An Interval Creation Approach to Construct Interval Type-2 Fuzzy Sets

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***Abstract*—Interval type-2 fuzzy sets (IT2 FSs) are more popular over type-1 fuzzy sets (T1 FSs) as they capture uncertainty in a better way in many real-world problems. Modelling uncertainty with an IT2 FS is relatively complex and solely depends on how to define its Footprint of Uncertainty (FOU)—an uncertainty region of IT2 FS bounded by its lower and upper membership functions. Existing methods either transform existing T1 FSs to IT2 FSs or design IT2 FSs from input data, captured by interval via constructing T1 FSs. In both cases, they require additional processing step and computational time and thus not useful for time-sensitive applications (e.g., intelligent transportation systems, online recommendation systems). To address this limitation, this paper puts forward a simple approach, termed as 'Interval Creation Approach' (ICA) to design IT2 FSs directly from the input data. Further, it is designed to work equally well for transforming existing T1 FS into IT2 FS. The new approach skips data pre-processing phase and the creation of T1 FS as an intermediary to define the FOU of the IT2 FS, thus contributing to faster execution. The paper provides a description of the ICA along with a comparison of its effectiveness against the state-of-the-art methods using both synthetic and real-world data sets.**

***Index Terms*—Interval type-2 fuzzy set, type-1 fuzzy set, interval creation approach, footprint of uncertainty**

I. Introduction

Type-2 fuzzy sets (T2 FSs) were developed by Zadeh [1] as an extension of type-1 fuzzy sets (T1 FSs) [2]. They include a third dimension as additional degrees of freedom to capture higher level of uncertainty [3]–[5]. Although T1 FSs are widely used form of FSs, their crisp membership functions (MFs) limit scope for modelling uncertainty and thus making them unsuitable for applications with multiple sources of complex and varying levels of uncertainty [1], [3], [4], [6]–[9]. Conversely, general T2 FSs (GT2 FSs) with fuzzy MFs provide better capturing of uncertainty and have potential to offer superior results for many real-world problems but have limited usage due to their complex structure and huge computational cost [3], [4], [10]. To overcome these limitations, interval type-2 FSs (IT2 FS) were developed [8], [11] with all secondary membership grades being equal to 1.0, which greatly reduces their computational cost. Particularly, an IT2 FS is defined as a collection of uncountable number of embedded T1 FSs [8] which constitutes the uncertainty region of its primary MF, known as the Footprint of Uncertainty (FOU), bounded by a lower MF (LMF) and an upper MF (UMF). The later definition allows to apply all T1 FS mathematics on IT2 FSs [8], [11], thus making them simple to handle than GT2 FSs. Figure. 1 presents three types of FSs where the shaded areas are the FOU of the IT2 and GT2 FSs.

Several studies have already demonstrated that the IT2 FSs capture uncertainty with higher accuracy and provide better performance than T1 FSs for a wide range of real-world problems [7], [8], [12]. Further, less complex and reduced computational cost of IT2 FSs lead to their enhanced application against GT2 FSs [4], [13], [14]. Many existing solutions leveraging T1 FSs are now adapted to their IT2 FSs equivalent for more reliable decision making (e.g. image thresholding, VANETs) [15]–[17]. This underscores the need to effectively transform T1 FSs to IT2 FSs. Further, easy handling of calculations with IT2 FSs motivate people to design IT2 FS-based solutions to real-world problems directly from the input data [8], [18].

Several methods have been proposed to construct IT2 FSs from existing T1 FSs where they mainly focus on how to define the FOU of IT2 FSs [9], [15]. On the contrary, some approaches are designed to work directly with the input data, captured as interval-valued, to generate IT2 FSs where they first create T1 FSs from the data and then convert them into IT2 counterparts [13], [19]. Among them, some methods consider specific T1 FSs (such as, trapezoidal and triangular FSs) and their conversion into IT2 FSs [13], [19], thus restricting their applications to wider community. In both cases, existing methodologies use additional pre-processing step [4], [14] and thus involves extra computational time and unsuitable for

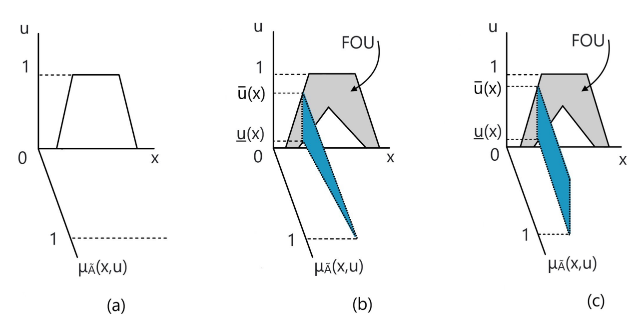


Fig. 1: (a) T1 FS, (b) GT2 FS, (c) IT2 FS (FOU=Footprint of Uncertainty).

use in time-sensitive real-world applications (e.g. intelligent transportation systems, online recommendation).

To address above-mentioned limitations, this paper puts forward a methodology, termed as 'Interval Creation Approach' (ICA) to uniformly handle the construction of IT2 FSs either from T1 FSs or interval-valued data. The new ICA method is designed to convert any T1 MFs and can efficiently handle the conversion of the left-shoulder and right-shoulder T1 FSs [13] into IT2 FSs. Further, it avoids data pre-processing phase, (i.e. no removal of outliers, often carrying valuable information about different opinions [14], [20]) as well as the creation of T1 FS as an intermediary to define the FOU of the IT2 FS from the input data, thus making the ICA method computationally faster. The paper provides a description of the ICA method. Further, it compares the performance of the ICA method using both synthetic and real-world data sets and demonstrates its effectiveness against the state-of-the-art methods in respect to real-world applications.

The paper is organized as follows: Section II describes some fuzzy concepts along with a brief review on the well-known methods of constructing IT2 FSs. Section III introduces the proposed 'Interval Creation Approach' (ICA) to construct IT2 FSs and Section IV demonstrates the performance of the proposed methodology against the state-of-the-art methods using synthetic and real-world data sets. Lastly, Section V concludes the paper and highlights future work.

II. Background and Related Work

This section first defines some fuzzy concepts used in the paper. Then, it briefly reviews well-known methods used to construct IT2 FSs from T1 FSs or interval-valued data. Here, we focus on the methods used to generate IT2 FSs as the latter provides the foundation for the extension to GT2 FSs.

