

Supplemental material: Vibrational dressing in kinetically constrained Rydberg spin systems

Paolo P. Mazza,¹ Richard Schmidt,^{2,3} and Igor Lesanovsky^{1,4}

¹*Institut für Theoretische Physik, University of Tübingen,
Auf der Morgenstelle 14, 72076 Tübingen, Germany*

²*Max-Planck-Institute of Quantum Optics, Hans-Kopfermann-Strasse, 1, 85748 Garching, Germany*

³*Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, 80799 München, Germany*

⁴*School of Physics and Astronomy and Centre for the Mathematics
and Theoretical Physics of Quantum Non-Equilibrium Systems,
The University of Nottingham, Nottingham, NG7 2RD, United Kingdom*

In this supplemental material we show step-by-step how the Hamiltonian (3) in the main text can be rewritten as in Eq. (7). We will also write explicitly all the interaction terms.

HAMILTONIAN IN THE EFFECTIVE SPACE

Let us start by considering the Hamiltonian describing Rydberg atoms in the effective “constrained” Hilbert space. This reads (see Eq. (4) in the main text):

$$\begin{aligned}
 H = & \Omega \left(\sum_{\alpha} |\alpha\rangle \langle \alpha| \mu^x + \mu^- |\alpha + 1\rangle \langle \alpha| + \text{h.c.} \right) \\
 & + \kappa \sum_{\alpha} \frac{\mu^z - \mathbb{1}}{2} |\alpha\rangle \langle \alpha| (a_{\alpha+1}^{\dagger} + a_{\alpha+1} - a_{\alpha}^{\dagger} - a_{\alpha}) \\
 & + \omega \sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha},
 \end{aligned} \tag{S1}$$

where the μ -operators are the ones defined in the main text. The first step is to move to the Fourier space for the bosonic modes of the harmonic traps. This is achieved by defining

$$a_m = \frac{1}{\sqrt{N}} \sum_{p=-N/2}^{N/2} A_p e^{i\frac{2\pi}{N}mp}. \tag{S2}$$

We thus see that the difference between the phonon creation operators appearing in the interaction term can be rewritten as

$$a_{m+1}^{\dagger} - a_m^{\dagger} = \frac{1}{\sqrt{N}} \sum_p \left[\left(e^{-i\frac{(m+1)2\pi}{N}p} - e^{-i\frac{m2\pi}{N}p} \right) A_p^{\dagger} \right]. \tag{S3}$$

As we showed in the main text (see Eq. (4)) this leads to the Hamiltonian

$$\begin{aligned}
 H = & \Omega \sum_{\alpha} |\alpha\rangle \langle \alpha| \mu^x + \Omega \sum_{\alpha} (\mu^- |\alpha + 1\rangle \langle \alpha| + \text{h.c.}) \\
 & + \frac{\kappa(\mu^z - \mathbb{1})}{2\sqrt{N}} \sum_p \left[\left(e^{-i\frac{2\pi}{N}p} - 1 \right) e^{-i\frac{2\pi}{N}\hat{\alpha}} A_p^{\dagger} + \text{h.c.} \right] \\
 & + \omega \sum_p A_p^{\dagger} A_p,
 \end{aligned} \tag{S4}$$

in which $\hat{\alpha} = \sum_{\alpha} \alpha |\alpha\rangle \langle \alpha|$. At this point we can apply the Lee-Low-Pines transformation, which is defined as

$$U = \exp \left[-i\hat{\alpha} \sum_p \frac{2\pi p}{N} A_p^{\dagger} A_p \right] \tag{S5}$$

$$U^{\dagger} = \exp \left[i\hat{\alpha} \sum_p \frac{2\pi p}{N} A_p^{\dagger} A_p \right]. \tag{S6}$$

