

Dynamics of Cayley Forms

KIRILL KRASNOV

Abstract: The most natural first-order PDE's to be imposed on a Cayley 4-form in eight dimensions is the condition that it is closed. As is well-known, this implies integrability of the $\text{Spin}(7)$ -structure defined by the Cayley form, as well as Ricci-flatness of the associated metric. We address the question as to what the most natural second-order in derivatives set of conditions is. We start at the linearised level, and construct the most general diffeomorphism-invariant second order in derivatives Lagrangian that is quadratic in the perturbations of the Cayley form. We find that there is a two-parameter family of such Lagrangians. We then describe a non-linear completion of the linear story. We parametrise the intrinsic torsion of a $\text{Spin}(7)$ -structure by a 3-form, and show that this 3-form is completely determined by the exterior derivative of the Cayley form. The space of 3-forms splits into two $\text{Spin}(7)$ irreducible components, and so there is a two-parameter family of diffeomorphism-invariant Lagrangians that are quadratic in the torsion, matching the linearised story. We then describe a first-order in derivatives version of the action functional, which depends on the Cayley 4-form and auxiliary 3-form as independent variables. There is a unique functional whose Euler-Lagrange equation for the auxiliary 3-form states that it is equal to the torsion 3-form. For any member of our family of theories, the Euler-Lagrange equations are written only using the operator of exterior differentiation of forms, and do not require the knowledge of the metric-compatible Levi-Civita connection. Geometrically, there is a preferred member in the family of Lagrangians, and we propose that its Euler-Lagrange equations are the most natural second-order equations to be satisfied by Cayley forms. Our construction also leads to a natural geometric flow in the space of Cayley forms, defined as the gradient flow of our action functional.

1. Introduction

A $\text{Spin}(7)$ -structure on an 8-dimensional manifold is defined to be a 4-form of a special algebraic type. Such a 4-form is known as a Cayley form, and its $\text{GL}(8, \mathbb{R})$ stabiliser is $\text{Spin}(7)$. An 8-manifold admits a $\text{Spin}(7)$ -structure if it

is spin. However, since most of the considerations in this paper are local, we do not need to concern ourselves with assumptions about M .

As is well-known since [1], a $\text{Spin}(7)$ -structure is integrable if and only if the associated Cayley 4-form Φ is closed $d\Phi = 0$. This in turn implies that the metric determined by Φ is Ricci-flat. It is clear that $d\Phi = 0$ gives the most geometrically motivated set of first-order PDE's on the Cayley form. In this paper we address the question of what the most natural second-order PDE's are. We describe a certain construction, inspired by the Plebanski formalism [2], see also [3] for a recent description most closely aligned with the motivations of the present paper. The result of the construction is a unique action functional for Φ , whose Euler-Lagrange equations are a set of second-order PDE's on it. As it will have become clear from the construction and the equations it results in, these equations possess some desirable properties. In particular, they are constructed solely from the operator of the exterior differentiation on forms, so one never needs to know the covariant derivative of the metric determined by Φ to write them down.

The main outcome of our construction is the action

$$(1) \quad S[\Phi, C] = \int_M \Phi \wedge (dC - 6C \wedge_\Phi C) + \frac{\lambda}{6} v_\Phi + \text{constr.}$$

Here $\Phi \in \Lambda^4(M)$ is a Cayley form, and we have included in the action a set of constraint terms whose purpose is to guarantee that Φ is of the correct algebraic type. These depend solely on Φ as well as some necessary Lagrange multipliers variation with respect to which imposes the constraints. An easy comparison between the dimension of the space of 4-forms $\dim(\Lambda^4) = 70$ and the dimension of the orbit $\dim(\text{GL}(8, \mathbb{R})/\text{Spin}(7)) = 43$ shows that there are 27 independent constraints to be satisfied. We will never need to specify these constraints explicitly, as only the variation of these terms with respect to Φ matters for the Euler-Lagrange equations, and this can be determined by a different argument, see below. The object $C \in \Lambda^3(M)$ is what we refer to as the auxiliary 3-form. The Euler-Lagrange equations for C are algebraic, and determine C in terms of the exterior derivative of Φ , see below. After this solution is substituted back into the action, one gets a second order in derivatives action for Φ only. The term λv_Φ is a 'cosmological constant' term, with $\lambda \in \mathbb{R}$ being a parameter and v_Φ being the volume form for Φ , which can be taken to be $v_\Phi = (1/14)\Phi \wedge \Phi$. Finally, $C \wedge_\Phi C$ is a 4-form constructed from two copies of C , as well as the (inverse) metric g^{ab} determined by Φ . In index notation that we will be using in this article, it is given by

$$(2) \quad (C \wedge_\Phi C)_{abcd} := C_{abp} C_{cdq} g^{pq}.$$

Even though (1) is the most natural action for Cayley forms for reasons to become clear below, there are two independent scalars, quadratic in $C \in \Lambda^3$ that can be constructed, which follows from the fact that there are precisely two $\text{Spin}(7)$ -irreducible representations in the decomposition of Λ^3 , see (27) and (28). A particular combination of these two independent invariants of C appears in (1). We can, however, consider a more general family of Lagrangians given by

$$(3) \quad S_\kappa[\Phi, C] = \int_M \Phi \wedge (dC - 6C \wedge_\Phi C) + \frac{\kappa}{6}(C)^2 v_\Phi + \frac{\lambda}{6} v_\Phi + \text{constr.}$$

Our analysis of the linearised theory below will show that there is a two-parameter family of diffeomorphism-invariant Lagrangians that are second order in derivatives and quadratic in perturbations of the Cayley form. One of these parameters can always be absorbed into the perturbation of the Cayley form, resulting in one significant parameter. We will verify that the linearisation of (3) reproduces the one-parameter family of linearised Lagrangians, thus showing that (3) gives the non-linear completion of the most general diffeomorphism-invariant linear Lagrangian. However, for reasons to be explained now, the $\kappa = 0$ action functional (1) is the geometrically preferred one. The argument that fixes this action proceeds through a series of propositions.

The fact that the dimension of the space where the intrinsic torsion of a $\text{Spin}(7)$ -structure lies is equal to the dimension of the space of 3-form is known. However, the paper [4], which was an important precursor to our construction, uses a different parametrisation. The analog of Lemma 2.10 of [4] in our parametrisation is the following statement:

Proposition 1.1. *The intrinsic torsion of a $\text{Spin}(7)$ -structure, measured by $\nabla_i \Phi_{abcd}$, where ∇ is the covariant derivative with respect to the Levi-Civita connection for the metric defined by Φ , lies in $\Lambda^1 \otimes \Lambda_7^4$. For the notation explaining Λ_7^4 and the decomposition of the space of forms into irreducible components see below. The intrinsic torsion can be parametrised by an object $T \in \Lambda^3$ so that*

$$(4) \quad \nabla_i \Phi_{abcd} = 4T_{i[a}{}^p \Phi_{p|bcd]}.$$

Here the index p of T_{aip} is raised with the metric determined by Φ .

We remark that the only metric that is used in this paper is the one defined by Φ .

It turns out that the torsion 3-form is completely determined by the exterior derivative $d\Phi$. This is the content of the following proposition:

Proposition 1.2. *The Hodge dual of the projection of (4) to the space of 5-forms can be written as*

$$(5) \quad \star d\Phi = \frac{2}{5} J_3(T),$$

where J_3 is a certain operator $J_3 : \Lambda^3 \rightarrow \Lambda^3$ defined by Φ , see (25). The operator J_3 is invertible, and so T is completely determined by $d\Phi$.

We now have the proposition linking the action (1) and the relation (5) between the torsion 3-form and the exterior derivative of the Cayley form:

Proposition 1.3. *The Euler-Lagrange equation arising from (1) by extremising it with respect to C is $C = T$.*

One can rephrase this by saying that (1) is precisely the first-order action dependent on both Φ, C that leads to $C = T$ as the C field equation. Importantly, there is no ambiguity in the construction of the action once we demand that $C = T$ is to follow. In contrast, the critical value of C for the more general $\kappa \neq 0$ action (3) is not T , but rather only related to T by a certain non-trivial transformation. We get, instead

$$(6) \quad C = \frac{6T + \kappa J_3(T)}{6 - (5 + \kappa)\kappa}.$$

The property of the action (1) that the value of C as determined by its corresponding Euler-Lagrange equation is the intrinsic torsion $C = T$ makes this action a precise analogue of the 4D Plebanski action, see [3]. In this sense, it is a preferred member in the more general family (3).

The next proposition describes the Euler-Lagrange equations resulting by varying (1) with respect to Φ :

Proposition 1.4. *The Euler-Lagrange equations resulting from extremisation of (1) with respect to Φ can be written as:*

$$(7) \quad \partial_{[a} T_{bcd]} - \frac{3}{2} T_{[ab}{}^p T_{cd]p} - \frac{1}{8} (TT)_{[a|e|} \Phi^e{}_{bcd]} + \frac{\lambda}{84} \Phi_{abcd} = \Psi_{[ab}{}^{pq} \Phi_{|pq|cd]},$$

where Ψ^{abcd} is an arbitrary matrix in $\text{Sym}_0^2(\Lambda_7^2)$, which is the space of symmetric tracefree matrices, with the trace defined as $\text{Tr}(\Psi) = \Psi_{ab}{}^{ab}$, and

$$(8) \quad (TT)_{ab} := \Phi^{ijkl} T_{ija} T_{klb} - \frac{1}{7} g_{ab} \Phi^{ijkl} T_{ij}{}^p T_{klp}$$

is a symmetric matrix quadratic in the torsion. We note that the factor of $1/7$ here is not a typo. We can also write the Euler-Lagrange equations in form notation as

$$(9) \quad d\Phi - 6T \wedge_{\Phi} T - \frac{1}{16}K(TT) + \frac{\lambda}{42}\Phi = \Psi\Phi.$$

Here K is the map from the space of symmetric tensors to Λ^4 described in (50). It is shown later in the text, see Section 2.9, that a general 4-form in Λ_{27}^4 can be parametrised as $\Psi\Phi$, and so the right-hand side of both (7) and (9) is a general element of Λ_{27}^4 .

Proposition 1.5. *An alternative way of writing the field equations is to project both sides on the Λ_{1+7+35}^4 component in Λ^4 . This gives the following set of equations*

$$(10) \quad \begin{aligned} & \frac{1}{4}\Phi_b{}^{pqr}\nabla_a T_{pqr} - \frac{3}{4}\Phi_b{}^{pqr}\nabla_r T_{apq} - \frac{3}{2}\Phi_b{}^{pqr}T_{ap}{}^s T_{qrs} \\ & - \frac{3}{4}\Phi^{pqrs}T_{apq}T_{brs} + \frac{1}{2}g_{ab}\left(\lambda + \frac{3}{8}\Phi^{pqrs}T_{pq}{}^p T_{rsp}\right) = 0. \end{aligned}$$

The equations here are written in terms of ∇ , but they have the same form with ∇ replaced by the partial derivative operator ∂ . We note that the left-hand side is not automatically ab symmetric, and the anti-symmetric part of these equations are non-trivial. The anti-symmetric part can be shown to lie in Λ_7^2 , and so the total number of independent second-order differential equations is $36 + 7 = 43$, the dimension of the space of Cayley forms.

We can also characterise what the field equations imply for the Riemann curvature of the metric defined by Φ . This is done in the main text. We will see that the metrics defined by Φ that are the critical points of the action functional (3) are not in general Einstein, irrespective of value of κ .

There is another Lagrangian in the family (3), namely one corresponding to $\kappa = -2$, which is special. As a computation shows, the linearised Lagrangian in this case is just that for a metric perturbation. The other component that parametrises the perturbation of Φ , namely one living in Λ_7^2 , does not receive any kinetic terms at this value of κ . One can rephrase this by saying that the linearisation of the $\kappa = -2$ Lagrangian is the same as the linearisation of the Einstein-Hilbert metric Lagrangian. One could then be led to believe that the non-linear theory for $\kappa = -2$ is just that describing Einstein metrics. This is, however, not the case, as is confirmed by calculations, see Section 6.10. There are very interesting differences only visible at the non-linear level, still to be better understood.

The set of second-order PDE's (7) is the main result of our construction. We propose these equations as the most natural set of second-order PDE's for a Cayley form to satisfy. There is also a natural geometric flow in the space of Cayley forms that our construction defines. The gradient of the action functional (1) with respect to Φ , with $C = T$, is given by the Hodge dual of the 4-form on the left-hand-side of (9), projected to the space Λ_{1+7+35}^4 , and this defines a certain Spin(7)-flow whose properties are yet to be understood.

It is through this gradient flow that our work connects to a wider context of flows of geometric structures [5]. Much work has been done in the context of such flows for the case of G_2 -structures, starting with [6] and culminating in the recent work [7], which was an important predecessor to this paper. A particular geometric flow that has received attention is the so-called harmonic flow [8], which is defined as the gradient flow of the squared norm of the torsion tensor. This flow has been studied in various settings in [9], [10], [11], [12] and very recently in the case of Spin(7)-structures in [13]. The functional generating the flow in [13] is $\int |T|^2$. In contrast, the functional (1) that we advocate in this work, after the auxiliary field C is solved for $C = T$ and substituted into the action, reduces to

$$(11) \quad S[\Phi] \sim \int_M \Phi \wedge T \wedge_{\Phi} T \sim \int_M v_{\Phi} \Phi^{abcd} g^{ef} T_{abe} T_{cdf}.$$

Unlike $\int |T|^2$, this functional does not have a definite sign, see (127). But it is this functional that is analogous to the one that plays the distinguished role in the case of SU(2) structures in four dimensions, see [3]. In fact, the constructions of this paper can be described as providing a generalisation of the Plebanski formalism for 4D General Relativity to eight dimensions.