A. Type-1 Fuzzy Set (T1 FS)

A T1 FS, A in the universe of discourse (X) is characterized by a type-1 membership function (T1 MF), $\mu_A(x)$ such that [2] (see Fig. 1(a)),

$$A = (x, \mu_A(x))|\ \forall x \in X, \mu_A(x) \in [0,1]. \quad (1)$$

B. General Type-2 Fuzzy Set (GT2 FS)

A GT2 FS, $\widetilde{A}$ in X is characterized by a type-2 membership function (T2 MF), $\mu_{\widetilde{A}}(x,u)$ such that [1], [11], [21]

$$\widetilde{A} = ((x,u), \mu_{\widetilde{A}}(x,u))|\ \forall x \in X, u \in J_x \subseteq [0,1], \quad (2)$$

where J_x is the primary membership of x and $\mu_{\widetilde{A}}(x,u) \in [0,1]$ is the secondary membership of x. $\widetilde{A}$ can alternatively be expressed as

$$\widetilde{A} = \int_{x\in X}\int_{u\in J_x} \mu_{\widetilde{A}}(x,u)/(x,u), J_x \subseteq [0,1], \quad (3)$$

where $\int$ denotes union. For the discrete X, $\widetilde{A}$ is again defined in (4).

$$\widetilde{A} = \sum_{x\in X}\sum_{u\in J_x} \mu_{\widetilde{A}}(x,u)/(x,u), J_x \subseteq [0,1]. \quad (4)$$

In $\widetilde{A}$, the two endpoints are associated with two T1 MFs referred to as lower membership function (LMF) and upper membership function (UMF) [21] (see Fig. 1(b)).

C. Interval Type-2 Fuzzy Set (IT2 FS)

An IT2 FS is a GT2 FS whose all secondary membership grades are equal to 1, defined in (5) (see Fig. 1(c)).

$$\widetilde{A} = ((x,u), \mu_{\widetilde{A}}(x,u))|\ \forall x \in X, u \in J_x \subseteq [0,1], \\ \mu_{\widetilde{A}}(x,u) = 1. \quad (5)$$

Following (3), an IT2 FS can be expressed [8] in (6).

$$\widetilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u), J_x \subseteq [0,1]. \quad (6)$$

D. Footprint of Uncertainty (FOU) of IT2 FS

An IT2 FS, $\widetilde{A}$ is completely described by its FOU [13] which graphically presents the uncertainty in the primary memberships of $\widetilde{A}$ (see Fig. 1(c)). It is the union of all primary membership grades of $\widetilde{A}$, defined in (7) [8].

$$FOU(\widetilde{A}) = \bigcup_{x\in X} J_x \quad (7)$$

Alternatively, the FOU of $\widetilde{A}$ can be described by its LMF and UMF [8], [22], [23], defined in (8),

$$FOU(\widetilde{A}) = \bigcup_{x\in X} [\underline{u}_{\widetilde{A}}(x), \overline{u}_{\widetilde{A}}(x)] \quad (8)$$

where $\underline{u}_{\widetilde{A}(x)}$ and $\overline{u}_{\widetilde{A}}(x)$ are the LMF and UMF of $\widetilde{A}$. With FOU, A IT2 FS $\widetilde{A}$ can be expressed as following,

$$\widetilde{A} = 1/FOU(\widetilde{A}) \quad (9)$$

Equation (8) drastically simplifies the computation with T2 FSs and enhances the application of IT2 FS in real-world problems. With the FOU, an 'embedded T1 FS' is closely related which is a T1 FS contained within the FOU. The union of all embedded T1 FSs makes the FOU and presents the IT2 FS, defined in (10),

$$\widetilde{A} = \bigcup_{i=1}^{m*} A^{(i)} \quad (10)$$

where A^i is the ith embedded T1 FS [11].[1]

E. Literature Review

Numerous studies have been conducted to construct IT2 FSs where some approaches were developed to transform existing T1 FSs to IT2 FSs, while others to generate IT2 FSs from the interval-valued data via constructing T1 FSs [13], [19]. In this regard, Tizhoosh [15] proposed a simple conversion approach of T1 MF to IT2 MFs (i.e. UMF and LMF) for better

[1] Representing IT2 FS using embedded T1 FSs is particularly useful as it allows to apply T1 FS mathematics on IT2 FSs [11], thereby reducing the complexity of using IT2 FS in real-world applications compared to GT2 FS.

thresholding in image processing. This approach calculates the UMF and LMF of IT2 FS from the T1 MF, $\mu(x)$ using (11).

$$\begin{aligned} \mu_U(x) &= [\mu(x)]^{\frac{1}{\alpha}} \\ \mu_L(x) &= [\mu(x)]^{\alpha} \end{aligned} \tag{11}$$

