Optimal internal pressurisation of cylindrical shells for maximising their critical bending load

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ABSTRACT

The paper studies the influence of internal pressure on circular thin-walled pipes ($D/t > 150$) subjected to pure bending. Both straight pipes and curved pipes are analysed. Both yield and buckling failures are considered. It is shown that internal pressure decreases the limiting load for yield but increases the limiting load for buckling.

The study is mainly FEA-based. A formula to predict critical moment given by linear buckling analysis is proposed. Comments on difference between linear and non-linear analysis results are given. It is shown that a pipe curvature opposite to the bending moment can increase the critical load. It is shown that cylindrical thin-walled shells have an optimal value of internal pressure to which limiting load for yield and critical buckling moment are equal, corresponding to an optimal use of material.

1. Introduction

Slender parts and thin-walled components are frequently used for limiting the structural weight. They are widely used in engineering and find many applications in aeronautical and aerospace field wherein lightness is key factor. However, such structures are naturally subjected to static stability problems under compression loads, i.e. buckling failure. Depending on their geometry, material and load conditions, buckling could be the most likely type of failure.

Buckling phenomenon has been widely investigated by several authors [1–3]. Some focused on thin cylindrical shells [4–6]. Specific studies have been conducted on curved pipes and showed that pipe curvature decreases the critical load [7]. Also the benefits in terms of buckling of internally pressurised cylindrical shells have long been known, in cases of axial loading [8], torsion [9] and bending [10,11].

The main novelty of this work is the exhibition of the fact that a cylindrical thin-walled shell under bending is characterised by an optimal value of internal pressure which maximises the critical applied bending moment. Such conditions correspond to a maximum exploitation of the structural potential of the shell.

The paper is organised as follows: Section 2 summarises the previous research on buckling of cylindrical thin-walled shells; Section 3 provides and an analytical treatise on limiting load for yield of pipes under bending and internal pressure; Section 4 describes the model used in FE analyses and presents the obtained results; Section 5 concludes the paper.

2. Background

The first studies of collapse of cylindrical thin-walled shells started analytically and focused on axial compressive loads. Tymoshenko and Gere [1] found that the critical stress for a long cylindrical shell simply supported at the ends is expressed by

$$\sigma_{cr} = \frac{Et}{r\sqrt{3(1-\nu^2)}} \tag{1}$$

where σ_{cr} is the critical stress, E is the Young Modulus, t is the wall thickness, r is the mean pipe radius and ν is the Poisson's ratio.

For cylindrical thin-walled shells bending moment and maximum stress are related as follows:

$$\sigma = \frac{M}{\pi r^2 t} \tag{2}$$

where σ is the maximum stress in the axial direction and M is the bending moment.

Assuming that a circular thin-walled pipe under pure bending buckles when the compressive stress reaches the value corresponding to buckling due to axial compressive load, the critical moment M_{cr} can be calculated by simply combining Eqs. (1) and (2). For

a steel with $\nu=0.3$ one obtains:

$$M_{cr} = \sigma_{cr}\pi r^2 t = 0.605\pi E r t^2 \qquad (3)$$

Tymoshenko and Gere [1], Yudo and Yoshikawa [7], Brazier [4], Chwalla [12] and other authors conducted studies on circular pipe under bending and their results range over values in between 0.55 and 1.3 times the value expressed by Eq. (3). The reason for this variability is that Eq. (3) provides a reference value for critical moment but it cannot represent the real value for circular pipes since their behaviour and their strength is strongly depending on other parameters, especially diameter-thickness ratio D/t [13].

A formula to calculate the buckling moment M_p in plastic region was proposed [14]:

$$M_p = \left(1.05 - 0.0015\frac{D}{t}\right)\sigma_y D^2 t \qquad (4)$$

where D is the pipe diameter and σ_y is the yield stress. This equation is widely accepted to be a good design criterion. Plastic buckling will not be taken into account in this paper since its occurring is usually due to a low value of yield stress and/or a low value of D/t ratio [3].