More work is needed to get better intuition about the properties of the critical points of both the $\kappa = 0$ and $\kappa \neq 0$ actions. We hope that this work will follow. It is worth remarking already at this point, however, that a construction similar to that described in this paper is possible also for other G -structures, in various dimensions. It would be particularly interesting to perform a similar analysis and construct actions for 3-forms in 7-dimensions, building on the work [7].

Many of the tensor computations in the paper are performed using symbolic manipulation with xAct Mathematica package [14]. For the convenience of the interested reader, we have placed a Mathematica notebook containing all the definitions and some example calculations in a GitHub repository [15]. The stated representation theoretic facts are obtained using the Mathematica package LieART [16].

2. Decomposition of the spaces of forms

Our index (anti-) symmetrisation conventions are as follows. Square brackets denote anti-symmetrisation, and are defined by

$$(12) \quad [a_1 \dots a_k] = \frac{1}{k!} \sum_p (-1)^p p(a_1 \dots a_k),$$

where the sum is taken over all permutations p and $(-1)^p$ is plus or minus identity depending on whether the permutation is even or odd. Symmetrisation is denoted by round brackets, and is defined similarly, apart from the absence of the minus signs. A vertical line around an index (or a set of indices) denotes the fact that this set of indices is not involved in (anti-) symmetrisation.

2.1. Basic algebra

Similar to [6] and [4], we use the index notation, which is very useful for encoding various relations satisfied by the Cayley form. The basic algebraic relation satisfied by the 4-form Φ_{abcd} is

$$(13) \quad \begin{aligned} \Phi_{ijkp} \Phi_{abcp} = & \\ & g_{ia} g_{jb} g_{kc} + g_{ib} g_{jc} g_{ka} + g_{ic} g_{ja} g_{kb} - g_{ia} g_{jc} g_{kb} - g_{ic} g_{jb} g_{ka} - g_{ib} g_{ja} g_{kc} \\ & - g_{ia} \Phi_{jkbc} - g_{ja} \Phi_{kibc} - g_{ka} \Phi_{ijbc} \\ & - g_{ib} \Phi_{jkca} - g_{jb} \Phi_{kica} - g_{kb} \Phi_{ijca} \\ & - g_{ic} \Phi_{jkab} - g_{jc} \Phi_{kiab} - g_{kc} \Phi_{ijab}. \end{aligned}$$

This identity, together with identities related to self-duality of Φ , are sufficient for most of the calculations one needs to do with Φ . A Mathematical notebook that can be used for algebraic computations with Φ based on this identity is available via [15].

One more contraction of the above identity gives

$$(14) \quad \Phi_{ijpq} \Phi_{abpq} = 6g_{ia} g_{jb} - 6g_{ib} g_{ja} - 4\Phi_{ijab}.$$

Yet one more contraction gives

$$(15) \quad \Phi_{ipqr} \Phi_{apqr} = 42g_{ia}.$$

The 4-form Φ is self-dual

$$(16) \quad \frac{1}{4!} \epsilon_{ijkl}{}^{abcd} \Phi_{abcd} = \Phi_{ijkl}.$$

Useful consequences of self-duality are

$$(17) \quad \epsilon^{ajklpqr} \Phi_{bpqr} = 30 \delta_b^{[a} \Phi^{ijkl]},$$

and

$$(18) \quad \epsilon^{ijklmnpq} \Phi_{abpq} = 60 \delta_a^{[i} \delta_b^j \Phi^{klmn]},$$

and

$$(19) \quad \epsilon^{ijklmnp} \Phi_{abcr} = 210 \delta_a^{[i} \delta_b^j \delta_c^k \Phi^{lmnp]}.$$

2.2. Identity

The following non-trivial identity

$$(20) \quad -2\Phi_{[ijk}{}^{[a}\Phi_l]{}^{bcd]} - 3\Phi_{[ij}{}^{[ab}\Phi_{kl]}{}^{cd]} + 42\Phi_{[ij}{}^{[ab}\delta_k^c\delta_l^d]} + \Phi_{ijkl}\Phi^{abcd} = 0$$

can be checked by multiplying with δ_a^i and using the identity (13) to check that the result is zero. Another useful check is to contract the left-hand side with Φ^{ijkl} , again producing zero. This identity does not seem to have appeared in the literature before. One can derive this identity as follows. An argument detailed in Section 2.5 below suggests that there must exist an identity of this sort. One can then take an arbitrary combination of the first three terms and equate this to the last term. Requiring the contraction with δ_a^i and with Φ^{ijkl} to give correct identities leads to the unique choice of the coefficients as above. This computation is spelled out in [15]. Another possibility is to look for an arbitrary linear combination of the first three terms that is self-dual with respect to both $abcd$ and $ijkl$. Again, this uniquely determines the coefficients as above.

2.3. Decomposition of Λ^2

The following material is standard, see e.g. [4]. Our notation for the operators introduced is different from that in [4], but, we hope, is systematic and convenient.

We introduce the following operator on 2-forms

$$(21) \quad J_2 : \Lambda^2 \rightarrow \Lambda^2, \quad J_2(\beta)_{ij} = \frac{1}{2} \Phi_{ij}{}^{ab} \beta_{ab}.$$

Using (14) we see that

$$(22) \quad (J_2)^2 = 3\mathbb{I} - 2J_2.$$

This means that the eigenvalues of J_2 are $-3, 1$. The eigenspace of eigenvalue -3 is Λ_7^2 , and eigenspace of eigenvalue 1 is Λ_{21}^2 . The two projectors are

$$(23) \quad \pi_7 = \frac{1}{4}(\mathbb{I} - J_2), \quad \pi_{21} = \frac{1}{4}(3\mathbb{I} + J_2).$$

For later purposes we note that

$$(24) \quad J_2^{-1} = \frac{1}{3}(2\mathbb{I} + J_2).$$

2.4. Decomposition of Λ^3

We introduce the following operator on 3-forms

$$(25) \quad J_3 : \Lambda^3 \rightarrow \Lambda^3, \\ J_3(\gamma)_{ijk} = \frac{1}{2}(\Phi_{ij}{}^{pq} \gamma_{kpq} + \Phi_{jk}{}^{pq} \gamma_{ipq} + \Phi_{ki}{}^{pq} \gamma_{jpq}) = \frac{3}{2} \Phi_{[ij}{}^{pq} \gamma_{k]pq}$$

A calculation using (13) gives

$$(26) \quad (J_3)^2 = 6\mathbb{I} - 5J_3.$$

This means that the eigenvalues of J_3 are $-6, 1$. The eigenspace of eigenvalue -6 is Λ_8^3 , and eigenspace of eigenvalue 1 is Λ_{48}^3 . The elements of the space Λ_8^3 are of the form

$$(27) \quad \Lambda_8^3 = \{X^p \Phi_{pijk}, X \in TM\},$$

and

$$(28) \quad \Lambda_{48}^3 = \{\gamma \in \Lambda^3 : \gamma \wedge \Phi = 0\}.$$

We note that

$$(29) \quad \pi_{48} = \frac{6}{7} \left(\mathbb{I} + \frac{1}{6} J_3 \right), \quad \pi_8 = \frac{1}{7} (\mathbb{I} - J_3).$$

We also note that

$$(30) \quad J_3^{-1} = \frac{1}{6} (J_3 + 5\mathbb{I}).$$

2.5. Decomposition of Λ^4

Unlike [4], which uses a more indirect approach, we decompose Λ^4 in a way completely analogous to what was done in the case Λ^2, Λ^3 . The only arising difficulty is that the operator that we need in this case satisfies a more complicated (fourth-order) relation.

We introduce the following operator on 4-forms

$$(31) \quad J_4 : \Lambda^4 \rightarrow \Lambda^4, \quad J_4(\sigma)_{ijkl} = 3\Phi_{[ij}{}^{pq}\sigma_{kl]pq} = \frac{1}{2}(\Phi_{ij}{}^{pq}\sigma_{pqkl} + \Phi_{ki}{}^{pq}\sigma_{pqjl} + \Phi_{il}{}^{pq}\sigma_{pqjk} + \Phi_{kl}{}^{pq}\sigma_{pqij} + \Phi_{jl}{}^{pq}\sigma_{pqki} + \Phi_{jk}{}^{pq}\sigma_{pqil}).$$

We remark that this is the map denoted by Λ_Φ in [4]. We have the following relation, also to be found in [4]

$$(32) \quad (J_4)^2(\sigma)_{ijkl} = \frac{1}{2}(\Phi_{ij}{}^{ab}\Phi_{kl}{}^{cd} + \Phi_{ki}{}^{ab}\Phi_{jl}{}^{cd} + \Phi_{il}{}^{ab}\Phi_{jk}{}^{cd})\sigma_{abcd} + 6\sigma_{ijkl} - 8J_4(\sigma)_{ijkl} = \frac{3}{2}\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 24\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + 6\sigma_{ijkl}.$$

We also have the following result for the cube of this operator

$$(33) \quad (J_4)^3(\sigma)_{ijkl} = -6\Phi_{[ijk}^a\Phi_{l]}^{bcd}\sigma_{abcd} - 15\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} + 258\Phi_{[ij}{}^{ab}\sigma_{kl]ab} - 24\sigma_{ijkl}.$$

Using the identity (20) we can rewrite this as

$$(34) \quad (J_4)^3(\sigma)_{ijkl} = -6\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} + 132\Phi_{[ij}{}^{ab}\sigma_{kl]ab} - 3\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} - 24\sigma_{ijkl}.$$

Finally, for the fourth power of this operator we have

$$(35) \quad (J_4)^4(\sigma)_{ijkl} = 87\Phi_{[ijk}^a\Phi_{l]}^{bcd}\sigma_{abcd} + \frac{345}{2}\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd}$$

$$-2643\Phi_{[ij}{}^{ab}\sigma_{kl]ab} + \frac{9}{2}\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} + 168\sigma_{ijkl}.$$

Using (20) we can rewrite this as

$$(36) \quad (J_4)^4(\sigma)_{ijkl} = 42\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\sigma_{abcd} - 816\Phi_{[ij}{}^{ab}\sigma_{kl]ab} \\ + 48\Phi_{ijkl}\Phi^{abcd}\sigma_{abcd} + 168\sigma_{ijkl}.$$

This shows that

$$(37) \quad (J_4)^4 + 16(J_4)^3 + 36(J_4)^2 - 144J_4 = 0,$$

or in other words

$$(38) \quad (J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I})(J_4 - 2\mathbb{I})J_4 = 0.$$

This shows that the operator J_4 has eigenvalues $-12, -6, 2, 0$, see also [4]. It is clear that the identity (20) is key to make this calculation work. In fact, this is how the identity (20) was derived in the first place. One knows that (37) must be true, which shows that an identity of the type (20) must be true. One then writes a general relation of the sort (20), and fixes the coefficients in such a way that its contraction gives a true statement. This uniquely fixes (20).

The eigenspaces are the irreducible components of the space of 4-forms

$$(39) \quad \Lambda_1^4 = \{\sigma \in \Lambda^4 : J_4(\sigma) = -12\sigma\}, \quad \Lambda_{27}^4 = \{\sigma \in \Lambda^4 : J_4(\sigma) = 2\sigma\}, \\ \Lambda_7^4 = \{\sigma \in \Lambda^4 : J_4(\sigma) = -6\sigma\}, \quad \Lambda_{35}^4 = \{\sigma \in \Lambda^4 : J_4(\sigma) = 0\}.$$

This will follow after we characterise each of the irreducible components below.

2.6. Projector on Λ_{27}^4

In what follows it will be useful to have the projector on Λ_{27}^4 explicitly. It is clear that it is a multiple of $J_4(J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I})$. Taking into account the eigenvalues of J_4 on different subspaces, it is not difficult to check that the required multiple is $1/224$. Thus, we have

$$(40) \quad \pi_{27} = \frac{1}{224}J_4(J_4 + 12\mathbb{I})(J_4 + 6\mathbb{I}) = \frac{1}{224}((J_4)^3 + 18(J_4)^2 + 72J_4).$$

Using (34) and (32) we get

$$(41) \quad \pi_{27}(\sigma)_{ijkl} = \frac{3}{32} \left(\Phi_{[ij}{}^{ab} \Phi_{kl]}{}^{cd} \sigma_{abcd} - 4\Phi_{[ij}{}^{ab} \sigma_{kl]ab} \right. \\ \left. - \frac{1}{7} \Phi_{ijkl} \Phi^{abcd} \sigma_{abcd} + 4\sigma_{ijkl} \right).$$

We have explicitly checked that this projector kills 4-forms of the form

$$(42) \quad H_{[i}{}^p \Phi_{jkl]p}, \quad H \in \Lambda^1 \otimes \Lambda^1,$$

which lie in Λ_{1+35+7}^4 . This characterisation of Λ_{1+35+7}^4 is subject of the next two subsections.

