# On the impossibility of protecting risk-takers*

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**Abstract**

Risk-neutral sellers can extract high profits from risk-loving buyers using lotteries. To limit risk-taking, gambling is heavily regulated in most countries. In this paper, I show that protecting risk-loving buyers is essentially impossible. Even if sellers are restricted from using mechanisms that resemble lotteries, sellers can still construct selling mechanisms that ensure unbounded profits as long as buyers are risk-loving, at least asymptotically.

Asymptotically risk-loving preferences are both sufficient and necessary for unbounded profits. Buyers are asymptotically risk-loving, for example, when they are globally risk-loving, when they have cumulative prospect theory preferences, or when their utility is bounded from below. Therefore, this paper characterizes an informative lower bound for the profits in several classes of preferences that have so far not been studied in the mechanism design literature.

*JEL*: D82, D44, C72, D81

*Keywords*: auctions, gambling, risk preferences, prospect theory

## 1 Introduction

Gambling is either illegal or heavily regulated in most countries. The primary economic reason[1] for gambling restrictions is to protect agents from harming themselves by making poor decisions. Namely, risk-neutral sellers can generate large profits through lotteries that exploit risk-loving agents.[2]

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[1] There are other reasons for regulating gambling, including protecting gambling addicts, collecting tax revenues, and preventing criminal activities and cheating.

[2] Azevedo and Gottlieb (2012) showed that a risk-neutral monopolist can generate infinite profits when selling lotteries to agents with prospect theory preferences.

In practice, the line between risky-but-legal sales mechanisms and gambling is unclear. Selling goods at posted prices is not considered harmful even when only a random fraction of buyers get to purchase the goods.[3] Similarly, most auctions are not considered forms of gambling, although auction outcomes are random from the individual bidder's perspective. There are many selling mechanisms whose connection to gambling is even less clear. For example, some sellers use auction formats that produce highly random outcomes and are highly profitable (for example, penny auctions and lowest-unique-bid auctions).[4]

This paper takes the perspective of a regulator who wants to know which types of selling mechanisms should be regulated by gambling laws to avoid the possibility of sellers extracting very large profits from risk-loving buyers. For example, US regulators define gambling as "any game, contest or promotion that combines the elements of prize, chance and consideration."[5] Currently, gambling laws only apply to selling mechanisms that have *all* three of the following elements: chance (which corresponds to random allocation or payments), prize (allocating something in addition to the object for sale), and consideration (charging buyers who do not receive the object for sale).

The main policy conclusion of this paper comes in the form of an impossibility result. The result shows that for a large class of preferences, selling mechanisms with *none* of the elements described above can still generate unboundedly large profits. This result requires only that buyers are asymptotically risk-loving, which means that their marginal utility of money converges to zero with infinitely large transfers to the seller. This is a mild condition that is satisfied, for example, when agents have prospect theory preferences[6] or their utility is bounded. [7] Moreover, this result shows that asymptotic risk-loving preferences are both necessary and sufficient for unbounded revenues. The class of mechanisms that allows for extraction of infinite profits is the class of non-random winner-pays auctions, where the highest bidder gets the object and pays a transfer that is a deterministic function of bids, and the other bidders pay nothing. Of course, unbounded profits also require unbounded transfers from buyers. However, the result also has a strong implication in the case of finite budgets: the profits are mostly limited by buyers' budget

[3] For example, when Led Zeppelin performed a concert in 2007, about 20 million people requested tickets, priced at £125, but there were only 16,000 tickets available. According to an online ticket reseller, the market value of a ticket was £7,425. `http://www.blabbermouth.net/news/average-led-zeppelin-ticket-price-for-reunion-show/`

[4] For example, penny auctions are reminiscent of English auctions but include a large bid fee, and are therefore, highly random in equilibrium (Hinnosaar, 2016). The Gambling Commission in the UK said that it "was not convinced that penny auctions amounted to gambling." `http://news.bbc.co.uk/2/hi/business/7793054.stm` Similarly, the European Commission spokesperson commented that penny auctions are governed by rules on auctions (rather than on gambling). `http://www.technologizer.com/2008/10/17/swoopo-seems-safe-from-legal-action-in-eu/`

[5] This is the definition given by the Federal Communications Commission for online gambling, but a similar definition applies to offline gambling. `https://www.fcc.gov/consumers/guides/broadcasting-contests-lotteries-and-solicitation-funds`

[6] When agents have a nonlinear probability weighting function, the condition on the value function needs to be adjusted with the weighting function. The combined sufficient condition is satisfied essentially by all functional forms used in the literature.

[7] In this paper I take a reduced form approach and assume that buyers in the static mechanism are guided by a utility function; I do not consider where such preferences come from. For example, they could come from a social safety net that ensures a minimal standard of living even if the agent loses everything.

constraints rather than their valuation of the object sold.

To see how the seller can obtain unbounded profits with auctions of this type, consider the following modification of the second-price auction, which I will call *$T$-mechanism*: Fix a large transfer $T > 0$. Let the highest bid be $b$. The highest bidder is the winner and pays either $T$ if the second highest bid is above some threshold $\gamma(b)$ or 0 if the second highest bid is below $\gamma(b)$, where $\gamma(b)$ is chosen so that bidding one's true value is optimal for a buyer of type $b$. For risk-neutral buyers, the mechanism would be equivalent (in terms of expected payments) to the second-price auction for any $T$. Increases in $T$ and the corresponding increases of $\gamma(b)$ would not affect expected transfers or profits.

However, a large $T$ makes the auction risky from the bidders' perspective because they essentially face a gamble where they may get the object for free with high probability or pay a large transfer with low probability. When buyers are asymptotically risk-loving, the expected payment increases fast enough to make profits arbitrarily large. For example, if the utility function is bounded below, then as $T$ gets large enough, its increase changes utility very little, and therefore the seller can raise the transfer almost without reducing the probability of agents paying it.