Here, $\alpha \in (1, \infty)$. Despite its simplicity, this approach often faces difficulty in determining a suitable value for α as it is application-dependent. Further, it often completely changes the shape of IT2 FS compared to the original T1 MF [24], thus failed to systematically comparing the performance between original T1 FS and the newly constructed IT2 FS, [9]. Moreover, it cannot produce the FOU of the IT2 FS for the crisp range of trapezoidal T1 FS with the membership degree of 1. Later, Aladi et al. [25] developed a new 'T1 FS to IT2 FS' conversion method (M1 method), along with its refinement in [9] (M2 method) to show the relationship between the size of FOU of IT2 FSs and the noise in data. The initial method focuses only on maintaining uniform FOU over the core area of fuzzy set (i.e. the support of the LMF) whereas the refined one gives emphasis on keeping the same structural configuration (i.e. if the T1 MF is triangular, both LMF and UMF of the IT2 FS also be triangular but both be trapezoidal in the initial approach). The former method mainly depends only on the selection of a fixed parameter, $c \in [0, 1]$ based on the uncertainty level in the dataset. The refined one introduces a new parameter, δ to be subtracted/added from the parameters of T1 FS MF to make the left and right end-points of the LMF and UMF while again maintaining the same structural outline in IT2 FS as T1 FS.[2] Unfortunately, both methods do not focus on the optimization while converting to IT2 FSs which may affect their ultimate results.[3] Further, these approaches were developed only for creating IT2 FSs from T1 FSs and did not provide any solution on how to generate IT2 FSs from the input data. Considering this, Liu et al. [13] introduced an 'Interval Approach' (IA) to generate IT2 FSs for interval-valued data for a word from a group of subjects. The IA maps each interval to either a symmetric triangular T1 MF with a mean value as peak, left shoulder T1 MF, or right shoulder T1 MF. Then, it interprets the T1 MFs as an embedded T1 FSs to construct the FOU of the IT2 FS from their union. It performs data pre-processing eliminating outliers, nonsensical data, data not maintaining a tolerance threshold, and non-overlapped intervals. Indeed, the IA is a systematic method to construct IT2 FSs, however, it may lead to create the left-shoulder and right-shoulder IT2 FSs without FOU for some regions. Further, for a single interval opinion, it generates IT2 FS which is similar to T1 FS, hence, additional higher-order uncertainty with IT2 FS cannot be achieved. Later, Wu et al. [19] refined the IA as 'Enhanced Interval Approach' (EIA) with an altered pre-processing stage and an improved procedure for computing the LMF. Although the EIA minimises the limitations of the IA, both IA and EIA depend on specific FS models, e.g. triangular or trapezoidal MFs [4] and involve substantial computational time due to their pre-processing phase. In addition, both approaches firstly convert the interval-valued data to T1 FSs as an intermediate step before generating IT2 FSs which further makes them computationally expensive. To address these challenges and ensuring faster execution, Miller et al. [14] presented a new approach to transform interval-valued survey responses from experts on multiple occasions into GT2 FSs. This approach firstly calculates the intra-expert uncertainty using T1 FSs and then combines these FSs to generate zSlices-based GT2 FS [27] to present both the intra- and inter-expert uncertainty in two separate domains of GT2 FS. In this approach, no data pre-processing is performed as it considers the entire data set to accurately portray the experts' opinions. Although the generation of zSlices-based GT2 FS substantially reduces design complexity and execution time, the creation of T1 FSs as intermediary in this case involves extra computation, thus not suitable to use in the dynamic environment like VANET where faster computation is a prerequisite [28], [29].

[2] In [9], the value of δ is taken arbitrarily depending on the application.

[3] Some applications, e.g. boost DC-DC converters show prominent results with optimization while converting T1 Fs to IT2 FS [26].

Following the above discussion, it becomes clear that there is a compelling need for establishing an approach to construct IT2 FSs directly from the interval-valued data without via T1 FSs to ensure faster execution. At the same time, the same approach should be applicable to transform any existing T1 FS to IT2 FS. The next section introduces such an uniform approach that effectively generate IT2 FSs from interval-valued data or T1 FSs.

III. The Interval Creation Approach (ICA)

This section proposes a new method, the Interval Creation Approach (ICA), for generating IT2 FSs from the interval-valued data as well as T1 FSs. The ICA omits the data pre-processing step as we do not want to remove any outlier as they often carry different perspective of experts' opinion [14], [20] and prefer to create IT2 FSs capturing the entire data set. Further, it is designed to deal with any T1 FS models and does not create any T1 FS as an intermediate step before generating IT2 FS from the input data.

A. *Factors to Consider*

The following factors are considered while designing the proposed ICA method.

- T1 fuzzy MFs (e.g. triangular, trapezoidal, Gaussian, or generalized bell-shaped) can be complete or incomplete [21] (see Figs. 3(a) and (b)).
- The left- and right-most intervals of a set of intervals are treated in the same way like incomplete T1 FS [13], [19].
- A set of intervals can be merged using union [30].

B. *Proposed Methodology*

Considering the completeness/incompleteness of T1 FSs and intervals, the proposed methodology to construct IT2 FS is discussed below. Figure 2 presents the overall working process of the ICA method for constructing the IT2 FS from a T1 FS and/or interval-valued data.

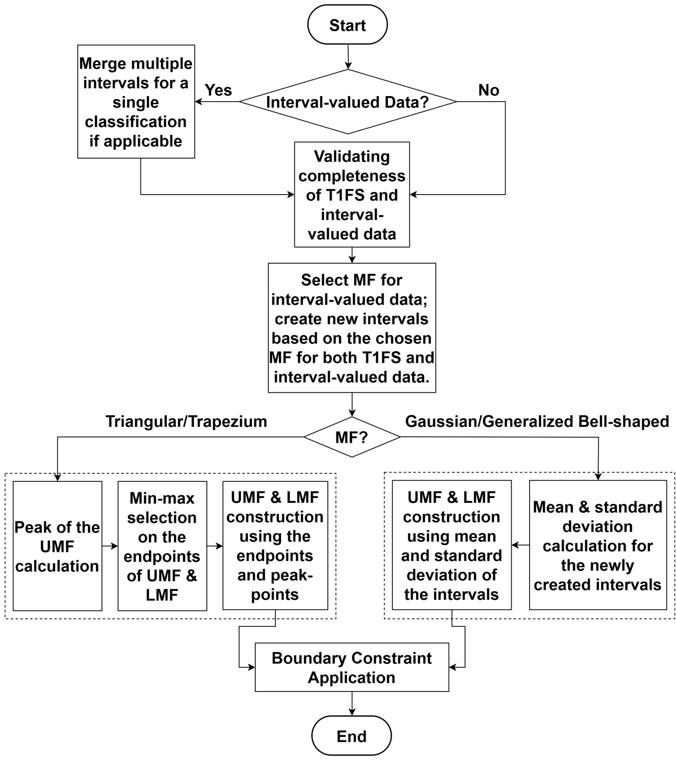
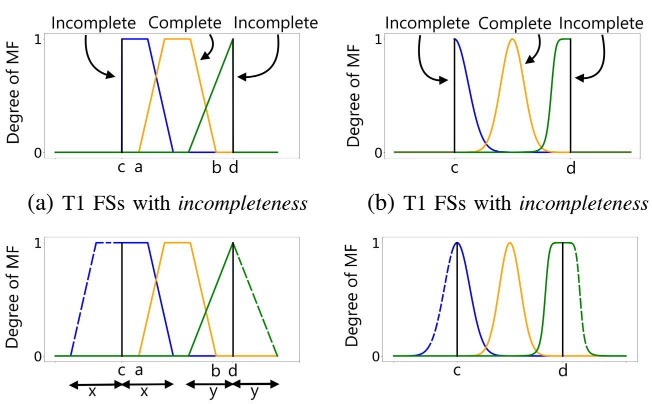


Fig. 2: Flowchart to construct IT2 FSs by the ICA method.