3. Limiting load for yield

Pure bending refers to a load condition wherein bending is the only load acting in a member. Despite the fact that there are no actual cases of structural members subjected to pure bending in reality, their study is relevant since it can provide a prediction of the type of failure and a limit allowable load for similar load condition.

A member subjected to bending will have a non-constant stress along the cross-section being partly under tension and partly under compression. Increasing the bending moment, the tensile part could reach yield whereas the compression part could buckle. Here limiting load for yield of a cylindrical internally pressurised thin-walled pipe is analytically studied.

In order to calculate the limiting load for yield of a cylindrical thin-walled shell subjected to internal pressure and bending moment, one has to take into account the stresses caused by these two different loads. For the sake of simplicity, it is assumed that the material is linear and isotropic and the cross-section does not change with loading.

Internal pressure will result in circumferential stress $\sigma_{\theta,p}$ and axial stress $\sigma_{z,p}$ where the subscript p indicates stress due to pressure.

$$\sigma_{\theta,p} = \frac{pD}{2t} \qquad (5)$$

$$\sigma_{z,p} = \frac{pD}{4t} \qquad (6)$$

Bending moment will result in a non-constant axial stress $\sigma_{z,m}$ along a cross-section where the subscript m indicates stress due to moment.

$$\sigma_{z,m} = \frac{M}{\pi r^2 t} \qquad (7)$$

When both the loads – pressure and moment – are acting, the total stresses are

$$\sigma_\theta = \sigma_{\theta,p} \qquad (8)$$

$$\sigma_z = \sigma_{z,p} + \sigma_{z,m} \qquad (9)$$

Assuming the yield as limit, the admissible stress σ_{adm} is expressed by using Von Mises criterion for plane stress:

$$\sigma_{adm} = \sqrt{\sigma_z^2 - \sigma_z\sigma_\theta + \sigma_\theta^2} = \sigma_y \qquad (10)$$

where σ_y is the yield stress.

Expressing σ_z as a function of σ_θ and σ_y Eq. (10) leads to Eq. (11):

$$\sigma_z = \frac{\sigma_\theta \pm \sqrt{4\sigma_y^2 - 3\sigma_\theta^2}}{2} \qquad (11)$$

Introducing Eqs. (5) and (8) in (11) one has

$$\sigma_z = \frac{\frac{pD}{2t} \pm \sqrt{4\sigma_y^2 - 3\left(\frac{pD}{2t}\right)^2}}{2} \qquad (12)$$

Eq. (12) represents the maximum admissible axial stress σ_z in a pipe with diameter D, thickness t, yield stress σ_y and internal pressure p before yield occurs. Just a part of σ_z calculated by Eq. (12) is available to resist moment, $\sigma_{z,m}$, since the pressure already results in the remaining part $\sigma_{z,p}$. Hence, combining Eqs. (6), (7), (9) and (12) one obtains the limiting load for yield in terms of maximum bending moment M_{max} as a function of pipe
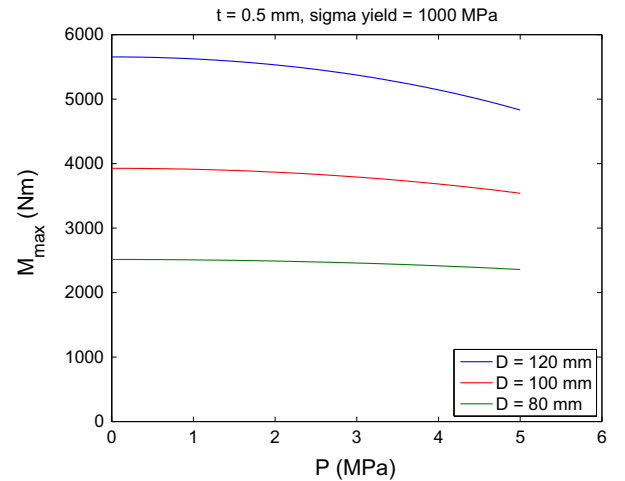

