\mu\phi + 2\gamma_r(-\beta_r\delta\mu + \phi)}$$

Corollary 1.5: Sensitivity of steady-state solution.

i. $\frac{\partial \bar{I}}{\partial C_0} < 0$, $\frac{\partial \bar{r}}{\partial C_0} < 0$.

ii. $Sign\left[\frac{\partial \bar{I}}{\partial p}\right] = Sign\left[\mu(\gamma_r + \phi)(\beta_r\gamma_p - \beta_p\gamma_r) + \beta_r\gamma_r\delta\right]$,
$Sign\left[\frac{\partial \bar{r}}{\partial p}\right] = Sign\left[\gamma_p(2\gamma_r - \beta_r\delta\mu + 2\phi) + \delta(\beta_r\delta - \mu\beta_p(\gamma_r + \phi))\right]$

iii. *When $p = C_0 + M$, where M is constant margin earned by the firm over the base cost C_0, we have* $Sign\left[\frac{\partial \bar{I}}{\partial C_0}\right] = Sign\left[\mu(\beta_r\gamma_p - \beta_p\gamma_r)\right]$, *and*
$Sign\left[\frac{\partial \bar{r}}{\partial C_0}\right] = Sign\left[\gamma_p(2\gamma_r - \beta_r\delta\mu + 2\phi) - \delta\beta_p\mu(\gamma_r + \phi)\right]$

iv. *When $p = C_0(1 + m)$, where m indicates a constant % margin earned by the firm over the base cost C_0, we have* $Sign\left[\frac{\partial \bar{I}}{\partial C_0}\right] = Sign\left[(1+m)\mu(\gamma_r+\phi)(\beta_r\gamma_p - \beta_p\gamma_r) + m\beta_r\gamma_r\delta\right]$,
and $Sign\left[\frac{\partial \bar{r}}{\partial C_0}\right] = Sign\left[(1 + m)\left[\gamma_p(2\gamma_r - \beta_r\delta\mu + 2\phi) - \delta\beta_p\mu(\gamma_r + \phi)\right] + m\beta_r\delta^2\right]$

To interpret the results of Corollary 1.5, we first note that under the assumption of $(2\gamma_r - 2\beta_r\delta\mu + \phi) \geq 0$, which is what we need to get a feedback solution in the centralized case, we will always have $(2\gamma_r - \beta_r\delta\mu + \phi) \geq 0$, and hence $(2\gamma_r - \beta_r\delta\mu + 2\phi) > 0$. With this in mind, we get the following insights. As the base unit cost increases due to factors such as for e.g. increase in raw material cost, but the retail price is unchanged, the firm's moves towards a lower long run innovation effort and lower long run goodwill. However, if the firm looks to keep a constant base margin ($(p - C_0)$ or p/C_0) and change the retail price accordingly with C_0, the changes in long run innovation effort and goodwill depend on various other model parameters. As mentioned earlier, in this context (Corollary 1.5 iii. and iv.), we refer to a product with a higher C_0 as a higher "value" product. The following observations can be made from points iii. and iv. in Corollary 1.5.

a.) If the innovation efforts increase the unit manufacturing cost ($\mu < 0$), then we find that $\left[\frac{\partial \bar{r}}{\partial C_0}\right] > 0$. In other words, the long run goodwill for a higher value product is always higher than that of a lower value product. $\frac{\partial \bar{I}}{\partial C_0} > 0$ when β_p is high and γ_p is low, and $\frac{\partial \bar{I}}{\partial C_0} < 0$ when β_p is low and γ_p is high. Thus we can say that when price has a high impact on demand but low impact on goodwill, in the long run we have high innovation and high goodwill for higher value products and low innovation and low goodwill for lower value products. On the other hand if consumer price has a high impact on goodwill but low impact on demand, then in the long run we have lower innovation but higher goodwill (due to higher price) for higher value products.

b.) If the innovation efforts lead to efficiency and reduce the unit manufacturing cost ($\mu < 0$), we can see that both $\frac{\partial \bar{I}}{\partial C_0}$ and $\frac{\partial \bar{r}}{\partial C_0}$ are > 0 when β_p is low and γ_p is high, and the opposite holds true when β_p is high and γ_p is low. Thus, we can say that when the impact of price on demand is low but on goodwill is high, we have high innovation and high goodwill for higher value products. On the other hand when price has a high impact on demand but low impact on goodwill, then long run innovation and goodwill are lower for a higher value product.

When the base cost is kept constant but the retail price changes (Corollary 1.5 ii.), the insights on the impact of retail price on the long run innovation and goodwill are similar to those on the impact of product value as highlighted in the points a.) and b.) above.

4.2 Decentralized channel

We now study the decentralized problem where the supplier as the Stackelberg leader announces her innovation effort policy in feedback form and the buyer follows by deciding his feedback innovation efforts policy. The optimal control problems of the supplier, and buyer, are given by (4), and (5), respectively, subject to the state dynamics given in (2). We can write the Hamilton-Jacobi-Bellman (HJB) equation for the supplier as

$$\phi V^s(r) = \max_{I_s} \left[(w + \mu(I_s + I_b) - C_0)(\alpha - p\beta_p + r\beta_r) - I_s^2 + V_r^s(p\gamma_p - r\gamma_r + (I_b + I_s)\delta)\right] \tag{20}$$

where $V_r^s = \frac{\partial V^s(r)}{\partial r}$, subject to the growth condition (see for reference Bensoussan et al. (2014))

$$\lim_{t\to\infty} e^{-\phi t} V^s(r(t)) = 0, \tag{21}$$

and the HJB equation for the buyer as

$$\phi V^b(r) = \max_{I_b} \left[(p-w)(\alpha - p\beta_p + r\beta_r) - I_b^2 + V_r^b(p\gamma_p - r\gamma_r + (I_b + I_s)\delta)\right] \tag{22}$$

where $V_r^b = \frac{\partial V^b(r)}{\partial r}$, subject to its growth condition

$$\lim_{t\to\infty} e^{-\phi t} V^b(r(t)) = 0. \tag{23}$$

The readers are referred to Bensoussan et al. (2014) for a verification theorem for feedback Stackelberg solutions in infinite horizon differential games. We follow the standard backward induction approach. Given the innovation effort policy by the supplier $I_s(r)$, solving the first-order condition (f.o.c.) w.r.t. $I_b(r)$ in the buyer's HJB equation (22), we get the buyer's optimal response as [3]

$$I_b(r) = \frac{\delta V_r^b}{2}. \tag{24}$$

Using (24) we rewrite the supplier's HJB equation in (20) as

$$\phi V^s(r) = (\alpha - p\beta_p + r\beta_r)(\mu I_s - C_0 + w + \frac{\mu\delta V_r^b}{2}) - Is^2 + V_r^s(p\gamma_p - r\gamma_r + \delta(I_s + \frac{V_r^b\delta}{2})). \tag{25}$$

We now apply f.o.c. in supplier's HJB equation in (25) w.r.t. I_s and get [4]

$$I_s(r) = \frac{\mu(\alpha - p\beta_p + r\beta_r) + \delta V_r^s}{2}. \tag{26}$$

Now using (24) and (26) in (22), and (25), we can write the p.d.e.'s for value functions of the buyer, and supplier, respectively, as follows.

$$\begin{aligned} 4\phi V^b(r) =& -4p^2\beta_p - 4w(\alpha + r\beta_r) + 4p(\alpha + w\beta_p + r\beta_r + V_r^b\gamma_p) \\ &+ V_r^b(-4r\gamma_r + 2\mu\delta(\alpha - p\beta_p + r\beta_r) + (V_r^b + 2V_r^s)\delta^2) \end{aligned} \tag{27}$$

$$\begin{aligned} 4\phi V^s(r) =& \, 4(w - C_0)(\alpha - p\beta_p + r\beta_r) + \mu^2(\alpha - p\beta_p + r\beta_r)^2 + 4V_r^s(p\gamma_p - r\gamma_r) \\ &+ 2\mu\delta(V_r^b + V_r^s)(\alpha - p\beta_p + r\beta_r) + \delta^2 V_r^s(2V_r^b + V_r^s) \end{aligned} \tag{28}$$

[3] It can be easily seen the second order condition (s.o.c.) is also satisfied as we get $\frac{\partial^2 V^b}{\partial I_b^2} = -\frac{2}{\phi} < 0$.

[4] In this case as well the s.o.c. is easily satisfied as we get $\frac{\partial^2 V^s}{\partial I_s^2} < 0$.

Similar to our approach in the centralized model, we conjecture the following forms of the value functions

$$V^s(r) = s_2 r^2 + s_1 r + s_0 \tag{29}$$

$$V^b(r) = b_2 r^2 + b_1 r + b_0, \tag{30}$$

where s_j and b_j, $j = 0, 1, 2$, are constants whose values depend on various model parameters. This gives us

$$V_r^s = 2s_2 r + s_1 \tag{31}$$

$$V_r^b = 2b_2 r + b_1. \tag{32}$$

Following an approach similar to section 3.1, we use (29)-(32) to rewrite the LHS and the RHS of the value function equations in (27) and (28), and then compare the coefficients of r^2, r, and the constant term on the LHS and RHS of (27) and (28). As a result we are able to write the non-linear algebraic equations in the constants s_j and b_j, $j = 0, 1, 2$. We can then solve these equations to get values of constants s_j and b_j, $j = 0, 1, 2$, in different cases, and can therefore obtain a feedback Stackelberg equilibrium, as summarized below. Please see Appendix B for further details.

Proposition 2:

(i.) *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) \geq 0$, *the optimal feedback policies of the two players are given by*

$$I_s(r) = r * [\frac{2\gamma_r + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta}] + Ks_1$$

$$I_b(r) = \frac{(p - w)\beta_r\delta}{\phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}.$$

(ii.) *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) < 0$, *the optimal feedback policies of the two players are given by*

$$I_s(r) = r * [\frac{2\gamma_r + \phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r + 6\mu\beta_r\delta + \phi)}}{6\delta}] + Ks_2$$

$$I_b(r) = r * [\frac{4\gamma_r + 2\phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r + 6\mu\beta_r\delta + \phi)}}{3\delta}] + Kb_2$$

Here, Ks_1, Ks_2, *and* Kb_2 *are constants whose values depend on exogenous model parameters,* *i.e.,* $\alpha, \beta_r, \beta_p, \gamma_r, \gamma_p, \delta, \phi, \mu, C_0, p, w$.

Details on deriving the result in Proposition 2 can be found in Appendix B. To summarize, we get two possible solutions depending on the model parameters. When innovation efforts increase unit costs ($\mu < 0$), or even when they reduce unit costs but the reduction in costs is not large enough ($0 \leq \mu \leq \frac{(2\gamma_r+\phi)}{2\beta_r\delta}$), we get solution (i.) as above, and when the reduction in unit costs due to innovation efforts is large enough ($\mu > \frac{(2\gamma_r+\phi)}{2\beta_r\delta}$), we get solution (ii.) above. Further investigation into results in Proposition 2 yields some interesting insights described below.

Corollary 2.1: Dependence of optimal innovation efforts on product goodwill.

i. *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) \geq 0$, *the seller's efforts are linear in goodwill* (r), *whereas the buyer's efforts are constant and independent of* r. *Moreover,* $Sign[\frac{\partial I_s(r)}{\partial r}] = Sign[\mu]$.

ii. *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) < 0$, *both players' efforts are linear in goodwill. Moreover,* $Sign[\frac{\partial I_s(r)}{\partial r}] > 0$ *and* $Sign[\frac{\partial I_b(r)}{\partial r}] < 0$, *always.*

Looking at the above result, we can conclude that the supplier's innovation efforts increase in goodwill when the innovation efforts reduce the manufacturing cost ($\mu > 0$), and decrease in goodwill when the they increase the unit manufacturing cost ($\mu < 0$). The buyer's response however is different. If the innovation efforts increase the unit cost or they decrease the unit cost but their impact is not large enough (i.e., $\mu \leq \frac{(2\gamma_r+\phi)}{2\beta_r\delta}$), the buyer's efforts do not change with goodwill. However, if innovation efforts lead to large enough reduction in unit cost ($\mu > \frac{(2\gamma_r+\phi)}{2\beta_r\delta}$), the buyer puts a lower innovation effort for a higher level of goodwill. Corollary 2.1 shows a very interesting result and clearly highlights the difference in the incentives and hence their optimal strategies of innovation efforts between the two players in the supply chain. Similar to the centralized firm's problem, it also again highlights how μ, i.e., innovation efforts' impact on unit cost, has an impact on optimal strategies feedback strategies. It is interesting to note that while the seller always follows a dynamic policy wherein the optimal innovation efforts evolve with goodwill (and hence time); the buyer may actually adopt a static policy wherein its innovation efforts are constant over time depending on model parameters, despite the clear impact of goodwill on the dynamics of the product goodwill. Furthermore, very interestingly, under no case the buyer's optimal innovation effort is increasing in goodwill, it is either a constant or decreasing with goodwill.

To study the sensitivity of optimal policies w.r.t. various model parameters we compute the partial derivatives w.r.t. various model parameters, and after a few steps of algebra obtain some useful results as discussed below.