We stress, again, that this transformation is important because it decouples the lattice degrees of freedoms from the phonons. Applying the transformation (S6) to the operators in Eq. (S4) we have:

$$U^\dagger A_p U = \exp \left\{ -i \frac{2\pi p}{N} \hat{\alpha} \right\} A_p, \quad (\text{S7})$$

and

$$U^\dagger |m+1\rangle \langle m| U = e^{+i \sum_p A_p^\dagger A_p \frac{2\pi p(m+1)}{N}} |m+1\rangle \langle m| e^{-i \sum_p A_p^\dagger A_p \frac{2\pi p m}{N}} = |m+1\rangle \langle m| e^{\frac{-i2\pi}{N} p \sum_p A_p^\dagger A_p}. \quad (\text{S8})$$

Therefore, Hamiltonian (S4) can be rewritten as

$$\begin{aligned} U^\dagger \hat{H} U = \Omega \sum_{\alpha} \left[|\alpha\rangle \langle \alpha| \mu^x + |\alpha+1\rangle \langle \alpha| e^{-i \frac{2\pi}{N} p \sum_p A_p^\dagger A_p} \mu^- + |\alpha\rangle \langle \alpha+1| e^{i \frac{2\pi}{N} \sum_p p A_p^\dagger A_p} \mu^+ \right] \\ + \frac{\kappa(\mu^z - \mathbb{1})}{2\sqrt{N}} \sum_x \left[\left(e^{-i \frac{2\pi}{N} p} - 1 \right) e^{-i \frac{2\pi}{N} \hat{\alpha}} A_p^\dagger + \text{h.c.} \right] + \omega \sum_p A_p^\dagger A_p. \end{aligned} \quad (\text{S9})$$

In order to get the rid of the lattice labels α we move to the Fourier space for the quasi-particles:

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{q=-N/2}^{N/2} e^{i \frac{\alpha 2\pi q}{N}} |q\rangle \quad (\text{S10})$$

We then obtain

$$\begin{aligned} \hat{H} = \Omega \sum_q |q\rangle \langle q| \left[\mu^x + \mu^- e^{-i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} + \mu^+ e^{+i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} \right] + \\ \frac{\kappa(\mu^z - \mathbb{1})}{2\sqrt{N}} \sum_x \left[\left(e^{-i \frac{2\pi}{N} p} - 1 \right) A_p^\dagger + \text{h.c.} \right] + \omega \sum_p A_p^\dagger A_p. \end{aligned} \quad (\text{S11})$$

Note, that Hamiltonian (S11) is diagonal in the quasi-particles momentum q . Hence, we can diagonalize for every q the free part of it, i.e. the Hamiltonian corresponding to $\kappa = 0$.

DIAGONALIZATION OF THE FREE PART

Let us rewrite Eq. (S11) in matrix form, i.e. writing explicitly the matrices $\mu^{x,\pm}$ and completing the squares for the bosonic part

$$\begin{aligned} \hat{H}_q = \Omega \left(\begin{array}{cc} 0 & e^{+i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} + 1 \\ e^{-i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} + 1 & 0 \end{array} \right) + \\ + \omega \sum_p \left[A_p + \frac{\kappa}{\omega\sqrt{N}} \left(e^{+i \frac{2\pi}{N} p} - 1 \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]^\dagger \left[A_p + \frac{\kappa}{\omega\sqrt{N}} \left(e^{-i \frac{2\pi}{N} p} - 1 \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] + \\ - \frac{\kappa^2}{\omega N} \sum_p 2 \left[1 - \cos \left(\frac{2\pi}{N} p \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]. \end{aligned} \quad (\text{S12})$$

Defining a *displacement* operator for the bosons, i.e.