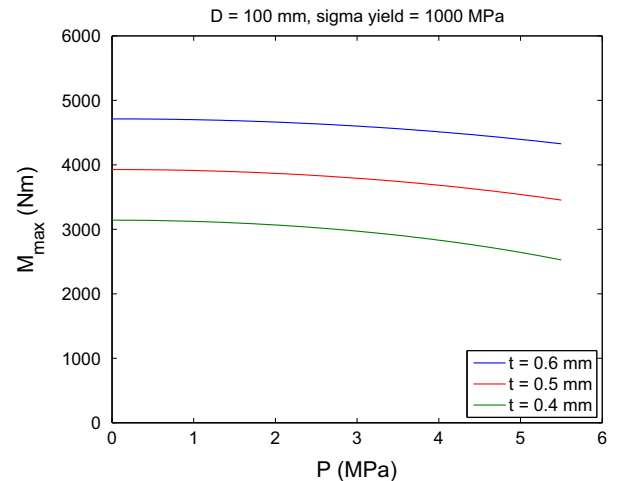**Fig. 1.** Yield strength for different values of D.
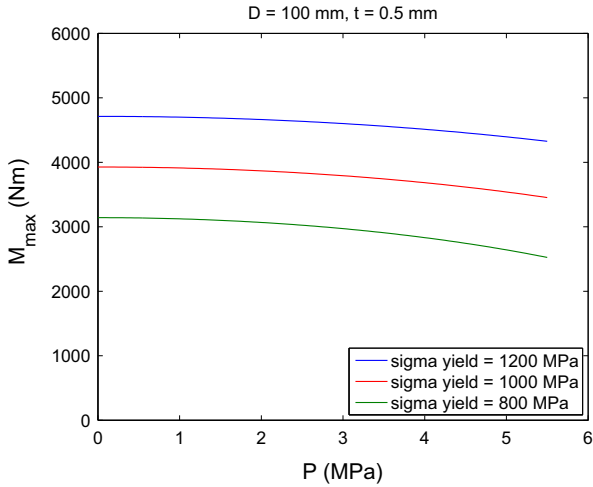

Fig. 2. Yield strength for different values of t.

Fig. 3. Yield strength for different values of sigma yield.

geometry and internal pressure:

$$M_{max} = \sigma_{z,m}\pi r^2 t = \left(\frac{\frac{pD}{2t} \pm \sqrt{4\sigma_y^2 - 3\left(\frac{pD}{2t}\right)^2}}{2} - \frac{pD}{4t} \right)\pi\frac{D^2}{4}t \qquad (13)$$

Curve trend of Eq. (13) is shown in Figs. 1, 2 and 3. It is clear that limiting load for yield decreases with internal pressure.

4. Numerical case studies

4.1. FE model

The FEA software used to perform buckling analysis is Ansys 14.5. The model consists in a cylindrical shell with its longitudinal axis coinciding with Z axis and bending moment acting in the X direction (YZ plane) (Fig. 4). Pure moments are applied to the two ends of the shell, which are constrained to behave as rigid planes in order to force buckling to take place in the middle part of the pipe.

The mid-section, with coordinate $Z=0$, is constrained to ensure the symmetry. The constraints allow the mid-section to be free to deform in plane but maintain it coincident with XY plane. All the nodes are constrained in the Z direction. Additionally, the 2 nodes in the YZ plane are constrained in the X direction and the 2 nodes in the XZ plane are constrained in the Y direction.