Corollary 2.2: *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) \geq 0$, *we can get the following results*

i. $\frac{\partial I_b(r)}{\partial C_0} = 0,$ *and* $\frac{\partial I_s(r)}{\partial C_0} = \frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial C_0} < 0,$ *always.*

ii. $\frac{\partial I_b(r)}{\partial p} > 0,$ *always. Sign* $\left[\frac{\partial I_s(r)}{\partial p}\right]$ *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial p}\right]$ *depend on model parameters.*

iii. $\frac{\partial I_b(r)}{\partial w} < 0,$ *and* $\frac{\partial I_s(r)}{\partial w} > 0,$ *always, and*
$Sign\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial w}\right] = Sign\left[(2\gamma_r+\phi)(\gamma_r-\beta_r\delta\mu) - \gamma_r\sqrt{(2\gamma_r+\phi)(2\gamma_r-2\beta_r\delta\mu+\phi)}\right]$

iv. $\frac{\partial I_b(r)}{\partial \beta_p} = 0,$ *and* $\frac{\partial I_s(r)}{\partial \beta_p} = \frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial \beta_p} = Sign\left[-\mu\right].$

v. $\frac{\partial I_b(r)}{\partial \beta_r} > 0,$ *always. Sign* $\left[\frac{\partial I_s(r)}{\partial \beta_r}\right]$ *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial \beta_r}\right]$ *depend on model parameters.*

vi. *When* $w = C_0 + M_s$ *and* $p = w + M_b = C_0 + M_s + M_b$, *where* M_s *and* M_b *are constant base margins earned by the seller and buyer, respectively, we have* $\frac{\partial I_b(r)}{\partial C_0} = 0,$ *and*
$Sign\left[\frac{\partial I_s(r)}{\partial C_0}\right] = Sign\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial C_0}\right] =$
$Sign\left[\gamma_p\left(2\gamma_r-\beta_r\delta\mu+\phi-\sqrt{(2\gamma_r+\phi)(2\gamma_r-2\beta_r\delta\mu+\phi)}\right)-\delta\beta_p\mu(\gamma_r+\phi)\right].$

vii. *When* $w = (1+m_s)C_0$ *and* $p = (1+m_b)w = (1+m_s)(1+m_b)C_0$, *where* m_s *and* m_b *indicate constant % base margins earned by the seller and buyer, respectively, we have* $\frac{\partial I_b(r)}{\partial C_0} > 0,$ *always. Sign* $\left[\frac{\partial I_s(r)}{\partial C_0}\right]$ *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial C_0}\right]$ *depend on complex terms which are functions of model parameters. However, we can conclude that* $Sign\left[\frac{\partial^2 I_s}{\partial\beta_p\partial C_0}\right] = Sign\left[\frac{\partial^2(I_b+I_s)}{\partial\beta_p\partial C_0}\right] = Sign\left[-\mu\right].$

Corollary 2.3: *When* $(2\gamma_r - 2\mu\beta_r\delta + \phi) < 0$, *i.e.,* $\mu > \frac{2\gamma_r+\phi}{2\beta_r\delta}$, *we can get the following results*

i. $\frac{\partial I_b(r)}{\partial C_0} > 0$ *and* $\frac{\partial I_s(r)}{\partial C_0} < 0$ *always.* $\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial C_0} \le 0$ *when* $\mu \in \left(\frac{2\gamma_r+\phi}{2\beta_r\delta}, \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}\right]$, *and* $\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial C_0} > 0$ *when* $\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$.

ii. *Sign* $\left[\frac{\partial I_b(r)}{\partial p}\right]$, *Sign* $\left[\frac{\partial I_b s(r)}{\partial p}\right]$, *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial p}\right]$ *depend on model parameters. However, the following can be concluded.*

- $\frac{\partial^2 I_b}{\partial\beta_p\partial p} > 0$. *Furthermore, in the special case when* $\gamma_p = 0$, $\frac{\partial I_b(r)}{\partial p} > 0,$ *always.*
- $\frac{\partial^2 I_s}{\partial\beta_p\partial p} < 0.$

iii. $\frac{\partial I_b(r)}{\partial w} < 0,$ $\frac{\partial I_s(r)}{\partial w} < 0,$ *and* $\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial w} < 0,$ *always.*

iv. *When $w = C_0 + M_s$ and $p = w + M_b$, where M_s and M_b are constant base margins earned by the seller and buyer, respectively, Sign* $\left[\frac{\partial I_b(r)}{\partial p}\right]$, *Sign* $\left[\frac{\partial I_b s(r)}{\partial p}\right]$, *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial p}\right]$ *depend on model parameters. However, the following can be concluded*

- $\frac{\partial^2 I_b}{\partial \beta_p \partial C_0} > 0$. *In the special case when* $\gamma_p = 0$, $\frac{\partial I_b(r)}{\partial C_0} > 0$,.
- $\frac{\partial^2 I_s}{\partial \beta_p \partial C_0} < 0$. *In the special case when* $\gamma_p = 0$, $\frac{\partial I_s(r)}{\partial C_0} < 0$,.
- $\frac{\partial^2 (I_b+I_s)}{\partial \beta_p \partial C_0} \leq 0$ *when* $\mu \in \left(\frac{2\gamma_r+\phi}{2\beta_r\delta}, \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}\right]$, *and* $\frac{\partial^2 (I_b+I_s)}{\partial \beta_p \partial C_0} > 0$ *when* $\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$. *Similar behaviour is observed for Sign* $\left[\frac{\partial (I_b+I_s)}{\partial C_0}\right]$ *in the special case when* $\gamma_p = 0$.

v. *When $w = (1+m_s)C_0$ and $p = (1+m_b)w = (1+m_s)(1+m_b)C_0$ where m_s and m_b indicate constant % base margins earned by the seller and buyer, respectively, Sign* $\left[\frac{\partial I_b(r)}{\partial p}\right]$, *Sign* $\left[\frac{\partial I_b s(r)}{\partial p}\right]$, *and Sign* $\left[\frac{\partial\left(I_b(r)+I_s(r)\right)}{\partial p}\right]$ *depend on model parameters. However, the following can be concluded*

- $\frac{\partial^2 I_b}{\partial \beta_p \partial C_0} > 0$.
- $\frac{\partial^2 I_s}{\partial \beta_p \partial C_0} < 0$.
- $\frac{\partial^2 (I_b+I_s)}{\partial \beta_p \partial C_0} \leq 0$ *when* $\mu \in \left(\frac{2\gamma_r+\phi}{2\beta_r\delta}, \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}\right]$, *and* $\frac{\partial^2 (I_b+I_s)}{\partial \beta_p \partial C_0} > 0$ *when* $\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$.

We describe key insights from our results in Corollary 2.2 and 2.3 below.

i. Impact of base cost C_0 : From Corollary 2.2 (i.) and 2.3 (i.) we can see that if the base unit cost increases and all the other parameters are kept constant, the supplier's effort always decreases, whereas its impact on the buyer's effort and aggregate SC effort depends on other model parameters as well. When innovation leads to increase in unit cost, or when the reduction in unit cost is not 'large' enough, i.e., ($\mu < \frac{2\gamma_r+\phi}{2\beta_r\delta}$), the buyer's innovation effort does not depend on base unit cost, and the total SC effort decreases with C_0. However, if the innovation leads to unit cost reduction and its impact is large enough, i.e., ($\mu > \frac{2\gamma_r+\phi}{2\beta_r\delta}$), the buyer's effort increases with base unit cost. The aggregate SC effort is increasing in C_0 only when the innovation leads to 'significant' reduction in unit cost, i.e., ($\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$), and for all other cases the SC aggregate innovation effort is lower for a higher base cost item.

ii. Impact of base cost C_0 under constant margins: If the wholesale price and retail price also increase with the unit base cost while keeping the 'base margins' constant for both the players, we have two sub-cases. As done in the centralized case, we can view a product with higher unit base cost C_0 (and therefore higher wholesale and retail prices) as a product with higher 'value'. We discuss the insights in these two cases below.

Case A: constant base margins in terms of dollar value (Corollary 2.2 (vi.) and 2.3 (iv.):

In the first case when the absolute margins for the two players, i.e., M_s, M_b are constant, we can observe the following. If innovation leads to increase in unit base cost or even when it reduces the base cost but its impact is not too high (i.e., $\mu < \frac{2\gamma_r+\phi}{2\beta_r\delta}$,) the buyer's innovation effort does not change with C_0. When the innovation efforts lead to large enough reduction in unit cost ($\mu > \frac{2\gamma_r+\phi}{2\beta_r\delta}$), the buyer's effort is likely to increase with C_0 when price has low impact on demand (i.e., low β_p); and is always increasing with C_0 in the special case when price has no impact on goodwill (i.e., $\gamma_p = 0$). When innovation efforts increase the unit cost ($\mu < 0$), the supplier's effort always increases with C_0. However, when innovation efforts reduce the unit cost ($\mu > 0$), the supplier's effort is likely to be decreasing in C_0 when β_p is high, and it is always decreasing in C_0 when $\gamma_p = 0$. The aggregate SC effort shows the same behaviour as supplier's effort w.r.t. changes in C_0, with the exception of the case when the cost reduction impact of innovation exceeds an even higher threshold ($\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$.) In such a scenario, the aggregate SC effort follows same behaviour as that of buyer's effort, i.e, likely to increase with C_0 when β_p is low, and always increasing with C_0 in the special case when $\gamma_p = 0$.

Case B: constant base markup in % (Corollary 2.2 (vii.) and 2.3 (v.):

In the second case, when the wholesale and retail prices change with base cost while keeping the base % margins constant for both the players, we can make the following observations. The buyer's effort is increasing in C_0 when $\mu < \frac{2\gamma_r+\phi}{2\beta_r\delta}$. When $\mu > \frac{2\gamma_r+\phi}{2\beta_r\delta}$, the buyer's effort is likely to be increasing in C_0 for high values of β_p. When innovation leads to increase in unit cost ($\mu < 0$), the supplier's effort is likely to be increasing in C_0 when β_p is high. On the other hand, the supplier's effort is likely to be decreasing in C_0 for high values of β_p when the innovation efforts reduce the unit cost ($\mu > 0$). The sensitivity of aggregate SC effort w.r.t. C_0 for different values of β_p follows similar pattern as in the first case, i.e., similar behaviour as supplier's effort when $\mu < \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$, and similar behaviour as buyer's effort when $\mu > \frac{2(\gamma_r+\phi)(\gamma_r+2\phi)}{\beta_r\delta(2\gamma_r+\phi)}$.

iii. Impact of demand sensitivity to price β_p and goodwill β_r (Corollary 2.2 (iv.) and (v.)): In the case when $(2\gamma_r - 2\mu\beta_r\delta + \phi) \geq 0$, we find that the buyer's innovation effort does not depend on demand sensitivity to price β_p. The dependence of seller's optimal effort on β_p however is same as that of a centralized firm (Section 3.1), i.e.; innovation effort increases with demand sensitivity when the innovation efforts increase unit cost ($\mu < 0$), and decrease with demand sensitivity when innovation efforts are cost-reducing ($\mu > 0$). The buyer's innovation effort however increases with the demand sensitivity to goodwill (β_r). We found that obtaining such insights through analytical means is a bit difficult in the case when $(2\gamma_r - 2\mu\beta_r\delta + \phi) < 0$, due to the complexity of expressions.

4.3 Comparison of Centralized and Decentralized Solutions

When $(2\gamma_r - 2\mu\beta_r\delta + \phi) \geq 0$, an analytical comparison can be done between the feedback innovation policy of a centralized firm with those of two firms in a decentralized channel. It is to be noted that when $(2\gamma_r - 2\mu\beta_r\delta + \phi) < 0$, we cannot obtain an optimal feedback policy for a centralized firm and hence a comparison cannot be made in that case. We compute the difference in total channel feedback innovation efforts as:

$$\Delta I = I(r) - (I_s(r) + I_b(r)) = \frac{(p-w)\beta_r\delta(2\gamma_r^2 - \mu\beta_r\delta\phi + \gamma_r(\phi - 2\mu\beta_r\delta - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}))}{(\phi + \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta+\phi)})(2\gamma_r^2 + 2\gamma_r(\phi - \mu\beta_r\delta) + \phi(\phi - \mu\beta_r\delta + \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta+\phi)}))}$$

Similarly, we can also compute the difference between the rate of change of goodwill in a centralized vs a decentralized channel. This difference is given by

$$\Delta\dot{r} = \dot{r}\mid_{centralized} - \dot{r}\mid_{decentralized} = \delta\Delta I.$$

We can now obtain the following result.

Corollary 2.4: *ΔI and $\Delta\dot{r}$ are independent of state variable r, and are constants depending on model parameters. Moreover, $Sign[\Delta I] = Sign[\Delta\dot{r}] = Sign[-\mu]$*

Corollary 2.4 indicates that when innovation efforts lead to reduction in unit manufacturing costs (as long as $\mu \leq \frac{(2\gamma_r + +\phi)}{2\beta_r\delta}$), decentralized contributions from the two channel members will lead to a higher overall innovation efforts and higher product goodwill. On the other hand, when the innovation efforts increase the unit production cost, a centralized decision maker would put overall more efforts and a centralized channel will lead to higher goodwill. To extend this argument, it can be said that when innovation is process focussed and leads to improved quality of conformance, a decentralized channel is better than a centralized one in terms of higher channel-wide innovation effort and higher product goodwill. However, when efforts are focussed on quality of design but lead to increase in unit costs, centralized decision making results in more overall innovation effort than a decentralized one.

