

EREMENKO'S CONJECTURE FOR FUNCTIONS WITH REAL ZEROS: THE ROLE OF THE MINIMUM MODULUS

D. A. NICKS, P. J. RIPPON, AND G. M. STALLARD

ABSTRACT. We consider the class of real transcendental entire functions f of finite order with only real zeros, and show that if the iterated minimum modulus tends to ∞ , then the escaping set $I(f)$ of f has the structure of a spider's web, in which case Eremenko's conjecture holds. This minimum modulus condition is much weaker than that used in previous work on Eremenko's conjecture. For functions in this class we analyse the possible behaviours of the iterated minimum modulus in relation to the order of the function f .

1. INTRODUCTION

Let f be a transcendental entire function and denote by f^n , $n = 0, 1, 2, \dots$, the n th iterate of f . The *escaping set*

$$I(f) = \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

plays a key role in complex dynamics with much recent work motivated by Eremenko's conjecture that all the components of the escaping set are unbounded. This work has led to a much better understanding of the structure of $I(f)$.

It is known that for many families of transcendental entire functions, including the exponential family, $I(f)$ has the structure of a *Cantor bouquet* consisting of uncountably many unbounded curves – see, for example, [26]. We have shown, however (for example in [22]), that there are many families of functions for which $I(f)$ has the structure of an infinite *spider's web*; that is, $I(f)$ is connected and there exist bounded simply connected domains G_n , $n \in \mathbb{N}$, such that

$$G_n \subset G_{n+1} \text{ and } \partial G_n \subset I(f), \text{ for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n=1}^{\infty} G_n = \mathbb{C}.$$

If $I(f)$ is a Cantor bouquet or a spider's web, then Eremenko's conjecture holds. Many results on $I(f)$ have been obtained by studying the *fast escaping set* $A(f)$, introduced in [3], which can be defined as follows; see [22]. First put

$$(1.1) \quad A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

where $M(r) = M(r, f) = \max\{|f(z)| : |z| = r\}$, $r > 0$, $M^n(r) = M^n(r, f)$ denotes the n th iterate of $r \mapsto M(r, f)$, and $R > 0$ is so large that $M(r) > r$ for $r \geq R$, and then put

$$A(f) = \{z : \text{for some } \ell \in \mathbb{N}, f^\ell(z) \in A_R(f)\}.$$

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Indeed, major progress on Eremenko's conjecture was made by showing that for many large classes of functions $A_R(f)$ is a spider's web, from which it follows that $I(f)$ is a spider's web, so Eremenko's conjecture holds (in a particularly strong way). These were the first transcendental entire functions for which it was known that $I(f)$ is connected; see [20], [22].

Subsequently, other methods of proving that $I(f)$ is connected were found. For example, Rempe-Gillen [19] showed that for the exponential function $f(z) = e^z$ the set $I(f)$ is connected, though it is known not to be a spider's web [17], whereas Evdoridou [6] showed that for Fatou's function $f(z) = z + 1 + e^{-z}$ the set $I(f)$ is a spider's web but $A_R(f)$ is not. Other functions for which $I(f)$ is a spider's web but $A_R(f)$ is not were given in [25].

Most proofs that $A_R(f)$ is a spider's web rely on the function f enjoying some form of classical relationship between the minimum modulus

$$m(r) = m(r, f) = \min\{|f(z)| : |z| = r\}$$

and the maximum modulus $M(r)$, which implies that there exist $r > R > 0$ such that

$$(1.2) \quad m^n(r) > M^n(R) \quad \text{and} \quad M^n(R) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

from which it follows that $A_R(f)$ is a spider's web; see [16].

Here we give a new approach which enables us to obtain a large family of functions f for which $I(f)$ is a spider's web, based only on the iterated minimum modulus. Recall that the *order* $\rho(f)$ of a transcendental entire function f is

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

and f is said to be *real* if

$$f(\bar{z}) = \overline{f(z)}, \quad \text{for } z \in \mathbb{C}.$$

Theorem 1.1. *Let f be a real transcendental entire function of finite order with only real zeros, and suppose*

$$(1.3) \quad \text{there exists } r > 0 \text{ such that } m^n(r) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then $I(f)$ is a spider's web and hence is connected.

Remark The property (1.3) was introduced in [16], where it was shown to imply that a certain superset of $I(f)$ is always connected. It is also shown there that many examples of entire functions satisfy (1.3), including all functions of order less than $1/2$. Note that there are functions of order less than $1/2$, and even of order 0, that do not satisfy the stronger condition (1.2); see [24].

The proof of Theorem 1.1 is in two parts depending on the genus of the function, a non-negative integer closely related to the order of the function, to be defined shortly. First, in Section 2 we prove Theorem 1.1 for all functions satisfying the hypotheses of the theorem with genus at most 1. In this case we show that $I(f)$ contains a spider's web consisting of points that escape to infinity at a uniform rate related to the minimum modulus. In particular, as we show in Section 3, for real functions of order less than $1/2$ with real zeros, a certain subset of $I(f)$ called the quite fast escaping set, $Q(f)$, contains a spider's web.

Then, in Section 4 we consider transcendental entire functions f of finite order with only real zeros and of genus at least 2. We use results from the value distribution theory of entire functions to show that in this case there is *no* value of r for which $m^n(r) \rightarrow \infty$ as $n \rightarrow \infty$, and so the hypotheses of Theorem 1.1 cannot be satisfied. In fact we prove the stronger result that, for such functions, there exists a ray from 0 on which $f(z) \rightarrow 0$ as $z \rightarrow \infty$. This result, which is of independent interest, completes the proof of Theorem 1.1.