C. Construction of IT2 FS from T1 FS

1) Completeness Check: We check whether the T1 FS is complete or incomplete. If it is incomplete, we convert it to a complete T1 FS by adding a similar configuration to its opposite side (i.e. mirror reflection [31]). Note that for the incomplete left-most T1 FS, a counter-clock wise mirror reflection is performed, while for the incomplete right-most T1 FS, a clock-wise mirror reflection is applied. Figure 3 shows such transformations for different T1 FSs. For the incomplete triangular and trapezoidal MFs, the support of the left-most T1 FS turns from the range $[c, c+x]$ to $[c-x, c+x]$ while the support of the right-most T1 FS from the range $[d-y, d]$ to $[d-y, d+y]$ (see Figs. 3(a) and (c)). As Gaussian and



(a) T1 FSs with *incompleteness* (b) T1 FSs with *incompleteness*

(c) T1 FSs with *completeness* (d) T1 FSs with *completeness*

Fig. 3: T1 FSs (a) trapezoidal and triangular, (b) Gaussian and generalized bell-shaped) and their completeness check.

generalized bell-shaped MFs are distributed within the range $[c, d]$, the active intervals for the supports of the left-most incomplete Gaussian, middle complete Gaussian and the right-most incomplete generalized bell-shaped are same, which is [c,d]. Thus, for the incomplete Gaussian and generalized bell-shaped T1 FSs, the supports for the left- and right-most T1 FSs are transformed to [c-(d-c),d] and [c,d+(d-c)] respectively (see Figs. 3(b) and (d)).

2) Interval Creation: For each complete T1 FS, we have the interval comprising its support. We discretise the interval at 0.01 value increment and calculate their mean (m) and standard deviation (s). For transforming an IT2 FS from a T1 FS, at least two intervals are needed to build its FOU [13], [19]. Equation (12) is used to generate such two intervals for the trapezoidal IT2 FS.

$$\begin{aligned} [a_{MF}^{(1)}, b_{MF}^{(1)}] &= [m-1\times s, m+2\times s], \\ [a_{MF}^{(2)}, b_{MF}^{(2)}] &= [m-2\times s, m+1\times s] \end{aligned} \tag{12}$$

For other MFs (triangle, Gaussian, and generalized bell-shaped), (13) is used to generated intervals for them.

$$\begin{aligned} [a_{MF}^{(1)}, b_{MF}^{(1)}] &= [m-1\times s, m+1\times s], \\ [a_{MF}^{(2)}, b_{MF}^{(2)}] &= [m-2\times s, m+2\times s] \end{aligned} \tag{13}$$

3) Calculation of the Peak Value for the UMF: For the trapezoidal T1 FS, the peak is modelled by an interval, $[e, f]$. To extend $[e, f]$ for IT2 trapezoidal FS, we compute the mean (m) and standard deviation (s) by discretising $[e, f]$ at 0.01 value increment and calculate the new range, $[e', f']$ with (14).

$$[e', f'] = [m-2\times s, m+2\times s] \tag{14}$$

For other T1 FSs (triangle, Gaussian, and generalized bell-shaped), the peak of the T1 FS (i.e. a single value) is considered as the peak of the IT2 FS.

4) Min-Max Selection: The left endpoints for the UMF and LMF are computed using (15).

$$\begin{aligned} \underline{a}_{MF} &= min\{a_{MF}^{(1)}, a_{MF}^{(2)}\}, \\ \overline{a}_{MF} &= max\{a_{MF}^{(1)}, a_{MF}^{(2)}\} \end{aligned} \tag{15}$$

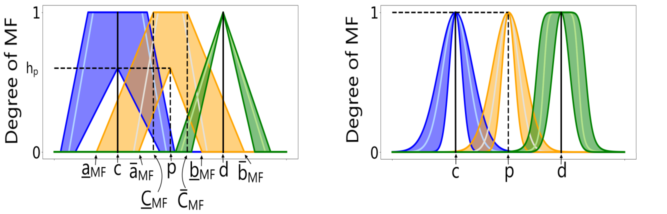
Similarly, the right endpoints for the LMF and UMF are computed using (16).

$$\begin{aligned} \underline{b}_{MF} &= min\{b_{MF}^{(1)}, b_{MF}^{(2)}\}, \\ \overline{b}_{MF} &= max\{b_{MF}^{(1)}, b_{MF}^{(2)}\} \end{aligned} \tag{16}$$

For trapezoidal IT2 FS, the peak for UMF will be in a range where the left and right endpoints are calculated with (17).

$$\underline{C}_{MF} = e', \; \overline{C}_{MF} = f' \tag{17}$$

For triangular IT2 FS, the peak values of the LMF and UMF are the same as that of T1 FS ($\underline{C}_{MF} = \overline{C}_{MF}$). However, for Gaussian and generalized bell-shaped IT2 FS, the peak for the UMF is calculated as the average of $\underline{a}_{MF}$ and $\overline{b}_{MF}$ and the peak for the LMF is calculated as the average of $\overline{a}_{MF}$ and $\underline{b}_{MF}$.



(a) Trapezoidal/Triangular IT2 FSs (b) Gaussian/G. Bell-shaped IT2 FSs

Fig. 4: Initial IT2 FSs from T1 FSs.

5) Creation of the UMF: Using straight lines, we connect the points, ($\underline{a}_{MF}$, 0), ($\underline{C}_{MF}$,1), ($\bar{C}_{MF}$,1), and ($\bar{b}_{MF}$,0) to create the UMF for the trapezoidal and triangular IT2 FSs. Besides, (18) and (19) are employed to generate the UMF of IT2 Gaussian and IT2 generalized bell-shaped FSs respectively.

$$f(x; s, m) = e^{-\frac{(x-m)^2}{2s^2}} \tag{18}$$

$$f(x; s, k, m) = \frac{1}{1 + |\frac{x-m}{s}|^{2k}} \tag{19}$$

In (18) and (19), x is a member of universal set. In (19), k is the skewness factor shaping the curve on both sides of the central plateau. Further, m and s are computed based on the newly created interval $[a^{(2)}_{MF}, b^{(2)}_{MF}]$.