Although I am (perhaps thankfully) not aware of sellers offering auctions reminiscent of these $T$-mechanisms in practice, their scarcity should not be taken as a sign that they would not be profitable. The mechanism highlighted here works as a worst-case bound when buyers' preferences are risk-loving *only* asymptotically and the seller is restricted to using *only* winner-pays non-random auctions. If in practice some buyers were more risk-loving or sellers could offer a wider class of mechanisms (for example, lotteries or all-pay auctions), it would be even easier to extract large profits.

A relevant question is whether anyone would accept the type of risk highlighted here. From the buyers' perspective, the auction creates a lottery that almost surely gives them the object for free but requires a very large payment with small probability. For obvious reasons, this type of choice cannot be tested experimentally. In practice, however, there are many situations where people are faced with similar trade-offs and are sometimes willing to accept such gambles. For example, millions of people in the US have no health insurance. With high probability, nothing happens to them, and they save money on insurance premiums; but if something unexpected happens, this choice could lead to personal bankruptcy or worse. Similar examples can be found with respect to risky investments, real estate ownership, and career choices.[8]

This paper contributes to two branches of literature. First, mechanism design literature has studied profit-maximizing mechanisms under various preferences. Optimal auctions with risk-neutral agents were characterized by Myerson (1981) and Riley and Samuelson (1981). Matthews (1983) and Maskin and Riley (1984) characterized optimal mechanisms with risk-averse buyers. In recent years, a new stream of literature has analyzed optimal mechanisms with non-standard preferences. For example, Di Tillio, Kos, and Messner (2014) and Bose and Renou (2014)

[8]It must be noted that these examples do not fit my model, as there is no auctioneer designing the format and the source of uncertainty is exogenous. However, from the individual's perspective, the trade-offs are similar.

characterized the optimal mechanisms when buyers are ambiguity averse, and Carbajal and Ely (2015) examine situations in which buyers are loss averse (with a piecewise linear function as in Kőszegi and Rabin (2006)). All these papers assume that the utility function is piecewise linear or concave, so marginal utility of transfers is bounded away from zero. The optimal mechanisms in all these papers ensure finite profits.

I show that results change drastically when agents are asymptotically risk-loving; profits from the optimal mechanism are unboundedly large, and the structure of optimal mechanisms is different with highly variable payment rules. Modeling agents as asymptotically risk-loving covers several important classes of preferences that are so far uncovered by mechanism design literature: risk-loving preferences, prospect theory preferences, and bounded utility. Moreover, the results also show that the optimal mechanisms are not robust to small changes in risk preferences. When risk preferences change from slightly risk-averse to slightly risk-loving, then both the profits increase discontinuously and the optimal mechanism itself changes completely.

Second, the paper also contributes to the discussion on the implications of prospect theory. Prospect theory, formulated by Kahneman and Tversky (1979; 1992), is widely supported by experimental evidence. However, in market settings, it has some undesirable features. Azevedo and Gottlieb (2012) showed that when sellers are able to offer arbitrary lotteries, profits are unbounded and Pareto-efficiency is not well defined.[9] Ebert and Strack (2015) proved that, due to skewed preferences over gains and losses and the probability weighting function, agents with prospect theory preferences in a dynamic context can potentially continue taking gambles until all of their resources are exhausted.

Instead of focusing on particular preferences, I characterize a necessary and sufficient condition for unbounded profits: asymptotically risk-loving preferences. As it turns out, this condition is satisfied by cumulative prospect theory preferences, as well as other types of preferences. The paper extends the results in two directions. First, for unbounded profits, it is sufficient that the seller can at least offer non-random winner-pays auctions. Second, because the randomness of the mechanism comes from the equilibrium bidding behavior rather than a lottery chosen by the seller, the implementation of the mechanism does not rely on details of the model.

# 2 Model

A monopolistic risk-neutral seller sells a single object at zero cost. There are $n \geq 2$ buyers with independent private types. Buyer $i$'s type, $\theta_i \in [\underline{\theta}, \overline{\theta}]$, is an indicator of his valuation of the object and is distributed according to the cumulative distribution function $F$ with full support. To shorten the notation, I denote the distribution of the highest value among all other buyers by $G(\theta_i) = F(\theta_i)^{n-1}$ and $g(\theta_i) = G'(\theta_i) = (n-1)f(\theta_i)F(\theta_i)^{n-2}$.

Buyers are expected utility maximizers (in Section 5 I extend the analysis to arbitrary probability weighting functions to accommodate cumulative prospect theory preferences). A

[9] Rieger and Wang (2006) pointed out that with (standard) prospect theory preferences, lotteries with finite expected value may lead to infinite utility. They proposed alternative specifications for value and probability weighting functions, but these specifications do not solve the problem highlighted here.

buyer who has type $\theta$ and receives a transfer $t$ gets Bernoulli utility $u(\theta, t)$ when he or she receives the object and $u_0(t_i)$ otherwise. Functions $u$ and $u_0$ are common for all buyers and continuously differentiable, such that $u_\theta(\theta, t) = \frac{\partial u(\theta,t)}{\partial \theta} > 0$, $u_t(\theta, t) = \frac{\partial u(\theta,t)}{\partial t} > 0$, and $u_0'(t) = \frac{\partial u_0(t)}{\partial t} > 0$. I normalize $u_0(0) = 0$ and assume that $u(\theta, 0) > 0$ and $\lim_{t \to -\infty} u(\theta, t) < 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

I now introduce two assumptions that are crucial for the main result: buyers are *asymptotically risk-loving* and the seller can use *non-random winner-pays auctions* or at least *non-random auctions*. Remember that agents are risk-loving when their utility function is convex. For the results in this paper, buyers do not need to be risk-loving everywhere, it suffices that they are risk-loving asymptotically. I call agents asymptotically risk-loving when their utility becomes convex asymptotically (i.e., with large payments). That is, if an agent receives a positive transfer or pays a small amount, the utility function can have an arbitrary positive slope (which could be concave or linear, for example), but in the limit, when the payment becomes large, the utility becomes convex. More formally, the utility function must be above any linear function with positive slope in the case of sufficiently large transfers.