The genus of a function is defined as follows. Recall that any finite order entire function has a Hadamard representation

$$(1.4) \quad f(z) = z^n e^{Q(z)} \prod_{k=1}^{\infty} E(z/a_k, m),$$

where Q is a polynomial, the a_k are the non-zero zeros of f , the Weierstrass primary factors are

$$E(z, 0) = 1 - z \quad \text{and} \quad E(z, m) = (1 - z)e^{z + \frac{z^2}{2} + \dots + \frac{z^m}{m}}, \quad m \geq 1,$$

and m is the smallest integer for which $\sum_{k=1}^{\infty} |a_k|^{-(m+1)}$ is convergent. The *genus* of f is

$$(1.5) \quad \max\{m, \deg Q\}.$$

The genus is thus an integer and it satisfies the inequalities

$$\rho(f) - 1 \leq \text{genus of } f \leq \rho(f);$$

see [8, pp. 24–29] or [27, pp. 250–253], for example.

In Section 5, we study real entire functions with only real zeros, and of genus less than 2 and order at least $1/2$. We construct examples of such functions for which property (1.3) holds, so Theorem 1.1 applies, and examples for which it does not hold.

A summary of our results relating the property (1.3) to the growth of real transcendental entire functions with only real zeros is given in the following theorem.

Theorem 1.2. (a) *Let f be a real transcendental entire function of finite order with only real zeros.*

- (i) *If $0 \leq \rho(f) < 1/2$, then property (1.3) always holds.*
- (ii) *If the genus of f is at least 2 (which includes the case that $\rho(f) > 2$), then property (1.3) never holds.*
- (b) *For any $\rho \in [1/2, 2]$ there are examples of real transcendental entire functions with real zeros that have order ρ for which property (1.3) holds, and also examples of such functions for which (1.3) does not hold.*

Remark In relation to Theorem 1.2 (a)(ii), note that any transcendental entire function f of at least order 2 mean type has genus at least 2; see [8, proof of Theorem 1.11], for example.

2. SPIDERS' WEBS IN $I(f)$

We begin our proof of Theorem 1.1 by letting f be a real transcendental entire function of genus at most 1 with only real zeros for which there exists $r > 0$ such that $m^n(r) \rightarrow \infty$ as $n \rightarrow \infty$. We show that, for such a function, $I(f)$ is a spider's web. In fact we prove the following stronger result.

Theorem 2.1. *Let f be a real transcendental entire function of genus at most 1 with only real zeros and suppose that there exists $r > 0$ such that*

$$(2.1) \quad m^n(r) \text{ is a strictly increasing sequence with } m^n(r) \rightarrow \infty \text{ as } n \rightarrow \infty, \\ \text{and } m(r)^{1/3} \geq |f(0)|. \text{ Then } I(f) \text{ contains a spider's web in}$$

$$\{z : |f^n(z)| \geq M^{-1}(m^n(r)^{1/3}) \text{ for all } n \in \mathbb{N}\}.$$

This result is sufficient to show that Theorem 1.1 holds for functions of genus at most 1. Indeed, whenever $I(f)$ contains a spider's web, $I(f)$ must be a spider's web, by [25, Lemma 4.5], and whenever (1.3) holds there exists $r > 0$ such that (2.1) holds, by [16, Theorem 2.1], which is stated as Lemma 3.1 in this paper.

The proof of Theorem 2.1 uses the sequence of closed sets

$$I_N = \{z : |f^n(z)| \geq M^{-1}(m^{n+N}(r)^{1/3}) \text{ for all } n \in \mathbb{N}\}, \quad N = 0, 1, 2, \dots,$$

and we show that I_N contains a spider's web, for all $N = 0, 1, \dots$, using proof by contradiction. The basic idea of the proof is that if I_N does *not* contain a spider's web for some N , then we can take a suitably long curve γ_0 in its complement. We then show that successive images of this curve must either experience repeated radial stretching escaping to infinity, or they must eventually wind round the origin meeting a component of the fast escaping set. In either case we are able to deduce that the curve γ_0 contains a point in I_N which is a contradiction.

The proof requires several results from earlier papers. To state these, we need the following notation. For $r > 0$, we write $C(r) = \{z : |z| = r\}$ and, for $0 < r_1 < r_2$, we write

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\} \quad \text{and} \quad \bar{A}(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}.$$

If γ is a plane curve that lies in a simply connected domain containing no zeros of f , then, for any pair of distinct points z_0, z'_0 in γ , we denote the net change in the argument of $f(z)$ as z traverses γ from z_0 to z'_0 by $\Delta \arg(f(\gamma); z_0, z'_0)$.

The main tool that we use to obtain winding is Theorem 2.2 below. This was originally stated for a continuum [15, Theorem 2.1] but here we only need to apply the result to a curve.

Also, although our original theorem was stated for certain entire functions of order less than 2, the proof only required f to be in the Laguerre–Pólya class (see [15, discussion in Section 5]). This class is the closure of the set of real polynomials with real zeros and, by the Laguerre–Pólya theorem ([11] and [18]), it consists of functions of the form

$$f(z) = \pm z^n e^{b_2 z^2 + b_1 z + b_0} \prod_{k=1}^{\infty} E(z/a_k, m),$$

where the a_k are real, $m = 0$ or 1 , $b_0, b_1 \in \mathbb{R}$ and $b_2 \leq 0$. In particular, all real entire functions with real zeros and genus at most 1 belong to this class.

Theorem 2.2. *Let f be a real transcendental entire function of genus at most 1 with only real zeros. There exists $R_0 = R_0(f) > 0$ such that, if s and a are positive real numbers with*

$$(2.2) \quad s \geq R_0 \quad \text{and} \quad \log s \geq \frac{64}{a^2} + \frac{80\pi}{a}$$

and γ is a curve in $\{z : \operatorname{Im} z \geq 0\}$ that meets both $C(s)$ and $C(s^{1+a})$ with

$$(2.3) \quad 1/M(s) \leq |f(z)| \leq M(s), \quad \text{for } z \in \gamma,$$

then there exist a curve $\Gamma \subset \gamma \cap \overline{A}(s, s^{1+a})$ and $z_0, z'_0 \in \Gamma$ such that

$$\Delta \arg(f(\Gamma); z_0, z'_0) \geq \frac{1}{10\pi} \log M(s) \log s^a.$$

We need the following corollary of Theorem 2.2.