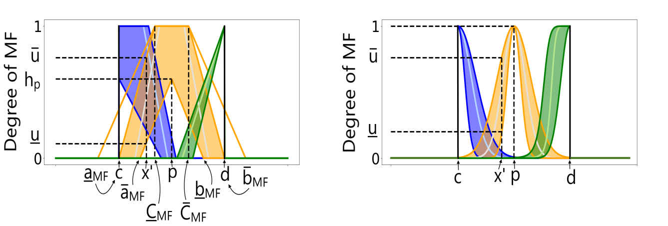
6) Creation of the LMF: For trapezoidal and triangular MFs, we calculate the peak of the LMF with (20) and (21) where the peak point is (p, h_p).

$$p = \frac{\underline{b}_{MF}(\bar{C}_{MF} - \bar{a}_{MF}) + \bar{a}_{MF}(\underline{b}_{MF} - \underline{C}_{MF})}{(\bar{C}_{MF} - \bar{a}_{MF}) + (\underline{b}_{MF} - \underline{C}_{MF})} \tag{20}$$

$$h_p = \frac{\underline{b}_{MF} - p}{\underline{b}_{MF} - \underline{C}_{MF}} \tag{21}$$

Using straight lines, we connect the points, ($\underline{a}_{MF}$, 0), ($\bar{a}_{MF}$,0), (p,h_p), ($\underline{b}_{MF}$, 0) and ($\bar{b}_{MF}$,0), resulting in a triangular LMF [13] (for trapezoidal MF, it is considered as truncated). Figure 4(a) shows the constructed IT2 FSs from the trapezoidal and triangular T1 FSs. For Gaussian and generalized bell-shaped T1 FSs, we compute m and s for $[a^{(1)}_{MF}, b^{(1)}_{MF}]$ and generate the LMFs using (18) (see Fig. 4(b)).

7) Boundary Constraints: If the endpoints of the left- and right-most IT2 FSs are out of the predefined range $[c, d]$, we bind them to the closet boundary values [32]. Thus, $\underline{a}_{MF}$, $\bar{a}_{MF}$ and $\underline{C}_{MF}$ are bounded by the left endpoint, c whereas $\underline{b}_{MF}$, $\bar{b}_{MF}$ and $\bar{C}_{MF}$ by the right endpoint, d. Further, if any portion of Gaussian or generalized bell-shaped IT2 FS is out of its corresponding range, we prune that portion [32]. Figures 5(a) and (b) show the resulting IT2 FSs for all T1 FSs. It is noted that the same process is also carried out for the IT2 FS related to the complete T1 FS.



(a) Trapezoidal/Triangular IT2 FSs (b) Gaussian/G. Bell-shaped IT2 FSs

Fig. 5: The resulting IT2 FSs using the ICA method.

D. Construction of IT2 FS from interval-valued data

Before constructing IT2 FS from the interval-valued data, we first merge the intervals under the same classifications using union to get single interval for each classification.

1) Completeness Check: We first decide on which MF is employed to model the set of intervals. The left- and right-most intervals are always considered incomplete [13], so we need to make them complete. Suppose, $[x, y]$ and $[m, n]$ present the left- and right-most intervals accordingly. If they are presented using triangular or trapezoidal MF, then after transformation to complete intervals, the left- and right-most intervals become $[x - (y - x), y]$ and $[m, n + (n - m)]$ respectively. In case of Gaussian or generalized bell-shaped MF, if the boundary interval is [c,d], then the complete left- and right-most intervals are $[c - (d - c), d]$ and $[c, d + (d - c)]$ respectively.

2) Interval Creation: We create new intervals for the interval-valued data with (12) and (13) with respect to the chosen MF.

3) Calculation of the Peak Value for UMF: For trapezoidal case, the average of the left and right endpoints of the interval, $[a^{(2)}_{MF}, b^{(2)}_{MF}]$ of (12) is e and the average of the left and right endpoints of the interval $[a^{(1)}_{MF}, b^{(1)}_{MF}]$ of (12) is f. The interval $[e, f]$ is extended using (14) which models the interval of peak for UMF of IT2 FS. For other MFs, the peak is the average of the left and right endpoints of the intervals in (13) where the average for both intervals is the same.

4) Min-Max Selection: Follow Section (III-C4).

5) Creation of the UMF: Follow Section (III-C5).

6) Creation of the LMF: Follow Section (III-C6).

7) Boundary Constraints: Follow Section (III-C7).

Claim: Figure 5 presents interval IT2 FSs.

Proof: If we take a vertical slice at any point $x = x'$ in Fig. 5(a)-(b), we obtain degree of MFs in an interval $[\underline{u}, \overline{u}]$. For $\forall x \in X$ has primary degree of membership expressed as an interval. Union of all intervals makes the shaded region between the LMF and UMF which satisfies the definition of FOU. We omit the third dimension in the T2 FS construction using the ICA, as it gives no new information. The secondary membership degree remains constant, 1, for all intervals associated with each $x \in X$. For any $x = x'$, we can write,

$$\mu_{\widetilde{A}}(x = x', u) \equiv \mu_{\widetilde{A}}(x = x') = \int_{u \in J_{x'}} 1/u \;\; ; J_x \subseteq [0, 1],$$

which satisfies the definition of IT2 FS. So, we can say that our constructed figure is an IT2 FS. ■

IV. Demonstration

This section evaluates the performance of the ICA using synthetic examples and two real-world data sets against the IA, EIA, and M2 methods, already discussed in Section II.

Two different metrics are used to evaluate the effectiveness of these methods (see Section IV-A).[4]

A. Evaluation Metrics

1) Root Mean Square Error (RMSE): It estimates error by considering the average deviation of the predicted values from the observed ones [20]. The RMSE is defined as,

$$RMSE = \sqrt{\frac{\Sigma(P_i - O_i)^2}{n}} \tag{22}$$

where P_i and O_i are the predicted and observed values respectively and n is the total number of observations. Lower the RMSE value, higher the accuracy of the system.

2) Coefficient of Determination (R^2): It determines what proportion of observed values reflected in the predicted values [33]. The R^2 is defined as,

$$R^2 = 1 - \frac{SSR}{SST} \tag{23}$$

where SSR is the sum of the squared differences between the observed and predicted values. SST is the sum of the squared differences between the observed values and their mean. Higher the R^2 value, higher the accuracy of the system.