The element used to create the mesh is SHELL181 which is a four-shell element implementing Kirchhoff–Love plate theory. This allows for having a high number of elements and trace the buckling shape without increasing the number of degrees of freedom and computational time excessively. The pipe diameter D is 100 mm and the pipe length L is such that the ends constraints do not have a significant impact on the buckling critical moment; after some preliminary analyses the length was chosen to be equal to 1000 mm, since a further increase in length leads to the same results. A mesh sensitivity study has shown that the appropriate element size is 5 mm in the axial direction (200 subdivisions in total) and 8.726 mm in the circumferential direction (36 subdivisions in total). The material behaviour is linear and isotropic. More values of wall thickness are analysed. For curved pipes, more values of pipe centerline radius are analysed. The centerline radius refers to the pipe curvature of the unloaded model, without accounting for the effects of the internal pressure and the bending moment. Table 1 summarises the model parameters.
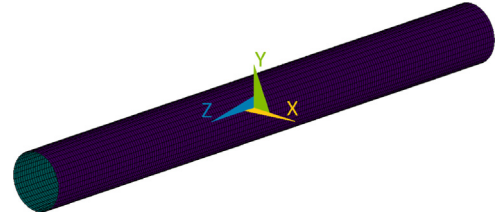

Fig. 4. FE model.

Table 1
FE model parameters.

Parameter	Value
Pipe diameter, D	100 mm
Pipe length, L	1000 mm
Wall thickness, t (D/t)	0.25 mm (400) – 0.50 mm (200)
Radius of curvature[a], R (R/D)	−5 m (−50) – 5 m (50)
Young Modulus, E	200 GPa
Poisson's ratio, ν	0.3

[a] Negative values represent pipe whose initial curvature is opposite to the curvature deriving from bending moment.

4.2. Linear buckling analysis

The first type of analysis performed is a linear analysis which consists in calculating the critical moment as a result of the eigenvalue problem. An insight on how to perform a linear buckling analysis with an applied bending moment and an internal pressure are in Appendix A. Fig. 5 shows results for pipes with 3 different values of thickness.

Results of further analyses conducted with different parameters (D, E, ν) have lead to Eq. (14) which well predicts the critical moment calculated by FEA for straight thin-walled pipes subjected to a bending moment and internal pressure:

$$M_{cr} = \frac{1.68}{\sqrt{3(1-\nu^2)}}EDt^2 + 0.203D^3p \qquad (14)$$

or, in terms of σ_{cr}, one obtains Eq. (15) by introducing Eqs. (1) in (14):

$$M_{cr} = 1.68D^2\frac{\sigma_{cr}}{2} + 0.203D^3p \qquad (15)$$

Eq. (14) consists of two terms:

- The first term represents the critical moment for no internal pressure, i.e. points on Y axis in Fig. 5. Its value is slightly higher than the value calculated by Eq. (3) (+7%). This is because in axial loaded pipes the entire cross-section undergoes the same compressive stress, whereas in bent pipes just a small part withstands high compressive stresses. Hence, a higher value of critical stress was expected.
- The second term represents the enhancement due to internal pressure. It is proportional to p, indeed curves in Fig. 5 are straight lines with slope equal to $0.203D^3$.