Because agents' utility functions may depend on their types and whether or not they receive the object, I define asymptotic risk-loving utility for a generic function $v : \mathbb{R} \to \mathbb{R}$ of money such that $v'$ and $v''$ denote the first and second derivative, respectively.

**Definition 1.** *Utility $v$ is asymptotically risk-loving if* $\lim_{t \to -\infty} \frac{1}{t} v(t) = 0$.

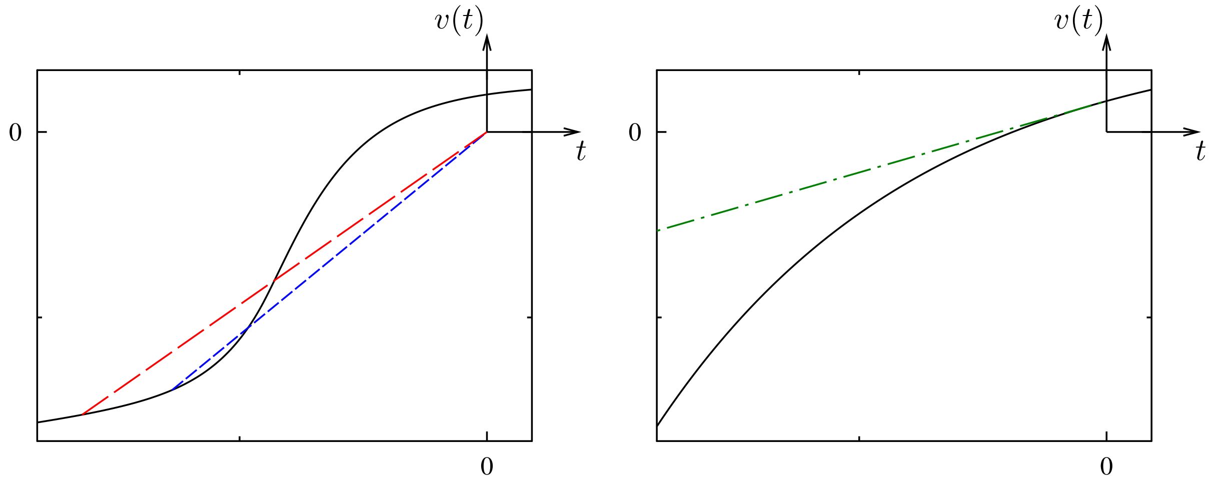
As asymptotically risk-loving preferences will be both sufficient and necessary for unbounded transfers, it is useful to also consider the opposite of asymptotic preferences. By Lemma 1 preferences are not asymptotically risk-loving if the utility as a function of negative transfers is bounded above by a linear function with a positive slope. Figure 1 gives a graphical comparison of utility functions that satisfy and do not satisfy the definition.

**Lemma 1.** *Utility $v$ is <u>not</u> asymptotically risk-loving if and only if there exists $\varepsilon > 0$ such that for all $t < 0$, $v(t) \leq v(0) + \varepsilon t$.*

The assumption is satisfied in a wide range of situations. Lemma 2 gives several sufficient conditions for Definition 1 that are perhaps easier to verify and relate to standard definitions. The assumption is satisfied when marginal utility from large negative transfers converges to zero; that is, the agent is almost indifferent between paying the large sum $T$ or paying $T + 1$.[10] For example, when utility is bounded from below, the agent is asymptotically risk-loving because the utility converges to a constant. Bounded utility may either be a property of preferences[11] or arise in situations where agents can declare bankruptcy after incurring losses that are too large. Agents are also asymptotically risk-loving whenever they become risk-lovers in the limit according to the Arrow-Pratt relative or absolute risk measures. For example, any agent who is risk loving globally, or at least in losses, is also asymptotically risk-loving.

[10] The assumption can also be interpreted as diminishing sensitivity toward losses (Wakker, 1994).

[11] For example, Savage's axioms imply bounded utility (Fishburn, 1970).




(a) Asymptotiocally risk-loving

(b) Not asymptotically risk-loving

Figure 1: An agent is asymptotically risk-loving when his utility from money is becoming convex in the left tail. Equivalently, the utility from negative transfers is not bounded above by a linear curve.

**Lemma 2.** *Utility $v$ is asymptotically risk-loving*

1. *if* $\lim_{t \to -\infty} v'(t) = 0$ , *or*

2. *if the utility function is bounded from below, or*

3. *if the second derivative with respect to transfer $v''(t)$ exists and either*

$$\lim_{t \to -\infty} \frac{t v''(\theta, t)}{v'(\theta, t)} > 0 \text{ or } \lim_{t \to -\infty} \frac{v''(\theta, t)}{v'(\theta, t)} < 0 \tag{1}$$

*(i.e., asymptotically risk-loving according to the Arrow-Pratt relative or absolute risk measure).*

The proofs of Lemmas 1 and 2 can be found in the online appendix.[12]

The most general class of mechanisms I consider in this paper are *selling mechanisms*, which require that participation is voluntary and net payments from the buyers to the seller are always non-negative. These conditions limit the seller's ability to create artificial lotteries by offering monetary prizes in addition to the value of the object.