For later purposes, we note that $\mathbb{I} - \pi_{27}$ projects out the Λ_{27}^4 component of any 4-form, and is given by

$$(43) \quad \mathbb{I} - \pi_{27} = \frac{1}{32} \left(20\sigma_{ijkl} - 3\Phi_{[ij}{}^{ab} \Phi_{kl]}{}^{cd} \sigma_{abcd} \right. \\ \left. + 12\Phi_{[ij}{}^{ab} \sigma_{kl]ab} + \frac{3}{7} \Phi_{ijkl} \Phi^{abcd} \sigma_{abcd} \right).$$

We also note that π_{27} can be understood in a simple way. Indeed, taking a 4-form $\sigma_{ijkl} \in \Lambda^4$, we can interpret this as an object in $\text{Sym}^2(\Lambda^2)$, and apply the projector π_7 on the indices ij and on the indices kl . After this the result can be projected back to Λ^4 by antisymmetrising the indices. The result of this operation is

$$(44) \quad \left((\pi_7 \sigma \pi_7) \Big|_{\Lambda^4} \right)_{ijkl} = \frac{1}{64} \left(\Phi_{[ij}{}^{ab} \Phi_{kl]}{}^{cd} \sigma_{abcd} - 4\Phi_{[ij}{}^{ab} \sigma_{kl]ab} + 4\sigma_{ijkl} \right).$$

This contains almost all the terms in $\pi_{27}(\sigma)$. The only term present in (41) and absent in $(\pi_7 \sigma \pi_7) \Big|_{\Lambda^4}$ is the third term in (41), whose purpose is to make the result tracefree. So, we can write

$$(45) \quad \pi_{27}(\sigma) = 6(\pi_7 \sigma \pi_7) \Big|_{\Lambda^4} - \text{trace},$$

where the last term just removes the trace of the first. This gives a simple and useful interpretation of the projector π_{27} .

2.7. An operator from Λ^2 to Λ_7^4 and its inverse

Let us introduce the following operator

$$(46) \quad \Lambda^2 \ni \beta_{ij} \rightarrow K(\beta)_{ijkl} = 4\beta_{[i|p|} \Phi^p{}_{jkl]} \in \Lambda^4.$$

We note that the introduced operator K is the diamond map from [4]. It can now be checked that

$$(47) \quad K \circ \pi_{21} = 0, \quad K \circ \pi_7 = K.$$

Both of these can be understood by noting that the image $K(\beta) \in \Lambda^4$ is precisely the orbit of the basic 4-form Φ under the action of the Lie algebra $\mathfrak{spin}(8)$. The statement that $K \circ \pi_{21} = 0$ is just the statement that Φ is $\text{Spin}(7)$ invariant. This means that $\text{Ker}(K) = \Lambda_{21}^2$, and the image is Λ_7^4 .

To find the inverse of K on Λ_7^4 let us consider

$$(48) \quad \Lambda^4 \ni \sigma_{ijkl} \rightarrow K'(\sigma)_{ij} = \frac{1}{2} \Phi_i^{pqr} \sigma_{jpqr} - \frac{1}{2} \Phi_j^{pqr} \sigma_{ipqr} \in \Lambda^2.$$

We then have

$$(49) \quad \pi_{21} \circ K' = 0, \quad K' \circ K = 96\pi_7.$$

This means that K' is (a multiple of) the inverse of K on Λ_7^4 .

The operator K to Λ^4 can be generalised and applied to a general tensor from $\Lambda^1 \otimes \Lambda^1$. In particular, it can be applied to a symmetric tensor $h_{ij} \in \text{Sym}^2(\Lambda^1)$

$$(50) \quad \text{Sym}^2(\Lambda^1) \ni h_{ij} \rightarrow K(h)_{ijkl} = 4h_{[i|p]} \Phi_{jkl]}^p \in \Lambda^4.$$

The image of $\text{Sym}^2(\Lambda^1)$ under the action of K can be seen to be $\Lambda_1^4 \oplus \Lambda_{35}^4$. Restricted to its image, the operator $K : \text{Sym}^2(\Lambda^1) \rightarrow \Lambda^4$ is invertible, with the inverse being a multiple of

$$(51) \quad K' : \Lambda^4 \rightarrow \text{Sym}^2(\Lambda^1), \quad K'(\sigma)_{ij} = \Phi_i^{pqr} \sigma_{j)pqr} \in \text{Sym}^2(\Lambda^1).$$

2.8. Characterisation of Λ_{1+35+7}^4

We can apply the map K to a general element $H_{ij} \in \Lambda^1 \otimes \Lambda^1$

$$(52) \quad K(H)_{ijkl} := 4H_{[i}^p \Phi_{jkl]p}.$$

We already know that $\pi_{27}(K(H)) = 0$, and so the image of this map lies in Λ_{1+35+7}^4 . We also know that the map K applied to the symmetric part of H lies in Λ_{1+35}^4 , and to the anti-symmetric part in Λ_7^4 . A computation gives the following result

$$(53) \quad J_4(K(H))_{ijkl} = -3(H_{[i}^p - H^p_{[i} \Phi_{jkl]p} - 6\Phi_{ijkl} H_p^p.$$

This shows that when H is symmetric tracefree $H_{ij} = H_{(ij)}$, $H_p^p = 0$ we have $J_4(K(H)) = 0$. This shows that Λ_{35}^4 is the eigenspace of J_4 of eigenvalue 0. When H_{ij} is anti-symmetric, we have $J_4(K(H)) = -6K(H)$, and so Λ_7^4 is eigenspace of eigenvalue -6 . When $H_{ij} = g_{ij}$, we have $J_4(K(H)) = -12K(H)$, and thus Λ_1^4 is eigenspace of eigenvalue -12 . This gives the characterisation described above in (39).

2.9. Characterisation of Λ_{27}^4

It will be useful to have an explicit parametrisation of a general element of Λ_{27}^4 , similar to have we already have a parametrisation of a general element of the other irreducible subspaces Λ_{1+35+7}^4 . We start with a definition

Definition 2.1. *We say that a matrix Ψ is an element of the space of symmetric tracefree matrices*

$$(54) \quad \Psi^{abcd} \in \text{Sym}_0^2(\Lambda_7^2),$$

if the following requirements are satisfied:

$$(55) \quad \Psi^{abcd} = \Psi^{[ab][cd]} = \Psi^{[cd][ab]}, \quad \Psi_{ab}{}^{ab} = 0, \quad \pi_7 \Psi = \Psi = \Psi \pi_7.$$

We can then construct a 4-form that we denote as $\Psi\Phi$ as

$$(56) \quad (\Psi\Phi)_{abcd} := \Psi_{[ab}{}^{pq} \Phi_{cd]pq}.$$

Let us show that the projection of this to Λ_{1+35+7}^4 vanishes. To compute this, we apply the map $K' : \Lambda^4 \rightarrow \Lambda^1 \otimes \Lambda^1$. A calculation gives

$$(57) \quad (\Psi\Phi)_{ipqr} \Phi^{apqr} = \delta_i^a \left(\Psi_{qr}{}^{qr} - \frac{1}{2} \Psi_{pqrs} \Phi^{pqrs} \right) + 4\Psi_{ip}{}^{ap} + \Phi_{ipqr} \Psi^{apqr} - \Psi_{ipqr} \Phi^{apqr}.$$

The first term here is zero because Ψ is tracefree $\Psi_{qr}{}^{qr} = 0$, and also $\pi_7 \Psi = \Psi$ means $\pi_{21} \Psi = 0$, which implies

$$(58) \quad 3\Psi_{ijkl} + \frac{1}{2} \Phi_{ij}{}^{pq} \Psi_{pqkl} = 0.$$

So, if $\Psi_{qr}{}^{qp} = 0$ then also $\Psi_{pqrs} \Phi^{pqrs} = 0$. On the other hand, if we contract jl in this expression we get

$$(59) \quad \Psi_k{}^{pqr} \Phi_{ipqr} = -6\Psi_{ipk}{}^p.$$

The right-hand side is ik symmetric, and thus the left-hand side must also be ik symmetric. This shows that the last two terms in (57) cancel. It remains to characterise $\Psi_{ip}{}^{ap}$. To do this, we compute $0 = \pi_{21}\Psi\pi_{21}$

$$(60) \quad 0 = 3\Psi_{ijkl} + \frac{3}{2}\Phi_{ij}{}^{pq}\Psi_{pqkl} + \frac{3}{2}\Psi_{ij}{}^{pq}\Phi_{pqkl} + \frac{1}{4}\Phi_{ij}{}^{pq}\Phi_{kl}{}^{rs}\Psi_{pqrs}.$$

Taking the jl contraction of this we get

$$(61) \quad 0 = 8\Psi_{ipk}{}^p + 2\Psi_k{}^{pqr}\Phi_{ipqr} + 2\Psi_i{}^{pqr}\Phi_{kpqr} \\ + \frac{1}{2}g_{ik}\left(\Psi_{pq}{}^{pq} - \frac{1}{2}\Psi_{pqrs}\Phi^{pqrs}\right).$$

Using $\Psi_{qr}{}^{qp} = 0$, $\Psi_{pqrs}\Phi^{pqrs} = 0$ as well as (59) here we see that $\Psi_{ipk}{}^p = 0$. All in all, all the terms in (57) are zero and the object (56) is in Λ_{27}^4 . This gives the desired parametrisation of a general element of Λ_{27}^4 .

3. Linearised theory

In this section we address the question as to what is the most general action invariant under diffeomorphisms that can be constructed for the fields living in the representations Λ_{1+7+35}^4 of the group $\text{Spin}(7)$. We use physics terminology here, in which a field is a tensor that transforms in some (not necessarily irreducible) representation of the relevant Lie group. In our case the fields in $\Lambda_{1+35}^4 \sim \text{Sym}^2(\Lambda^1)$ encode perturbations h_{ab} of a metric tensor, and $\Lambda_7^4 \sim \Lambda_7^2$ is an additional field. It is well-known that there is a unique quadratic in h_{ab} and second-order in derivatives diffeomorphism invariant action. This holds true in any dimension, and the argument to this effect will be given below. We will see that, in contrast, there is no longer a unique diffeomorphism-invariant action for $\text{Sym}^2(\Lambda^1)$ and Λ_7^2 fields. There are two independent possible linearised actions that can be constructed. Non-linear completion of the theories described here is the subject of the following sections.

3.1. The usual metric only case

This story is standard, and works in exactly the same way in any dimension. We review it for completeness, and for establishing the main idea of the calculation to follow in the $\text{Spin}(7)$ case. For concreteness, we do calculations in dimension eight, but the story repeats itself with no changes in any dimension.

In the usual gravity case one considers fields transforming with respect to the Lorentz group $\text{SO}(8)$. The metric perturbation contains two irreducible

representations $\mathbf{1}, \mathbf{35}_v$. The subscript v stands for 'vector', to distinguish them from also possible spinor representations. This is a standard notation at least in some literature. As is also standard, we refer to the irreducible representations by their dimensions written in bold face. Let us denote the fields in representations $\mathbf{1}, \mathbf{35}_v$ by h, \tilde{h}_{ab} respectively. We use the same letter to refer to both fields because later it will be convenient to combine them together into a single symmetric tensor h_{ab} . We are interested in an action that contains two derivatives. It will be useful to think in terms of Fourier transform, and denote the derivative by its Fourier transform p^a at intermediate stages of the computation. The two most obvious action terms one can construct are of the type $p^2 h^2, p^2 (\tilde{h}_{ab})^2$, where the notations used are $p^2 = p^c p_c$ and $(\tilde{h}_{ab})^2 = \tilde{h}_{ab} \tilde{h}^{ab}$. To analyse the other possible terms we need to decompose the product of two derivatives into irreducibles, taking into account that they commute. We have

$$(62) \quad \mathbf{8}_v \otimes_S \mathbf{8}_v = \mathbf{1} + \mathbf{35}_v.$$

The trivial representation here corresponds to p^2 , and the other representation is $p_a p_b$ with the trace removed. The p^2 terms were already taken into account, so we only need to consider the possible couplings between $p_a p_b$ and the two other factors of either h or \tilde{h} . There is no term with $p_a p_b$ and two factors of h . There is clearly a mixed term $h p^a p^b \tilde{h}_{ab}$. To determine possible terms with two factors of \tilde{h} we need

$$(63) \quad \mathbf{35}_v \otimes_S \mathbf{35}_v = \mathbf{1} + \mathbf{35}_v + \mathbf{294}_v + \mathbf{300}.$$

There is only a single occurrence of $\mathbf{35}_v$ here, which means that there is only a single term that does not reduce to $p^2 (\tilde{h}_{ab})^2$, and this is $(p^a \tilde{h}_{ab})^2$.

All in all, there are just 4 possible terms in the action that one can write. We now go back to the notation that uses the derivative operators, and write the linear combination of the above four terms with arbitrary coefficients

$$(64) \quad \mathcal{L} = \frac{1}{2} \tilde{h}^{bc} \partial^a \partial_a \tilde{h}_{bc} + \frac{\alpha}{2} h \partial^a \partial_a h + \beta h \partial^a \partial^b \tilde{h}_{ab} + \gamma (\partial^a \tilde{h}_{ab})^2.$$

A note is in order about our notations here. The notation $(t_{a_1 \dots a_k})^2$, where $t_{a_1 \dots a_k}$ is an arbitrary tensor, means $t_{a_1 \dots a_k} t^{a_1 \dots a_k}$. This explains the last term, and similar type Lagrangian terms that will be written below. We have chosen the coefficient in front of the first term to be $1/2$, which we can always do by changing an overall coefficient in front of the Lagrangian \tilde{h}_{ab} . At this stage it

will be more convenient to introduce the fields

$$(65) \quad h_{ab} := \tilde{h}_{ab} + \frac{1}{8}\eta_{ab}h,$$

so that \tilde{h}_{ab} is the tracefree part of h_{ab} and $h = \eta^{ab}h_{ab}$. It is clear that the Lagrangian retains the same general form, except that the coefficients change. We will give the new coefficients the same name, hoping it will not lead to any confusion. The Lagrangian in terms of h_{ab} is

$$(66) \quad \mathcal{L} = \frac{1}{2}(\partial_a h_{bc})^2 + \frac{\alpha}{2}(\partial_a h)^2 - \beta h \partial^a \partial^b h_{ab} - \gamma(\partial^a h_{ab})^2.$$

We now demand diffeomorphism invariance of the action, with the field transformation properties being

$$(67) \quad h_{ab} = \partial_{(a}\xi_{b)}.$$

Note that this encapsulates transformation properties of both h, \tilde{h}_{ab} . In particular $\delta h = \partial^c \xi_c$.