4.3. Non-linear buckling analysis

The second type of FE analysis which has been performed is a non-linear buckling analysis. In this case the approach is not an eigenvalue problem but a load–deflection problem: loads are applied by incremental steps with the stress-stiffness matrix being recalculated at every step. When small load increments induce large deflections (deflection derivative tends to infinity), buckling
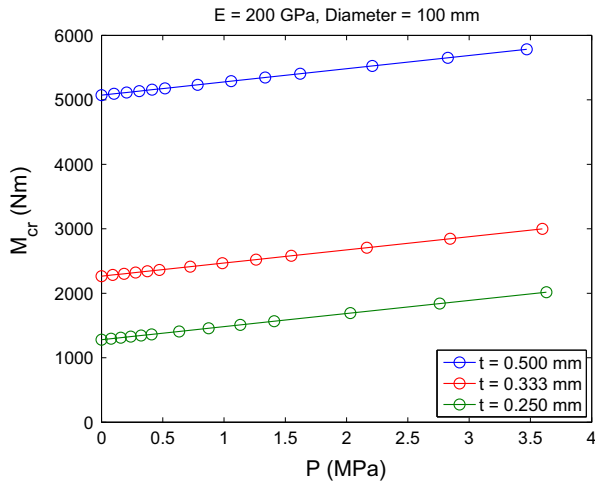
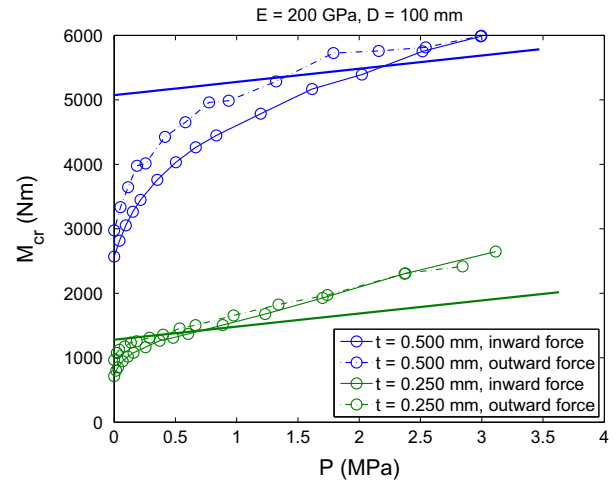**Fig. 5.** Linear buckling analysis of straight pipes.

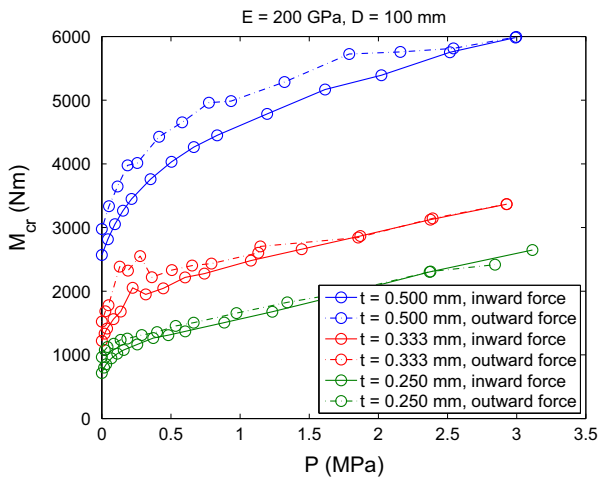**Fig. 7.** Linear and non-linear buckling analysis comparison.

Fig. 6. Non-linear buckling analysis of straight pipes.

occurs. Note that this nonlinearity is deformation induced and is not related to the material behaviour. This analysis is usually more accurate than the linear analysis and leads to lower critical loads.

In order to trigger structural instability, a small perturbation radial force was added to the model applied to the node where buckling is expected to start (bottom node at mid-section). The value of perturbation force is 2.5% the value of bending moment, e. g. 100 N of force in a model loaded with 4000 Nm of bending moment. Fig. 6 exhibits results for pipes with three different values of thickness. Fig. 7 shows a comparison between linear analysis results (Fig. 5) and non-linear analysis results (Fig. 6).

The following was observed:

- Inward perturbation forces cause a lower critical moment with respect to outward perturbation forces for low values of internal pressure. This suggests that in thin-walled circular pipes the point which first exhibits instability (bottom node at mid-section) has the tendency to buckle inward. This tendency disappears with increasing internal pressure until the perturbation direction will no longer play any relevant role.
- Linear buckling analysis overestimates the critical moment when no internal pressure is applied. The reason is that the linear analysis does not take into account large deflections during loading, therefore it neglects ovalisation of the cross-section and the consequent reduction of inertial moment.