**Definition 2.** *A selling mechanism is an ex ante individual rational mechanism where the realized transfers from the buyers to the seller are non-negative.*

Selling mechanisms still allow the seller to randomize both the allocation and the transfers, and thus create lotteries. They also allow positive transfers from buyers who do not receive the object, and therefore, all-pay auctions, for example, are allowed. The unbounded revenue result holds even if the seller's ability to offer mechanisms with these properties is limited.

[12] Available at `http://toomas.hinnosaar.net/impossibility_supplementary.pdf`

**Definition 3.** *A non-random auction satisfies the following rules:*

1. *Bidders simultaneously submit bids.*
2. *The highest bid gets the object if the bid is higher than some minimal level.*
3. *Bidders pay transfers to the seller, where the transfers are deterministic functions of the bids.*

**Definition 4.** *A non-random winner-pays auction is a non-random auction where only the winner pays a positive transfer.*

For example, first-price auctions and second-price auctions with or without reserve prices are non-random winner-pays auctions, whereas the rules exclude lotteries (where transfers are random), and Definition 4 excludes all-pay auctions (where losers pay positive amounts). Essentially, the restrictions exclude all mechanisms that resemble lotteries and other risky selling mechanisms used in practice, but they still allow for efficient and profit-maximizing mechanisms when agents are risk-neutral[13]. This class of mechanisms is as restrictive as possible without putting restrictions on the particular transfer functions that the seller can use. To specify transfer functions, regulators would need information they do not have, or they would need to exclude some mechanisms that are necessary for ensuring efficiency with risk-neutral or risk-averse buyers.

# 3 Main result

**Theorem 1.** *Depending on asymptotic properties of $u$ and $u_0$, there are three cases*

1. *If $u$ is asymptotically risk-loving, then for any $M > 0$ there exists a non-random winner-pays auction where all types $\theta_i \in (\underline{\theta}, \overline{\theta}]$ of buyers pay more than $M$ in expected transfers.*
2. *If $u_0$ is asymptotically risk-loving, then for any $M > 0$ there exists a non-random auction where all types $\theta_i \in (\underline{\theta}, \overline{\theta}]$ of buyers pay more than $M$ in expected transfers.*
3. *If neither $u$ or $u_0$ is asymptotically risk-loving, then there exists $\varepsilon > 0$ such that the expected transfer from type $\theta_i$ from any selling mechanism in any equilibrium is less than $\frac{1}{\varepsilon}u(\theta_i, 0)$.*

**Corollary 1.** *The profits from optimal mechanisms are infinite if and only if buyers are asymptotically risk-loving.*

As the profits are the sums of expected transfers, the boundedness and unboundedness of expected transfers clearly imply the same for the profits. The proofs of the first two parts are constructive. For the first part, I construct a particular non-random winner-pays auction that

[13] In fact, it is already too restrictive for profit maximization with risk-averse buyers, as shown by Maskin and Riley (1984) and Matthews (1983).

I call $T$*-mechanism.* In this mechanism, the winner's payment is either 0 or $T$, depending on whether the second-highest bid is above or below a certain threshold. To guarantee the incentives for truthful bidding, I choose different thresholds for each type $\theta$, denoted by $\gamma(\theta)$.

**Definition 5.** *$T$-mechanism is defined as follows. Fix a large transfer $T > 0$ and minimal bid $b^*$. Suppose the bids are $\mathbf{b} = (b_1 \ldots, b_n) \in [\underline{b}, \overline{b}]^n$. If buyer $i$ has the highest bid (so that $b_i > \max_{j \neq i} b_j = b_{-i}$), then $i$ gets the object if $b_i \geq b^*$. Other bidders do not get the object or pay anything, and buyer $i$ gets transfer*

$$\bar{t}(\mathbf{b}) = \begin{cases} 0 & \text{if } b_{-i} < \gamma(b_i), \\ -T & \text{if } b_{-i} \geq \gamma(b_i), \end{cases} \tag{2}$$

*where $\gamma : [\underline{b}, \overline{b}] \to [\underline{b}, \overline{b}]$ is a strictly increasing function that is chosen so that bidding one's own type is an equilibrium.*

The proof then consists of two lemmas. The first lemma shows that there exists a threshold function, $\gamma$, such that bidding one's own value is a Bayes-Nash equilibrium in each $T$-mechanism. The choice of $\gamma$ balances the trade-off between two forces. By bidding higher, the buyer could increase the chances of winning but also increase the probability of paying the large transfer. The second lemma shows that with this particular choice of $\gamma$, as $T$ increases, the expected transfers from $T$-mechanism can be made larger than any fixed bound. The expected transfer from this mechanism is $T$ times the probability of receiving this payment. Under the assumption of asymptotic risk-loving buyers, the realized transfer can be raised more quickly than the probability of paying the transfer has to be decreased.

The proof for the second part is analogous to the first part with one exception: if buyers are asymptotically risk-loving only when they do not get the object, the construction requires that one or more of the lower bidders pay large amounts. A simple modification of the $T$-mechanism gives the desired result.

Finally, the third part relies on the fact that when buyers are not asymptotically risk-loving, their utility functions are bounded above by some linear function (with slope at least $\varepsilon > 0$). The bound simply says that agents will never pay an expected value that is greater than their utility of getting the object for free, $u(\theta_i, 0)$, divided by $\varepsilon$, the lower bound for the marginal utility of money.

The proof can be found in Appendix A.

### 3.1 Examples

**Example 1.** Suppose that buyers are risk-neutral, i.e., the knife-edge case, where buyers are not yet asymptotically risk-loving. As $T$-mechanism is incentive compatible, individually rational, and assigns the object to the highest type (possibly subject to a reserve price), it is equivalent to a second-price auction and gives finite profit. It is an implementation of the optimal auction.