Corollary 2.3. *Let f be a real transcendental entire function of genus at most 1 with only real zeros. There exists $R_1 = R_1(f) \geq R_0(f)$ such that, if t and ε are positive real numbers with*

$$(2.4) \quad t^{1-\varepsilon} \geq R_1 \quad \text{and} \quad \varepsilon \in [10/\sqrt{\log t}, 1),$$

and γ is a curve in $\{z : \operatorname{Im} z \geq 0\}$ that meets both $C(t^{1-\varepsilon})$ and $C(t)$ with

$$(2.5) \quad 1/M(t^{1-\varepsilon}) \leq |f(z)| \leq M(t^{1-\varepsilon}), \quad \text{for } z \in \gamma,$$

then there exist a curve $\Gamma \subset \gamma \cap \overline{A}(t^{1-\varepsilon}, t)$ and $z_0, z'_0 \in \Gamma$ such that

$$\Delta \arg(f(\Gamma); z_0, z'_0) \geq 2\pi.$$

Proof. We claim that we can apply Theorem 2.2 to the curve γ with $s = t^{1-\varepsilon}$ and $s^{1+a} = t$. It follows from (2.5) that (2.3) holds. To show that (2.2) holds, we note that $a = \varepsilon/(1-\varepsilon)$ and so, by (2.4),

$$\begin{aligned} \frac{64}{a^2} + \frac{80\pi}{a} &= \frac{64(1-\varepsilon)^2}{\varepsilon^2} + \frac{80\pi(1-\varepsilon)}{\varepsilon} \\ &< \left(\frac{64}{100} \log t + \frac{80\pi}{10} \sqrt{\log t} \right) (1-\varepsilon) \\ &= \left(\frac{64}{100} + \frac{8\pi}{\sqrt{\log t}} \right) (1-\varepsilon) \log t \\ &< (1-\varepsilon) \log t = \log s, \end{aligned}$$

provided $\sqrt{\log t} > 200\pi/9$. Thus (2.2) holds, provided R_1 is sufficiently large.

So it follows from Theorem 2.2 that there exist a curve

$$\Gamma \subset \gamma \cap \overline{A}(s, s^{1+a}) = \gamma \cap \overline{A}(t^{1-\varepsilon}, t)$$

and $z_0, z'_0 \in \Gamma$ such that

$$\begin{aligned} \Delta \arg(f(\Gamma); z_0, z'_0) &\geq \frac{1}{10\pi} \log M(t^{1-\varepsilon}) \log t^\varepsilon \\ &\geq \frac{1}{10\pi} \log M(t^{1-\varepsilon}) \log(10\sqrt{\log t}) \\ &\geq \frac{1}{10\pi} \log(10\sqrt{\log t}) \geq 2\pi, \end{aligned}$$

by (2.4), provided R_1 is sufficiently large. $\square$

We also use the following result about the fast escaping set, proved in [25, Lemma 4.4].

Lemma 2.4. *Let f be a transcendental entire function. There exists $R_2 = R_2(f) > 0$ such that if $R \geq R_2$, then there is a component of $A_{R/2}(f)$ that meets $\{z : |z| < R\}$ and is unbounded.*

The next result we need concerns uniform rates of escape of quite a general form. This result was proved in [6, Theorem 1.4]. Note that, although the statement of [6, Theorem 1.4] assumes that the sequence (a_n) satisfies $a_{n+1} \leq M(a_n)$, for $n \in \mathbb{N}$, the proof there only uses the consequence of this assumption that $a_n \leq M^n(R)$ for $n \in \mathbb{N}$ and some $R > 0$, and we now state the result in that more general form.

Lemma 2.5. *Let f be a transcendental entire function and let (a_n) be a positive increasing sequence with $a_n \rightarrow \infty$ as $n \rightarrow \infty$, $a_n \leq M^n(R)$ for $n \in \mathbb{N}$ and some $R > 0$, and a_1 sufficiently large that the disc $D(0, a_1)$ contains a periodic cycle of f . Let*

$$I(f, (a_n)) = \{z : |f^n(z)| \geq a_n \text{ for all } n \in \mathbb{N}\}.$$

If $I(f, (a_n))^c$ has a bounded component, then $I(f, (a_n))$ contains a spider's web.

We also use the following result on the convexity of the maximum modulus function; see [21, Lemma 2.2].

Lemma 2.6. *Let f be a transcendental entire function. There exists $R_3 = R_3(f) > 0$ such that if $r > R_3$ and $c > 1$ then*

$$M(r^c) \geq M(r)^c.$$

Finally, we use the following technical lemma.

Lemma 2.7. *Let r_n be a sequence satisfying*

$$(2.6) \quad r_0 \geq \exp(1600) \quad \text{and} \quad r_{n+1} \geq r_n, \quad \text{for } n \geq 0.$$

Suppose further that there exists a subsequence r_{n_k} such that

$$(2.7) \quad r_{n_{k+1}} \geq r_{n_k}^{16}, \quad \text{for } k \in \mathbb{N}.$$

Now let (L_n) be a sequence such that

$$(2.8) \quad L_0 = 3, \quad L_{n_{k+1}} = L_{n_k}(1 - \delta_{n_k}), \quad \text{for } k \in \mathbb{N}, \quad \text{and } L_n = 3 \text{ otherwise,}$$

where

$$(2.9) \quad \delta_{n_k} = 10/\sqrt{\log r_{n_k}}, \quad \text{for } k \in \mathbb{N}.$$

Then $L_n \geq 2$ for all $n \in \mathbb{N}$.