- Non-linear buckling analysis is more sensitive to the applied pressure. Enhancements due to internal pressure – i.e. slope of curves – are significant, especially for low values of pressure. This can be explained considering pressure has two positive effects in non-linear analysis: material stiffening due to stress (as in linear analysis) and cross-section shape changing.

Since inward perturbation forces lead to lower critical moment values, from this point forward all results of non-linear buckling analysis will be referred to inward perturbation force if not otherwise specified.

4.4. Effect of pipe curvature

A series of non-linear buckling analysis have been performed for curved pipes in order to evaluate how the pipe's curvature influences critical moment. Fig. 8 shows the results obtained. Curvature k is defined as the ratio of pipe diameter D over radius of centerline curvature R. Negative values represent curvatures opposite to the bending moment action.

It can be seen that small curvatures have an important impact on the critical moment. Curved pipes have the same tendency as the straight one, exhibiting higher critical moments due to internal pressure. However, it is when the pressure values are low that curved pipes exhibit the most significant differences. Fig. 9 shows the critical moment with no pressure M_{cr0} (points on Y axis in Fig. 8) as a function of curvature for three values of ratio D/t and with inward and outward perturbation forces.

Fig. 9 reveals that negative curvatures can increase the critical moment for thin-walled circular pipes. In particular, there is an optimal value of curvature which maximises the critical moment. Some comments:

- The optimal curvature is a negative one, which can compensate the curvature caused by bending moment.
- The optimal curvature decreases (or increases, only looking at the absolute value) with increasing the ratio D/t.
- The benefits due to curvature increase with increasing the ratio D/t. For pipe with $D/t=200$ and inward perturbation force, critical moment at optimal pressure is 82% higher than critical moment for the corresponding straight pipe.
- For a certain pipe, the non-linear analysis with inward perturbation force leads to an optimal curvature slightly lower (or higher, only looking at the absolute value) than the optimal curved obtained with outward perturbation force.
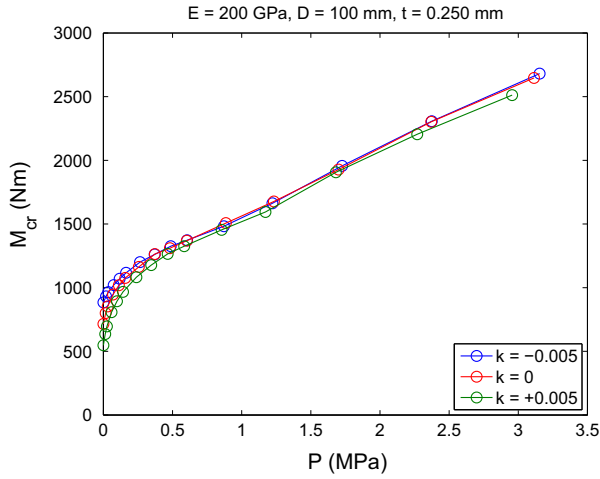
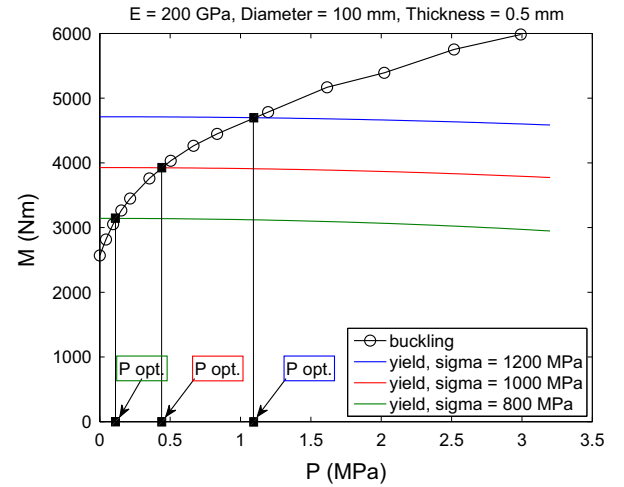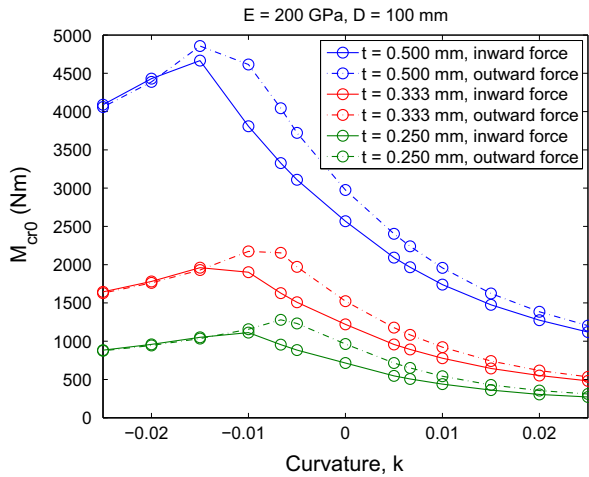**Fig. 8.** Non-linear buckling analysis of pipes with curvature *k*.