We now perform the variation, and set coefficients in front of independent terms to zero, allowing integration by parts. This results in the following set of coefficients

$$(68) \quad \gamma = \beta = 1, \quad \alpha = -1.$$

Thus, the unique (modulo field rescaling) Lagrangian that is diffeomorphism-invariant reads

$$(69) \quad \mathcal{L}_{GR} = \frac{1}{2}(\partial_a h_{bc})^2 - \frac{1}{2}(\partial_a h)^2 - h \partial^a \partial^b h_{ab} - (\partial^a h_{ab})^2,$$

which is the standard result.

3.2. Gauge-fixing

Let us also derive the standard gauge-fixed form of the Lagrangian. Completing the square in the $(\partial^a h_{ab})^2$ part, we can rewrite the Lagrangian as

$$(70) \quad \mathcal{L}_{GR} = \frac{1}{2}(\partial_a h_{bc})^2 - \frac{1}{4}(\partial_a h)^2 - (\partial^a (h_{ab} - \frac{1}{2}\eta_{ab}h))^2.$$

If we gauge-fix the diffeomorphisms by setting

$$(71) \quad \partial^a (h_{ab} - \frac{1}{2}\eta_{ab}h) = 0,$$

we get a simple linear combination of the terms containing $\partial^a \partial_a$ only.

3.3. The case of Spin(7)-structures

Let us now consider 3 fields in irreducible representations of Spin(7) given by **1**, **7**, **35**. These are precisely the representations appearing in a tangent vector to the GL(8) orbit of Cayley forms. We will refer to these fields as h, ξ, \tilde{h} respectively. The decomposition into irreducibles is now dictated by the Spin(7) representation theory. There are again terms involving p^2 , which are $p^2 h^2, p^2 \xi^2, p^2 \tilde{h}^2$. To determine other possible terms we need to consider the (symmetric) product of two derivatives. We have

$$(72) \quad \mathbf{8} \otimes_S \mathbf{8} = \mathbf{1} + \mathbf{35},$$

which is unchanged from the Spin(8) case. The trivial representation here corresponds to p^2 , and so we only need to consider the **35** representation. This must couple to the product of two fields from the list h, ξ, \tilde{h} . The non-trivial such decompositions are

$$(73) \quad \begin{aligned} \mathbf{7} \otimes_S \mathbf{7} &= \mathbf{1} + \mathbf{27}, \\ \mathbf{7} \otimes \mathbf{35} &= \mathbf{21} + \mathbf{35} + \mathbf{189}, \\ \mathbf{35} \otimes_S \mathbf{35} &= \mathbf{1} + \mathbf{27} + \mathbf{35} + \mathbf{105} + \mathbf{168} + \mathbf{294}. \end{aligned}$$

We are looking for every occurrence of the representation **35** here. We already know that the terms $h p^a p^b \tilde{h}_{ab}, (p^a \tilde{h}_{ab})^2$ are possible. The second line above shows that there is a new term of the type $p^a p^b \xi \tilde{h}$. It is easy to write down this term by noting that the representation **7** appears in the anti-symmetric part of the tensor product

$$(74) \quad \mathbf{35} \otimes_A \mathbf{35} = \mathbf{7} + \mathbf{21} + \mathbf{35} + \mathbf{189} + \mathbf{378}.$$

We already know that the best way to describe a field in representation **7** is by using a field in Λ_7^2 . Thus, let us introduce an object

$$(75) \quad \xi_{ab} \in \Lambda_7^2.$$

We can then construct the term coupling ξ_{ab}, \tilde{h}_{ab} as $p_a p^c h_{cb} \xi^{ab}$.

Going back to the notation that involves partial derivatives, it is now clear that there are just two terms that can be constructed from ξ_{ab} . These can be written as $\xi^{ab} \partial^c \partial_c \xi_{ab}$ and $\partial_b \tilde{h}^{ba} \partial^c \xi_{ca}$. Note that we can also write the

second term as $\partial_b h^{ba} \partial^c \xi_{ca}$, because the trace part of h_{ab} does not couple to ξ_{ab} . As before, we now write a general linear combination of all the possible terms, with arbitrary coefficients:

$$(76) \quad \mathcal{L} = \frac{\rho}{2} (\partial_a h_{bc})^2 + \frac{\alpha}{2} (\partial_a h)^2 - \beta h \partial^a \partial^b h_{ab} - \gamma (\partial^a h_{ab})^2 \\ + \frac{\lambda}{2} (\partial_a \xi_{bc})^2 - \mu \partial_b h^{ba} \partial^c \xi_{ca}.$$

As before the notation $(t_{a_1 \dots a_k})^2$ for any tensor means the complete contraction of two copies of this tensor. It will be convenient to put an arbitrary coefficient ρ also in front of the first term. This will allow us to write the general diffeomorphism-invariant Lagrangian as a linear combination of two separately invariant Lagrangians.

3.4. Some identities

Let us now explain why the term $(\partial^a \xi_{ab})^2$ is not added to the Lagrangian (76). The representation theory tells us that there is no representation **35** in the decomposition $\mathbf{7} \otimes_S \mathbf{7}$, and so this term must be a multiple of $\xi^{bc} \partial^a \partial_a \xi_{bc}$. Let us confirm that. Using the fact that $\xi \in \Lambda_7^2$ we have

$$(77) \quad \xi_{ab} = -\frac{1}{6} \Phi_{ab}{}^{pq} \xi_{pq},$$

and so

$$(78) \quad \xi_b{}^a \xi_{ca} = \frac{1}{36} \left(-28 \xi_b{}^a \xi_{ca} + 8 g_{bc} (\xi_{pq})^2 \right).$$

From this we get

$$(79) \quad \xi_b{}^a \xi_{ca} = \frac{1}{8} g_{bc} (\xi_{pq})^2.$$

This explains why the term $(\partial^a \xi_{ab})^2$ is already contained in the $\xi^{bc} \partial^a \partial_a \xi_{bc}$ term in the Lagrangian (76) and does not need to be added as a separate term.

3.5. Gauge-fixing

We note that there is a gauge in which the general Lagrangian is given by a sum of terms only involving the Laplacians. Indeed, for $\gamma \neq 0$ we can rewrite

the Lagrangian as

$$(80) \quad \mathcal{L} = \frac{\rho}{2}(\partial_a h_{bc})^2 + \left(\frac{\alpha}{2} + \frac{\beta^2}{4\gamma}\right)(\partial_a h)^2 + \left(\frac{\lambda}{2} + \frac{\mu^2}{32\gamma}\right)(\partial_a \xi_{bc})^2 \\ - \gamma(\partial^a(h_{ab} - \frac{\beta}{2\gamma}\eta_{ab}h + \frac{\mu}{2\gamma}\xi_{ab}))^2.$$

3.6. The transformation properties under diffeomorphisms

To determine the diffeomorphism transformation rules for all the fields we recall that h_{ab} and ξ_{ab} appear from a certain projection of the perturbation of the 4-form. If we call this perturbation $\phi \in \Lambda^4$, the fact that this 4-form is a tangent vector to the orbit of Cayley 4-forms means that $\phi \in \Lambda_{1+7+35}^4$. Let us define the fields h_{ab}, ξ_{ab} as

$$(81) \quad \tilde{h}_{ab} = \frac{1}{24}(\phi_{[a}{}^{pqr}\Phi_{b]pqr} - \frac{1}{8}\eta_{ab}\phi^{pqrs}\Phi_{pqrs}), \\ \xi_{ab} = \frac{1}{24}\phi_{[a}{}^{pqr}\Phi_{b]pqr}, \quad h = \frac{1}{168}\phi^{pqrs}\Phi_{pqrs}.$$

We emphasise that here and till the end of this section Φ is the background, which is assumed to be constant, and ϕ is the perturbation. A calculation shows that the inverse of this map is the following parametrisation of ϕ_{abcd}

$$(82) \quad \phi_{abcd} = -4(h_{[a}{}^p + \frac{1}{4}\xi_{[a}{}^p)\Phi_{bcd]p},$$

and this formula explains the choice of prefactors in (81). We note that

$$(83) \quad \frac{1}{96}\phi^{abcd}\phi_{abcd} = h^{ab}h_{ab} + \frac{3}{4}h^2 + \frac{1}{4}\xi^{ab}\xi_{ab},$$

where it is used that $\xi_{ab} \in \Lambda_7^2$.

Under diffeomorphisms

$$(84) \quad \delta\phi = i_\xi d\Phi + di_\xi\Phi.$$

We assume that the background 4-form Φ is closed (in fact constant), so that there is only the second term. Then

$$(85) \quad \delta\phi_{abcd} = -4\partial_{[a}\xi^p\Phi_{bcd]p},$$

and so $(1/4)\delta\xi_{ab} = \pi_7(\partial_{[a}\xi_{b]})$ giving

$$(86) \quad \delta h_{ab} = \partial_{(a}\xi_{b)}, \quad \delta\xi_{ab} = \partial_{[a}\xi_{b]} - \frac{1}{2}\Phi_{ab}{}^{pq}\partial_p\xi_q.$$

3.7. Determining the diffeomorphism-invariant Lagrangian

The variation of the Lagrangian (76), modulo surface terms, is given by

$$(87) \quad \delta\mathcal{L} = (\rho - \gamma - \frac{\mu}{2})\partial^a h_{ab}\partial^2\xi^b + (-\beta + \gamma - \frac{\mu}{2})\partial^a\partial^b h_{ab}(\partial\xi) \\ - (\alpha + \beta)\partial^2 h(\partial\xi) + (4\lambda - \frac{\mu}{2})\partial^a\xi_{ab}\partial^2\xi^b.$$

Here $\partial^2 = \partial^a\partial_a$ and $(\partial\xi) = \partial^a\xi_a$. We have used the fact that ξ_{ab} is in Λ_7^2 , and so $(1/2)\Phi_{ab}{}^{pq}\xi_{pq} = -3\xi_{ab}$. Setting to zero the coefficients in front of the independent parts we get a system of equations. The solution depends on two of the parameters, for which we can take ρ, μ . Then

$$(88) \quad \alpha = -\rho + \mu, \quad \beta = \rho - \mu, \quad \gamma = \rho - \frac{\mu}{2}, \quad \lambda = \frac{\mu}{8}.$$

It is clear that the resulting diffeomorphism-invariant Lagrangian is the sum of two separately invariant terms

$$(89) \quad \mathcal{L} = \rho\mathcal{L}_{GR} + \mu\mathcal{L}',$$

where

$$(90) \quad \mathcal{L}' = \frac{1}{2}(\partial_a h)^2 + h\partial^a\partial^b h_{ab} + \frac{1}{2}(\partial^a h_{ab})^2 + \frac{1}{16}(\partial_c\xi_{ab})^2 - \partial_b h^{ba}\partial^c\xi_{ca}.$$

We thus observe that the linearised action in the case of Spin(7)-structures is not unique. There are two linearly independent such actions, and the general action is given by their linear combination. One of the parameters can always be absorbed into the perturbation of the 4-form field, but the other parameter remains.

It is interesting to remark that the story we described parallels precisely the story one finds in the case of SU(2) structures in four dimensions, see [3] section 6. In four dimensions there are also two diffeomorphism-invariant terms that can be written down for perturbations of an SU(2) structure. The main difference with four dimensions is that in that case there is an additional symmetry that can be invoked, namely SU(2) gauge transformations, that

allows to eliminate one of the two independent terms. In the case of four dimensions the origins of this extra symmetry are in the fact that $\mathfrak{so}(4) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, and \mathfrak{g}^\perp is also a Lie algebra. It is the requirement of gauge invariance with respect to \mathfrak{g}^\perp that eliminates one of the two possible terms in the case of 4D. In the present case of eight dimensions \mathfrak{g}^\perp is not a Lie algebra, and no similar requirement of gauge invariance is possible. However, it is not impossible that for some special value of the ratio μ/ρ an extra gauge symmetry arises in the theory described by (89), and this selects a preferred member of the family of theories. At the moment of writing this remark we do not know whether this is the case, but this possibility is under investigation.

4. Intrinsic torsion

We now proceed to our construction of the non-linear theories completing the linear story described above. The purpose of this section is to recall the definition of the intrinsic torsion of a Spin(7)-structure and establish some facts that are necessary for the following.

4.1. Characterisation of the intrinsic torsion

We start with the following proposition, whose proof can also be found in [4].

Proposition 4.1. *The intrinsic torsion of a Spin(7)-structure, measured by $\nabla_a \Phi_{ijkl}$, where ∇_a is the metric-compatible covariant derivative, takes values in $\Lambda^1 \otimes \Lambda_7^4$. Using the isomorphism $\Lambda_7^2 \sim \Lambda_7^4$ provided by the operator K , see (46), the intrinsic torsion can be parametrised by an object in $\Lambda^1 \otimes \Lambda_7^2$. Explicitly,*

$$(91) \quad \nabla_a \Phi_{ijkl} = T_{a;ip} \Phi^p_{jkl} - T_{a;jp} \Phi^p_{kli} + T_{a;kp} \Phi^p_{lij} - T_{a;lp} \Phi^p_{ijk},$$

$$T_{a;ij} \in \Lambda^1 \otimes \Lambda_7^2.$$

We remark that we use the same notation for the intrinsic torsion $T_{a;ij}$ as in [4]. The semi-colon here should not be confused with the symbol denoting the covariant derivative, which is standard in some physics literature. We never use this notation for the covariant derivative in this paper, and thus we hope that no confusion arises.