Proof. It follows from (2.8) that

$$(2.10) \quad L_n \geq 3 \prod_{k \in \mathbb{N}} (1 - \delta_{n_k}), \quad \text{for } n \in \mathbb{N}.$$

Also, it follows from (2.6) and (2.7) that

$$\log r_{n_k} \geq 16^{k-1} \log r_{n_1} \geq 16^{k-1} \log r_0 \geq 100 \times 16^k, \quad \text{for } k \in \mathbb{N}.$$

Together with (2.9), this implies that

$$(2.11) \quad \delta_{n_k} \leq \frac{10}{\sqrt{100 \times 16^k}} = \frac{1}{4^k}, \quad \text{for } k \in \mathbb{N}.$$

It follows from (2.10) and (2.11) that

$$L_n \geq 3 \prod_{k \in \mathbb{N}} \left(1 - \frac{1}{4^k}\right) \geq 3 \left(1 - \sum_{k \in \mathbb{N}} \frac{1}{4^k}\right) = 2, \quad \text{for } n \in \mathbb{N},$$

as claimed. □

We now prove the main result of this section.

Proof of Theorem 2.1. First recall that $r > 0$ satisfies (2.1) and $m(r)^{1/3} \geq |f(0)|$.

As stated earlier, we shall show that, under the given hypotheses, all the sets

$$(2.12) \quad I_N = \{z : |f^n(z)| \geq M^{-1}(m^{n+N}(r)^{1/3}) \text{ for all } n \in \mathbb{N}\}, \quad N = 0, 1, 2, \dots,$$

contain a spider's web, which is sufficient to prove Theorem 2.1.

We assume then that there exists some positive integer N_0 such that the set I_{N_0} does *not* contain a spider's web, and show that this assumption gives a contradiction. Since, by (2.1), the sets I_N have the property that

$$(2.13) \quad I_{N_2} \subset I_{N_1}, \quad \text{for } N_2 > N_1 \geq 0,$$

our assumption implies that, for all $N \geq N_0$, the set I_N does not contain a spider's web.

We now choose $N \geq N_0$ so large that

$$(2.14) \quad \text{the disc } D(0, M^{-1}(m^{N+1}(r)^{1/3})) \text{ contains a periodic cycle of } f,$$

$$(2.15) \quad m^{N+2}(r)^{1/3} \geq \max\{R_1, R_3, M(R_2), M(1), \exp(1600)\}$$

and

$$(2.16) \quad M(s) \geq s^{32}, \quad \text{for } s \geq (m^{N+2}(r))^{1/2}.$$

Next we choose R so large that if $a_n = M^{-1}(m^{n+N}(r)^{1/3})$, then $a_n \leq M^n(R)$ for $n \in \mathbb{N}$. Indeed, if we choose $R \geq m^{N-1}(r)$, then for all $n \in \mathbb{N}$ we have

$$m^{n+N}(r) = m^{n+1}(m^{N-1}(r)) \leq M^{n+1}(R), \quad \text{so } a_n \leq M^n(R).$$

Thus such a choice of R is possible. Also, by (2.14), the disc $D(0, a_1)$ contains a periodic cycle of f .

Then the hypotheses of Lemma 2.5 are satisfied for this sequence (a_n) . Therefore, since I_N does not contain a spider's web, I_N^c must have at least one unbounded component and hence there must be an unbounded curve $\Gamma_0 \subset I_N^c$.

Next note that if we increase N , then (2.14)–(2.16) remain true, as does the statement that $\Gamma_0 \subset I_N^c$ by (2.13).

We now increase N if necessary to ensure that the curve Γ_0 meets both the circles $C(M^{-1}(m^{N+2}(r)^{1/3}))$ and $C(m^{N+2}(r))$. We can then choose a subcurve γ_0 of Γ_0 such that

$$\gamma_0 \subset \overline{A}(M^{-1}(m^{N+2}(r)^{1/3}), m^{N+2}(r))$$

and

$$\gamma_0 \text{ meets both } C(M^{-1}(m^{N+2}(r)^{1/3})) \text{ and } C(m^{N+2}(r)).$$

The idea of the proof is to show that if such a curve γ_0 exists, then we can construct a sequence of curves $\gamma_n \subset f^n(\gamma_0)$ and positive sequences (r_n) and (L_n) such that, for $n \geq 0$, we have

$$(2.17) \quad f(\gamma_n) \supset \gamma_{n+1},$$

$$(2.18) \quad \gamma_n \subset \overline{A}(M^{-1}(r_n^{1/L_n}), r_n), \text{ and } \gamma_n \text{ meets both } C(M^{-1}(r_n^{1/L_n})) \text{ and } C(r_n),$$

where

$$(2.19) \quad r_n \geq m^{N+n+2}(r) \quad \text{and} \quad 2 \leq L_n \leq 3,$$

and also

$$(2.20) \quad L_{n+1} = 3 \quad \text{or} \quad L_{n+1} \geq L_n(1 - 10/\sqrt{\log r_n}) \quad \text{and} \quad r_{n+1} \geq M(r_n^{1/2}) \geq r_n^{16}.$$

(These conditions formalise what we mean by saying that the images of the curve γ_0 experience ‘repeated radial stretching’.)

We can then deduce from (2.17), (2.18), (2.19), and (2.20), that there is a point $z_0 \in \gamma_0$ such that, for $n \geq 0$,

$$(2.21) \quad f^n(z_0) \in \gamma_n, \quad \text{so} \quad |f^n(z_0)| \geq M^{-1}(r_n^{1/3}).$$

By (2.12) and (2.19), this implies that $z_0 \in I_N$ which is a contradiction since $z_0 \in \gamma_0 \subset I_N^c$.