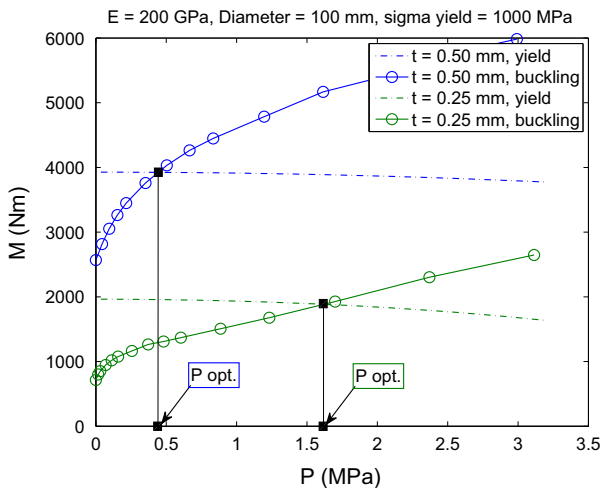**Fig. 9.** Critical moment with no pressure as function of pipe curvature.

Fig. 10. Optimal internal pressure for straight pipes with different ratio *D/t*.

4.5. Optimal internal pressurisation

In the previous section it was shown that internal pressure plays a significant role with regard to mechanical behaviour of cylindrical thin-walled shells subjected to flexural excitations. Increasing internal
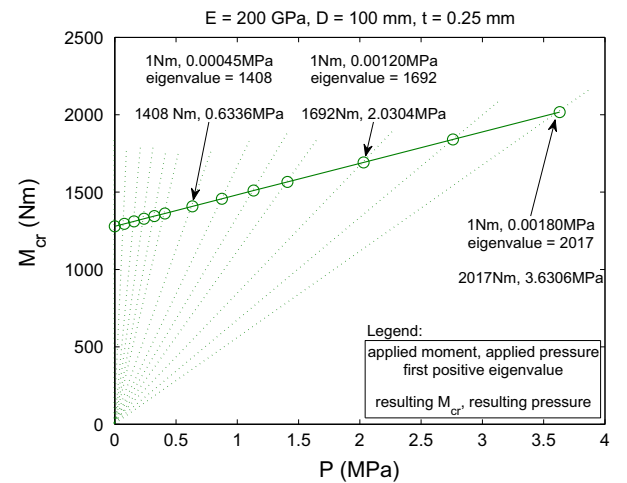

Fig. 11. Optimal internal pressure for straight pipes with different yield stress.

Fig. 12. Insights on linear buckling analysis of straight pipes with three of calculated points highlighted. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this paper.)

pressure induces a lower limiting load for yield (Eq. (13) and Figs. 1–3). Increasing internal pressure results in improved structural stability properties (Figs. 5–8). Hence, an optimal pressure will exist at which the limiting load for yield and buckling becomes maximum.