Proof. The proof of this proposition consists in showing that the projections of $\nabla_a \Phi_{ijkl} \in \Lambda^1 \otimes \Lambda^4$ to all other irreducible components of Λ^4 apart from Λ_7^4 vanish. It is given in [4], and similar computations in the case of G_2 structures are spelled out in [7]. We spell out an alternative, completely explicit proof,

which is made possible by our knowledge of the projections to Λ_{35+1}^4 and the expression (41) for the projector to Λ_{27}^4 . The projection to Λ_{35+1}^4 is obtained by computing

$$(92) \quad 2\Phi_{(i}{}^{pqr}\nabla_{|a|}\Phi_{j)pqr} = \nabla_a\Phi_i{}^{pqr}\Phi_{jpqr} = 42\nabla_ag_{ij} = 0.$$

For the projection on Λ_{27}^4 the computation is a bit more involved. First, we need some identities. We have, on one hand

$$\begin{aligned} & \nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\Phi_{abcd}) = \\ & \Phi_{[kl}{}^{cd}\Phi_{|abcd}\nabla_{|p|}(\Phi_{ij]}{}^{ab}) + \Phi_{[ij}{}^{ab}\Phi_{|abcd}\nabla_{|p|}(\Phi_{kl]}{}^{cd}) + \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = \\ & 12\delta_{[k}^a\delta_{l]}^b\nabla_{|p|}\Phi_{ij]ab} - 4\Phi_{[kl}{}^{ab}\nabla_{|p|}\Phi_{ij]ab} + 12\delta_{[i}^a\delta_{j]}^b\nabla_{|p|}\Phi_{kl]ab} - 4\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab} \\ & + \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = 24\nabla_p\Phi_{ijkl} - 8\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab} + \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd}. \end{aligned}$$

On the other hand

$$(93) \quad \nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\Phi_{abcd}) = 28\nabla_p\Phi_{ijkl}.$$

Thus, we have

$$(94) \quad \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} = 4\nabla_p\Phi_{ijkl} + 8\Phi_{[ij}{}^{ab}\nabla_{|p|}\Phi_{kl]ab}.$$

We also have

$$(95) \quad \nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]ab}) = 2\Phi_{[ij}{}^{ab}(\nabla_{|p|}\Phi_{kl]ab}).$$

On the other hand,

$$(96) \quad \nabla_p(\Phi_{[ij}{}^{ab}\Phi_{kl]ab}) = -4\nabla_p\Phi_{ijkl},$$

and so

$$(97) \quad \begin{aligned} \Phi_{[ij}{}^{ab}(\nabla_{|p|}\Phi_{kl]ab}) &= -2\nabla_p\Phi_{ijkl}, \\ \Phi_{[ij}{}^{ab}\Phi_{kl]}{}^{cd}\nabla_p\Phi_{abcd} &= -12\nabla_p\Phi_{ijkl}. \end{aligned}$$

Using (41), these identities, as well as $\Phi^{abcd}\nabla_p\Phi_{abcd} = 0$, it is easy to see that

$$(98) \quad \pi_{27}(\nabla_p\Phi_{ijkl}) = 0.$$

Finally, to establish (91) we just need to recall that a general element of Λ_7^4 can be parametrised as $K(\beta), \beta \in \Lambda_7^2$, where $K : \Lambda_7^2 \rightarrow \Lambda_7^4$ is the map introduced in (46). We thus have

$$(99) \quad \nabla_a \Phi_{ijkl} = -4T_{a;[i|p|} \Phi_{jkl]}^p,$$

where $T_{a;ij} \in \Lambda^1 \otimes \Lambda_7^2$. This is precisely the formula (91). \square

4.2. Parametrisation by the torsion 3-form

As is known, see e.g. [18] Example 3.4., the spaces $\Lambda^1 \otimes \Lambda_7^2$ and Λ^3 are isomorphic. We can make this isomorphism explicit, in one direction, by parametrising the intrinsic torsion $T_{a;ij}$ as follows

$$(100) \quad T_{a;ij} = \pi_7(T_{aij}) = \frac{1}{4}T_{aij} - \frac{1}{8}\Phi_{ij}{}^{kl}T_{akl}, \quad T_{aij} \in \Lambda^3.$$

An explicit relation in the other direction is

$$(101) \quad T_{aij} = \frac{4}{3}T_{a;ij} + 4T_{[a;ij]} + T_{[a;kl]}\Phi_{ij}{}^{kl} + \frac{2}{9}T_{k;lm}\Phi^{klm}{}_{[i}g_{j]a}.$$

Using this parametrisation, we can rewrite (91) in terms of the torsion 3-form.

Proposition 4.2. *In the parametrisation of the intrinsic torsion by a torsion 3-form, we have*

$$(102) \quad \nabla_a \Phi_{ijkl} = T_{aip}\Phi_{pjkl} - T_{ajp}\Phi_{pkli} + T_{akp}\Phi_{plij} - T_{alp}\Phi_{pijk}, \\ T_{aij} \in \Lambda^3.$$

A proof is by explicit verification, substituting (100) into (91). Note that this is the same formula for the covariant derivative of the basic 4-form, but now with the torsion 3-form instead of the object $T_{a;ij} \in \Lambda^1 \otimes \Lambda_7^2$. We would like to emphasise that the approach of this paper to a large extent depends on the existence of the formula (102).

4.3. Connection with skew-symmetric torsion

As is known, see [17], [18], any $\text{Spin}(7)$ -structure on an 8-dimensional manifold admits a unique connection with totally skew-symmetric torsion. Such a connection is given by

$$(103) \quad \tilde{\nabla}_a X_i = \nabla_a X_i - T_{aip}X_p.$$

It is then clear that the relation (102) can be interpreted as the statement that the 4-form is parallel with respect to $\tilde{\nabla}$

$$(104) \quad \tilde{\nabla}_a \Phi_{ijkl} = 0.$$

The known existence of a unique connection with totally skew-symmetric torsion thus gives an alternative justification why the formula (102) must exist.

4.4. Torsion 3-form from the exterior derivative of the Cayley form

Proposition 4.3. *The torsion 3-form is completely determined by the exterior derivative $d\Phi$. Explicitly, we have*

$$(105) \quad T = \frac{5}{2} J_3^{-1}(\star(d\Phi)),$$

where J_3 is the operator in 3-forms introduced in (25), and $\star(d\Phi)$ is the Hodge dual of $d\Phi$.

Proof. On one hand, we have

$$(106) \quad \star(d\Phi)_{mnr} = \frac{1}{5!} \epsilon_{mnr}{}^{ajkl} \partial_a \Phi_{ijkl}.$$

On the other hand, substituting here the right-hand-side of (102) we have

$$(107) \quad \frac{1}{5!} \epsilon_{mnr}{}^{ajkl} \partial_a \Phi_{ijkl} = \frac{1}{30} \epsilon_{mnr}{}^{ajkl} T_{aip} \Phi_{pjkl}.$$

Now, using (17) we get

$$(108) \quad \begin{aligned} \epsilon_{mnr}{}^{ajkl} T_{aip} \Phi_{pjkl} = \\ 6(\Phi_{mn}{}^{pq} T_{rpq} + \Phi_{nr}{}^{pq} T_{mpq} + \Phi_{rm}{}^{pq} T_{npq}) = 12 J_3(T)_{mnr}. \end{aligned}$$

This means we have

$$(109) \quad \star(d\Phi)_{mnr} = \frac{2}{5} J_3(T)_{mnr}.$$

Now, the operator J_3 is invertible, with inverse given by (30). This proves the proposition. \square

5. Riemann curvature identities

Having described the intrinsic torsion and its relation with the covariant and exterior derivatives of the Cayley form, we can obtain very useful characterisations of (some parts of) the Riemann curvature. This material is standard, see for example Theorem 2.10 in [13]. The difference in our treatment is that we use the parametrisation of the torsion by a 3-form.

5.1. Irreducible components of the Riemann tensor

This material is well-known, see e.g. [13]. The Riemann tensor is an object with values in $\text{Sym}^2(\Lambda^2)$, with Λ^4 removed. Given that $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{21}^2$, it is easy to compute the decomposition of $\text{Sym}^2(\Lambda^2)$ into irreducibles using the well-known facts about the tensor products of irreducible representations of $\text{Spin}(7)$. We denote representations using the corresponding dimension written in bold face. We need the following tensor product decompositions:

$$\begin{aligned} (110) \quad & \mathbf{7} \otimes_S \mathbf{7} = \mathbf{1} \oplus \mathbf{27}, \\ & \mathbf{7} \otimes \mathbf{21} = \mathbf{105} \oplus \mathbf{35} \oplus \mathbf{7}, \\ & \mathbf{21} \otimes_S \mathbf{21} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{35} \oplus \mathbf{168}. \end{aligned}$$

Taking into account that

$$(111) \quad \Lambda^4 = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27} \oplus \mathbf{35},$$

we see that Riemann curvature gets decomposed into the following irreducible components

$$(112) \quad \text{Riemann} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{35} \oplus \mathbf{105} \oplus \mathbf{168}.$$

Of these the Ricci part is

$$(113) \quad \text{Ricci} = \mathbf{1} \oplus \mathbf{35},$$

and the Weyl part is

$$(114) \quad \text{Weyl} = \mathbf{27} \oplus \mathbf{105} \oplus \mathbf{168}.$$

Our next task is to characterise which parts of the Riemann curvature can be extracted from the intrinsic torsion.

5.2. Part of Riemann curvature from the torsion

We now take the commutator of two covariant derivatives applied to the basic 4-form to get

$$(115) \quad 4R_{ab[i}{}^p\Phi_{|p|jkl]} = 2\nabla_{[a}\nabla_{b]}\Phi_{ijkl} = 8\nabla_{[a}(T_{b][i|p|}\Phi^p{}_{jkl]}).$$

Applying the product rule and using (102) one more time we get

$$(116) \quad 4R_{ab[i}{}^p\Phi_{|p|jkl]} = 4\nabla_a(T_{b[i|p|})\Phi^p{}_{jkl]} - 4\nabla_b(T_{a[i|p|})\Phi^p{}_{jkl]} \\ + 4T_{a[i}{}^pT_{|bp|}{}^q\Phi_{jkl]q} - 4T_{b[i}{}^pT_{|ap|}{}^q\Phi_{jkl]q}.$$

This is what is known as the Bianchi identity in the literature, see e.g. [4] Theorem 4.2.

5.3. Identity for the divergence of the torsion 3-form

Before we proceed any further, a useful consequence of this identity is obtained by multiplying it with $\epsilon^{mnabijkl}$, and using (17). On the left-hand side we get identically zero, by properties of the Riemann curvature. The right-hand side is non-trivial and so we get

$$(117) \quad 2\nabla_a T_{bi[m}\Phi_n]{}^{abi} + \nabla^a T_{abi}\Phi^{bi}{}_{mn} \\ + 2T_{ai}{}^p T_{bp[m}\Phi_n]{}^{abi} - 2T_a{}^{pq} T_{bpq}\Phi_{mn}{}^{ab} = 0.$$

We would now like to extract from here the divergence $\nabla^a T_{amn}$ of the torsion 3-form in terms of other quantities. Applying to this expression $(1/4)(\mathbb{I} + J_2)$, we get

$$(118) \quad \nabla^a T_{amn} = \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_n]T_{abc} - \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{|a|}T_{n]bc} \\ - \Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}.$$

We can rewrite this in a different form, by applying the projection to Λ_7^2 . We get

$$(119) \quad \nabla^a T_{amn} - \frac{1}{2}\Phi_{mn}{}^{pq}\nabla^a T_{apq} = \\ \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_n]T_{abc} - \frac{3}{2}\Phi_{[m}{}^{abc}\nabla_{|a|}T_{n]bc} - 2\Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}.$$

It is useful to rewrite this as the divergence of the original torsion. Using (102) we have

$$(120) \quad 4\nabla^a T_{a;mn} = \nabla^a T_{amn} - \frac{1}{2}\Phi_{mn}{}^{pq}\nabla^a T_{apq} - \Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp},$$

and thus

$$(121) \quad 4\nabla^a T_{a;mn} = \frac{1}{2}\Phi_{[m}{}^{abc}\nabla_{n]}T_{abc} - \frac{3}{2}\Phi_{[m}{}^{abc}\nabla_{[a}T_{n]bc} - 3\Phi_{[m}{}^{abc}T_{n]a}{}^p T_{bcp}.$$

5.4. Component of the Riemann curvature

Thinking about R_{abcd} as an object in $\Lambda^2 \otimes_S \Lambda^2$ (with a copy of Λ^4 removed), and recalling the operator K introduced in (46), we see that the object on the left-hand side of (116) is valued in $\Lambda^2 \otimes \Lambda_7^4$. We can then apply the inverse operator K' to obtain

$$(122) \quad R_{abij} - \frac{1}{2}\Phi_{ij}{}^{pq}R_{abpq} = \nabla_a T_{bij} - \nabla_b T_{a ij} - \frac{1}{2}\Phi_{ij}{}^{cd}\nabla_a T_{bcd} + \frac{1}{2}\Phi_{ij}{}^{cd}\nabla_b T_{acd} + T_{ai}{}^p T_{bjp} - T_{bi}{}^p T_{ajp} - \Phi_{ij}{}^{pq}T_{ap}{}^k T_{bqk}.$$

Both sides of this equality can be checked to be in Λ_7^2 with respect to indices ij , by applying the projector to Λ_{21}^2 and seeing that the result is identically zero. This computation makes it obvious that all apart from the **168** part of the Weyl curvature are determined by the intrinsic torsion. Indeed, all parts but this one come from $\Lambda^2 \otimes \Lambda_7^2$, and this is precisely what the part of the Riemann curvature tensor that the intrinsic torsion determines.