**Example 2.** Suppose that there are two buyers, with types distributed independently and uniformly in $[0,1]$, buyers are expected utility maximizers, and the utility functions are of constant absolute risk-aversion, i.e., $u(\theta_i, t_i) = {}^1\!/\!_r[1 - e^{-r[\theta_i + t_i]}]$ for a buyer who receives the object and $u_0(t_i) = {}^1\!/\!_r[1 - e^{-rt_i}]$ for a buyer who does not. In this example an optimal mechanism with risk-neutral buyers ($r = 0$) would be the second price auction with reserve price $\theta^* = {}^1\!/\!_2$, and the maximized profit with risk-neutral buyers is ${}^5\!/\!_{12}$. As the profits from the second price auction are independent of risk preferences, this is a lower bound for profits for any $r$. Profits with risk-averse buyers are increasing in $r$, but always less than the full surplus of $\mathbb{E}[\max\{\theta_i\}] = {}^2\!/\!_3$.

Now, consider a $T$-mechanism described above, with a fixed $T \geq 1$. The expected transfer from types $\theta_i \geq \theta^*$ from $T$-mechanism with reserve price $\theta^*$ is

$$t(\theta_i) = \frac{\frac{1}{r}[e^{r\theta_i} - r\theta_i] + \frac{1}{r}e^{r\theta^*}[r\theta^* - 1]}{\frac{1}{T}[e^{rT} - 1]} \tag{3}$$

Notice that for any $r < 0$ (risk-loving buyers), the expression is strictly increasing in $T$ and in the limit as $T \to \infty$, it converges to $\infty$. On the other hand, with $r > 0$ (risk-averse buyers), it is strictly decreasing in $T$ and converges to 0. Profit from $T$-mechanism is

$$\pi = 2\int_{\theta^*}^{1} t(\theta_i) d\theta_i. \tag{4}$$

In particular, since $T$-mechanism provides a lower bound for profits, it follows that as long as $r < 0$ and an arbitrarily large $T$ can be chosen, the profit from optimal mechanisms must be unbounded.

However, the example illustrates that even with limited budgets the profits from selling to risk-loving buyers can be very large. Namely, $T$ can be interpreted as the buyers' maximal budget. As Figure 2 shows, profits from selling to risk-loving buyers are always higher than profits from the second price auction. Moreover, even with relatively small budgets and small degrees of risk-lovingness, the profits rise above the full surplus and even above the maximal valuation of the object.

## 4 Detail-free implementation

In the proof of Theorem 1, I construct a $T$-mechanism to show that if buyers are asymptotically risk-loving, a direct mechanism with unboundedly large transfers exists. A potential limitation of this approach is that constructing this mechanism requires fixing a payment rule that depends on the details of the distribution of types, the utility functions, and the number of buyers. The following result shows that if we relax the direct implementation requirement, the seller can, in fact, implement the same outcomes without knowing any of these details.

I define *detail-free $T$-mechanism* in the same way as $T$-mechanism, but instead of a particularly chosen function $\gamma(b_i)$, I use a linear function $\overline{\gamma} b_i$, where $\overline{\gamma}$ can be an arbitrary fraction in $(0, 1)$.

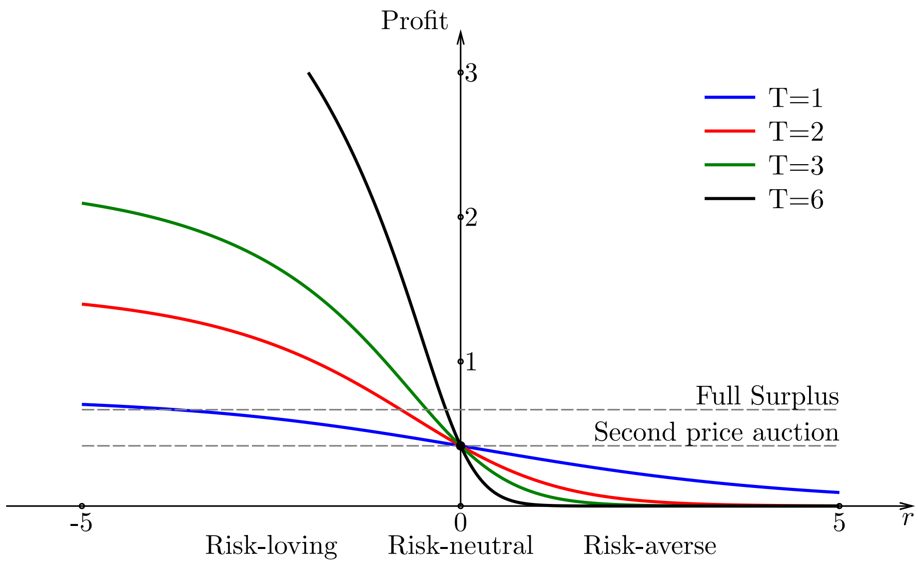



Figure 2: Profits from $T$-mechanisms from two buyers with CARA utility, uniformly distributed private valuations, and limited budgets.

**Definition 6.** *Fix $\overline{\gamma} \in (0,1)$, a reserve bid $b^*$, and a large transfer $T$. A detail-free $T$-mechanism is a non-random winner-pays auction, where the winner's payment rule is*

$$\bar{t}(\mathbf{b}) = \begin{cases} 0 & \text{if } b_{-i} < \overline{\gamma} b_i, \\ -T & \text{if } b_{-i} \geq \overline{\gamma} b_i. \end{cases} \tag{5}$$

**Theorem 2.** *For any $\overline{\gamma} \in (0,1)$, $b^*$, and $T$, the detail-free $T$-mechanism has an equilibrium that is payoff-equivalent to the truthful equilibrium in the $T$-mechanism.*

The proof in Appendix A is constructive. It constructs a symmetric equilibrium bidding function $b : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ that induces the same probabilities of winning and transfers as the truthful equilibrium in a corresponding $T$-mechanism.