5 Model with pricing decisions

We now analyse the general model summarized in Figure 1 to include the wholesale and retail pricing decisions in addition to the innovation efforts for both the players. Here as well, we focus on feedback Stackelberg equilibrium and obtain optimal policies as functions of state variable. We first consider the problem of a centralized channel.

5.1 Centralized channel

In a centralized channel, the optimal control problem of the decision maker is

$$V(r) = \max_{p(t),I(t)\geq 0} \left\{ \int_0^\infty e^{-\phi t} \left[(p(t) - c(t))D(t) - I^2(t) \right] dt \right\},$$

subject to the state equation in (6). The HJB equation for the centralized decision maker's can be written as

$$\phi V(r) = \max_{I_i} \left[(p + \mu I_i - C_0)(\alpha - p\beta_p + r\beta_r) - I_i^2 + V_r(p\gamma_p - r\gamma_r + I_i\delta) \right] \tag{33}$$

subject to the boundary condition

$$\lim_{t\to\infty} e^{-\phi t} V(r(t)) = 0. \tag{34}$$

One can see that a special case of our centralized problem when $\mu = 0$, i.e., when innovation efforts have no impact on unit manufacturing cost, is very similar to Fruchter (2009) with one difference being in how demand is modeled. Fruchter (2009) considered a general demand function to begin with and then used a special case where the goodwill and price terms are multiplicative separable, we however consider a demand function that is linear in price and goodwill.

We can apply the first order conditions in (33) w.r.t. I and p and can obtain the optimal values as shown below [5]

$$p^*(r) = \frac{2(\alpha + C_0\beta_p + r\beta_r + V_r\gamma_p) - V_r\beta_p\delta\mu - \beta_p\mu^2(\alpha + r\beta_r)}{\beta_p(4 - \beta_p\mu^2)} \tag{35}$$

$$I_i^*(r) = \frac{2Vr\delta + \alpha\mu - C_0\beta_p\mu + r\beta_r\mu - V_r\gamma_p\mu}{4 - \beta_p\mu^2}. \tag{36}$$

We then follow similar approach as used in previous section. We use optimal decisions in (35) and (36) in (33) to write the full expression for the value function. We then conjecture a value function quadratic in state, i.e., $V(r) = a_2r^2 + a_1r + a_0$, where the constants $a_2, a_1,$

[5] To investigate the second order conditions we find that $\frac{\partial^2 V}{\partial p^2} = -\frac{2\beta_p}{\phi}$, $\frac{\partial^2 V}{\partial I^2} = -\frac{2}{\phi}$, and $(\frac{\partial^2 V}{\partial p^2}) * (\frac{\partial^2 V}{\partial I^2}) - (\frac{\partial^2 V}{\partial I \partial p})^2 = \frac{\beta_p(4-\beta_p\mu^2)}{\phi^2}$. We then assume that the exogenous parameters β_p and μ are such that $(4 - \beta_p\mu^2) > 0$, which then ensures that the Hessian matrix is negative definite and hence the second order condition is satisfied.

and a_0 are obtained by solving a set of non-linear algebraic equations (see Appendix C). We can now present the following result.

Proposition 3: *The optimal feedback pricing and innovation efforts policy of the centralized decision maker are linear in the state variable and are given by*

$$p^*(r) = r * \left[\frac{(2\gamma_r+\phi)(2\gamma_p+\beta_p\mu\delta) + 2\beta_r\delta(\delta-\mu\gamma_p) + (\beta_p\mu\delta-2\gamma_p)\sqrt{(2\gamma_r+\phi)^2 - \frac{4\beta_r^2(\delta-\gamma_p\mu)^2+4\beta_r(\beta_p\delta\mu+\gamma_p(2-\beta_p\mu^2))(2\gamma_r+\phi)}{\beta_p(4-\beta_p\mu^2)}}}{4(\gamma_p^2+\beta_p\delta^2-\beta_p\gamma_p\delta\mu)} \right]$$
$$+ L$$

$$I^*(r) = r * \left[\frac{2\beta_r\gamma_p(\gamma_p\mu-\delta) + \beta_p(\gamma_p\mu-2\delta)\left(2\gamma_r+\phi-\sqrt{(2\gamma_r+\phi)^2 - \frac{4\beta_r^2(\delta-\gamma_p\mu)^2+4\beta_r(\beta_p\delta\mu+\gamma_p(2-\beta_p\mu^2))(2\gamma_r+\phi)}{\beta_p(4-\beta_p\mu^2)}}\right)}{4(\gamma_p^2+\beta_p\delta^2-\beta_p\gamma_p\delta\mu)} \right]$$
$$+ Q$$

where L and Q are constants whose value depend on exogenous model parameters, i.e., $\alpha, \beta_r, \beta_p, \gamma_r, \gamma_p, \delta, \phi, \mu, C_0$.

For further details on Proof of Proposition 3, the readers are referred to Appendix C.

We try to investigate how the optimal decisions change with the goodwill r, and similar to previous section, we compute the first partial derivative of the optimal policies in Proposition 3 w.r.t. r. However, given the complexity of expressions in this case, we are able to obtain analytical result for a special case when the goodwill dynamics is not impacted by price and only innovation effort by the firm has an impact on the product goodwill, i.e., when $\gamma_p = 0$. We present our result below.

Corollary 3.1: *In the special case when $\gamma_p = 0$, we have $\frac{\partial I(r)}{\partial r} > 0$. Furthermore, $\frac{\partial p(r)}{\partial r} \geq 0$ for $\mu \in \left(-\frac{2}{\sqrt{\beta_p}}, \frac{2}{\sqrt{\beta_p}} - \frac{2\beta_r\delta}{\beta_p(2\gamma_r+\phi)}\right]$, and $\frac{\partial p(r)}{\partial r} < 0$ for $\mu \in \left(\frac{2}{\sqrt{\beta_p}} - \frac{2\beta_r\delta}{\beta_p(2\gamma_r+\phi)}, \frac{2}{\sqrt{\beta_p}}\right)$.*

Corollary 3.1 indicates that when price has no impact on perceived quality of the product or the goodwill, the innovation efforts are always higher (lower) for a higher (lower) level of goodwill. Furthermore, when $\mu \in \left(-\frac{2}{\sqrt{\beta_p}}, \frac{2}{\sqrt{\beta_p}} - \frac{2\beta_r\delta}{\beta_p(2\gamma_r+\phi)}\right]$, price also is higher (lower) for a higher (lower) level of product goodwill. Thus, in this scenario, depending on the model parameters, it is likely that the firm will take either the route of a 'high end product' (high quality, high price, and high goodwill), or that of a 'low end product' (low quality, low price, and low goodwill). On the other hand, when $\mu \in \left(\frac{2}{\sqrt{\beta_p}} - \frac{2\beta_r\delta}{\beta_p(2\gamma_r+\phi)}, \frac{2}{\sqrt{\beta_p}}\right)$, retail price is set relatively lower (higher) for a higher (lower) level of product goodwill.

5.2 Decentralized channel

We now analyze the decentralized channel in which both players decide their optimal price and innovation effort strategies. The optimal control problems of the two players are given in (4) and (5), subject to the state constraint in (1). We first write the supplier's HJB equation.

$$\phi V^s(r) = \max_{I_s,w} \left[(w + \mu(I_s + I_b) - C_0)(\alpha - p\beta_p + r\beta_r) - I_s^2 + V_r^s(p\gamma_p - r\gamma_r + (I_b + I_s)\delta)\right] \tag{37}$$

where $V_r^s = \frac{\partial V^s(r)}{\partial r}$, subject to the growth condition

$$\lim_{t\to\infty} e^{-\phi t} V^s(r(t)) = 0, \tag{38}$$

The HJB equation for the buyer is

$$\phi V^b(r) = \max_{I_b,p} \left[(p - w)(\alpha - p\beta_p + r\beta_r) - I_b^2 + V_r^b(p\gamma_p - r\gamma_r + (I_b + I_s)\delta)\right] \tag{39}$$

where $V_r^b = \frac{\partial V^b(r)}{\partial r}$, subject to its growth condition

$$\lim_{t\to\infty} e^{-\phi t} V^b(r(t)) = 0. \tag{40}$$

Similar to the approach in Section 3.2, we follow the standard backward induction approach. Given the wholesale price and innovation effort policy by the supplier, i.e., $w(r)$, and $I_s(r)$, we solve the first-order condition (f.o.c.) w.r.t. $p(r)$ and $I_b(r)$ in the buyer's HJB equation (39). We get the buyer's optimal response as [6]

$$p(r) = \frac{\alpha + \beta_p w(r) + r\beta_r + \gamma_p V_r^b}{2\beta_p} \tag{41}$$

$$I_b(r) = \frac{\delta V_r^b}{2}. \tag{42}$$

[6] The s.o.c. for buyer's problem is easily satisfied as we have $\frac{\partial^2 V^b}{\partial p^2} = -\frac{2\beta_p}{\phi}$, $\frac{\partial^2 V^b}{\partial I_b^2} = -\frac{2}{\phi}$, and $\frac{\partial^2 V^b}{\partial p \partial I_b} = 0$.

Using (41)-(42), we rewrite the supplier's HJB equation (37) and then applying the first order conditions in supplier's problem w.r.t. w and I_s, we get the following [7]

$$w(r) = \frac{4(\alpha + C_0\beta_p + r\beta_r - V_r^b\gamma_p + V_r^s\gamma_p) - 2\beta_p\delta\mu(V_r^b + V_r^s) - \beta_p(\alpha + r\beta_r - V_r^b\gamma_p)\mu^2}{\beta_p(8 - \beta_p\mu^2)} \tag{43}$$

$$I_b(r) = \frac{8V_r^s\delta - (-2\alpha + 2C_0\beta_p - 2r\beta_r + 2(V_r^b + V_r^s)\gamma_p - V_r^b\beta_p\delta\mu)\mu}{2(8 - \beta_p\mu^2)}. \tag{44}$$

We use (41)-(44) to rewrite value function equations in (37) and (39). We then once again conjecture value functions that are quadratic in state (see equations (29)-(32)) and follow the same approach as used in previous section. We obtain the following set of non-linear algebraic equations in the value function coefficients $s_j, b_j, j = 0, 1, 2$, which characterizes a feedback Stackelberg equilibrium.