Fig. 10 shows optimal pressures for straight pipes with 2 different values of *D/t*, obtained by superimposing curves of Section 3 and Section 4.3. Fig. 11 shows optimal pressure for a straight pipe with *D/t* equal to 200 and different values of sigma yield.

Higher values of σ_y allow applying higher values of internal pressure before reaching yield. In other terms, for a certain pipe $p_{optimal}$ increases with increasing σ_y.

5. Concluding remarks

The present paper confirms that internal pressure increases the critical moment of thin-walled pipes under bending. Curved pipes behave as straight pipes, with an increment of critical moment due to pressure, however they show a somewhat different value of buckling limit with respect to straight pipes when no pressure or low values of pressure are applied. In these cases, a slight negative curvature increases the critical moment and positive curvature decreases it.

Internal pressure improves the performance against buckling but facilitates yield since it leads to higher stress in the tensile side of the pipe. There exists an optimal pressure to which the two types of failure have the same limit and that allows to fully exploit the structural properties of the pipe. Such optimal value of pressure increases with increasing of material performance, i.e. σ_y.

These findings show that internal pressurisation is an effective way to enhance structural properties. Such techniques can induce significant advantages for thin-walled structures. The application is to be seen in conjunction with the use of high performance materials, ideally composites, thus presenting great potential for aeronautical and aerospace engineering applications.

Acknowledgements

Appendix A. Appendix

The linear buckling analysis predicts the theoretical buckling strength of a structure, that is the bifurcation point of an ideal linear elastic problem. The buckling problem is formulated as an eigenvalue problem [15]:

$$(\mathbf{K} + \lambda_i \mathbf{S})\boldsymbol{\psi}_i = \mathbf{0} \tag{16}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{S}$ is the stress stiffness matrix, λ_i is the ith eigenvalue (used to multiply the loads which generated $\mathbf{S}$) and $\boldsymbol{\psi}_i$ is the ith eigenvector of displacements.

Linear buckling problems have several admitted solutions. A solution is given by an eigenvalue λ_i and the corresponding eigen-vector $\boldsymbol{\psi}_i$. The term critical buckling load usually refers to the solution with the minimum eigenvalue.

When the bending moment is the only acting load, for each eigenvalue which satisfies Eq. (16) it will exist another eigenvalue with opposite sign and same absolute value. For instance, the analysed pipe with $D = 100$ mm and $t = 0.25$ mm subjected to a bending load equal to $+1$ Nm, provided as minimum eigenvalues $+1279$ and -1279, that means the minimum critical moments is $+1279$ Nm (which rotates the end downward) and -1279 Nm (which rotates the end upward). If the same analysis was repeated with a bending moment equal to 2 Nm, the resulting eigenvalues would be ± 639.5, since an eigenvalue is just a multiplying factor of loads.

When more loads are acting, it needs to be considered that eigenvalue multiplies all of them. Hence, applying bending moment and internal pressure, the minimum eigenvalue (in terms of absolute value) given by the software is a negative eigenvalue, that corresponds to external pressure which decreases the critical bending moment. To overcome this problem, the option of setting the initial search point close to the expected critical load value of the solution is used and the desired first positive eigenvalue is thus calculated. For instance, the previous pipe was then analysed with a bending moment equal to $+1$ Nm and an internal pressure equal to 0.00045 MPa and the first positive eigenvalue was 1408. Multi-plying the applied loads into 1408 one obtains the pair of values $M_{cr} = 1408$ Nm and internal pressure $P = 0.6336$ MPa which corre-sponds to one of the green points in Fig. 5.

The pair of acting loads, e.g. 1 Nm and 0.00045 MPa, can be seen as a combination of bending moment and internal pressure which represents one of the dashed lines in Fig. 12. The ratio moment/pressure indicates the line slope of these dashed lines and eigenvalues are a measure of the distance between the origin and the markers (green circles in Fig. 12).

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