## 5 Cumulative prospect theory preferences

Cumulative prospect theory was introduced by Tversky and Kahneman (1992) and is widely supported by experimental evidence. It is a modification of the expected utility theory and postulates that the decision maker evaluates risky options with respect to some reference point according to some value function (which replaces the utility function) and perceives probabilities according to some weighting function which may be different for gains and losses.

In this section, I modify the model introduced above to accommodate cumulative prospect theory preferences. A natural reference point is the initial allocation, i.e., the situation where a buyer does not get the object and does not pay or receive anything. In this case, we can reinterpret $u$ and $u_0$ as the value functions. In addition to this, the buyers may have nonlinear probability weighting functions $w^+ : [0,1] \to [0,1]$ and $w^- : [0,1] \to [0,1]$ for gains and losses, respectively. I assume that functions $w^+$ and $w^-$ are strictly increasing and continuously differentiable.

**Definition 7.** *A prospect theory agent with value function $v$ and probability weighting function $w^-$ for losses is asymptotically risk-loving if for all $c > 0$, $\lim_{t \to -\infty} w^-\left(\frac{-c}{t}\right) v(t) = 0$.*

With linear weighting function (i.e., in the case of an expected utility maximizer) $w^-(p) = p$, Definition 7 is equivalent to Definition 1. Prospect theory literature assumes that the weighting function is concave in small probabilities. In this case, Definition 7 is stronger than Definition 1.

Lemma 3 gives sufficient conditions that may be easier to verify than the condition in Definition 7. Again, any agent with a bounded value function is asymptotically risk-loving. The second and the third part of the lemma show that if either $\lim_{p \to 0} \frac{w^-(cp)}{w^-(p)}$ or $\lim_{t \to -\infty} \frac{v(ct)}{v(t)}$ is finite for all $c > 0$, then it suffices to have $\lim_{p \to 0} w^-(p) v\left(-\frac{1}{p}\right) = 0$. This condition was used by Azevedo and Gottlieb (2012), who verified that this condition is satisfied by virtually all models proposed in prospect theory literature. Since essentially all models proposed in prospect theory literature[14] use either a bounded or power utility function (which has a finite $\lim_{t \to \infty} \frac{v(ct)}{v(t)}$), we can conclude that with the functional forms used in the literature, prospect theory agents are asymptotically risk-loving.

**Lemma 3.** *A prospect theory agent with $v$ and $w^-$ is asymptotically risk-loving*

1. *if $v(t) \geq \underline{u} > -\infty$ for all $t \in \mathbb{R}$, or*
2. *if $\lim_{p \to 0} w^-(p) v\left(-\frac{1}{p}\right) = 0$ and $\lim_{p \to 0} \frac{w^-(cp)}{w^-(p)} \in \mathbb{R}$ for all $c > 0$, or*
3. *if $\lim_{p \to 0} w^-(p) v\left(-\frac{1}{p}\right) = 0$ and $\lim_{t \to -\infty} \frac{v(ct)}{v(t)} \in \mathbb{R}$ for all $c > 0$.*

**Theorem 3.** *Suppose that buyers are asymptotically risk-loving cumulative prospect theory agents. Then there exists a non-random winner-pays auction where almost all types of buyers pay unboundedly large expected transfers.*

The proof in the online appendix is analogous to the proof of Theorem 1.

**Theorem 4.** *When selling to cumulative prospect theory agents, for any $\overline{\gamma} \in (0,1)$, $b^*$, and $T$, the detail-free $T$-mechanism has an equilibrium that is payoff-equivalent to the truthful equilibrium in $T$-mechanism.*

The proof of Theorem 2 shows that we can always construct an equilibrium in the detail-free $T$-mechanism that induces the same probabilities of winning and paying 0 or $T$, respectively. As the probabilities are unchanged, the weighted probabilities are unchanged, too; therefore, the proof of Theorem 2 applies here, as well.

[14] Examples include Tversky and Kahneman (1992), Prelec (1998), and Rieger and Wang (2006).

## 6 Discussion

Profits from optimal mechanisms are unbounded if and only if buyers are asymptotically risk-loving. Of course, to achieve unbounded profits, buyers need to have access to unlimited funds. The result also has strong implications for cases with limited budgets. Namely, the profits from the optimal mechanism with asymptotically risk-loving buyers are mostly determined by buyers' budgets rather than their valuations of the object. As Example 2 illustrated, even with relatively small budgets the profits from risky mechanisms rise quickly above the full surplus and even the maximal valuation of the object.

Prohibiting all known types of gambling and selling mechanisms that resemble lotteries is not sufficient to deter the seller from extracting unbounded profits from asymptotically risk-loving buyers. In addition, a regulator would have to limit specific functional forms that the payments can take. Doing so would either require situation-specific information, which would mean regulating all non-standard auctions with gambling laws or excluding some mechanisms that are necessary to achieve efficiency with risk-neutral and risk-averse agents. In particular, it is sufficient to set an upper bound for the payments, but setting an upper bound would either create inefficiencies (e.g., making it impossible to sell valuable objects) or require more information than the regulator has. After all, auctions are used precisely because the seller does not know the buyers' valuations.

## References


Azevedo, E. M., and D. Gottlieb (2012): "Risk-neutral firms can extract unbounded profits from consumers with prospect theory preferences," *Journal of Economic Theory*, 147(3), 1291–1299.