B. Synthetic Examples

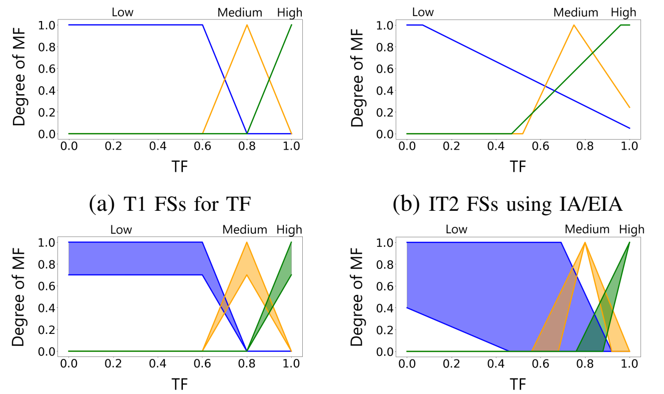
Three synthetic T1 FSs are considered from [16], [17], created for the time-sensitive, uncertain Vehicular Ad Hoc Network (VANET). Here, each T1 FS corresponds to 'Trust Factor' (TF)—one of the properties of vehicles used to identify malicious vehicles and ensure a secured communication through VANET [16], [17]. Figure 6(a) shows such three T1 FSs—Low, Medium, and High for TF. We apply the IA, EIA, M2, and ICA on them to create IT2 FSs and the resulting FSs are shown in Figs.- 6(b)-(d). From Fig. 6(b), it is seen that both IA and EIA cannot generate IT2 FSs based on the single intervals related to the T1 FSs and result in a new set of T1 FSs, different from the original ones. Note that both IA and EIA provide identical IT2 FS in these cases, thus shown together in Fig. 6(b). Figure 6(c) shows that for $\delta = 0$ and $c = 0.3$, the M2 generates good resulting IT2 FSs. However, the M2 completely depends on the randomly selected δ, as a result a different selection of δ may reduce its performance (see Fig. 9 with $\delta = 0.2$ and $c = 0.3$ in Appendix). Figure 6(d) shows that the ICA defines reasonable FOU and generates better IT2 FSs capturing the T1 FSs. Note that Table I shows all steps used to generate the IT2 FSs using the ICA method.[5]

C. Real-world Data Sets

Two real-world data sets are used to analyse the performance of the proposed ICA against other three methods.

[4] Note that due to page constraint, we avoid discussion on the time-complexity of these methods but will provide in our future journal publication.

[5] The proposed ICA method is designed to apply for any MFs including Gaussian as well as Generalized bell-shaped MFs. As both IA and EIA methods are specific to trapezoidal and triangular MFs, in Section IV-B, we consider synthetic examples with only trapezoidal and triangular MFs.



(a) T1 FSs for TF

(b) IT2 FSs using IA/EIA

(c) IT2 FSs using M2

(d) IT2 FSs using ICA

Fig. 6: T1 FSs presenting 'Trust Factor' (TF) and their conversion to IT2 FSs using the IA/EIA, M2, and ICA methods.

TABLE I: Generation of IT2 FSs from T1 FSs for 'Trust Factor' (TF) using the ICA method.

Steps	MFs		
	Low	Medium	High
Completeness Check	T1 FS is incomplete, new interval $=[(0-0.8), 0.8]$ $= [-0.8, 0.8]$	T1 FS is complete	T1 FS is incomplete, new interval $= [0.8, (1-0.8)+1]$ $= [0.8, 1.2]$
Interval Analysis	$[-0.8, 0.8]$, $c = 0, \sigma = 0.46$	$[0.6, 1]$, $c = 0.8, \sigma = 0.12$	$[0.8, 1.2]$, $c = 1, \sigma = 0.12$
Interval Creation	$[0 - 2 \times 0.46,$ $0 + 1 \times 0.46]$, $[0 - 1 \times 0.46,$ $0 + 2 \times 0.46]$ $= [-0.92, 0.46]$, $[-0.46, 0.92]$	$[0.8 - 1 \times 0.12,$ $0.8 + 1 \times 0.12]$, $[0.8 - 2 \times 0.12,$ $0.8 + 2 \times 0.12]$ $= [0.68, 0.92]$, $[0.56, 1.04]$	$[1 - 1 \times 0.12,$ $1 + 1 \times .12]$, $[1 - 2 \times 0.12,$ $1 + 2 \times 0.12]$ $= [0.88, 1.12]$, $[0.76, 1.24]$
Peak-Points Analysis of UMF	After completion, $[e, f]$= $[(0-0.6), 0.6]$ $= [-0.6, 0.6]$ $c^{'} = 0, \sigma^{'} = 0.346$ $[e', f'] = [-0.692, 0.692]$ $\underline{C}_{MF} = -0.692 = e$ $\overline{C}_{MF} = 0.692 = f$	$\underline{C}_{MF} = \overline{C}_{MF}$ $= 0.8$, same peak-points as the T1 FS	$\underline{C}_{MF} = \overline{C}_{MF}$ $= 1$, same peak-points as the T1 FS
Min-Max Selection	$\underline{a}_{MF} = min\{-0.92,$ $-0.46\} = -0.92$ $\overline{a}_{MF} = max\{-0.92,$ $-0.46\} = -0.46$ $\underline{b}_{MF} = min\{0.46,$ $0.92\} = 0.46$ $\overline{b}_{MF} = max\{0.46,$ $0.92\} = 0.92$	$\underline{a}_{MF} = min\{0.68,$ $0.56\} = 0.56$ $\overline{a}_{MF} = max\{0.68,$ $0.56\} = 0.68$ $\underline{b}_{MF} = min\{0.92,$ $1.04\} = 0.92$ $\overline{b}_{MF} = max\{0.92,$ $1.04\} = 1.04$	$\underline{a}_{MF} = min\{0.88,$ $0.76\} = 0.76$ $\overline{a}_{MF} = max\{0.88,$ $0.76\} = 0.88$ $\underline{b}_{MF} = min\{1.12,$ $1.24\} = 1.12$ $\overline{b}_{MF} = max\{1.12,$ $1.24\} = 1.24$
Peak-Points Analysis of LMF	$p =$ $\{0.46 \times$ $(0.692 - (-0.46)\}$ $+(-0.46) \times$ $(0.46 - (-0.692))\}/$ $\{(0.692 - (-0.46))+$ $(0.46 - (-0.692))\}$ $= 0$ $, h_p =$ $(0.46 - 0)/$ $\{0.46 - (-0.692)\}$ $= 0.4$	$p =$ $\{0.92(0.8 - 0.68)+$ $0.68(0.92 - 0.8)\}/$ $\{(0.8 - 0.68)+$ $(0.92 - 0.8)\}$ $= 0.8$ $, h_p =$ $(0.92 - 0.8)/$ $\{0.920.8\}$ $= 1$	$p =$ $\{1(1 - 0.88)+$ $0.88(1 - 1)\}/$ $\{(1 - 0.88)+$ $(1 - 1)\}$ $= 1$ $, h_p =$ $(1.12 - 1)/$ $\{1.12 - 1\}$ $= 1$
Boundary Constraints	$\underline{a}_{MF} < 0$, so,$\underline{a}_{MF} = 0$ $\overline{a}_{MF} < 0$, so $\overline{a}_{MF} = 0$ $\underline{C}_{MF} < 0$, so $\underline{C}_{MF} = 0$	$\overline{b}_{MF} > 1$, so $\overline{b}_{MF} = 1$	$\underline{b}_{MF} > 1$, so,$\underline{b}_{MF} = 1$ $\overline{b}_{MF} > 1$, so $\overline{b}_{MF} = 1$