$$\begin{aligned}
\beta_p(8 - \beta_p\mu^2)\phi s_2 =& \beta_r^2 + 4(b_2 + s_2)^2\gamma_p^2 - 16s_2\beta_p\gamma_r + 8s_2(2b_2 + s_2)\beta_p\delta^2 - 4(b_2 + s_2)^2\beta_p\gamma_p\delta\mu + \beta_p^2(2s_2\gamma_r + b_2^2\delta^2)\mu^2 \\
& - 2\beta_r(2(b_2 - 3s_2)\gamma_p - (b_2 + s_2)\beta_p\delta\mu + s_2\beta_p\gamma_p\mu^2) \qquad (45)\\
\beta_p(8 - \beta_p\mu^2)\phi s_1 =& 2(\alpha(\beta_r - 2b_2\gamma_p + 6s_2\gamma_p) + C_0\beta_p(-\beta_r + 2(b_2 + s_2)\gamma_p) + \gamma_p(-(b_1 - 3s_1)\beta_r + 2(b_1 + s_1)(b_2 + s_2)\gamma_p) - 4s_1\beta_p\gamma_r) \\
& + 8(b_2s_1 + (b_1 + s_1)s_2)\beta_p\delta^2 - \beta_p(-(b_1 + s_1)\beta_r + 2(b_2 + s_2)(\alpha - C_0\beta_p - 2(b_1 + s_1)\gamma_p))\delta\mu \\
& - \beta_p(2s_2\alpha\gamma_p + s_1\beta_r\gamma_p - s_1\beta_p\gamma_r - b_1b_2\beta_p\delta^2)\mu^2 \qquad (46)\\
4\beta_p(8 - \beta_p\mu^2)\phi s_0 =& 4\alpha^2 + 4C_0^2\beta_p^2 + 4b_1^2\gamma_p^2 + 8b_1s_1\gamma_p^2 + 4s_1^2\gamma_p^2 + 16b_1s_1\beta_p\delta^2 + 8s_1^2\beta_p\delta^2 - 4b_1^2\beta_p\gamma_p\delta\mu - 8b_1s_1\beta_p\gamma_p\delta\mu + 4s_1^2\beta_p\gamma_p\delta\mu + b_1^2\beta_p^2\delta^2\mu^2 \\
& - 4C_0(b_1 + s_1)\beta_p(-2\gamma_p + \beta_p\delta\mu) - 4\alpha(2C_0\beta_p + 2(b_1 - 3s_1)\gamma_p - (b_1 + s_1)\beta_p\delta\mu + s_1\beta_p\gamma_p\mu^2) \qquad (47)\\
\beta_p(8 - \beta_p\mu^2)^2\phi b_2 =& 4(\beta_r^2 + 4(7b_2 - s_2)\beta_r\gamma_p + 4(b_2^2\gamma_p^2 + 2b_2s_2\gamma_p^2 + s_2^2\gamma_p^2 - 8b_2\beta_p\gamma_r + 4b_2(b_2 + 2s_2)\beta_p\delta^2)) \\
& - 8(3b_2 + s_2)\beta_p(-\beta_r + 2(b_2 + s_2)\gamma_p)\delta\mu + 4\beta_p(b_2(-8\beta_r\gamma_p + 8\beta_p\gamma_r) + (b_2 - s_2)^2\beta_p\delta^2)\mu^2 \\
& + 2b_2\beta_p^2(-\beta_r + 2(b_2 + s_2)\gamma_p)\delta\mu^3 - b_2\beta_p^2(-2\beta_r\gamma_p + 2\beta_p\gamma_r + b_2\beta_p\delta^2)\mu^4 \qquad (48)\\
\beta_p(8 - \beta_p\mu^2)\phi b_1 =& 8(7b_1\beta_r\gamma_p - s_1\beta_r\gamma_p + 2b_1b_2\gamma_p^2 + 2b_2s_1\gamma_p^2 + 2b_1s_2\gamma_p^2 + 2s_1s_2\gamma_p^2 + \alpha(\beta_r + 14b_2\gamma_p - 2s_2\gamma_p) + C_0\beta_p(2(b_2 + s_2)\gamma_p - \beta_r) \\
& + 4\beta_p(-8b_2\alpha\gamma_p - 4b_1\beta_r\gamma_p + 4b_1\beta_p\gamma_r + (b_1 - s_1)(b_2 - s_2)\beta_p\delta^2)\mu^2 - 8b_1\beta_p\gamma_r + 8(b_2(b_1 + s_1) + b_1s_2)\beta_p\delta^2) \\
& + \beta_p^2(2b_2(-\alpha + C_0\beta_p + 2b_1\gamma_p + s_1\gamma_p) - b_1(\beta_r - 2s_2\gamma_p))\delta\mu^3 + \beta_p^2(2b_2\alpha\gamma_p + b_1\beta_r\gamma_p - b_1\beta_p\gamma_r - b_1b_2\beta_p\delta^2)\mu^4 \\
& - 4\beta_p(-2(3b_2 + s_2)(\alpha - C_0\beta_p) - (3b_1 + s_1)\beta_r + 4(3b_1b_2 + 2b_2s_1 + 2b_1s_2 + s_1s_2)\gamma_p)\delta\mu \qquad (49)\\
4\beta_p(8 - \beta_p\mu^2)^2\phi b_0 =& 16\alpha^2 + 16C_0^2\beta_p^2 + 16b_1^2\gamma_p^2 + 32b_1s_1\gamma_p^2 + 16s_1^2\gamma_p^2 + 64b_1^2\beta_p\delta^2 + 128b_1s_1\beta_p\delta^2 - 48b_1^2\beta_p\gamma_p\delta\mu \\
& - 64b_1s_1\beta_p\gamma_p\delta\mu - 16s_1^2\beta_p\gamma_p\delta\mu + 4b_1^2\beta_p^2\delta^2\mu^2 - 8b_1s_1\beta_p^2\delta^2\mu^2 + 4s_1^2\beta_p^2\delta^2\mu^2 + 4b_1^2\beta_p^2\gamma_p\delta\mu^3 + 4b_1s_1\beta_p^2\gamma_p\delta\mu^3 - b_1^2\beta_p^3\delta^2\mu^4 \\
& + 4C_0\beta_p(8(b_1 + s_1)\gamma_p - 4(3b_1 + s_1)\beta_p\delta\mu + b_1\beta_p^2\delta\mu^3) \\
& + 4\alpha(8(7b_1 - s_1)\gamma_p + \beta_p(-8C_0 + \mu(4(3b_1 + s_1)\delta - 16b_1\gamma_p\mu - b_1\beta_p\delta\mu^2 + b_1\beta_p\gamma_p\mu^3))) \qquad (50)
\end{aligned}$$

We can now state the following result.

Proposition 4: *A feedback Stackelberg equilibrium is obtained in which the wholesale price*

[7] To investigate the s.o.c. for supplier's problem, we find that $\frac{\partial^2 V^s}{\partial w^2} = -\frac{\beta_p}{\phi}$, $\frac{\partial^2 V^s}{\partial I_s^2} = -\frac{2}{\phi}$, and $(\frac{\partial^2 V^s}{\partial w^2}) * (\frac{\partial^2 V^s}{\partial I_s^2}) - (\frac{\partial^2 V^s}{\partial I_s \partial w})^2 = \frac{\beta_p(8-\beta_p\mu^2)}{4\phi^2}$. Given our assumption in centralized problem, i.e., $(4 - \beta_p\mu^2) > 0$, we will always have $(8 - \beta_p\mu^2) > 0$. The Hessian matrix for the supplier's problem will be negative definite and hence supplier's s.o.c. is also satisfied.

and innovation efforts policy of the supplier is given by the following two equations

$$
\begin{aligned}
w(r) = \frac{1}{\beta_p(8-\beta_p\mu^2)} \Bigg[& r * \Big(4\beta_r - 8(b_2 - s_2)\gamma_p - 4(b_2+s_2)\beta_p\delta\mu - \beta_p(\beta_r - 2b_2\gamma_p)\mu^2\Big) \\
& - 4(\alpha + C_0\beta_p + (s_1 - b_1)\gamma_p) + 2(b_1+s_1)\beta_p\delta\mu + \beta_p(\alpha - b_1\gamma_p)\mu^2 \Bigg], \\
I_s(r) = \frac{1}{(8-\beta_p\mu^2)} \Bigg[& r * \Big(s_2(8\delta - 2\gamma_p\mu) + \mu(\beta_r - 2b_2\gamma_p + b_2\beta_p\delta\mu)\Big) \\
& + \frac{8s_1\delta - 2(-\alpha + C_0\beta_p + (b_1+s_1)\gamma_p)\mu + b_1\beta_p\delta\mu^2}{2} \Bigg],
\end{aligned}
$$

and the retailer's retail price and innovation efforts policy is given by

$$
\begin{aligned}
p(r) = \frac{1}{\beta_p(8-\beta_p\mu^2)} \Bigg[& r * \Big(6\beta_r + 4(b_2+s_2)\gamma_p - 2(b_2+s_2)\beta_p\delta\mu - \beta_p\beta_r\mu^2\Big) \\
& + 2C_0\beta_p + (b_1+s_1)(2\gamma_p - \beta_p\delta\mu) + \alpha(6 - \beta_p\mu^2) \Bigg], \\
I_b(r) = rb_2\delta + \frac{b_1\delta}{2}. &
\end{aligned}
$$

The value functions of the two players are quadratic in the state variable and can be written in the form (29)-(30), where the coefficients $s_i, b_i, i = 0, 1, 2,$ are constants that depend on model parameters. The values of these coefficients are obtained by solving the non-linear algebraic equations in (45)-(50), and must satisfy the following condition in (51). [8]

$$
\frac{6\beta_r\gamma_p - 8\beta_p\gamma_r + 4(b_2+s_2)(\gamma_p^2 + 2\beta_p\delta^2 - \beta_p\gamma_p) + \beta_p\beta_r\delta\mu - \beta_p(\beta_r\gamma_p - \beta_p\gamma_r)\mu^2}{\beta_p(8-\beta_p\mu^2)} < \frac{\phi}{2}. \quad (51)
$$

In the next subsection we discuss some of our key observations from numerical analysis.

5.3 Numerical Analysis

In our paper, we have obtained several insights through analytical means, particularly in the case of exogenously determined prices (Section 3). However, when prices are also decision variables, given the complexity of the expressions for optimal policies and the non-linear set of algebraic equations that need to be solved, we have to resort to numerical analysis to obtain some more key insights, particularly on the sensitivity of optimal policies w.r.t.

[8] The condition in (51) is required to satisfy the growth conditions of the value functions, i.e., (38) and (40). The readers may refer to the proofs of Propositions 1, 2, and 3 in the Appendix for similar approach. To summarize, the state equation can written in the form $\dot{r} = A_1 * r + A_0$, where A_1 and A_0 are constants. To satisfy the growth conditions of value functions, we must have $A_1 < \phi/2$. The L.H.S. in equation (51) is A_1 written in terms of coefficients s_2 and b_2, and hence we get this condition in (51).

different model parameters. In our experiments, we fixed the values of all parameters except the one parameter of interest, and then changed the value of that parameter to understand how optimal decisions change with that parameter, given any goodwill level r. We conducted numerical experiments for a wide array of model parameters and in this section, and report some results that are representative of the larger insights obtained across these sets of model parameters that we used. We first report some results for the centralized problem.

Figures 2 - 10 depict the changes in optimal feedback policies w.r.t. changes in different parameters in a centralized channel. Optimal feedback innovation effort $I(r)$ and retail price $p(r)$ are linear in state r. For this section, we denote p_r and I_r as the slope, i.e., first derivative of $p(r)$ and $I(r)$, respectively, w.r.t. r; and p_0 and I_{i_0} as the constant terms in $p(r)$ and $I(r)$, respectively. Thus, the optimal feedback decisions can be written as $p(r) = r * p_r + p_0$, and $I(r) = r * I_r + I_0$. p_r, I_r, p_0, and I_{i_0}, are therefore constant coefficients whose values depend on model parameters and Figures 2 - 10 show sensitivity of these coefficients w.r.t. different model parameters. We summarize some of our observations below.

We found that as the base cost C_0 increases (Figure 2), the firm may charge a higher retail price to compensate for the higher cost, however the innovation effort does not seem to have a significant change or a clear trend. As base value of demand (α) increases (Figure 3), the firm has a larger market potential. The firm then looks to charge a higher retail price to take advantage of higher demand and its innovation effort might also increase slightly. Figures (4) and (5) show a clear trend of the firm decreasing its price as well as its innovation efforts as the customer demand becomes more sensitive to price (higher (β_p)), whereas it increases price and innovation efforts when the market is more sensitive to goodwill or perceived quality (higher β_r). In terms of sensitivity w.r.t. goodwill-dynamics parameters, i.e., γ_p, γ_r, and δ (Figures 6, 7, and 9), we find that both price and innovation efforts are increasing in γ_p and δ, and both are decreasing in γ_r. Figure 8 shows the sensitivity w.r.t. μ, and here we observe that while the innovation efforts increase with μ, the retail price decreases with μ when μ is positive. Thus, when innovation leads to a higher efficiency in unit costs, the firm increases its innovation efforts but simultaneously reduces its retail price. On the other hand when innovation might increase unit production cost ($\mu < 0$), while the innovation efforts follow same trend, i.e., increasing in μ; the retail price however might not change significantly. In other words, when innovation efforts increase unit production costs, the firm puts a lower innovation effort as their cost-increasing impact gets higher (i.e., μ taking a more negative value). Furthermore, when μ is negative and a very low value, i.e., when innovation leads to significant increase in unit costs, we find that $I_r < 0$, which means that the innovation efforts are decreasing in goodwill. Finally, as the firm's discount rate ϕ increases, its prices and innovation efforts decrease. Thus, a firm which tends to put more weight-age on instantaneous profits rather than long term cash flow (hence higher discount rate), will invest less in innovation and will charge less price from customers.

Figures 11 - 18 show the sensitivity of optimal policies w.r.t. different model parameters in a two-echelon decentralized channel, for a given goodwill level r. On similar lines as above, we denote $p_{r1} = \frac{\partial p(r)}{\partial r}, w_{r1} = \frac{\partial w(r)}{\partial r}, I_{b_{r1}} = \frac{\partial I_b(r)}{\partial r}$, and $I_{s_{r1}} = \frac{\partial I_s(r)}{\partial r}$. Similarly, $p_{r0}, w_{r0}, I_{b_{r0}}$, and $I_{s_{r0}}$, denote the constant terms in the optimal feedback pricing and innovation efforts policies. Once again, while we conducted numerical analysis for a wide set of parameters, Figures 11 - 18 present representative results depicting typical trends that we observed. In these figures, we we set the base value of the parameters as follows: $C_0 = 1, \alpha = 1, \beta_p = 0.1, \beta_r = 0.1, \gamma_p = 0.1, \gamma_r = 0.1, \delta = 0.1, \mu = 0.1$, and $\phi = 0.1$. We then changed the values of each parameter one by one while keeping the other parameters at the same value as above. We summarize some of our key findings below.