To construct the sequences (L_n) , (r_n) and (γ_n) , we proceed as follows. Start by putting $r_0 = m^{N+2}(r)$, so γ_0 , r_0 and $L_0 = 3$ have the required properties. Next, suppose that, for $k = 0, 1, \dots, n$, we have chosen curves γ_k and positive numbers r_k and L_k such that (2.18) and (2.19) are satisfied with n replaced by k and, in addition, (2.17) and (2.20) hold, with n replaced by k for $k = 0, 1, \dots, n-1$. We then show that we can choose a curve γ_{n+1} and positive numbers r_{n+1} and L_{n+1} so that (2.17) and (2.20) hold, and (2.18) and (2.19) hold with n replaced by $n+1$.

We begin by putting

$$(2.22) \quad r_{n+1} = \max_{z \in \gamma_n} |f(z)|$$

and let q be the largest integer such that $m^q(r) \leq r_n$. It follows from (2.19) that $q \geq N + n + 2$. Also, it follows from (2.18) that γ_n must meet $C(m^q(r))$, since

$$m^q(r) \geq M^{-1}(m^{q+1}(r)) > M^{-1}(r_n) \geq M^{-1}(r_n^{1/L_n}).$$

Thus

$$(2.23) \quad r_{n+1} \geq m^{q+1}(r) \geq m^{N+n+3}(r),$$

so r_{n+1} satisfies the first inequality in (2.19).

Next we suppose that there exists $z \in \gamma_n$ with $|f(z)| \leq M^{-1}(r_{n+1}^{1/3})$. Then it is clear that there exists a curve γ_{n+1} having the properties (2.17)–(2.20), with $L_{n+1} = 3$.

If such a z does not exist, then we must have

$$(2.24) \quad |f(z)| > M^{-1}(r_{n+1}^{1/3}), \quad \text{for } z \in \gamma_n.$$

We now put

$$(2.25) \quad \varepsilon_n = 10/\sqrt{\log r_n}$$

and note that, by (2.15) and (2.19), we have

$$(2.26) \quad \varepsilon_n \leq 1/2.$$

We consider two cases. First suppose that

$$(2.27) \quad r_{n+1} \geq M(r_n^{1-\varepsilon_n}).$$

In this case we have $r_{n+1} > M(r_n^{1/2}) \geq r_n^{16}$, by (2.26), (2.19) and (2.16). We put

$$(2.28) \quad L_{n+1} = L_n(1 - \varepsilon_n)$$

and note that r_{n+1} and L_{n+1} satisfy (2.20). Also, by (2.15), it follows from Lemma 2.7 that $L_{n+1} \geq 2$. Thus L_{n+1} satisfies the second inequality in (2.19).

We can now show that there exists a curve γ_{n+1} with the required properties. First, it follows from (2.18) that there exists $z \in \gamma_n$ with

$$(2.29) \quad |f(z)| \leq r_n^{1/L_n}.$$

Also, it follows from (2.27), Lemma 2.6 (in view of (2.15) and (2.19)), and (2.28) that

$$(2.30) \quad r_{n+1} \geq M(r_n^{1-\varepsilon_n}) \geq M(r_n^{1/L_n})^{L_n(1-\varepsilon_n)} = M(r_n^{1/L_n})^{L_{n+1}},$$

so $r_n^{1/L_n} \leq M^{-1}(r_{n+1}^{1/L_{n+1}})$. Together with (2.29) and (2.22), this is sufficient to show that there exists a curve γ_{n+1} satisfying (2.17) and (2.18) with n replaced by $n+1$.

It remains to consider the case when

$$(2.31) \quad r_{n+1} < M(r_n^{1-\varepsilon_n}).$$

We show that this leads to a contradiction and so cannot occur. A key fact needed to obtain this contradiction is that f has the symmetry property

$$(2.32) \quad f(\bar{z}) = \overline{f(z)}, \quad \text{for } z \in \mathbb{C}.$$

We will obtain a contradiction by applying Corollary 2.3, with

$$t = r_n \quad \text{and} \quad \varepsilon = \varepsilon_n,$$

to a curve γ'_n meeting $C(r_n^{1-\varepsilon_n})$ and $C(r_n)$, chosen such that $\gamma'_n \subset \{z : \operatorname{Im} z \geq 0\}$ and

$$(2.33) \quad \gamma'_n \subset \gamma_n \cup \gamma_n^*,$$

where $*$ denotes reflection in the real axis.

We check that the hypotheses of Corollary 2.3 are satisfied. First we have

$$r_n^{1-\varepsilon_n} \geq r_n^{1/2} \geq R_1,$$

by (2.15) and (2.19). Next, it follows from (2.22), (2.31), (2.32) and (2.33) that

$$|f(z)| \leq M(r_n^{1-\varepsilon_n}), \quad \text{for } z \in \gamma'_n,$$

and, from (2.24), (2.32), (2.33), and also (2.23) and (2.15), that

$$(2.34) \quad |f(z)| \geq M^{-1}(r_{n+1}^{1/3}) > 1 > 1/M(r_n^{1-\varepsilon_n}), \quad \text{for } z \in \gamma'_n.$$

Therefore, by Corollary 2.3, there exists a curve $\Gamma \subset \gamma'_n$ and points $z_0, z'_0 \in \Gamma$ and such that

$$(2.35) \quad \Delta \arg(f(\Gamma); z_0, z'_0) \geq 2\pi.$$

We also have, from (2.34), that

$$|f(z)| \geq M^{-1}(r_{n+1}^{1/3}), \quad \text{for } z \in \Gamma.$$

Together with (2.35), (2.15), (2.32) and Lemma 2.4, this implies that

$$f(\Gamma) \cap A_{M^{-1}(r_{n+1}^{1/3})/2}(f) \neq \emptyset.$$

So, by (2.32) and (2.33), there exists $z_n \in \gamma_n$ such that

$$(2.36) \quad |f(z_n)| \geq M^{-1}(r_{n+1}^{1/3}) \quad \text{and} \quad f(z_n) \in A_{M^{-1}(r_{n+1}^{1/3})/2}(f).$$