Bose, S., and L. Renou (2014): "Mechanism design with ambiguous communication devices," *Econometrica*, 82(5), 1853–1872.

Carbajal, J. C., and J. C. Ely (2015): "A Model of price discrimination under loss aversion and state-contingent reference points," *Theoretical Economics, forthcoming.*

Di Tillio, A., N. Kos, and M. Messner (2014): "The design of ambiguous mechanisms," *Available at SSRN 2106097.*

Ebert, S., and P. Strack (2015): "Until the bitter end: on prospect theory in a dynamic context," *American Economic Review, forthcoming.*

Fishburn, P. C. (1970): *Utility theory for decision making*, no. 18. Research Analysis Corporation.

Hinnosaar, T. (2016): "Penny auctions," *International Journal of Industrial Organization*, 48, 59–87.

Kahneman, D., and A. Tversky (1979): "Prospect theory: an analysis of decision under risk," *Econometrica*, 47(2), 263–291.

Kőszegi, B., and M. Rabin (2006): "A model of reference-dependent preferences," *The Quarterly Journal of Economics*, 121(4), 1133–1165.

Maskin, E., and J. Riley (1984): "Optimal auctions with risk averse buyers," *Econometrica*, 52(6), 1473–1518.

Matthews, S. A. (1983): "Selling to risk averse buyers with unobservable tastes," *Journal of Economic Theory*, 30(2), 370–400.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6(1), 58–73.

Prelec, D. (1998): "The probability weighting function," *Econometrica*, 66(3).

Rieger, M., and M. Wang (2006): "Cumulative prospect theory and the St. Petersburg paradox," *Economic Theory*, 28(3), 665–679.

Riley, J. G., and W. F. Samuelson (1981): "Optimal auctions," *The American Economic Review*, 71(3), 381–392.

Tversky, A., and D. Kahneman (1992): "Advances in prospect theory: Cumulative representation of uncertainty," *Journal of Risk and Uncertainty*, 5(4), 297–323.

Wakker, P. (1994): "Separating marginal utility and probabilistic risk aversion," *Theory and Decision*, 36(1), 1–44.

# Appendices

## A Proofs

### Proof of Theorem 1

1. Lemma 4 shows that for any $T$, there exists $\gamma : [\underline{\theta}, \overline{\theta}] \to [\underline{\theta}, \overline{\theta}]$ such that bidding one's own type is a Bayes-Nash equilibrium; that is, $b_i(\theta_i) = \theta_i$ is optimal for each buyer. Lemma 5 shows that if $u$ is asymptotically risk-loving, then for any $\theta^* > \underline{\theta}$ with sufficiently large $T$, the $T$-mechanism ensures arbitrarily large expected transfers from all types $\theta \geq \theta^*$.

2. To prove the result, it suffices to construct a modification of the $T$-mechanism with the following changes in the transfers: (1) the highest bidder pays 0, and (2) the second highest bidder pays 0 when the highest bid is above $\gamma(\theta_i)$ and $T$ if the highest bid is between $\theta_i$ and $\gamma(\theta_i)$. Of course, $\gamma(\theta_i) \geq \theta_i$, and $\gamma(0) = 0$. Analogous steps with Lemmas 4 and 5 show

that $\gamma$ can always be constructed and the mechanism ensures unboundedly large transfers for large $T$.

3. By the revelation principle, we can focus on direct mechanisms $(\mathbf{q}, \overline{\mathbf{t}}, \underline{\mathbf{t}})$, where $q_i(\boldsymbol{\theta})$ is the probability buyer $i$ receives the object when the types are $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ and when $\overline{t}_i(\boldsymbol{\theta}) \geq 0$ and $\underline{t}_i(\boldsymbol{\theta}) \geq 0$ are the transfers to the seller when the agent gets or does not get the object, respectively. (Both $\overline{t}_i$ and $\underline{t}_i$ can be random.)

   By Lemma 1, there exists an $\varepsilon > 0$ such that for all $\theta_i \in [\underline{\theta}, \overline{\theta}]$ and $t_i < 0$, $u(\theta_i, t_i) \leq u(\theta_i, 0) + \varepsilon t_i$ and $u_0(t_i) \leq u_0(0) + \varepsilon t_i = \varepsilon t_i$. The expected utility of the agent is

$$U(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[q_i(\boldsymbol{\theta}) u(\theta_i, \overline{t}_i(\boldsymbol{\theta})) + [1 - q_i(\boldsymbol{\theta})] u_0(\underline{t}_i(\boldsymbol{\theta}))\right] \leq \overline{q}(\theta_i) u(\theta_i, 0) + \varepsilon t(\theta_i), \tag{6}$$

   where $\overline{q}(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[q_i(\boldsymbol{\theta})] \in [0, 1]$ is the interim probability that type $\theta_i$ gets the object and $t(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[q_i(\boldsymbol{\theta})\overline{t}_i(\boldsymbol{\theta}) + [1 - q_i(\boldsymbol{\theta})]\underline{t}_i(\boldsymbol{\theta})]$ is the expected transfer from type $\theta_i$ to the seller.

   Individual rationality requires that $U(\theta_i) \geq u_0(0) = 0$; therefore, $-t(\theta_i) \leq \frac{1}{\varepsilon} u(\theta_i, 0)$.