We find that as the base cost C_0, increases (Figure 11), the seller charges a higher wholesale price and consequently the retailer also charges a higher retail price. Interestingly, both the firms also reduce their innovation efforts. With regard to μ, (Figure 12), we find that both the prices as well as the buyer's innovation effort do not seem to change much w.r.t. μ, whereas the supplier's innovation effort is much more sensitive to it and increases as the cost savings due to innovation efforts increases. Similar to the centralized channel, we find that when innovation increases unit cost and its impact on the cost is very high ($\mu < 0$ with a high absolute value), the seller's innovation effort is actually decreasing in goodwill. Similar to the pattern observed in the policies of a centralized firm, we find that both the firm's prices and innovation efforts are decreasing in demand sensitivity to price β_p (Figure 13), increasing in demand sensitivity to goodwill β_r (Figure 14), decreasing in price's impact on goodwill γ_p (Figure 15), and decreasing goodwill decay factor γ_r (Figure 16). As the overall innovation efforts' impact on goodwill (δ) increases (Figure 17), we find that innovation efforts by both the firms increase. Interestingly however, we find that as δ increases, the buyer increases its innovation effort, whereas we did not find a significant change in the seller's innovation effort. Finally, similar to the result in centralized channel, we find that as the common discount factor for both the firms increases, both the firms' prices and innovation efforts decrease. (Figure 18)

6 Discussion and Concluding remarks

We consider a supply chain consisting of a seller and a buyer with a vertical collaboration framework in innovation efforts. We first consider a case when the two parties make their respective dynamic innovation effort decisions with product goodwill as the state variable, and then extend this to include dynamic pricing decisions as well. We model the problem as a Stackelberg differential game where the seller first announces its wholesale price and innovation effort over time and the buyer responds with its retail price and innovation effort over time. We focus on the feedback Stackelberg solution and despite the analytical complexity of the model are able to obtain the feedback policies for a centralized firm, and the same for the two firms in a decentralized channel. The optimal feedback policies for innovation and pricing decisions obtained are linear in the state variable. We obtain some very useful managerial insights in different scenarios using both analytical as well as numerical means. When prices are exogenous, we find that innovation efforts policies in centralized as well as decentralized channel are very much dependent on how innovation efforts affect the unit cost (i.e., do they lead to unit cost reduction or increment?). We compare total innovation efforts in a centralized vs a decentralized channel. We find that when innovation efforts are cost reducing, a decentralized channel has higher aggregate innovation efforts; whereas when innovation efforts increase unit cost, a centralized channel offers higher overall innovation efforts. When prices are also decision variables, due to the complexity of the the model, we use numerical analysis to study the sensitivity of optimal pricing and innovation policies w.r.t. various model parameters

In terms of manegerial insights, some of the main contributions of paper are: to understand the incentives of the supply chain firms for continuous innovation and improvement efforts over time, and to understand the dependence of innovation efforts on various model parameters such as the base cost, value of the product (when prices are exogenous), demand sensitivity to price etc. In addition to this, our paper attempts to address these manegerial questions for a centralized supply chain structre as well. When prices are exogenous, we note the clear difference between how the innovation policies of the two firms (supplier and buyer) differ with regard their feedback on the product goodwill and threfore the difference in how the two firms' polcies may evolve over time. While the supplier's response to the goodwill will depend on what impact does innovation has on its unit cost (positive), the buyer's response in no scenario increases with goodwill. In some cases it is optimal for the buyer to have a steady effort over time, and in other cases it is optimal for the buyer to decrease its innovation effort as goodwill increases. Thus, if suppose the initial goodwill of the product is low enough, it is all but intuitive to expect that the goodwill will very likely increase with time. Furthermore, if innovation is cost-reducing, one will find that the supplier will have an incentive to increase its innovation effort with goodwill and effectively increase its efforts with time. The buyer, however, will have an incentive to either have a steady policy or even to reduce its efforts over time, depending on the model parameters. In our opinion, this particular aspect also highlights the value of considering a dynamic model as opposed to a static one in which the two firms's policies will not change with time.

When the firms are engaged in pricing decisions along with innovation decisions, we obtain some interesting results via numerical analysis. Some of the findigs from our numerical experiments may yield some ineresting insights into how the firms (and the supply chain overall) may look to strategically position the product over time in the market, and might help in identifying circumstances under which the firms would eventually place the prouct as a high-price and high-innovation product vs a low-price low innovation product. This may depend on factors which are market and consumer driven (for e.g. demand and consumer goodwill parameters), operational parameters (such as base cost, and μ,), as well as financial parameters (such as discount rate ϕ). For e.g., if the market is more sensitive to price (high β_p) and less sensitive to brand goodwill (low β_r), the firms' actions may eventually lead to a scenario where the we have a lower priced product with low innovation levels. In marketing literature, it is widely understood that many times price plays an important role in how some customers perceive the quality of the product, and this effect is very much a function of product category, geography, and culture etc. Relating this to our model, one can say that if a market or product category is such that the impact of price on perceived quality (or goodwill), i.e., γ_p is very high, along with high impact of innovation as well (high δ), we are likely to see a scenario where the supply chain wide innovation levels are high and the product is also priced at a higher level. On the operational side, if we have a scenario where the innovation also leads to significant cost efficiencies (high μ), one might find a scenario which is actually very favourable for the consumer welfare, i.e., high innovation levels coupled with potentially small decrements in retail price. Such observations may help the decision makers in predicting the overall strategic and marketing focus of firms.

In our model we do not account for any manufacturing or processing cost at the downstream firm, i.e., the buyer's end. However, our model could be easily extended to explicitly model the situation where, for e.g., the buyer is a manufacturer, who buys a part from the supplier, enhances its value through further processing or uses it as a sub-assembly, incurs a positive processing/assembly cost and then sells the final product to the customers. Examples of such scenarios could be a cellphone manufacturer (like Samsung or Apple) working with the manufacturer of the camera, or a car manufacturer working with a seat seat supplier. In this case one can simply write the total unit cost for the downstream firm as the sum of wholesale price plus its own processing cost, and then the buyer will set a retail price that is higher than this total unit cost. One can argue that as long as the buyer's processing cost is fixed, i.e., does not change with time or depend upon the two firms' decisions; then without loss of generality one can assume it to be zero, which is essentially the case in our model. Thus, our insights should hold for the case when the buyer incurs a fixed and constant non-zero processing cost. However, there could be a case where the innovation efforts might impact unit cost at the buyer's end as well, for e.g., changes in design at the upstream end may bring about changes in assembly process at the buyer, and hence changes the buyer's assembly and handling cost. We have attempted such a scenario but in a dynamic model as ours, it further increases the complexity of the model and a closed form feedback solution becomes very difficult to obtain.

In this paper, we study the difference between total innovation effort in a centralized channel with that of a decentralized channel with vertical collaboration. We would like to add a remark that while making this comparison, one would be advised to recall that given the nature of our model, the cost structure in the two scenarios is slightly different. In a centralized channel, the instantaneous cost of the innovation efforts at any time is the square of total innovation effort by the centralized firm. However, in a decentralized channel the total cost of the innovation efforts in the supply chain is the sum of costs incurred by two players and that is going to be less than the square of the total channel innovation effort ($I_s^2 + I_b^2 \leq (I_s + I_b)^2$). While one could understandably argue against a direct comparison of the two scenarios because of this, we think that despite the difference in the cost structure, there is some value in this comparison. An outcome of this difference in cost in two channels is that if the total channel innovation efforts is same in the two types of channels, i.e., $I = I_b + I_s$, then by the very nature of how cost is accounted for (quadratic in each firm's innovation effort), a centralized channel will incur a higher overall cost. In other words, a decentralised channel, due to sharing of innovation efforts, could achieve same level of overall innovation as a centralized firm but with a lower overall cost, provided all the other factors are the same. The larger issue of whether a firm should take control of development and production of innovative components thus creating a centralized structure, vs a firm collaborating with its supplier is an important strategic question. This is highlighted in the Tesla and Panasonic example where as mentioned previously, Tesla's recent efforts are towards developing and making its own battery cells and on cutting down on collaboration with its supplier Panasonic. This obviously depends on a number of factors, and through this comparison, our paper attempts to explore the role of some of these factors. Thus, our insights hold in an environment when we have the following two conditions working together: i) the overall impact of innovation by two firms depends on the sum of their independently determined efforts (say for e.g. on the total man-hours in an R&D joint venture committed by two firms independently), and ii) the cost of these innovation efforts by a firm yields marginally diminishing returns. On the other hand, it is possible that a firm may face a significant initial cost in innovation and R&D, and is then able to observe some 'economics of scale' in its efforts. In such an environment, our cost assumptions will not hold and the insights in centralized vs decentralized channels could be very different.

Finally, in conclusion, this paper adds to the current literature in theoretical terms (in the domain of examples of feedback Stackelberg equilibria in supply chain); as well as in application by presenting optimal strategies and insights to analyse the dynamics and economics of innovation along with pricing decisions in a supply chain.

Acknowledgement:

Funding: This work was partially supported by the Saint-Étienne Métropole gouvernment, France, as well as by PHC CAI YUANPEI Programme, France–China [grant number 30246UK] and Chinese Academy of Sciences visiting professorship for Senior International Scientists [grant number 2013T2J0054].

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Appendix

A: Obtaining feedback policy in Proposition 1

To ensure that the solution in equations (14)-(16) yield a feedback policy and the result in Proposition 1, this solution must satisfy the boundary (growth) condition of the value function in (8). Using (13), we can write the state equation in (6) as $\dot{r} = A_1 r + A_0$, where $A_1 = -\gamma_r + a2\delta^2 + (\beta_r\delta\mu)/2$ is a constant. To satisfy the boundary condition (8), we essentially need $2A_1 - \phi < 0$. Solving equation (14) gives us 2 values in a_2. These are: $a_2 = \frac{2\gamma_r - \mu\beta_r\delta + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta^2}$ and $a_2 = \frac{2\gamma_r - \mu\beta_r\delta + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta^2}$. It can be seen that the first value of a_2 does not satisfy $2A_1 - \phi < 0$, whereas the second value of a_2 does. Hence we use $a_2 = \frac{2\gamma_r - \mu\beta_r\delta + \phi - \sqrt{(2\gamma_r+\phi)(2\gamma_r - 2\mu\beta_r\delta + \phi)}}{2\delta^2}$ to obtain the feedback policy. Constants a_1 and a_0 can be obtained using this value of a_2 in the equations (15)-(16).

B: Proof of Proposition 2

As discussed in section 3.2 in the main text, we use (29)-(32) to rewrite value functions in (27)-(28), and then compare these value function expressions to (29)-(30). Consequently, we write the following non-linear algebraic equations in the constants s_j and b_j, $j = 0, 1, 2$.

$$4\phi s_2 = -8s_2\gamma_r + (\mu\beta_r + 2s_2\delta)(\mu\beta_r + 2(2b_2 + s_2)\delta) \tag{52}$$

$$2\phi s_1 = 2(w - C_0)\beta_r + \mu^2(\alpha - p\beta_p)\beta_r + 4ps_2\gamma_p - 2s_1\gamma_r + \mu(2(b_2 + s_2)(\alpha - p\beta_p) + (b_1 + s_1)\beta_r)\delta + 2(b_2 s_1 + (b_1 + s_1)s_2)\delta^2 \tag{53}$$

$$4\phi s_0 = (\alpha - p\beta_p)(4w - 4C_0 + 2\mu(b_1 + s_1)\delta) + \mu^2(\alpha - p\beta_p)^2 + 4ps_1\gamma_p + s_1(2b_1 + s_1)\delta^2 \tag{54}$$

$$\phi b_2 = b_2(-2\gamma_r + \delta(\mu\beta_r + (b_2 + 2s_2)\delta)) \tag{55}$$

$$2\phi b_1 = 2p(\beta_r + 2b_2\gamma_p) - 2(w\beta_r + b_1\gamma_r) + \mu(2b_2(\alpha - p\beta_p) + b_1\beta_r)\delta + 2(b_2 s_1 + b_1(b_2 + s_2))\delta^2 \tag{56}$$

$$4\phi b_0 = -4(p - w)(-\alpha + p\beta_p) + 4b_1 p\gamma_p + 2b_1\mu(\alpha - p\beta_p)\delta + b_1(b_1 + 2s_1)\delta^2 \tag{57}$$

We can solve the equations (52)-(57) to get values of constants s_j and b_j, $j = 0, 1, 2$, in different cases and can therefore obtain a feedback Stackelberg equilibrium. Solving equations (52) and (55) in constants s_2 and b_2, we get the following 4 solutions: namely, solution 1, 2, 3, and 4, in equations (58), (59), (60), and (61), respectively.

$$s_{2a} = \frac{2\gamma_r - \beta_r\delta\mu + \phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)}}{2\delta^2}, \qquad b_{2a} = 0 \tag{58}$$

$$s_{2b} = \frac{2\gamma_r - \beta_r\delta\mu + \phi + \sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)}}{2\delta^2}, \qquad b_{2b} = 0 \tag{59}$$

$$s_{2c} = \frac{(2\gamma_r - 3\beta_r\delta\mu + \phi) - \sqrt{(4\gamma_r^2 + 6\beta_r\delta\mu\phi + \phi^2 + 4\gamma_r(3\beta_r\delta\mu + \phi))}}{6\delta^2}, \quad b_{2c} = \frac{2(2\gamma_r + \phi) + \sqrt{(4\gamma_r^2 + 6\beta_r\delta\mu\phi + \phi^2 + 4\gamma_r(3\beta_r\delta\mu + \phi))}}{3\delta^2} \tag{60}$$

$$s_{2d} = \frac{(2\gamma_r - 3\beta_r\delta\mu + \phi) + \sqrt{(4\gamma_r^2 + 6\beta_r\delta\mu\phi + \phi^2 + 4\gamma_r(3\beta_r\delta\mu + \phi))}}{6\delta^2}, \quad b_{2c} = \frac{2(2\gamma_r + \phi) - \sqrt{(4\gamma_r^2 + 6\beta_r\delta\mu\phi + \phi^2 + 4\gamma_r(3\beta_r\delta\mu + \phi))}}{3\delta^2} \tag{61}$$