$$\hat{D} = \exp \left[-\frac{\kappa}{\sqrt{N}\omega} \sum_p \left(e^{-i \frac{2\pi}{N} p} - 1 \right) A_p^\dagger - \text{h.c.} \right] \quad (\text{S13})$$

such that $\hat{D}^\dagger \tilde{A}_p \hat{D} = A_p$, with $\tilde{A}_p = A_p + \frac{\kappa}{\omega\sqrt{N}} \left(e^{-i \frac{2\pi}{N} p} - 1 \right)$, we can cast Eq. (S12) in the following form:

$$\hat{D}^\dagger \hat{H}_q \hat{D} = \omega \sum_p \tilde{A}_p^\dagger \tilde{A}_p + \Omega \left(\begin{array}{cc} 0 & e^{+i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} + 1 \\ e^{-i \frac{2\pi(\sum_p p A_p^\dagger A_p + q)}{N}} + 1 & 0 \end{array} \right) - 2 \frac{\kappa^2}{\omega} \hat{n} + \tilde{H}_{\text{int}}^{\text{II}} \quad (\text{S14})$$

Note, that the effect of the interaction between the lattice and the phonons is only in the argument of the displacement operator. We can now diagonalize the off-diagonal matrix appearing in (S14). Casting $\theta = \sum_p (pA_p^\dagger A_p + q)$, the matrix we want to diagonalize has therefore the form

$$\begin{pmatrix} 0 & e^{2i\theta\pi/N} + 1 \\ e^{-2i\theta\pi/N} + 1 & 0 \end{pmatrix}. \quad (\text{S15})$$

Its eigenvectors are $\begin{pmatrix} -e^{i\theta\pi/N} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} e^{i\theta\pi/N} \\ 1 \end{pmatrix}$, therefore the unitary matrix S which implements the diagonalization is

$$S = \begin{pmatrix} -e^{i\theta\pi/N} & e^{i\theta\pi/N} \\ 1 & 1 \end{pmatrix}. \quad (\text{S16})$$

The diagonalization induces a mixing between the states $|q, \mu^z = 1\rangle$ and $|q, \mu^z = 2\rangle$.

The term $\hat{H}_{\text{int}}^{\text{II}} = S^\dagger \tilde{H}_{\text{int}}^{\text{II}} S$ is obtained by the action of the displacement operator \hat{D} , defined in Eq (S13), on the Rabi part of the Hamiltonian Eq. (S12). We want to derive an effective expression for this interaction term in the perturbative limit. In the limit of small κ we can rewrite the displacement operator as

$$\hat{D} = e^{-\kappa \sum_p \alpha_p \hat{A}_p^\dagger + \kappa \sum_p \alpha_p^* \hat{A}_p} \simeq \left[\mathbb{1} - \kappa \sum_p (\alpha_p \hat{A}_p^\dagger - \alpha_p^* \hat{A}_p) + \frac{\kappa^2}{2} \left(\sum_p (\alpha_p^* \hat{A}_p - \alpha_p \hat{A}_p^\dagger) \right)^2 \right] + \dots, \quad (\text{S17})$$

where $\alpha_p = \frac{1}{\omega\sqrt{N}} (e^{-i\frac{2\pi}{N}p} - 1)$. Therefore, we have

$$\begin{aligned} \hat{D}^\dagger H_f \hat{D} &\simeq \\ &\left[\mathbb{1} - \kappa \sum_p (-\alpha_p \hat{A}_p^\dagger + \alpha_p^* \hat{A}_p) + \frac{\kappa^2}{2} \left(\sum_p (-\alpha_p^* \hat{A}_p + \alpha_p \hat{A}_p^\dagger) \right)^2 \right] H_f \left[\mathbb{1} - \kappa \sum_p (\alpha_p \hat{A}_p^\dagger - \alpha_p^* \hat{A}_p) + \frac{\kappa^2}{2} \left(\sum_p (\alpha_p^* \hat{A}_p - \alpha_p \hat{A}_p^\dagger) \right)^2 \right]. \end{aligned} \quad (\text{S18})$$