□

**Lemma 4.** *For a big $T$, there exists a function $\gamma$ such that bidding one's own value is an equilibrium in the $T$-mechanism.*

**Proof of Lemma 4** Fix an arbitrary $T$-mechanism. The expected utility for a bidder with value $\theta_i$ who bids $\theta_i'$ is

$$U(\theta_i'|\theta_i) = [G(\theta_i') - G(\gamma(\theta_i'))] u(\theta_i, -T) + G(\gamma(\theta_i')) u(\theta_i, 0). \tag{7}$$

The condition for the optimality of bidding one's own type is

$$g(\gamma(\theta_i))\gamma'(\theta_i) = \frac{-u(\theta_i, -T)}{u(\theta_i, 0) - u(\theta_i, -T)} g(\theta_i), \quad \forall \theta_i \in [\underline{\theta}, \overline{\theta}]. \tag{8}$$

Integrating this condition from $\underline{\theta}$ to $\theta_i$ and observing that obviously $\gamma(\underline{\theta}) = 0$ gives

$$G(\gamma(\theta_i)) = \int_{\underline{\theta}}^{\theta_i} \frac{-u(\theta_i', -T)}{u(\theta_i', 0) - u(\theta_i', -T)} dG(\theta_i'), \quad \forall \theta_i \in [\underline{\theta}, \overline{\theta}]. \tag{9}$$

Because $G$ is strictly increasing, Equation (9) defines a unique $\gamma(\theta_i)$ for each $\theta_i \in [\underline{\theta}, \overline{\theta}]$. Moreover, $\gamma$ is a continuous and strictly increasing function.

When $T$ is sufficiently large such that $u(\theta_i', -T) < 0$, the necessary condition equation (8) is also the sufficient condition under which reporting one's own value is the unique maximizer of expected utility because if $\gamma$ satisfies Equation (8), then[15]

$$\frac{d^2U(\theta_i'|\theta_i)}{d\theta_i d\theta_i'} = g(\theta_i')\left[\frac{u(\theta_i',0)}{-u(\theta_i',-T)}u_\theta(\theta_i,-T) + u_\theta(\theta_i,0)\right] > 0, \quad \forall \theta_i, \theta_i'. \tag{10}$$

□

**Lemma 5.** *If $u$ is asymptotically risk-loving, then with $T$-mechanisms, the seller can ensure unboundedly large expected transfers from all types $\theta_i \geq \theta^*$.*

**Proof of Lemma 5** The expected transfer from type $\theta_i$, denoted by $t(\theta_i)$, is

$$t(\theta_i) = T[G(\theta_i) - G(\gamma(\theta_i))] = T\int_{\underline{\theta}}^{\theta_i} \frac{u(\theta_i',0)}{u(\theta_i',0) - u(\theta_i',-T)} dG(\theta_i'). \tag{11}$$

Because of the asymptotic risk-lovingness, in the limit, for all $\theta_i' \in [\underline{\theta}, \theta_i]$,

$$\lim_{T\to\infty} T\frac{u(\theta_i',0)}{u(\theta_i',0) - u(\theta_i',-T)} = \lim_{T\to\infty} \frac{u(\theta_i',0)}{\lim_{T\to\infty}\frac{1}{T}u(\theta_i',0) - \lim_{T\to\infty}\frac{1}{T}u(\theta_i',-T)} = \infty. \tag{12}$$

Therefore, $\lim_{T\to\infty} t(\theta_i) = \infty$. □

**Proof of Theorem 2** Let $\gamma$ be the function that induces truthful equilibrium in the $T$-mechanism. It is sufficient to verify that there exists a bidding function $b : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ which, from the perspective of individual bidders, induces exactly the same probabilities of winning and paying 0 or $T$, respectively. As long as $b$ is strictly increasing, the highest type wins in both cases, so the only condition to check is

$$b(\gamma(\theta)) = \overline{\gamma} b(\theta), \quad \forall \theta \in [\underline{\theta}, \overline{\theta}]. \tag{13}$$

I construct such a function $b : [\underline{\theta}, \overline{\theta}] \to [0, \hat{T}]$ in the following steps

1. Fix $b(\overline{\theta}) = \hat{T}$. Then $b(\gamma(\overline{\theta})) = \overline{\gamma}\hat{T}$.
2. For any $\theta \in [\gamma(\overline{\theta}), \overline{\theta}]$, fix linear $b$; that is, $b(\theta) = \overline{\gamma}\hat{T} + (1-\overline{\gamma})\hat{T}\frac{\theta - \gamma(\overline{\theta})}{\overline{\theta} - \gamma(\overline{\theta})}$.
3. Define $b$ recursively as follows. Suppose that for $k \in \mathbb{N}$, the function $b$ is defined in the interval $[\gamma^k(\overline{\theta}), \gamma^{k-1}(\overline{\theta})]$. Then for all $\theta \in [\gamma^{k+1}(\overline{\theta}), \gamma^k(\overline{\theta})]$, the value $\gamma^{-1}(\theta) \in [\gamma^k(\overline{\theta}), \gamma^{k-1}(\overline{\theta})]$; thus, $b(\gamma^{-1}(\theta))$ is defined. Fix $b(\theta) = \overline{\gamma} b(\gamma^{-1}(\theta))$.
4. Finally, fixing $b(\underline{\theta}) = 0$ ensures that $b$ is strictly increasing and continuous for all $\theta$.

□

[15] Note that Equation (8) implies that $\frac{dU(\hat{\theta}_i|\hat{\theta}_i)}{d\theta_i'} = 0$; thus, Equation (10) guarantees that $\frac{dU(\hat{\theta}_i|\theta_i)}{d\theta_i'} = \int_{\hat{\theta}_i}^{\theta_i} \frac{d^2U(\hat{\theta}_i|t)}{d\theta_i' d\theta_i} dt$ is positive if and only if $\hat{\theta}_i < \theta_i$. Therefore, $\hat{\theta}_i = \theta_i$ is the unique maximizer of $U(\hat{\theta}_i|\theta_i)$.