Using (24), (26), (31), and (32), we can write the state equation as $\dot{r} = A_1 r + A_0$, where A_1 and A_0 are constants, and we have $A_1 = -\gamma_r + (b_2 + s_2)\delta^2 + (\beta_r\delta\mu)/2$. To satisfy the growth conditions in (21) and (23), essentially, we need to ensure that $\lim_{t\to\infty} e^{-\phi t} r^2(t) = 0$, which then translates to $2A_1 - \phi < 0$ or $A_1 < \phi/2$. We use all the above 4 solutions to calculate $A_1 - \phi/2$ and find that we always have $A_1 > \phi/2$ in solutions 2 and 3. However, using solution 1, we get $A_{1a} - \phi/2 = -(1/2)\sqrt{(2\gamma_r + \phi)(2\gamma_r - 2\beta_r\delta\mu + \phi)}$; and using solution 4, we get $A_{1d} - \phi/2 = 1/6(4\gamma_r + 2\phi - \sqrt{(2\gamma_r + \phi)(2\gamma_r + 6\beta_r\delta\mu + \phi)})$. Thus, we have $A_{1a} < \phi/2$ always as long as $A_1 a$ is real, i.e., $(2\gamma_r - 2\beta_r\delta\mu + \phi) > 0$. Moreover, it can be shown that for solution 4 to satisfy the growth constraint, i.e., to have $A_{1d} < \phi/2$, we require $(2\gamma_r - 2\beta_r\delta\mu + \phi) < 0$, in which case solution 1 does not give a real value. Furthermore, it can be seen that $(2\gamma_r - 2\beta_r\delta\mu + \phi) < 0$ requires $\mu > 0$, which in turn guarantees a real value in solution 4. Thus, the equilibrium is given by solution 1 when $(2\gamma_r - 2\beta_r\delta\mu + \phi) > 0$, and solution 4 when $(2\gamma_r - 2\beta_r\delta\mu + \phi) < 0$. We then use the respective values of s_2 and b_2 in equations (52)-(57) to obtain the values of remaining coefficients, and then use them in (24) and (26) to obtain feedback equilibrium innovation efforts as discussed in Proposition 2.

C: Further details on proof of Proposition 3

We follow similar steps as used to derive the results in Proposition 1 and 2, as described in their respective Proofs in the Appendix. Similar to Section 3.1, we obtain the following set of non-linear algebraic equations in the coefficients a_2, a_1, and a_0.

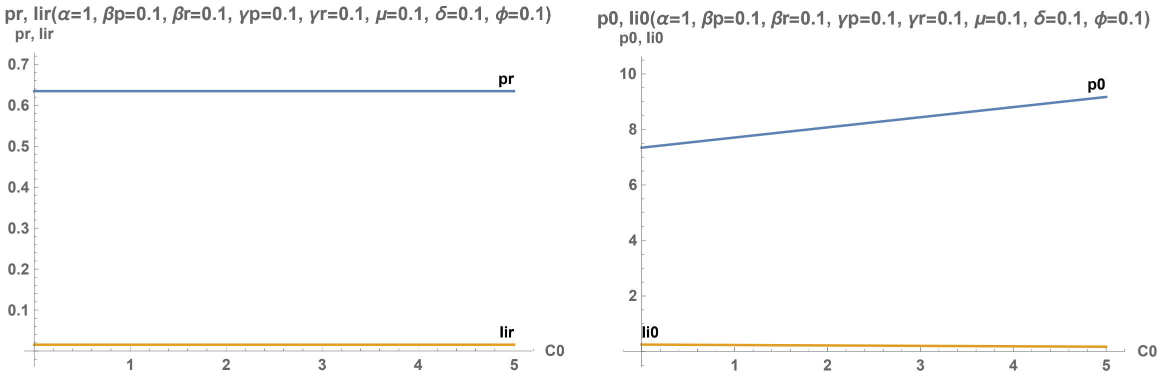
$$\begin{aligned}
\beta_p(4-\beta_p\mu^2)\phi a_2 &= \beta_r^2 + 2a_2\beta_r(\beta_p\delta\mu + \gamma_p(2-\beta_p\mu^2)) \\
&\quad + 2a_2(2a_2(\gamma_p^2 + \beta_p\delta^2 - \beta_p\gamma_p\delta\mu) + \beta_p\gamma_r(-4+\beta_p\mu^2)) \qquad (62) \\
\beta_p(4-\beta_p\mu^2)\phi a_1 &= 2\alpha(\beta_r + a_2(2\gamma_p + \beta_p\delta\mu - \beta_p\gamma_p\mu^2)) - 2C_0\beta_p(\beta_r - 2a_2\gamma_p + a_2\beta_p\delta\mu) \\
&\quad - a_1(-2\gamma_p(\beta_r + 2a_2\gamma_p) + 4\beta_p\gamma_r - 4a_2\beta_p\delta^2 - \beta_p(\beta_r - 4a_2\gamma_p)\delta\mu + \beta_p(\beta_r\gamma_p - \beta_p\gamma_r)\mu^2) \qquad (63) \\
\beta_p(4-\beta_p\mu^2)\phi a_0 &= \alpha^2 + C_0^2\beta_p^2 - a_1C_0\beta_p(-2\gamma_p + \beta_p\delta\mu) + a_1^2(\gamma_p^2 + \beta_p\delta^2 - \beta_p\gamma_p\delta\mu) \\
&\quad - \alpha(-2a_1\gamma_p + \beta_p(2C_0 + a_1\mu(-\delta + \gamma_p\mu))) - a_2\beta_p(4-\beta_p\mu^2)\sigma^2 \qquad (64)
\end{aligned}$$

To obtain feedback policies, we solve for parameters a_2, a_1, and a_0 in equations (62)-(64). Once again, we can write the state equation in (6) as $\dot{r} = A_1 r + A_0$, where A_1 and A_0 are constants and we have in this case

$$A_1 = \frac{2\beta_r\gamma_p - 4\beta_p\gamma_r + 4a_2(\gamma_p^2 + \beta_p\delta^2) + \beta_p(\beta_r - 4a_2\gamma_p)\delta\mu - \beta_p(\beta_r\gamma_p - \beta_p\gamma_r)\mu^2}{\beta_p(4-\beta_p\mu^2)}.$$

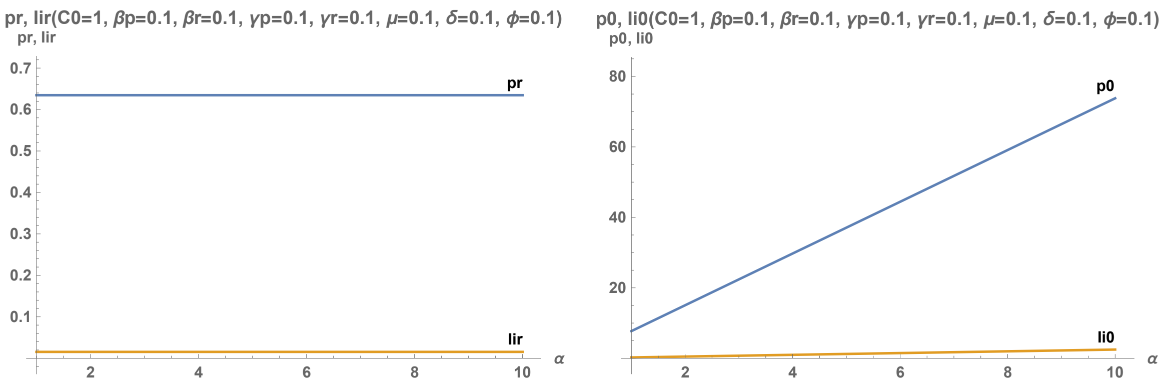
As discussed earlier, to satisfy the growth condition in (34), we need to have the value of a_2 such that $A_1 < \phi/2$. Solving equation (62) we get two values for a_2. Only one of those two values satisfies the condition $A_1 < \phi/2$, and we use that value to obtain the optimal feedback policy.

D: Figures from Numerical Analysis



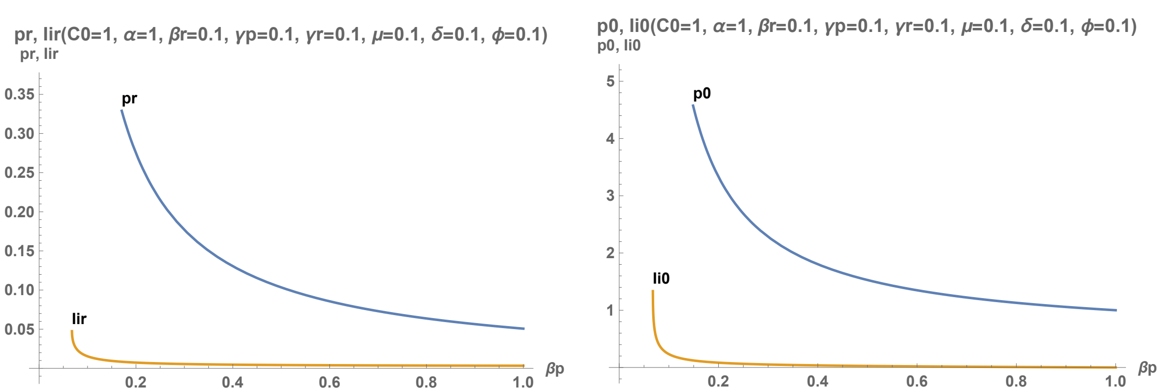
(a) p_r, Ii_r vs C_0 (b) p_0, Ii_0 vs C_0