From which we obtain, order by order in κ

$$\begin{aligned} H_f + V &= H_f - \kappa \left[\sum_p (-\alpha_p \hat{A}_p^\dagger + \alpha_p^* \hat{A}_p) H_f + H_f \sum_p (\alpha_p \hat{A}_p^\dagger - \alpha_p^* \hat{A}_p) \right] + \\ &\kappa^2 \left[\sum_p (\alpha_p \hat{A}_p^\dagger - \alpha_p^* \hat{A}_p) H_f \sum_p (-\alpha_p \hat{A}_p^\dagger + \alpha_p^* \hat{A}_p) \right] + \\ &\frac{\kappa^2}{2} \left[\left(\sum_p (\alpha_p \hat{A}_p^\dagger - \alpha_p^* \hat{A}_p) \right)^2 H_f + H_f \left(\sum_p (-\alpha_p \hat{A}_p^\dagger + \alpha_p^* \hat{A}_p) \right)^2 \right] + \dots \end{aligned} \quad (\text{S19})$$

At this point we can diagonalize H_f in Eq. (S19) obtaining

$$- \Omega \cos \left[\frac{\pi}{N} \left(\sum_p p A_p^\dagger A_p + q \right) \right] \mu^z + S^\dagger V S. \quad (\text{S20})$$

The interaction term $S^\dagger V S$ is quite complicated, however as long as we are interested in the first order correction on the ground state, we have that

$$H_{\text{int, GS}}^{\text{II}}(q) = \frac{\Omega \kappa^2}{\omega^2 N} \sum_p |\alpha_p|^2 A_p \cos(p A_p^\dagger A_p + q) A_p^\dagger. \quad (\text{S21})$$

This term contributes to the energy correction reported in the main text.

The complete Hamiltonian is therefore

$$H_q = \omega \sum_p \tilde{A}_p^\dagger \tilde{A}_p \mathbb{1} - \Omega \cos \left[\frac{\pi}{N} \left(\sum_p p A_p^\dagger A_p + q \right) \right] \mu^z + H_{\text{int}}^{\text{I}} + H_{\text{int}}^{\text{II}}, \quad (\text{S22})$$

with

$$H_{\text{int}}^{\text{I}} = -\frac{\kappa^2}{\omega}(\mathbb{1} - \mu^x) \quad (\text{S23})$$

and,

$$H_{\text{int}}^{\text{II}} = S^\dagger V S \quad (\text{S24})$$

For the leading order correction to the energy of the ground state band $E_{\text{GS}}(q)$ the only terms which gives a non-zero contribution are $H_{\text{int}}^{\text{I}}$ and the $H_{\text{int, GS}}^{\text{II}}$ (see discussion in main text).

EXPANSION OF THE POTENTIAL

In this section we justify the approximation reported in Eq. (2) in the main text. Let us consider a generic potential of a one-dimensional lattice embedded in two dimensions. This means that we can have fluctuations around the equilibrium position in two directions that we will call z for the longitudinal one and y for the transverse one. Without loss of generality we can suppose that the interaction depends only on the relative distance between two atoms, i.e.

$$V(\mathbf{r}_i, \mathbf{r}_j) = V(|\mathbf{r}_i - \mathbf{r}_j|) = V(|r_{i,z} - r_{j,z}|, |r_{i,y} - r_{j,y}|) = V(r_z, r_y) \quad (\text{S25})$$

where $r_z = r_{i,z} - r_{j,z}$ and $r_y = r_{i,y} - r_{j,y}$. For one-dimensional lattices, considering only nearest-neighbours interaction, the equilibrium positions of the atoms are $r_z = a$, with a the lattice spacing and $r_y = 0$. Performing the expansion we obtain

$$V(r_z, r_y) = V(a, 0) + \left. \frac{\partial V(r_z, r_y)}{\partial r_z} \right|_{r_z=a} \delta r_z + \left. \frac{\partial V(r_z, r_y)}{\partial r_y} \right|_{r_y=0} \delta r_y + \dots \quad (\text{S26})$$

As we reported in the main text, we can rewrite the displacement $\delta r_{z,y}$ in terms of the bosonic operators, the coupling is proportional to oscillator length, i.e.

$$\delta r_\mu = l_\mu (a_i^\dagger + a_i - a_j^\dagger - a_j) \quad \mu = z, y. \quad (\text{S27})$$