5.5. Ricci curvature scalar

Before we use the facts above to obtain a formula for the Ricci curvature, let us note that there are two different ways to extract the Ricci scalar from here. One is to contract the indices with $g^{ai}g^{bj}$. The other is to contract it with $-(1/6)\Phi^{abij}$. Both of these give

$$(123) \quad R = -\Phi^{abcd}\nabla_a T_{bcd} + T^{abc}T_{abc} + \Phi^{abcd}T_{ab}{}^p T_{cdp}.$$

Note that this only depends on the exterior derivative dT of the torsion 3-form.

5.6. Extracting the Ricci curvature

We can extract the Ricci tensor from (122) by multiplying with Φ_c^{bij} , and applying the Bianchi identity $R_{a[bij]} = 0$ to get

$$(124) \quad -\frac{1}{2}\Phi_{ij}{}^{pq}R_{abpq}\Phi_c^{bij} = -6R_{ac}.$$

Doing the same operations with the right-hand side and we get

$$(125) \quad R_{ab} = -\nabla^c T_{abc} - \frac{1}{2}\Phi_b{}^{ijk}\nabla_a T_{ijk} + \frac{1}{2}\Phi_b{}^{ijk}\nabla_i T_{ajk} \\ + T_a{}^{pq}T_{bpq} + \Phi_b{}^{ijk}T_{ai}{}^p T_{jkp}.$$

This is not explicitly symmetric in ab , and must therefore become symmetric when T_{ijk} is given by its expression (105). And indeed, the anti-symmetric part of the right-hand side vanishes in view of (118). Thus, the Ricci curvature is given by

$$(126) \quad R_{ab} = -\frac{1}{2}\Phi_{(a}{}^{ijk}\nabla_{b)}T_{ijk} + \frac{1}{2}\Phi_{(a}{}^{ijk}\nabla_{|i}T_{b)jk} \\ + T_a{}^{pq}T_{bpq} + \Phi_{(a}{}^{ijk}T_{b)i}{}^p T_{jkp}.$$

We have now proven the result known since [1]: when $d\Phi = 0$ the metric is Ricci-flat. Indeed, by (105) $d\Phi = 0$ implies $T = 0$, which in turn gives $R_{ab} = 0$ by (126). Note that, unlike the Ricci scalar (123), the Ricci tensor depends on the full covariant derivative of the torsion 3-form.

6. The action

This section is central to the whole paper. We consider a one-parameter family of action functionals of the Cayley form Φ and an auxiliary 3-form C . The Lagrangians we consider are first-order in derivatives, and the Euler-Lagrange equations for the auxiliary field C are algebraic. There is a member in the family of actions for which the Euler-Lagrange equation for C equates it with the intrinsic torsion T of the $\text{Spin}(7)$ -structure. Once the Euler-Lagrange equation for C is solved and C is obtained in terms for the derivative of Φ , this value of C can be substituted back into the Lagrangian, resulting in a second-order in derivatives Lagrangian that depends solely on Φ . As we will see, the Lagrangians arising this way are basically a linear combination of the two invariants that can be constructed from the intrinsic torsion of Φ . We

will analyse the actions arising this way, and derive the arising Euler-Lagrange equations for Φ .

There are some limitations to our construction. First, our motivation is to mimic what happens in Plebanski formalism that gives an efficient description of 4D General Relativity [3]. Plebanski action is similarly a functional that depends on an $SU(2)$ structure (encoded into a triple of 2-forms), and an auxiliary (connection) field. The action is first-order in derivatives and quadratic in the auxiliary field. After the connection is solved for from its Euler-Lagrange equations and substituted back into action, one obtains a second-order action, which is just one of the two possible invariants that can be built from the intrinsic torsion of the $SU(2)$ structure. Our construction generalises all this to the case of $Spin(7)$ structures. Motivated by this example of Plebanski formalism, we do not allow more complicated than quadratic dependence of the action on the auxiliary field C , as we would like to retain the possibility to solve for C explicitly. It is clear that more involved first-order actions depending on Φ, C can be constructed, with more complicated dependence on C than quadratic, but this is not pursued in the present paper.

6.1. A one-parameter family of action functionals

The action we want to construct is a functional of $\Phi \in \Lambda^4$ and $C \in \Lambda^3$. It will contain a term imposing the constraints that guarantee that Φ is of algebraic type of the Cayley form. We will never need to specify what these constraints are, as we will only need their consequences. Given Φ, C there is a natural top form that can be constructed, which is $\Phi \wedge dC$. We take the integral of this to be our 'kinetic', i.e. containing derivatives term. Lagrangians of this type are well-known in the context of topological field theories. Thus, the theory with the action $\int \Phi \wedge dC$, with no additional terms, is a topological field theory known as (Abelian) BF theory. Our Lagrangian, however, contains other terms which render the theory non-topological. Apart from the terms imposing the constraints on Φ , we also want the Lagrangian to contain terms quadratic in C , such that the variation of the action with respect to C gives a set of linear equations for C .

The representation theoretic fact $\Lambda^3 = \Lambda_8^3 \oplus \Lambda_{48}^3$ implies that there are two linearly independent quadratic invariants that can be constructed from a 3-form C . A computation gives

$$(127) \quad \pi_8(C)_{abc} C^{abc} = \frac{1}{7} (C_{abc})^2 - \frac{3}{14} \Phi^{abcd} C_{ab}{}^p C_{cdp},$$

$$\pi_{48}(C)_{abc}C^{abc} = \frac{6}{7}(C_{abc})^2 + \frac{3}{14}\Phi^{abcd}C_{ab}{}^pC_{cdp}.$$

This shows that the two linearly independent quadratic invariants constructed from C can be taken to be $(C_{abc})^2$ and $\Phi^{abcd}C_{ab}{}^pC_{cdp}$. The coefficient in front of one of these can always be chosen as desired by rescaling the C field. This leads us to consider the following one-parameter family of action functionals

$$(128) \quad S[\Phi, C] = \int \Phi \wedge (dC - 6C \wedge_{\Phi} C) + \frac{\kappa}{6}(C)^2 v_{\Phi} + \frac{\lambda}{6}v_{\Phi} + \text{constr.}$$

The choice of coefficients here will be convenient for what follows. The constant λ is a 'cosmological constant' term that can be set to zero if desired. The object $C \wedge_{\Phi} C$ is the 4-form

$$(129) \quad C \wedge_{\Phi} C := \frac{1}{4!}(C \wedge_{\Phi} C)_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \\ (C \wedge_{\Phi} C)_{ijkl} = g^{pq}C_{ijp}C_{klq}$$

and $v_{\Phi} = (1/14)\Phi \wedge \Phi$ is the volume form. Written in index notation the action becomes

$$(130) \quad S[\Phi, C] = \frac{1}{3!} \int \left(\frac{1}{4!} \tilde{\epsilon}^{ijklabcd} \Phi_{ijkl} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \right. \\ \left. + \kappa (C_{abc})^2 v_g + \lambda v_g \right) d^8 x.$$

The constraint terms are omitted for brevity. The object $\tilde{\epsilon}^{ijklabcd}$ is the density weight one totally anti-symmetric tensor. This exists on any orientable manifold, and does not need a metric for its definition. We emphasise that $\tilde{\epsilon}^{ijklabcd}$ is independent of any metric, in particular the metric defined by Φ , to make it clear that this tensor is not subject to variation when Euler-Lagrange equations are derived below. Using the self-duality (16) of Φ we can see that the two scalars added to the Lagrangian are indeed $(C_{abc})^2$ and $\Phi^{abcd}C_{ab}{}^pC_{cdp}$.

6.2. The variation with respect to the 3-form

The variation of the action with respect to C is given by

$$(131) \quad \delta_C S = \frac{1}{3!} \int v_g \left(-5 \frac{1}{5!} \tilde{\epsilon}^{ijklabcd} \partial_a \Phi_{ijkl} - 3 \Phi^{aecd} C_{ae}{}^b \right. \\ \left. + 2\kappa C^{bcd} \right) \delta C_{bcd} d^8 x,$$

where v_g is the volume form for g , and we used the self-duality of the basic 4-form in the second term. The resulting Euler-Lagrange equation is therefore

$$(132) \quad 5 \frac{1}{5!} \epsilon_{bcd}{}^{ijkl} \partial_a \Phi_{ijkl} - 2J_3(C)_{bcd} + 2\kappa C_{bcd} = 0.$$

When $\kappa = 0$, comparing to (109), we see that $C = T$. The coefficient in front of the second term in the action was selected so that this happens. In general we have

$$(133) \quad J_3(T) = J_3(C) - \kappa C.$$

For a general κ this relation can be inverted

$$(134) \quad C = \frac{6T + \kappa J_3(T)}{6 - (5 + \kappa)\kappa},$$

which shows that $\kappa = 1, -6$ are the values when the relation cannot be inverted. These are of course also the eigenvalues of J_3 . We are particularly interested in the case when $\kappa = 0$, where $C = T$, and $\kappa = -2$ where

$$(135) \quad C = \frac{1}{2}T - \frac{1}{6}J_3(T).$$

6.3. Variation of the metric with respect to the 4-form

To vary the action with respect to the 4-form, we need a formula for the variation of g^{ij} with respect to the 4-form Φ_{ijkl} . This is standard, see e.g. [4]. We provide the full derivation in our notations for convenience. The best way to obtain a relation between the variations is to consider a variation of the metric, thought of as an $\text{GL}(8, \mathbb{R})$ transformation. As we have already discussed in (50), such a transformation effected by a symmetric 8×8 matrix h_{ij} induces a change in the basic 4-form given by

$$(136) \quad K(h)_{ijkl} = 4h_{[i|p|} \Phi^p{}_{jkl]}.$$

It will be more convenient, however, to consider the variation of Φ^{ijkl} . We have

$$(137) \quad \delta \Phi_{ijkl} = 4\alpha \delta g_{[i|p|} \Phi^p{}_{jkl]}.$$

The coefficient of proportionality α should be fixable by taking the variation of any of the algebraic relations satisfied by Φ . For example we have

$$(138) \quad \Phi_{abcd}\Phi_{ijkl}g^{ia}g^{jb}g^{kc}g^{ld} = 336.$$

Varying this gives

$$(139) \quad 2\delta\Phi_{ijkl}\Phi^{ijkl} + 4 \cdot 42\delta g^{ia}g_{ia} = 0,$$

where we used (15). Using (137) we have

$$(140) \quad 4\alpha\delta g^{[i|p|}\Phi_p{}^{jkl]}\Phi_{ijkl} + 2 \cdot 42\delta g_{ia}g^{ia} = 0.$$

Using (15) again this becomes

$$(141) \quad 2\alpha\delta g_{ij}g^{ij} + \delta g^{ij}g_{ij} = 0,$$

which shows that $\alpha = 1/2$. Thus, we have

$$(142) \quad \delta\Phi_{ijkl} = 2\delta g_{[i|p|}\Phi_p{}^{jkl]}.$$

As a check of consistency of these expressions, we also compute

$$(143) \quad \begin{aligned} \delta(\Phi^{ijkl}) &= \delta(g^{ia}g^{jb}g^{kc}g^{ld}\Phi_{abcd}) = \\ &g^{ia}g^{jb}g^{kc}g^{ld}\delta\Phi_{abcd} + 4\delta^{[i|a}g^{jb}g^{kc}g^{l]d}\Phi_{abcd} = \\ &2\delta g^{[i|p|}\Phi_p{}^{jkl]} - 4\delta g^{[i|p|}\Phi_p{}^{jkl]} = -2\delta g^{[i|p|}\Phi_p{}^{jkl]} = -g^{ia}g^{jb}g^{kc}g^{ld}\delta\Phi_{abcd}. \end{aligned}$$

This is analogous to the relation that we have for the metric

$$(144) \quad \delta g^{ij} = -g^{ia}g^{jb}\delta g_{ab}.$$

We now extract δg_{ij} in terms of $\delta\Phi_{ijkl}$. To do so we multiply the above expression by $\Phi^a{}_{jkl}$. We get

$$(145) \quad \delta\Phi_{(i|jkl|}\Phi_a{}^{jkl)} = 12\delta g_{ia} + 9\delta g_{pq}g^{pq}g_{ia}.$$

One more contraction gives

$$(146) \quad \delta g_{pq}g^{pq} = \frac{1}{84}\delta\Phi_{ijkl}\Phi^{ijkl},$$

and so

$$(147) \quad \delta g_{ij} = \frac{1}{12}(\delta\Phi_{(i|pqr}|\Phi_j)^{pqr} - \frac{3}{28}g_{ij}\delta\Phi_{pqr s}\Phi^{pqr s}).$$

Because the variation of the 4-form with all upper indices is given by minus the variation of the form with the lower indices, and the same is true for the metric variation, we can also write

$$(148) \quad \delta g^{ij} = \frac{1}{12}(\delta\Phi^{(i|pqr}|\Phi_j)^{pqr} - \frac{3}{28}g^{ij}\delta\Phi_{pqr s}\Phi^{pqr s}),$$

which is the form of the relation that will be used later.