Figure 2: Price, Innovation efforts vs C_0



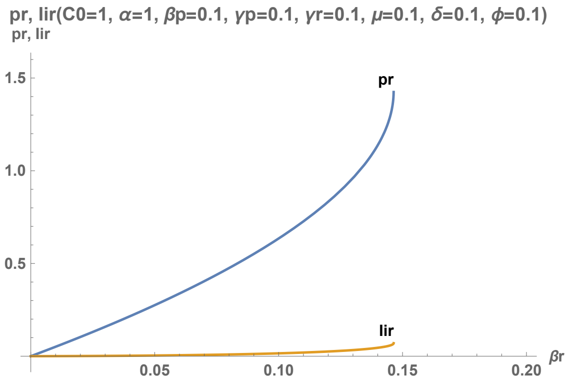
(a) p_r, Ii_r vs α (b) p_0, Ii_0 vs α

Figure 3: Price, Innovation efforts vs α

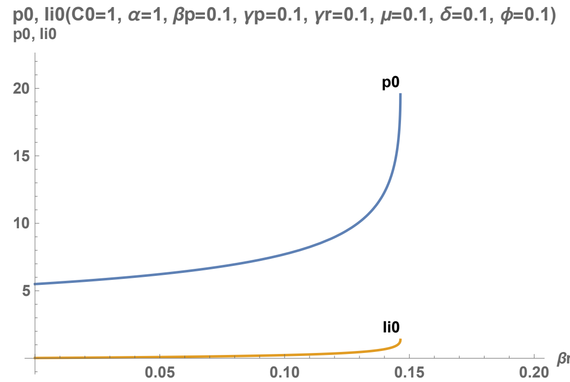


(a) p_r, Ii_r vs β_p (b) p_0, Ii_0 vs β_p

Figure 4: Price, Innovation efforts vs β_p

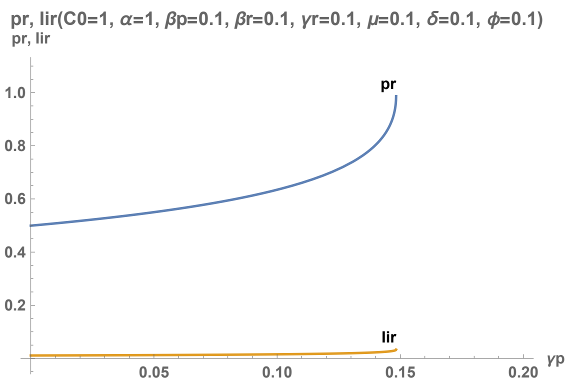


(a) p_r, Ii_r vs β_r

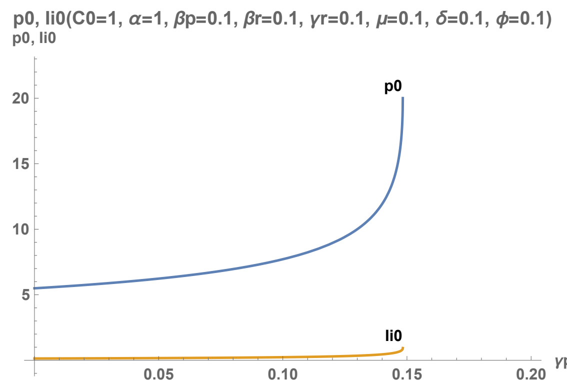


(b) p_0, Ii_0 vs β_r

Figure 5: Price, Innovation efforts vs β_r

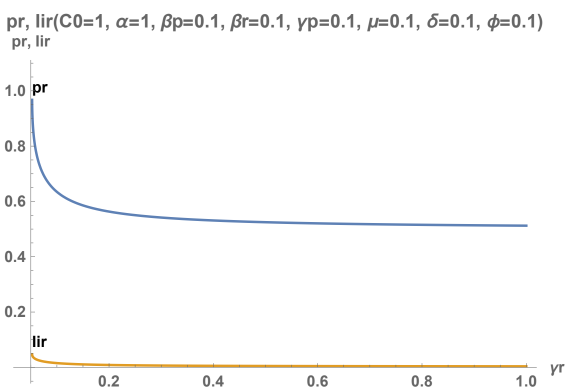


(a) p_r, Ii_r vs γ_p

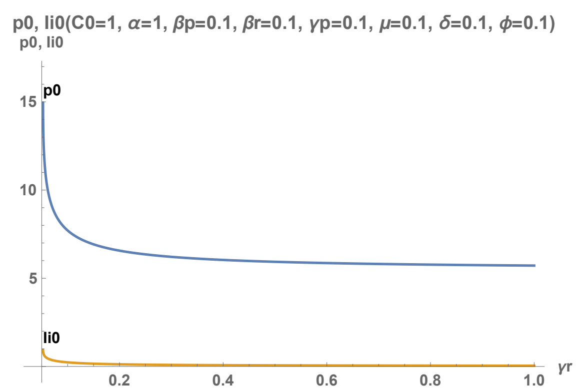


(b) p_0, Ii_0 vs γ_p

Figure 6: Price, Innovation efforts vs γ_p

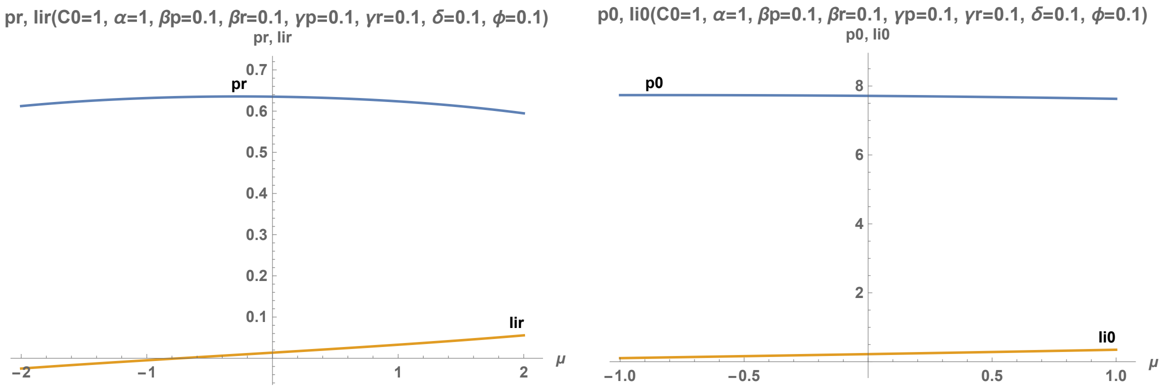


(a) p_r, Ii_r vs γ_r



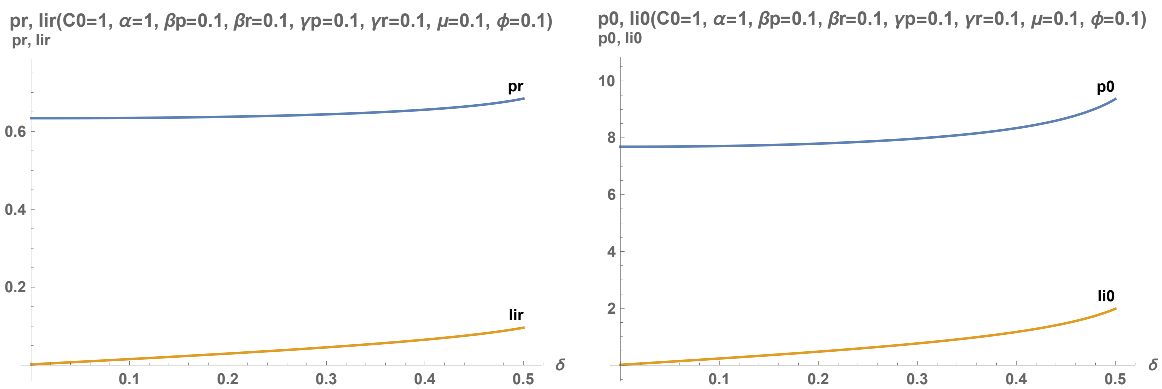
(b) p_0, Ii_0 vs γ_r

Figure 7: Price, Innovation efforts vs γ_r



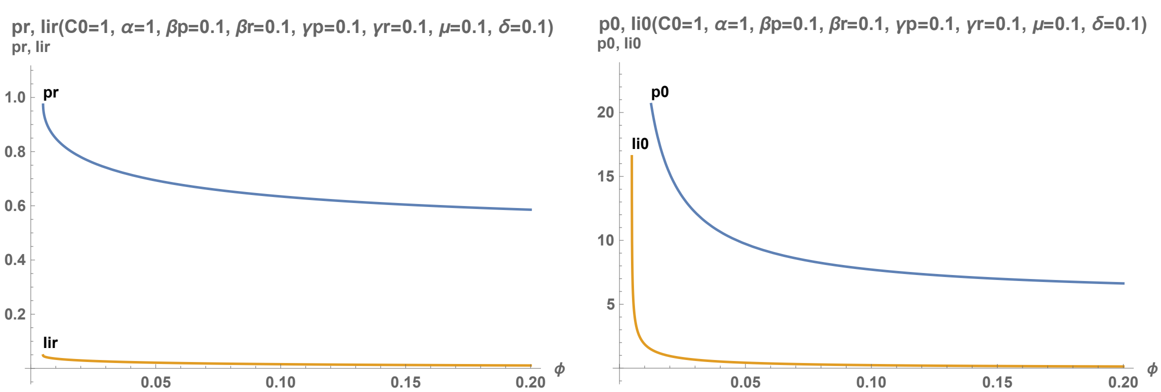
(a) p_r, Ii_r vs μ (b) p_0, Ii_0 vs μ

Figure 8: Price, Innovation efforts vs μ



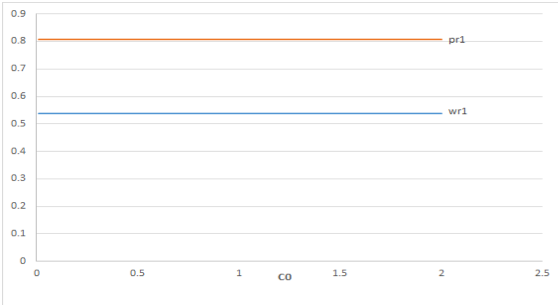
(a) p_r, Ii_r vs δ (b) p_0, Ii_0 vs δ

Figure 9: Price, Innovation efforts vs δ

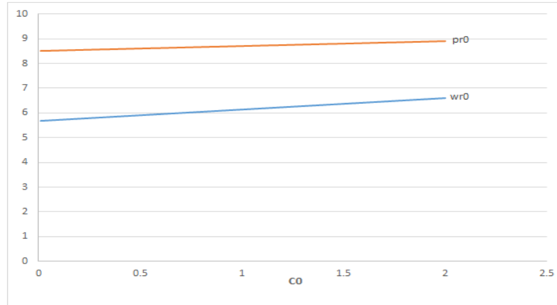


(a) p_r, Ii_r vs ϕ (b) p_0, Ii_0 vs ϕ

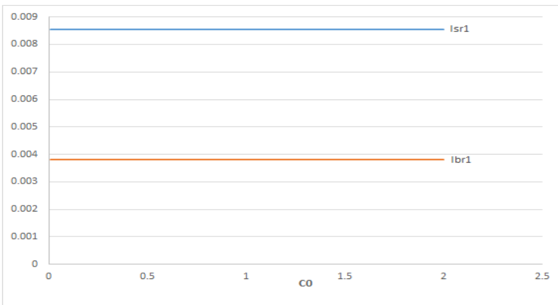
Figure 10: Price, Innovation efforts vs ϕ



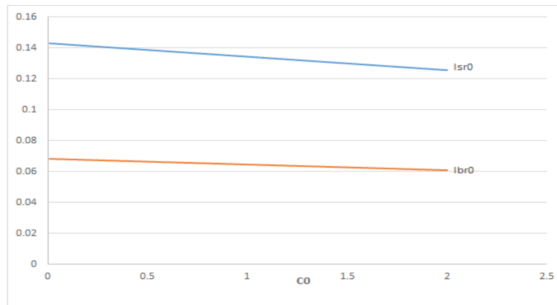
(a) w_{r1}, p_{r1} vs C_0



(b) w_{r0}, p_{r0} vs C_0

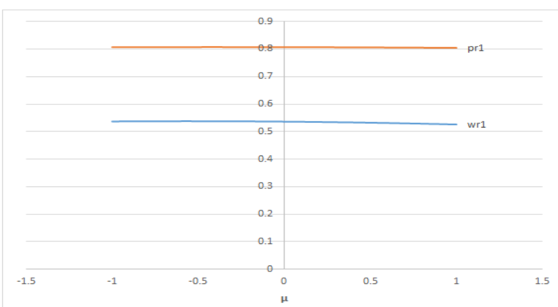


(c) Is_{r1}, Ib_{r1} vs C_0

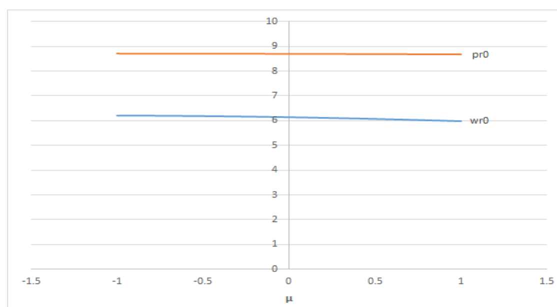


(d) Is_{r0}, Ib_{r0} vs C_0

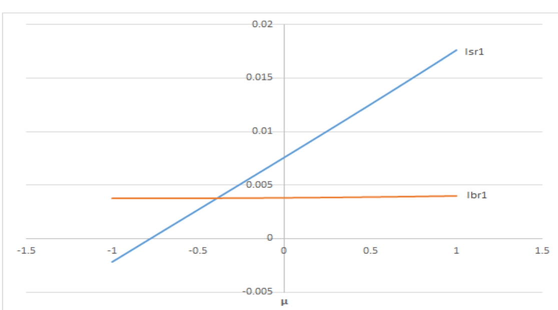
Figure 11: Price, Innovation efforts vs C_0



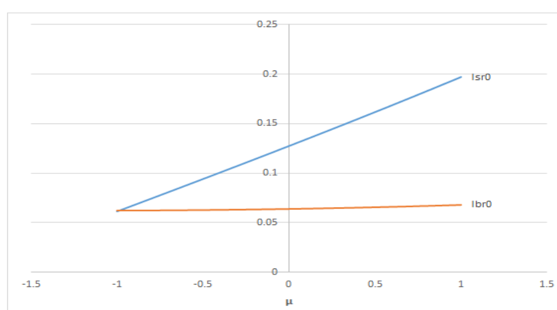
(a) w_{r1}, p_{r1} vs μ



(b) w_{r0}, p_{r0} vs μ

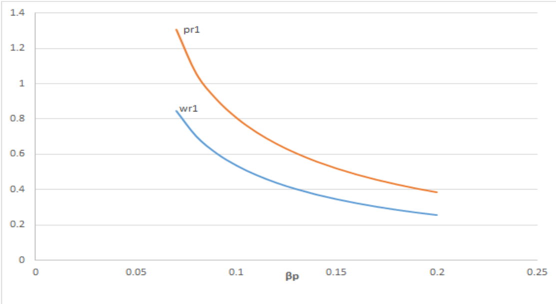


(c) Is_{r1}, Ib_{r1} vs μ

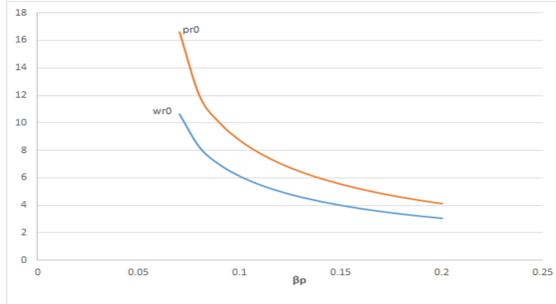


(d) Is_{r0}, Ib_{r0} vs μ

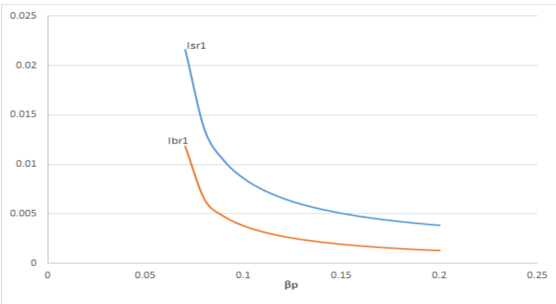
Figure 12: Price, Innovation efforts vs μ