From the construction of the curves γ_k , for $1 \leq k \leq n$, it follows that there exists $z_0 \in \gamma_0$ such that $f^k(z_0) \in \gamma_k$, for $0 \leq k \leq n$, and $f^n(z_0) = z_n$. Therefore, by (2.18), (2.19) and (2.23), we have

$$|f^k(z_0)| \geq M^{-1}(r_k^{1/3}) \geq M^{-1}(m^{N+k+2}(r)^{1/3}), \quad \text{for } 0 \leq k \leq n+1,$$

and, by (2.36), (1.1), (2.16) and then (2.23), we have

$$|f^{n+2}(z_0)| = |f^2(z_n)| \geq M(M^{-1}(r_{n+1}^{1/3})/2) \geq M^{-1}(r_{n+1}^{1/3}) \geq M^{-1}(m^{N+n+3}(r)^{1/3}).$$

By similar reasoning, we have, for $k > n+2$,

$$\begin{aligned} |f^k(z_0)| &= |f^{k-n-1}(f(z_n))| \geq M^{k-n-1}(M^{-1}(r_{n+1}^{1/3})/2) \geq M^{n+k-2}(M^{-1}(r_{n+1}^{1/3})) \\ &\geq M^{k-n-2}(M^{-1}(m^{N+n+3}(r)^{1/3})) = M^{-1}(M^{k-n-2}(m^{N+n+3}(r)^{1/3})), \end{aligned}$$

and hence, by (2.16),

$$|f^k(z_0)| \geq M^{-1}(M^{k-n-3}(m^{N+n+3}(r))) \geq M^{-1}(m^{N+k}(r)) > M^{-1}(m^{N+k}(r)^{1/3}).$$

Together these three estimates show that $z_0 \in I_N$, which is a contradiction since $z_0 \in \gamma_0 \subset I_N^c$. It follows that the case considered in (2.31) cannot occur and this completes the proof. $\square$

3. SPIDERS' WEBS IN $V(f)$ AND $Q(f)$

In this section we prove two results which follow from Theorem 2.1 and which show that, for many classes of real transcendental entire functions with only real zeros, the sets $V(f)$ and $Q(f)$ each contain a spider's web. These subsets of the escaping set were considered in earlier papers in connection with Eremenko's conjecture and a conjecture of Baker. In particular, we show here that for all real entire functions of order less than $1/2$ with only real zeros, the quite fast escaping set $Q(f)$ contains a spider's web.

The set $V(f)$, introduced in [16], is the set of points whose iterates grow at least as fast as the iterated minimum modulus; that is,

$$V(f) = \{z : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq \tilde{m}^n(R) \text{ for all } n \in \mathbb{N}\},$$

where

$$\tilde{m}(r) = \max_{0 \leq s \leq r} m(s), \quad r > 0,$$

and $R > 0$ is any number such that $\tilde{m}(r) > r$ for $r \geq R$. The existence of such an R follows from (1.3) by a result about escaping points of real functions [16, Theorem 2.1], which we have already used at the start of Section 2 and which we state here in full for the reader's convenience.

Lemma 3.1. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be continuous and put $\tilde{\varphi}(t) = \max_{0 \leq s \leq t} \varphi(s)$, $t > 0$. Then the following statements are equivalent.*

- (a) *There exists $t > 0$ such that $\varphi^n(t) \rightarrow \infty$ as $n \rightarrow \infty$.*
- (b) *There exists $t' > 0$ such that the set $\{\varphi^n(t') : n \in \mathbb{N}_0\}$ is unbounded.*
- (c) *There exists $T > 0$ such that $\tilde{\varphi}(t) > t$, for $t \geq T$ (or equivalently there exists $t > 0$ such that $\tilde{\varphi}^n(t) \rightarrow \infty$ as $n \rightarrow \infty$).*

(d) There exist $t \geq T > 0$ such that

$$\varphi^n(t) \text{ and } \tilde{\varphi}^n(T) \text{ increase strictly with } n \text{ to } \infty,$$

and

$$\varphi^n(t) \in [\tilde{\varphi}^n(T), \tilde{\varphi}^{n+1}(T)], \quad \text{for } n \in \mathbb{N}_0.$$

(e) There exists a sequence (t_n) of positive real numbers such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\varphi(t_n) \geq t_{n+1}, \quad \text{for } n \in \mathbb{N}_0.$$

By the equivalence of parts (a) and (c) of Lemma 3.1, property (1.3) holds if and only if there exists $R > 0$ such that $\tilde{m}(r) > r$ for $r \geq R$. Moreover, in this situation there must exist $r > 0$ such that $m^n(r) > \tilde{m}^n(R)$ for $n \in \mathbb{N}$ by part (d).

As mentioned in the introduction, in [16] it is shown that there are large classes of functions for which there exist $r > R > 0$ such that

$$m^n(r) > M^n(R) \text{ and } M^n(R) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

For such functions, $V(f)$ is equal to the fast escaping set $A(f)$ and this set is a spider's web. It is known, however, that this equality is not true in general for functions such that (1.3) holds, even for functions of order less than $1/2$. In [16] it was asked whether $V(f)$ contains a spider's web for all functions such that (1.3) holds.

As a consequence of Theorem 2.1, we have the following partial result in this direction.

Theorem 3.2. *Let f be a real transcendental entire function of genus at most 1 with only real zeros. If there exist $R > 0$ and $p \in \mathbb{N}$ such that*

$$(3.1) \quad \tilde{m}^p(s) \geq M(s), \quad \text{for } s \geq R,$$

then $V(f)$ contains a spider's web.

Proof. First, it follows from (3.1) that we can assume that $\tilde{m}^n(R) \rightarrow \infty$ as $n \rightarrow \infty$. Hence, by the equivalence of parts (c) and (d) of Lemma 3.1, there exists $r \geq R$ such that $m^n(r)$ is strictly increasing and $m^n(r) \geq \tilde{m}^n(R)$ for $n \in \mathbb{N}$.