6.4. Variation of the action with respect to the 4-form

We now derive the other half of the Euler-Lagrange equations. We first rewrite the action in terms of Φ^{abcd}

$$(149) \quad S[\Phi, C] = \frac{1}{3!} \int \left(v_g \Phi^{abcd} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \right. \\ \left. + \kappa (C_{abc})^2 v_g + \lambda v_g \right) d^8 x.$$

and then vary with respect to Φ^{abcd} . We have

$$(150) \quad \delta_\Phi S[\Phi, T] = \frac{1}{3!} \int v_g \left(\delta\Phi^{abcd} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \right. \\ \left. - \Phi^{abcd} \frac{3}{2} \delta g^{pq} C_{abp} C_{cdq} - \frac{1}{2} \delta g^{pq} g_{pq} \Phi^{abcd} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \right. \\ \left. - \frac{1}{2} \delta g^{pq} g_{pq} (\kappa (C_{abc})^2 + \lambda) \right) d^8 x.$$

The terms containing $\delta g^{pq} g_{pq}$ are from the variation of the volume form. We now substitute (148). The last term in the first line becomes

$$(151) \quad \left(-\frac{1}{8} \Phi^{ijkl} C_{ija} C_{kle} \Phi^e{}_{bcd} + \frac{3}{8 \cdot 28} \Phi^{ijkl} C_{ij}{}^p C_{klp} \Phi_{abcd} \right) \delta\Phi^{abcd}.$$

Thus, the variation of the action with respect to Φ^{abcd} is

$$(152) \quad E_{abcd} = \partial_{[a} C_{bcd]} - \frac{3}{2} C_{[ab}{}^p C_{cd]p} - \frac{1}{8} \Phi^{ijkl} C_{ij[a} C_{|kle|} \Phi^e{}_{bcd]} \\ + \frac{3}{8 \cdot 28} \Phi^{ijkl} C_{ij}{}^p C_{klp} \Phi_{abcd} - \frac{1}{2 \cdot 84} \Phi_{abcd} \left(\Phi^{ijkl} (\partial_i C_{jkl} - \frac{3}{2} C_{ij}{}^p C_{klp}) + \kappa (C_{ijk})^2 + \lambda \right).$$

This does not need to be zero, as the action also contains terms imposing the constraints guaranteeing that Φ_{abcd} is of the correct algebraic type. The constraint terms produce a variation that is an arbitrary tensor in Λ_{27}^4 . So, we can only deduce that the Λ_{35+1}^4 and Λ_7^4 projection of the above vanishes. Before we extract these projections, it is worth evaluating the trace of the field equations. We have

$$(153) \quad \Phi^{abcd} E_{abcd} = -2\lambda - 2\kappa(C_{abc})^2 - \Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{4}C_{ab}{}^p C_{cdp}).$$

This is the projection of the field equations onto Λ_1^4 , which must vanish. We therefore get the following consequence of the field equations

$$(154) \quad \Phi^{abcd}(\partial_a C_{bcd} - \frac{3}{4}C_{ab}{}^p C_{cdp}) + 2\lambda + 2\kappa(C_{abc})^2 = 0.$$

We can use this to simplify E_{abcd} . We have

$$(155) \quad \begin{aligned} \Phi^{ijkl}(\partial_i C_{jkl} - \frac{3}{2}C_{ij}{}^p C_{klp}) + \kappa(C_{ijk})^2 + \lambda = \\ -\lambda - \kappa(C_{ijk})^2 - \frac{3}{4}\Phi^{ijkl}C_{ij}{}^p C_{klp}, \end{aligned}$$

and so we can rewrite

$$(156) \quad \begin{aligned} E'_{abcd} = \partial_{[a} C_{bcd]} - \frac{3}{2}C_{[ab}{}^p C_{cd]p} - \frac{1}{8}\Phi^{ijkl}C_{ij[a}C_{|kle|}\Phi^e{}_{bcd]} \\ + \frac{1}{56}\Phi_{abcd}\Phi^{ijkl}C_{ij}{}^p C_{klp} + \frac{1}{2 \cdot 84}\Phi_{abcd}(\lambda + \kappa(C_{ijk})^2). \end{aligned}$$

The Λ_{35+1+7}^4 projections of this vanish when the Λ_{35+1+7}^4 projections of E_{abcd} vanish and vice versa, so $E'_{abcd} = 0$ gives an equivalent encoding of field equations.

6.5. Extracting Λ_{35+1+7}^4 projections

To understand the implications of the field equations we extract the Λ_{35+1}^4 and Λ_7^4 projections. This gives

$$(157) \quad \begin{aligned} \Phi_b{}^{pqr} E'_{apqr} = \frac{1}{4}\Phi_b{}^{pqr}\nabla_a C_{pqr} - \frac{3}{4}\Phi_b{}^{pqr}\nabla_r C_{apq} \\ - \frac{3}{2}\Phi_b{}^{pqr}C_{ap}{}^s C_{qrs} - \frac{3}{4}\Phi^{pqrs}C_{apq}C_{brs} + \frac{1}{4}g_{ab}\left(\lambda + \kappa(C_{pqr})^2 + \frac{3}{4}\Phi^{pqrs}C_{pq}{}^p C_{rsp}\right). \end{aligned}$$

Its ab symmetrisation and anti-symmetrisation compute the Λ_{35+1}^4 and Λ_7^4 parts respectively. We wrote the derivatives here as the covariant derivatives, for the computations to follow.

6.6. Rewriting the $\kappa = 0$ field equations - antisymmetric part

For $\kappa = 0$ we have $C = T$. Let us understand the arising field equations. We start with the anti-symmetric part. Taking (twice) the anti-symmetric part of the field equations (157) we get

$$(158) \quad \frac{1}{2}\Phi_{[a}{}^{pqr}\nabla_{b]}T_{pqr} - \frac{3}{2}\Phi_{[a}{}^{pqr}\nabla_{|r|}T_{b]pq} - 3\Phi_{[a}{}^{pqr}T_{b]p}{}^sT_{qrs} = 0.$$

With the help of the curvature identity (121) we can rewrite this as

$$(159) \quad \nabla^r T_{r;ab} = 0,$$

which is just vanishing of the divergence of the original torsion. This also makes it manifest that this equation is Λ_7^2 valued. Note also that this equation does not hold automatically. It is a non-trivial field equation to be imposed, and it becomes a second order PDE on the original 4-form. It can be interpreted as the evolution equation for the Λ_7^4 part of the Cayley form perturbation, as is confirmed by the linearised analysis below.

6.7. Rewriting the $\kappa = 0$ field equations - symmetric part

For the analysis of the symmetric part, we take (twice) the symmetric part of (157), also writing it with the opposite sign

$$(160) \quad -\frac{1}{2}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} + \frac{3}{2}\Phi_{(a}{}^{pqr}\nabla_{|r|}T_{b)pq} \\ + 3\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} + \frac{3}{2}\Phi^{pqrs}T_{apq}T_{brs} + \frac{3}{8}g_{ab}\Phi^{ijkl}T_{ij}{}^pT_{klp} + \lambda g_{ab} = 0.$$

Contract the resulting equation with g^{ab} we get (154). Comparing this with (123) we see that this is *not* the condition that the Ricci scalar is constant. Rather, using (123), we can rewrite this equation as

$$(161) \quad R = T_{abc}T^{abc} + \frac{1}{4}\Phi^{abcd}T_{ab}{}^pT_{cdp} + 4\lambda.$$

A computation shows that this can be rewritten as

$$(162) \quad R = T^{abc}(T + \frac{1}{6}J_3(T))_{abc} = \frac{7}{6}(T_{abc}^{48})^2 + 4\lambda.$$

Here $T_{48} = \pi_{48}(T)$ is the Λ_{48}^3 part of the torsion 3-form. We thus see that the curvature scalar is sourced just by this part of the torsion.

For the complete symmetric part of the equation, comparing this with (126), we can see that the second order part here does not reduce to that in R_{ab} . The comparison with (126) suggests that we can rewrite (160) as

$$(163) \quad 3R_{ab} + \Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} - 3T_a{}^{pq}T_{bpq} + \frac{3}{2}\Phi^{pqrs}T_{apq}T_{brs} + \frac{3}{8}g_{ab}\Phi^{ijkl}T_{ij}{}^pT_{klp} + \lambda g_{ab} = 0.$$

We thus see that the field equations *do not* state that the metric is Einstein. Instead, there are extra contributions coming from the torsion 3-form, and its derivatives. Note that the covariant derivative appears in this equation in such a way that, while both R_{ab} and $\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr}$ do depend on it, the specific combination of these terms that appears does not depend on ∇ . This will become more pronounced once we rewrite the field equations as a condition that a certain 4-form vanishes.

6.8. Different ways of writing the field equations

We note that we can introduce a symmetric tensor

$$(164) \quad H_{ab} := \Phi^{ijkl}T_{ija}T_{klb} - \frac{1}{7}g_{ab}\Phi^{ijkl}T_{ij}{}^pT_{klp}.$$

The 4-form encoding the field equations can then be written very compactly as

$$(165) \quad E'_{abcd} = \partial_{[a}T_{bcd]} - \frac{3}{2}T_{[ab}{}^pT_{cd]p} - \frac{1}{8}H_{[a|e|}\Phi^e{}_{bcd]} + \frac{\lambda}{84}\Phi_{abcd}.$$

Recall that E'_{abcd} is the tensor encoding the field equations of the theory, see (156). The field equations are then the statement that this equals to an arbitrary tensor in Λ_{27}^4 , which we know can be parametrised as (56). So, we get one of the possible ways of writing the field equations

$$(166) \quad \partial_{[a}T_{bcd]} - \frac{3}{2}T_{[ab}{}^pT_{cd]p} - \frac{1}{8}H_{[a|e|}\Phi^e{}_{bcd]} + \frac{\lambda}{84}\Phi_{abcd} = \Psi_{[ab}{}^{pq}\Phi_{|pq|cd]},$$

where Ψ^{abcd} is an arbitrary symmetric tracefree matrix in $\text{Sym}_0^2(\Lambda_7^2)$.

6.9. Yet another rewriting of the field equations

Yet another way of writing the field equations, potentially useful, is obtained by computing $\Phi_{abc}{}^s \Phi_s{}^{pqr} E'_{dpqr}$, and anti-symmetrising on $abcd$. This gives a 4-form that is projected onto the Λ_{35+1}^4 and Λ_7^4 parts, eliminating the Λ_{27}^4 part of E'_{abcd} that does not need to be zero. For a general 4-form we have

$$(167) \quad \frac{1}{6} \Phi_{abc}{}^s \Phi_s{}^{pqr} \sigma_{dpqr} = (\mathbb{I} - \frac{1}{2} J_4)(\sigma)_{abcd},$$

explicitly showing that the Λ_{27}^4 component is projected away. We now apply this projector to the 4-form E'_{abcd} to get the following 4-form field equations

$$(168) \quad \begin{aligned} & \nabla_{[a} T_{bcd]} - \frac{3}{4} \Phi_{[ab}{}^{pq} \nabla_c T_{d]pq} - \frac{3}{4} \Phi_{[ab}{}^{pq} \nabla_{|p} T_{cd|q} \\ & - \frac{3}{2} T_{[ab}{}^p T_{cd]p} + \frac{3}{4} \Phi_{[ab}{}^{pq} T_{cd]}{}^r T_{pqr} - \frac{1}{8} \Phi_{[abc}{}^p \Phi^{ijkl} T_{d]ij} T_{klp} - \frac{3}{2} \Phi_{[ab}{}^{pq} T_{c|p]}{}^r T_{d]qr} \\ & + \frac{1}{32} \Phi_{abcd} \Phi^{ijkl} T_{ij}{}^p T_{klp} + \frac{\lambda}{12} \Phi_{abcd} = 0. \end{aligned}$$

Since the first line here can be rewritten as

$$(169) \quad (\mathbb{I} - \frac{1}{2} J_4)(\nabla_{[a} T_{bcd]}),$$

we see that the operator that appears in the field equations is built from the usual partial derivative, rather than the covariant one.

6.10. Analysis of the $\kappa = -2$ field equations

In the general κ case, we can rewrite the field equations (157) in terms of the intrinsic torsion 3-form T , using the relation between C and T . However, the arising general κ results are too cumbersome. Using as the motivation the computation of the linearised action in the last section, we now specialise to the particularly interesting case $\kappa = -2$, when the linearised action coincides with that of General Relativity. Our intention is to see whether the full non-linear equations of the theory in this case also reduce to the Einstein condition.

We substitute C in the form (135) to (157) and whenever the derivatives get applied to the basic 4-form, evaluate them using (102). The resulting field equations are as follows

$$\frac{1}{4} \Phi_b{}^{pqr} \nabla_a T_{pqr} - \frac{1}{2} \Phi_b{}^{pqr} \nabla_r T_{apq} + \frac{1}{4} \Phi_a{}^{pqr} \nabla_r T_{bpq} + \frac{1}{2} (\nabla^p T_{abp} - \frac{1}{2} \Phi_{ab}{}^{cd} \nabla^p T_{cdp})$$

$$\begin{aligned}
& -\frac{23}{24}\Phi_b^{pqr}T_{ap}{}^sT_{qrs} - \frac{5}{24}\Phi_a^{pqr}T_{bp}{}^sT_{qrs} - \frac{1}{4}\Phi^{pqr}sT_{apq}T_{brs} + \frac{1}{6}T_a{}^{pq}T_{bpq} \\
& - \frac{1}{24}\Phi^{pqr}sT_{abp}T_{qrs} + \frac{1}{48}\Phi_{ab}{}^{pq}\Phi^{ijkl}T_{pqi}T_{jkl} - \frac{1}{24}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk} \\
& + \frac{1}{4}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{17}{24}\Phi^{pqr}sT_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqr}s\nabla_p T_{qrs}\right) = 0.
\end{aligned}$$

We now use (119) to simplify the first line. We also separate the symmetric and anti-symmetric parts. We get

$$\begin{aligned}
(170) \quad & \frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} - \frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_r T_{b)pq} \\
& - \frac{7}{6}\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} - \frac{1}{4}\Phi^{pqr}sT_{apq}T_{brs} + \frac{1}{6}T_a{}^{pq}T_{bpq} - \frac{1}{24}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk} \\
& + \frac{1}{4}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{17}{24}\Phi^{pqr}sT_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqr}s\nabla_p T_{qrs}\right) = 0
\end{aligned}$$

for the symmetric part and

$$(171) \quad -\frac{1}{4}\Phi_{[a}{}^{pqr}T_{b]p}{}^sT_{qrs} - \frac{1}{24}\Phi^{pqr}s(T_{abp} - \frac{1}{2}\Phi_{ab}{}^{cd}T_{cdp})T_{qrs} = 0$$

for the anti-symmetric part.