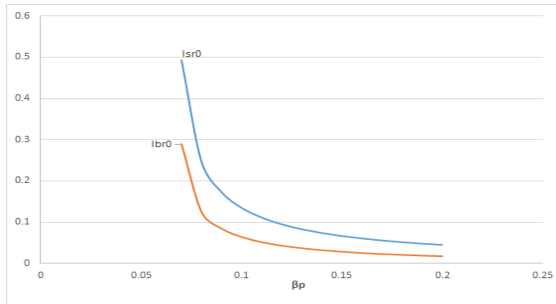
(a) w_{r1}, p_{r1} vs β_p



(b) w_{r0}, p_{r0} vs β_p

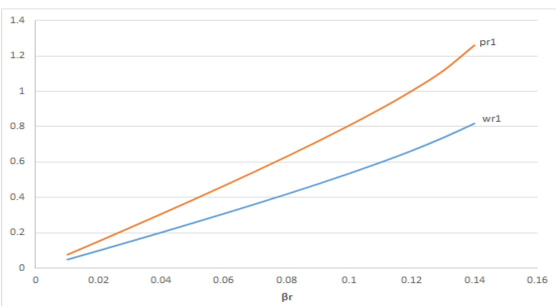


(c) Is_{r1}, Ib_{r1} vs β_p

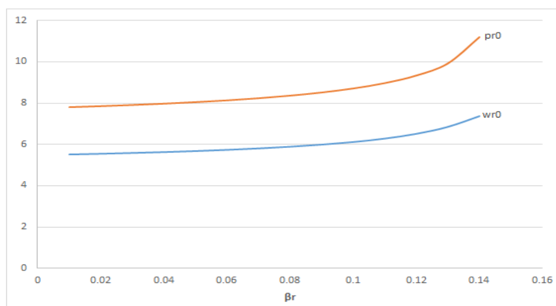


(d) Is_{r0}, Ib_{r0} vs β_p

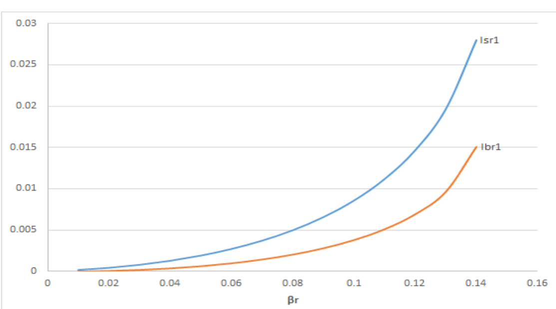
Figure 13: Price, Innovation efforts vs β_p



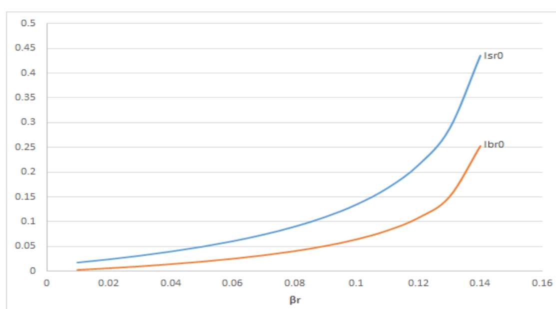
(a) w_{r1}, p_{r1} vs β_r



(b) w_{r0}, p_{r0} vs β_r

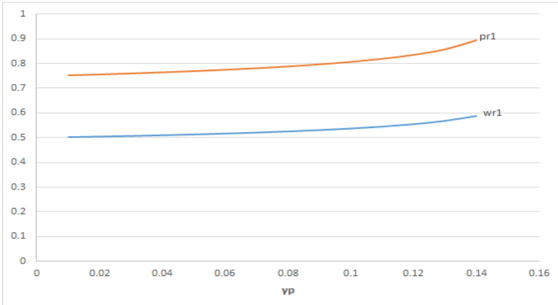


(c) Is_{r1}, Ib_{r1} vs β_r

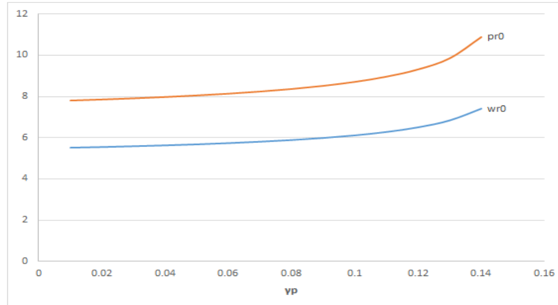


(d) Is_{r0}, Ib_{r0} vs β_r

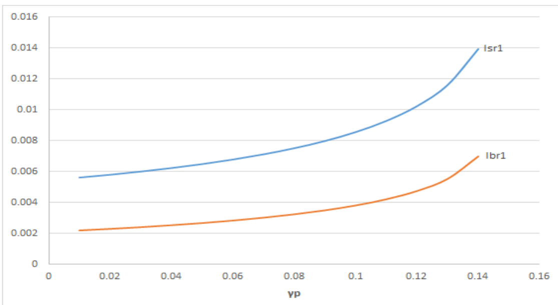
Figure 14: Price, Innovation efforts vs β_r



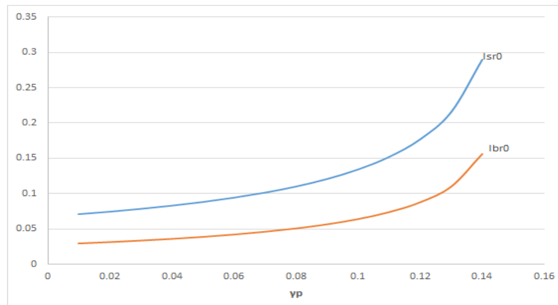
(a) w_{r1}, p_{r1} vs γ_p



(b) w_{r0}, p_{r0} vs γ_p

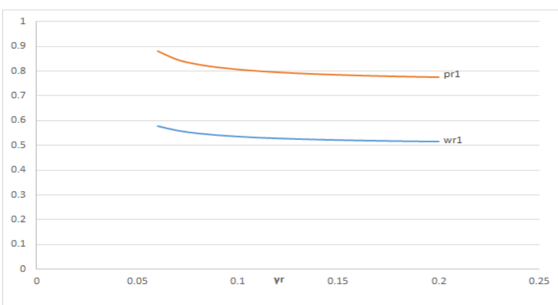


(c) Is_{r1}, Ib_{r1} vs γ_p

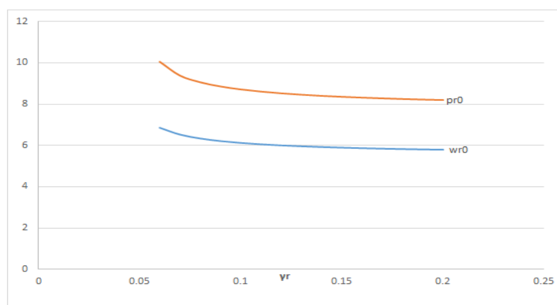


(d) Is_{r0}, Ib_{r0} vs γ_p

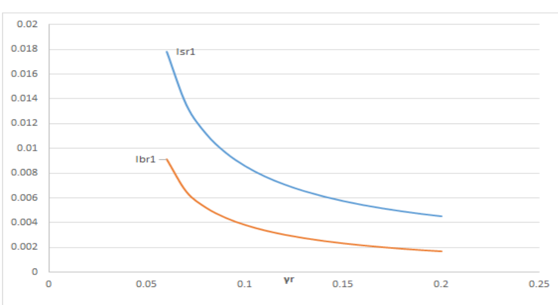
Figure 15: Price, Innovation efforts vs γ_p



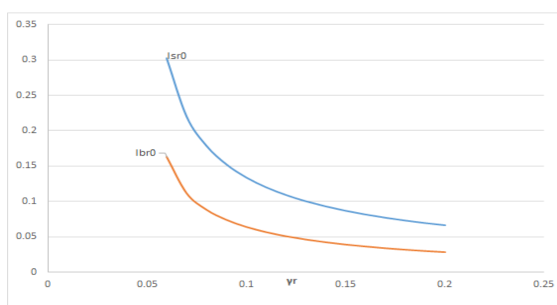
(a) w_{r1}, p_{r1} vs γ_r



(b) w_{r0}, p_{r0} vs γ_r

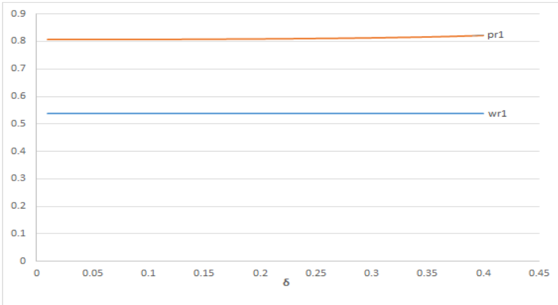


(c) Is_{r1}, Ib_{r1} vs γ_r

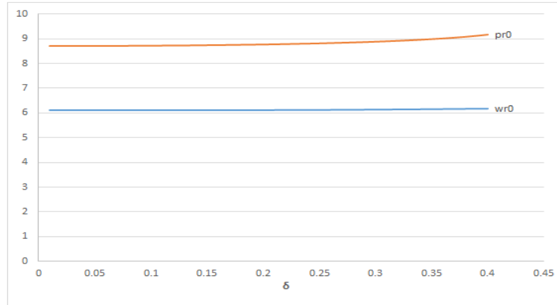


(d) Is_{r0}, Ib_{r0} vs γ_r

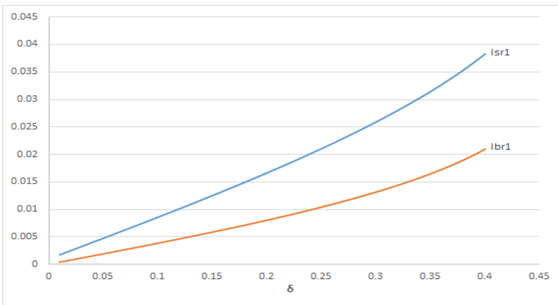
Figure 16: Price, Innovation efforts vs γ_r



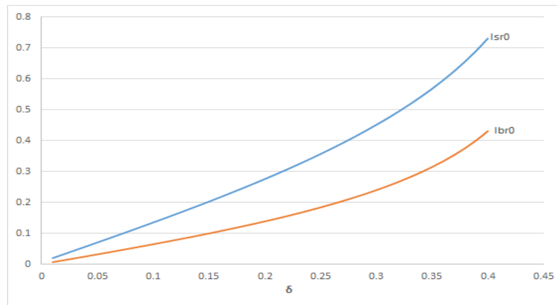
(a) w_{r1}, p_{r1} vs δ



(b) w_{r0}, p_{r0} vs δ

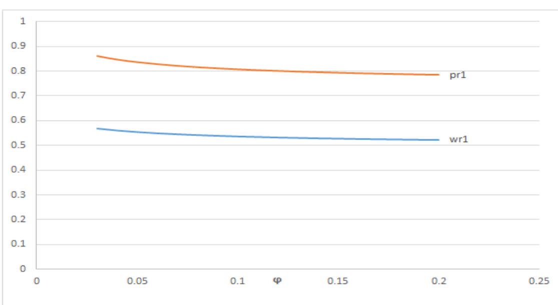


(c) Is_{r1}, Ib_{r1} vs δ

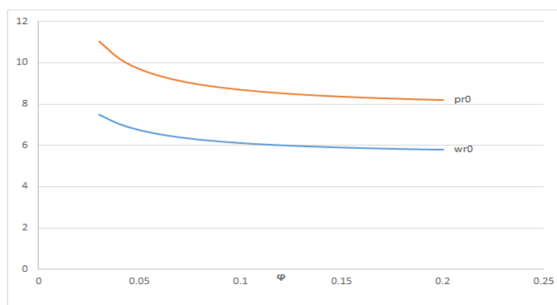


(d) Is_{r0}, Ib_{r0} vs δ

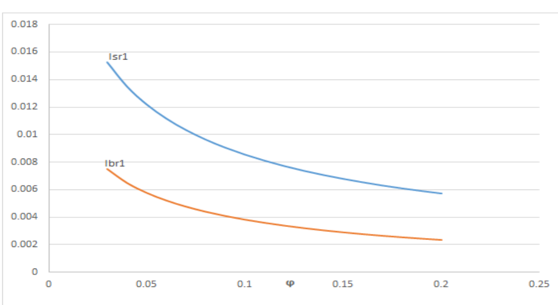
Figure 17: Price, Innovation efforts vs δ



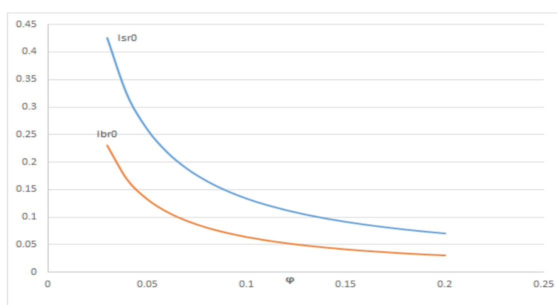
(a) w_{r1}, p_{r1} vs ϕ



(b) w_{r0}, p_{r0} vs ϕ



(c) Is_{r1}, Ib_{r1} vs ϕ



(d) Is_{r0}, Ib_{r0} vs ϕ

Figure 18: Price, Innovation efforts vs ϕ