If we now take $N \geq 2p+2$, sufficiently large that $M(r) \geq r^3$ for $r \geq M(\tilde{m}^{N-2p}(R))$, then, for $n \in \mathbb{N}$, we have

$$\begin{aligned} M^{-1}(m^{n+N}(r)^{1/3}) &\geq M^{-1}(\tilde{m}^{n+N}(R)^{1/3}) \\ &\geq M^{-1}((M^2(\tilde{m}^{n+N-2p}(R)))^{1/3}) \\ &\geq M^{-1}(M(\tilde{m}^{n+N-2p}(R))) \\ &= \tilde{m}^{n+N-2p}(R). \end{aligned}$$

So it follows from Theorem 2.1 that $V(f)$ contains a spider's web. $\square$

We end this section by showing that, in the special case that f is a real transcendental entire function with only real zeros and order less than $1/2$, Theorem 3.2 together with the $\cos \pi \rho$ theorem imply that the *quite fast escaping set* $Q(f)$ contains a spider's web. Recall that, for each $\varepsilon > 0$, we define

$$Q_\varepsilon(f) = \{z : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq \mu_\varepsilon^n(R) \text{ for all } n \in \mathbb{N}\},$$

where

$$\mu_\varepsilon(r) = M(r)^\varepsilon, \quad r > 0,$$

and $R > 0$ is so large that $\mu_\varepsilon(r) > r$, for $r \geq R$, and then put

$$Q(f) = \bigcup_{\varepsilon \in (0,1)} Q_\varepsilon(f).$$

In an earlier paper [25], we saw other families of functions of order less than $1/2$ for which $Q(f)$ contains a spider's web. For many of these, $Q(f) = A(f)$ (see [23]), but there are examples for which $Q(f)$ contains $A(f)$ strictly (see [24]).

We know of no examples of functions of order less than $1/2$ for which $Q(f)$ does not contain a spider's web, and we make the following conjecture.

Conjecture 3.3. *If f is a transcendental entire function of order less than $1/2$, then the quite fast escaping set $Q(f)$ contains a spider's web, from which it follows that $I(f)$ is a spider's web and hence is connected.*

We have the following partial result in this direction.

Theorem 3.4. *Let f be a transcendental entire function of order less than $1/2$. Then $V(f) \subset Q(f)$. If, in addition, f is real with only real zeros, then $Q(f)$ contains a spider's web.*

Proof. Let f be a transcendental entire function of order less than $1/2$. It follows from the $\cos \pi \rho$ theorem (see [1], [2] or [9, page 331]) that there exists $\varepsilon \in (0, 1)$ such that, for sufficiently large $r > 0$,

$$\text{there exists } s \in (r^\varepsilon, r) \text{ such that } m(s) \geq M(r^\varepsilon).$$

Therefore, for sufficiently large $r > 0$, we have

$$(3.2) \quad \tilde{m}(r) \geq M(r^\varepsilon) \text{ and hence } \tilde{m}^n(r) \geq (\mu_\varepsilon^n(r^\varepsilon))^{1/\varepsilon} \geq \mu_\varepsilon^n(r^\varepsilon), \quad \text{for } n \in \mathbb{N}.$$

Hence $V(f) \subset Q(f)$. The fact that $Q(f)$ contains a spider's web now follows from Theorem 3.2, by using (3.2) with $n = 2$ and Lemma 2.6. $\square$

4. FUNCTIONS OF GENUS AT LEAST 2

In this section we complete the proof of Theorem 1.1 by showing that if f is a real transcendental entire function of finite order with only real zeros and genus at least 2, then its minimum modulus satisfies

$$m(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

so f does not satisfy (1.3). Actually, we prove the following much stronger result.

Theorem 4.1. *Let f be a transcendental entire function of finite order with only real zeros and genus at least 2. Then*

(a) *there exists $\theta \in [0, 2\pi]$ such that*

$$f(re^{i\theta}) \rightarrow 0 \quad \text{as } r \rightarrow \infty;$$

(b) *0 is a deficient value of f .*

Either (a) or (b) implies that $m(r) \rightarrow 0$ as $r \rightarrow \infty$, so f does not satisfy (1.3).

The conclusion that 0 is a deficient value states that the *defect* of f at 0,

$$(4.1) \quad \delta(0, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r)}{T(r)} > 0.$$

Here,

$$N(r) = N(r, 0, f) = \int_0^r \frac{(n(t, f) - n(0, f))}{t} dt,$$

in which $n(t, f)$ is the number of zeros of f in $\{z : |z| \leq t\}$ counted according to multiplicity, and

$$T(r) = N(r) + \frac{1}{2\pi} \int_0^{2\pi} \log^+(1/|f(re^{i\theta})|) d\theta,$$

where $\log^+ t = \max\{\log t, 0\}$. It is clear that if 0 is a deficient value of f , then $m(r) \rightarrow 0$ as $r \rightarrow \infty$, since

$$\liminf_{r \rightarrow \infty} \left(\frac{1}{2\pi T(r)} \int_0^{2\pi} \log^+(1/|f(re^{i\theta})|) d\theta \right) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r)}{T(r)} \right) > 0,$$

so

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+(1/|f(re^{i\theta})|) d\theta \rightarrow \infty \quad \text{as } r \rightarrow \infty.$$

Theorem 4.1 has the following corollary for functions with only non-negative zeros.

Corollary 4.2. *Let f be a transcendental entire function of finite order with only non-negative zeros and genus at least 1. Then*

(a) *there exists $\theta \in [0, 2\pi]$ such that*

$$f(re^{i\theta}) \rightarrow 0 \quad \text{as } r \rightarrow \infty;$$

(b) *0 is a deficient value of f .*

Hence $m(r) \rightarrow 0$ as $r \rightarrow \infty$, so f does not satisfy (1.3).