1) With 'Wear Analysis' Data Set [33]: This data set is used to investigate the wear behavior of electroless Ni-P coating under lubricated condition [33]. A T1 fuzzy logic system was designed using this data set to predict the wear depth of electroless Ni-P coating under the mentioned condition where each of three input variables, *Load (N)*, *Speed (rpm)*,

TABLE II: Comparison of the *Wear-depth* prediction performance between T1 fuzzy logic system [33] and IT2 fuzzy logic systems using the ICA and M2 methods.

Evaluation metrics	T1 fuzzy logic system	IT2 fuzzy logic system		
		M2 with δ[6]		
		$c = 0$	$c = 0.5$	ICA
RMSE	0.425	0.636	0.704	**0.393**
R^2	96.5%	94.5%	93.6%	**97%**

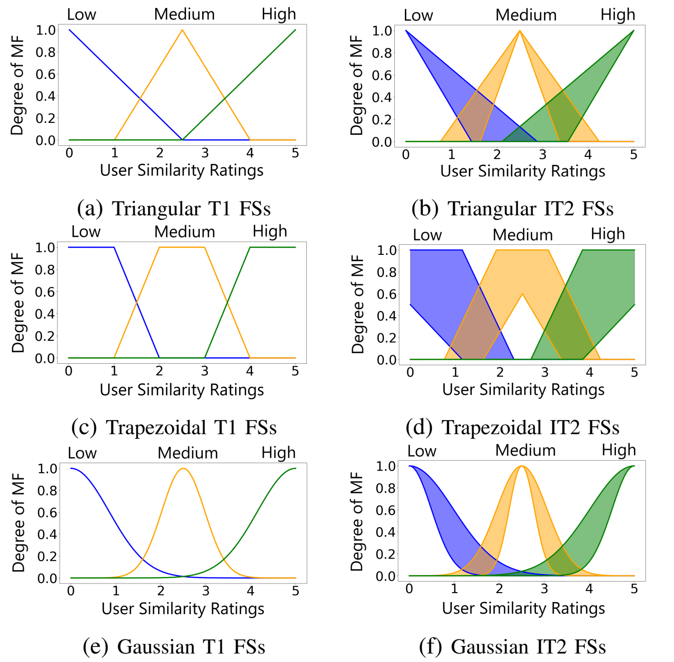
and *Time (min)* has three T1 triangular MFs—low, medium, and high, whereas the output variable, *Wear-depth (μm)* has nine T1 triangular MFs—extremely low, very low, low, low medium, medium, high medium, high, very high, and extremely high [33]. Further, 27 fuzzy if-then rules are applied to predict the *Wear-depth*. Based on this data set, we first construct IT2 fuzzy logic systems by employing the ICA and M2 (with different c and δ values[6]) methods to transform each of the triangular T1 FSs to triangular IT2 FSs (as both IA and EIA methods are not designed to convert T1 FS to IT2 FS) along with a minor modification of 27 if-then rules [33]. Then, we compare the prediction performance of the constructed IT2 fuzzy logic systems with that of the prior T1 fuzzy logic system [33] with respect to *RMSE* and R^2. A comparative results of these fuzzy logic systems are presented in Table II. From the results, we see that the IT2 fuzzy logic system with the ICA method provides the best performance with minimum *RMSE* and maximum R^2 in predicting the *Wear-depth*.

2) 'MovieLens' Data Set [34]: Siddiquee et al. [35] proposed a 'movie recommendation system' using T1 fuzzy logic for the *'MovieLens'* data set [34] with one million movie ratings (on a scale from 1 to 5) from 6000 users on 4000 movies. They considered three types of T1 FSs (triangular, trapezoidal, and Gaussian) to design the T1 fuzzy logic system for recommending movies to users. In each case, two input parameters (user similarity ratings and acceptance ratings, calculated using k-neighbourhood with $K = 5, 10, 20$) are used based on a set of nine if-then rules to calculate recommendation of a movie to a target user.

We first compare the recommendation performance of the T1 fuzzy logic system with that of the IT2 fuzzy logic systems constructed by the ICA and M2 methods. Figure 7 shows the resulting IT2 FSs from the ICA for different T1 FSs for the user similarity ratings (whereas for the M2, see Fig. 10 in Appendix). Table III exhibits the values of c and δ used in the M2 method and Table IV provides a comparative analysis on the movie recommendation performance between the T1 fuzzy logic system [35] and the IT2 fuzzy logic systems using the ICA and M2 methods with respect to *RMSE*. From the result, it is seen that the IT2 fuzzy logic based movie recommendation system outperforms with minimum *RMSE* in most of the cases.

As mentioned earlier, both IA and EIA are developed to construct IT2 FS only from the interval-valued data. Hence, to assess their performance in recommending movies to users,

[6]We consider $\delta = 5, 2, 1$ for *Load*, *Speed*, and *Time* respectively to present the system with M2 method more compatible with real-world scenario.



(a) Triangular T1 FSs (b) Triangular IT2 FSs (c) Trapezoidal T1 FSs (d) Trapezoidal IT2 FSs (e) Gaussian T1 FSs (f) Gaussian IT2 FSs

Fig. 7: T1 FSs showing user similarity ratings' MFs [35] and their conversion to IT2 FSs using the ICA method.