Here $l_\mu = \sqrt{\hbar/(m\omega_\mu)}$ is the harmonic oscillator length. It is possible to observe how tuning the trapping frequency in a different way in the two directions leads to a different coupling with the transverse and longitudinal modes. For the general case of a power-law decaying potential we have that:

$$V(r_z, r_y) \propto \frac{1}{(r_z^2 + r_y^2)^{\frac{\alpha}{2}}}, \quad (\text{S28})$$

therefore,

$$\left. \frac{\partial V(r_z, r_y)}{\partial r_y} \right|_{r_y=0} = \frac{\alpha r_y}{(r_z^2 + r_y^2)^{\frac{\alpha}{2}+1}} \Big|_{r_y=0} = 0. \quad (\text{S29})$$

This shows that, at first order, the contribution of the transverse modes to the longitudinal interaction is zero.

EXPERIMENTAL CONSIDERATIONS

In this section we give some remarks concerning the parameters of a possible experimental realisation of the system. We focus here on ^{87}Rb and ^{133}Cs . However, the order of magnitude of the parameters is comparable to that of other experiment conducted e.g. with ^{39}K and ^7Li . We will also explain more in detail how the observables discussed in the paper can be detected in an experiment.

Let us start by giving some typical values for the trap parameters that are usually set in optical tweezers experiments. The lattice constant a , i.e. the distance between the Rydberg atoms, is $a \approx 5\mu\text{m}$. The life-time of the Rydberg state with high principal quantum number n , $n \simeq 40 - 50$, is approximately $\tau_R \simeq 2 \cdot 10^{-5}\text{s}$. The trapping frequency ω is typically $\omega \simeq 2\pi \cdot 300\text{kHz}$, the Rabi frequency Ω can be

tuned until a maximum value of $\Omega_{max} \simeq 2\pi \cdot 10\text{MHz}$. The Van der Waals constant between ns -states scales with the Rydberg principal number n as $C_6([ns]) = n^{11}(c_0 + c_1n + c_2n^2)\text{au}$ [S1]. For ^{87}Rb we have $c_0 = 11.97$, $c_1 = -0.8486$ and $c_3 = 3.385 \cdot 10^{-3}$, for ^{133}Cs , instead, $c_0 = 10.64$, $c_1 = -0.6249$ and $c_3 = 2.33 \cdot 10^{-3}$. This leads, for $n = 43$, to an interaction strength between nearest neighbours of $V_{int,\text{Rb}} \approx 1\text{MHz}$ and $V_{int,\text{Cs}} \approx 0.65\text{MHz}$. In the case studied in this paper, what matters is not the interaction in itself (since we are in the facilitation regime) but its gradient, i.e. $G = -6\frac{V_{int}}{a}$. Considering the same parameters as before we obtain that $G_{\text{Rb}} = -1.2 \cdot 10^3\text{kHz}\mu\text{m}^{-1}$ and $G_{\text{Cs}} = -7.8 \cdot 10^2\text{kHz}\mu\text{m}^{-1}$. The interaction constant κ is related to the gradient via the harmonic oscillator length, i.e. $\kappa = -\frac{\hbar\omega}{\sqrt{2}}G$. With the previous data, we have: $\kappa_{\text{Rb}} \approx 16\text{kHz}$ and $\kappa_{\text{Cs}} \approx 9\text{kHz}$. Experimentally, these coupling constants can be controlled using microwave-dressing of Rydberg s - and p -states, as discussed in Ref. [S2]. This procedure enable us to tune independently the gradient from the interaction.

The many-body dynamics can be characterised by measuring the spin (Rydberg) density and the phonon density, as shown in Fig. 3 in the main text. The spin density can be detected by counting the atoms in the Rydberg state, this can be achieved using projective measurements (see for example [S3]). However, in these experiments we can also detect the phonon density, which is particularly interesting because it enables us to measure directly the effect of the dressing of the excitations. This can be done using side-band spectroscopy (as shown in Ref. [S4]). As stated in the main text, the combination of the detection methods and the exaggerated length scales offer unique opportunities for investigating polaron physics.

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