6.11. The trace

It will be useful for the later to compute the trace of the field equations. We get

$$(172) \quad 2\lambda - \frac{11}{4}(T_{abc})^2 - \frac{1}{8}\Phi^{abcd}T_{ab}{}^pT_{cdp} + \frac{3}{2}\Phi^{abcd}\nabla_a T_{bcd} = 0.$$

Using (123) we can rewrite this as

$$(173) \quad \frac{3}{2}R = 2\lambda - \frac{5}{4}(T_{abc})^2 + \frac{11}{8}\Phi^{abcd}T_{ab}{}^pT_{cdp}.$$

6.12. An identity

Contracting (20) with $T_{bcd}T^{jkl}$ we get the following identity

$$(174) \quad \frac{1}{4}\Phi_a{}^{pqr}\Phi_b^{ijk}T_{pqi}T_{rjk} =$$

$$-2\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} + \frac{1}{2}g_{ab}\Phi^{ijkl}T_{ij}{}^pT_{klp} - \frac{1}{2}\Phi^{pqrs}T_{apq}T_{brs} + \frac{1}{12}\Phi_a{}^{pqr}\Phi_b{}^{ijk}T_{pqr}T_{ijk}.$$

Using this in the symmetric part of the field equations we can transform it to

$$(175) \quad \begin{aligned} & \frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_{b)}T_{pqr} - \frac{1}{4}\Phi_{(a}{}^{pqr}\nabla_rT_{b)pq} \\ & - \frac{5}{6}\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} - \frac{1}{6}\Phi^{pqrs}T_{apq}T_{brs} + \frac{1}{6}T_a{}^{pq}T_{bpq} - \frac{1}{72}\Phi_a{}^{pqr}\Phi_b{}^{ijk}T_{pqr}T_{ijk} \\ & + \frac{1}{4}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{3}{8}\Phi^{pqrs}T_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqrs}\nabla_pT_{qrs}\right) = 0. \end{aligned}$$

We can now rewrite this in terms of the Ricci tensor using (126). We get

$$(176) \quad \begin{aligned} R_{ab} = & -\frac{2}{3}\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} - \frac{1}{3}\Phi^{pqrs}T_{apq}T_{brs} + \frac{4}{3}T_a{}^{pq}T_{bpq} \\ & - \frac{1}{36}\Phi_a{}^{pqr}\Phi_b{}^{ijk}T_{pqr}T_{ijk} + \frac{1}{2}g_{ab}\left(\lambda - \frac{17}{12}(T_{pqr})^2 + \frac{3}{8}\Phi^{pqrs}T_{pq}{}^pT_{rsp} + \frac{1}{2}\Phi^{pqrs}\nabla_pT_{qrs}\right). \end{aligned}$$

We can also use (123) to rewrite this as

$$(177) \quad \begin{aligned} R_{ab} + \frac{1}{4}g_{ab}R = & -\frac{2}{3}\Phi_{(a}{}^{pqr}T_{b)p}{}^sT_{qrs} - \frac{1}{3}\Phi^{pqrs}T_{apq}T_{brs} + \frac{4}{3}T_a{}^{pq}T_{bpq} \\ & - \frac{1}{36}\Phi_a{}^{pqr}\Phi_b{}^{ijk}T_{pqr}T_{ijk} + \frac{1}{2}g_{ab}\left(\lambda - \frac{11}{12}(T_{pqr})^2 + \frac{7}{8}\Phi^{pqrs}T_{pq}{}^pT_{rsp}\right). \end{aligned}$$

This makes it clear that the $\kappa = -2$ non-linear equations *do not coincide* with Einstein equations. Rather, these are Einstein equations with 'stress-energy' tensor sourced by the intrinsic torsion. Better understanding of these equations requires further work.

6.13. Rewriting the $\kappa = -2$ field equations - antisymmetric part

$$(178) \quad \Phi_{[a}{}^{pqr}T_{b]p}{}^sT_{qrs} + \frac{1}{6}\Phi^{pqrs}(T_{abp} - \frac{1}{2}\Phi_{ab}{}^{cd}T_{cdp})T_{qrs} = 0.$$

The expression in brackets contains a multiple of the projector π_7 , so it is in Λ_7^2 . The first term can also be checked to be in Λ_7^2 by computing the π_{21} projection and verifying that it is identically zero. Moreover, it can be checked that the above expression is invariant under the change

$$(179) \quad T_{abc} \rightarrow T_{abc} + \Phi_{abcd}V^d,$$

for any V^a , which means that it only depends on the Λ_{48}^3 part of T_{abc} . This means we can write this equation as

$$(180) \quad \Phi_{[a}{}^{pqr} \tilde{T}_{b]p}{}^s \tilde{T}_{qrs} = 0.$$

where $\tilde{T} = \pi_{48}(T)$. There is precisely one copy of the **7** representation in the tensor product $\mathbf{48} \otimes \mathbf{48}$, and the field equation (180) states that $\pi_7(\tilde{T} \otimes \tilde{T}) = 0$.

7. Linearisation

We now compute the linearisation of the general action (128) and verify that it gives the most general diffeomorphism-invariant linearised theory (89).

7.1. Linearisation of the non-linear action

We start with the full action without the cosmological constant part, and without the constraint terms, which we assume to be satisfied

$$(181) \quad S[\Phi, C] = \frac{1}{3!} \int \left(\frac{1}{4!} \Phi_{ijkl} (\partial_a C_{bcd} - \frac{3}{2} g^{pq} C_{abp} C_{cdq}) \epsilon^{ijklabcd} \right. \\ \left. + \kappa (C_{abc})^2 v_g \right),$$

where the volume element d^8x is omitted for compactness. We will then linearise around the background given by $C = 0$ and a constant Φ . Denoting the variation of Φ by ϕ and of C by c we get for the second variation

$$(182) \quad S^{(2)}[\phi, c] = \frac{1}{3!} \int \left(\frac{1}{4!} \epsilon^{ijklabcd} \phi_{ijkl} \partial_a c_{bcd} - \frac{3}{2} \Phi^{abcd} c_{ab}{}^p c_{cdp} \right. \\ \left. + \kappa (c_{abc})^2 \right),$$

A calculation shows

$$(183) \quad J_3(c)_{abc} c^{abc} = \frac{3}{2} \Phi^{abcd} c_{ab}{}^p c_{cdp},$$

which makes it easy to derive the Euler-Lagrange equation for c_{abc} , which is given by

$$(184) \quad \frac{1}{4!} \epsilon_{bcd}{}^{aijkl} \partial_a \phi_{ijkl} = 2J_3(c)_{bcd} - 2\kappa c_{bcd}.$$

We can now integrate by parts in (182) in the first term to rewrite it in terms of the tensor c_{abc} . We see that it is given by $(2J_3(c)_{bcd} - 2\kappa c_{bcd})c^{bcd}$. This means that the linearised action written in terms of ϕ only is given by

$$(185) \quad S^{(2)}[\phi] = \frac{1}{6} \int J_3(c)_{abc} c^{abc} - \kappa (c_{abc})^2,$$

where c_{abc} is given by (184). We note that the linearised action is manifestly diffeomorphism-invariant. Indeed, the linearised 4-form transforms as (85). This is clearer in the form notation

$$(186) \quad \delta\phi = di_\xi\Phi, \quad \xi \in TM.$$

As before, in this linearised calculation the background Φ is assumed closed, and ϕ is the perturbation. Thus, the variation of ϕ under diffeomorphisms is an exact form. The formula (184) shows that the tensor c_{abc} is obtained from the Hodge dual of the exterior derivative of the linearised 4-form, which is clearly diffeomorphism-invariant. So, any linearised action written in terms of c_{abc} is diffeomorphism-invariant.

7.2. Rewriting of the linearised action

We now rewrite the linearised action explicitly in terms of ϕ . The tensors c_{abc} and the linearised torsion t_{abc} are linearly related. We need to express c_{abc} in terms of t_{abc} . The linearised torsion t_{abc} is given by

$$(187) \quad J_3(t_{abc}) = \frac{1}{48} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl}.$$

We also have

$$(188) \quad J_3\left(\frac{1}{48} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl}\right) = \frac{1}{2} \Phi_{[a}{}^{pqr} \partial_b \phi_{c]pqr} + \frac{3}{4} \Phi_{[a}{}^{pqr} \partial_{|p|} \phi_{bc]qr}.$$

This means that

$$(189) \quad t_{abc} = \frac{5}{288} \epsilon_{abc}{}^{pijkl} \partial_p \phi_{ijkl} + \frac{1}{12} \Phi_{[a}{}^{pqr} \partial_b \phi_{c]pqr} + \frac{1}{8} \Phi_{[a}{}^{pqr} \partial_{|p|} \phi_{bc]qr}.$$

For completeness, we state the result of computation of the Λ_8^3 part of the torsion 3-form. We have

$$(190) \quad t_{abc} \Phi^{mabc} = \frac{1}{48} \Phi^{abcd} \partial^m \phi_{abcd} + \frac{1}{12} \Phi^{abcd} \partial_a \phi_{bcd}{}^m.$$

A long calculation (using algebraic manipulation) based on (19) gives

$$(191) \quad \int J_3(t_{abc})t^{abc} = \int \frac{5}{96}\phi^{abcd}\partial^p\partial_p\phi_{abcd} + \frac{1}{32}\Phi^{abcd}\phi_{ab}{}^{pq}\partial^e\partial_e\phi_{cdpq} \\ + \frac{5}{24}(\partial^a\phi_{abcd})^2 + \frac{1}{16}\Phi^{abcd}\partial^i\phi_{abip}\partial^j\phi_{cdj}{}^p - \frac{1}{8}\Phi^{abci}(\partial_i\partial^r\phi_{crpq})\phi_{ab}{}^{pq}.$$

This can be simplified by passing to the parametrisation of ϕ by fields h, ξ .

7.3. Evaluation of the linearised action

We now use the parametrisation (82). In this parametrisation, using (17) gives

$$(192) \quad J_3(t)_{abc} := \frac{1}{48}\epsilon_{abc}{}^{pijkl}\partial_p\phi_{ijkl} = \\ -\frac{1}{2}\Phi_{abc}{}^p\partial^i(h_{ip} - \frac{1}{4}\xi_{ip}) + \frac{1}{2}\Phi_{abc}{}^i\partial_i h - \frac{3}{2}\Phi_{[bc}{}^{ip}\partial_p(h_{a]i} - \frac{1}{4}\xi_{a]i}).$$

We have introduced the notation $J_3(t)$ for this quantity. We also have

$$(193) \quad J_3(t) = J_3(c) - \kappa c,$$

and so we need to compute $J_3(t)_{abc}c^{abc}$. We have

$$(194) \quad c = \frac{6t + \kappa J_3(t)}{6 - 5\kappa - \kappa^2}.$$

This means that we can write the linearised action as

$$(195) \quad S^{(2)}[\phi] = \int \mathcal{L}^{(2)} = \frac{1}{6(6 - 5\kappa - \kappa^2)} \int \kappa J_3(t)J_3(t) + 6J_3(t)t = \\ \frac{1}{6(6 - 5\kappa - \kappa^2)} \int \kappa J_3(t)J_3(t) + J_3(t)(J_3 + 5\mathbb{I})J_3(t).$$

A computation gives

$$(196) \quad \left(1 - \frac{5\kappa}{6} - \frac{\kappa^2}{6}\right) \mathcal{L}^{(2)} = \frac{1}{2}\left(1 + \frac{\kappa}{6}\right)(\partial_a h_{bc})^2 - \frac{1}{6}\left(1 - \frac{\kappa}{2}\right)(\partial_a h)^2 \\ - \frac{1}{3}\left(1 - \frac{\kappa}{2}\right)h\partial^a\partial^b h_{ab} - \frac{2}{3}(\partial^a h_{ab})^2 + \frac{1}{24}\left(1 + \frac{\kappa}{2}\right)(\partial_a \xi_{bc})^2 - \frac{2}{3}\left(1 + \frac{\kappa}{2}\right)\partial_b h^{ba}\partial^c \xi_{ca}.$$

This is the diffeomorphism-invariant Lagrangian of the type (76) with (88) and

$$(197) \quad \rho = 1 + \frac{\kappa}{6}, \quad \mu = \frac{2}{3}\left(1 + \frac{\kappa}{2}\right).$$

This shows that the linearisation of our general action gives the linearisation of the Einstein-Hilbert Lagrangian for $\kappa = -2$. We can also write down what the Lagrangian (80) becomes with this choice of the parameters. We get

$$(198) \quad \mathcal{L} = \frac{\kappa + 6}{12}(\partial_a h_{bc})^2 + \frac{(\kappa + 6)(\kappa - 2)}{96}(\partial_a h)^2 \\ + \frac{(\kappa + 6)(\kappa + 2)}{192}(\partial_a \xi_{bc})^2 - \frac{2}{3}(\partial^a(h_{ab} - \frac{2 - \kappa}{8}\eta_{ab}h + \frac{2 + \kappa}{4}\xi_{ab}))^2.$$

This shows that another interesting point in the theory space is $\kappa = 2$, when there is no separate kinetic term for the trace of the metric in the linearised Lagrangian. The value $\kappa = -6$ is also special. As is clear from (194), for this value of κ the tensor c_{abc} can no longer be solved for in terms of t_{abc} . All these special cases need to be studied further to understand their significance.

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Kirill Krasnov

School of Mathematical Sciences, University of Nottingham, Nottingham,
NG7 2RD, UK

E-mail: kirill.krasnov@nottingham.ac.uk, ORCID: 0000-0003-2800-3767