TABLE III: Different δ and c values for the M2 method for the IT2 fuzzy logic based 'Movie recommendation system'.

	MFs	c			
		0.2	0.4	0.8	1
δ	Triangular	0.02	0.045	0.12	0.182
	Trapezoidal	0.015	0.033	0.088	0.127
	Gaussian	0.004	0.009	0.017	0.021

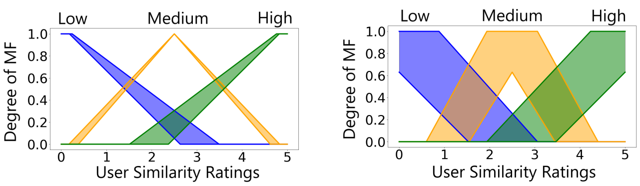
TABLE IV: Comparison of the movie recommendation performance between T1 fuzzy logic system [35] and IT2 fuzzy logic systems using the ICA and M2 methods.

MFs	K	T1 fuzzy logic system	IT2 fuzzy logic system				
			M2 with δ from Table-III				
			c = 0.2	c = 0.4	c = 0.8	c = 1	ICA
Trapezoidal	5	0.978	0.964	0.951	0.888	0.984	**0.871**
	10	0.989	0.978	0.966	0.912	0.97	**0.867**
	20	0.917	0.905	0.893	0.838	0.89	**0.809**
Triangular	5	0.925	0.918	0.91	**0.883**	1.07	0.907
	10	0.933	0.927	0.92	**0.9**	1.049	**0.9**
	20	0.845	0.839	0.833	**0.81**	0.977	**0.81**
Gaussian	5	0.934	0.931	0.93	0.916	1.696	**0.899**
	10	0.947	0.944	0.942	0.927	1.679	**0.907**
	20	0.868	0.864	0.862	0.844	1.623	**0.829**

we first compute different intervals from the low, medium and high MFs designed for the prior T1 fuzzy logic system. For instance, from Figs. 7(a), (c) and (e), we can compute the interval set {[0, 2.5], [0, 2], [0, 2.65]} to represent the low T1 FSs. With the computed interval sets, we again construct the IT2 fuzzy logic systems using the IA and EIA methods along with the ICA (as it is compatible to build IT2 FS from both intervals and T1 FS). Figure 8 exhibits the IT2 FSs generated using the IA, EIA (8(a)) and ICA (8(b)) methods. Table V shows the comparison between the IT2 fuzzy logic system using the IA/EIA method and that of using the ICA

TABLE V: Comparison of the movie recommendation performance between IT2 fuzzy logic system using the ICA method and those of using the IA/EIA methods with respect to *RMSE* and Execution time (ET) in seconds.

	IT2 fuzzy logic system			
	IA/EIA		ICA	
K	*RMSE*	ET (s)	*RMSE*	ET (s)
5	0.931	7.2	**0.878**	**5.64**
10	0.949	10.06	**0.884**	**8.375**
20	0.873	14.5	**0.805**	**12.5**



(a) IT2 FSs using IA/EIA (b) IT2 FSs using ICA

Fig. 8: IT2 FSs with the IA/EIA and ICA methods from the interval sets generated from Fig. 7.

method for movie recommendation with respect to *RMSE* and Execution time (ET) in seconds. The results reveal that the IT2 fuzzy logic system using the ICA once again shows the best performance with minimum *RMSE* and ET (s).

V. Conclusion and Future Works

The contribution of the paper centres on proposing an easy to use technique, referred to as 'Interval Creation Approach' (ICA) to construct IT2 FS from T1 FS as well as from interval-valued data. The proposed approach is designed such that it works on any T1 MF and does not involve data pre-processing phase (thus keeping outlier in the data set to portray different opinions) and/or creation of T1 TS before generating IT2 FS from interval data and hence reduces extra processing as well as computational cost. At an experimental level, the paper provides a detailed investigation contrasting the performance of the proposed ICA method vis-a-vis the IA, EIA, and M2 methods using both synthetic examples and real-world data sets. The exhaustive analyses have confirmed that the new method shows better performance with respect to evaluation metrics, such as *RMSE* and R^2 against other methods in recommending real-world applications. In the future, we plan to test the ICA method to generate IT2 fuzzy logic systems for time-sensitive, highly uncertain real-world applications (e.g. VANET communication). We further aim to extend this approach to find a faster way to construct GT2 FSs so that they can be used to efficiently solve complex real-world problems.

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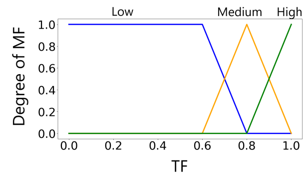
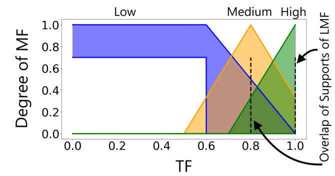
APPENDIX

Degraded Performance of the M2 method: Figure 9 exhibits the lower performance of the M2 methods with $\delta = 0.2$ and $c = 0.3$ in generating the IT2 FSs for the T1 FSs of Fig. 6 modelling ‘Trust Factor’ (TF).



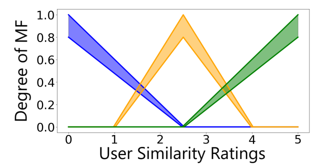
(a) T1 FSs for TF



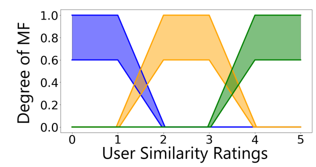
(b) IT2 FS using M2

Fig. 9: T1 FSs presenting ‘Trust Factor’ (TF) and their conversion to IT2 FSs using the M2 method.

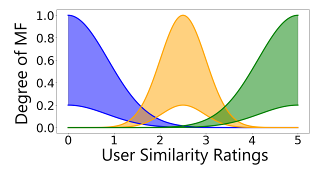
IT2 FSs using the M2 Method for the T1 FSs of Fig. 7:



(a) Triangular IT2 FS



(b) Trapezoidal IT2 FS



(c) Gaussian IT2 FS

Fig. 10: IT2 FSs using the M2 method for the T1 FSs of Fig. 7 showing user similarity ratings’ MFs with (a) $c = 0.2, \delta = 0.02$, (b) $c = 0.4, \delta = 0.033$, and (c) $c = 0.8, \delta = 0.017$.