Proof. The function $g(z) = f(z^2)$ has finite order, its zeros all lie on the real axis, and its genus is at least 2. Hence, by Theorem 4.1, g has limit 0 along a ray to ∞ , and 0 is a deficient value of g . It follows immediately that f has limit 0 along a ray to ∞ , and also that 0 is a deficient value of f , since, for $r \geq t > 0$, we have

$$n(t, g) = 2n(t^2, f), \quad \text{so} \quad N(r, 0, g) = N(r^2, 0, f),$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+(1/|g(re^{i\theta})|) d\theta = 2 \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+(1/|f(r^2 e^{i\theta})|) d\theta \right). \quad \square$$

The proof of Theorem 4.1 uses the following two lemmas. The first, used in the proof of part (b), is a major result of Edrei, Fuchs and Hellerstein [4, Corollary 1.2].

Lemma 4.3. *Suppose that f is an entire function with only real zeros, a_n say, such that*

$$(4.2) \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^2} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^\xi} < \infty,$$

for some $\xi \in (2, \infty)$. Then 0 is a deficient value of f .

The second lemma, used in the proof of part (a), concerns the asymptotic behaviour of Weierstrass primary factors of the form

$$E(z, m) = (1 - z)e^{z + \frac{z^2}{2} + \dots + \frac{z^m}{m}}, \quad m \geq 2.$$

Lemma 4.4. *Given an integer $m \geq 2$, there exists an angle θ such that*

$$(4.3) \quad \begin{aligned} &\bullet \text{ if } m \text{ is even, then } \cos m\theta < 0 \text{ and } \cos(m+1)\theta = 0; \\ &\bullet \text{ if } m \text{ is odd, then } \cos(m-1)\theta < 0, \cos m\theta = 0 \text{ and } \cos(m+1)\theta > 0. \end{aligned}$$

Moreover, for such θ and all $T \in \mathbb{R}$, we have

$$(4.4) \quad |E(Te^{i\theta}, m)| \leq 1,$$

and there exist $C, T_0 > 0$ such that, if $T \in \mathbb{R}$ and $|T| > T_0$, then

$$(4.5) \quad \log |E(Te^{i\theta}, m)| \leq \begin{cases} -C|T|^m, & \text{if } m \text{ is even,} \\ -C|T|^{m-1}, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. First, for $m \geq 2$ and m even, and any angle of the form

$$\theta = \frac{(4k-1)\pi}{2(m+1)}, \quad k = 1, 2, \dots, \frac{1}{2}m,$$

we have

$$(m+1)\theta = 2k\pi - \frac{\pi}{2} \quad \text{and} \quad m\theta = \left(2k\pi - \frac{\pi}{2}\right) \frac{m}{m+1} \in \left(2k\pi - \frac{3\pi}{2}, 2k\pi - \frac{\pi}{2}\right).$$

The even case of (4.3) follows from this, with m choices of the angle $\theta \in [0, 2\pi]$, by adding π to each of the $\frac{1}{2}m$ choices above.

The odd case of (4.3) follows similarly with

$$\theta = \frac{(4k-1)\pi}{2m}, \quad k = 1, 2, \dots, \frac{1}{2}(m-1),$$

giving $m-1$ choices of θ , by adding π to each of these $\frac{1}{2}(m-1)$ choices. It is helpful in the odd case to note that when $\cos m\theta = 0$, the conditions $\cos(m-1)\theta < 0$ and $\cos(m+1)\theta > 0$ are equivalent.

For any integer $m \geq 2$, such an angle θ , and $T \in \mathbb{R}$, we have

$$(4.6) \quad \log |E(Te^{i\theta}, m)| = \operatorname{Re} \left(Te^{i\theta} + \frac{(Te^{i\theta})^2}{2} + \dots + \frac{(Te^{i\theta})^m}{m} \right) + \log |1 - Te^{i\theta}|.$$

Now,

$$\begin{aligned} \frac{d}{dT} \log |E(Te^{i\theta}, m)| &= \operatorname{Re} \left(e^{i\theta} \left(\frac{(Te^{i\theta})^m - 1}{Te^{i\theta} - 1} \right) \right) + \frac{T - \cos \theta}{(1 - T \cos \theta)^2 + T^2 \sin^2 \theta} \\ &= \frac{T^{m+1} \cos m\theta - T^m \cos(m+1)\theta}{(1 - T \cos \theta)^2 + T^2 \sin^2 \theta}. \end{aligned}$$

By our choice of θ , this last expression is positive when $T < 0$ and is negative when $T > 0$. It follows that $\log |E(Te^{i\theta}, m)| \leq \log |E(0, m)| = 0$ for all $T \in \mathbb{R}$.

We also see from (4.6) that, as $|T| \rightarrow \infty$,

$$\log |E(Te^{i\theta}, m)| = \frac{T^m \cos m\theta}{m} + \frac{T^{m-1} \cos(m-1)\theta}{m-1} + O(|T|^{m-2}).$$

The final claim of the lemma follows, again using our choice of θ . $\square$

Proof of Theorem 4.1. We first write f in the form (1.4), where the zeros a_k are real and Q is a polynomial of degree d_Q . The proof when f has a finite number of zeros is a simpler version of the proof when f has infinitely many zeros, so we assume the latter.

We then write

$$P(z) = \prod_{k=1}^{\infty} E(z/a_k, m), \quad \text{so} \quad f(z) = z^n e^{Q(z)} P(z),$$

and we recall that m is the least integer for which $\sum |a_k|^{-(m+1)}$ is convergent. We also recall from (1.5) that the genus of f is $\max\{m, d_Q\}$.

First we prove part (a). The proof splits into two cases depending on whether or not d_Q is greater than m . In some sense, when $m \geq d_Q$ the growth/decay of $P(z)$ dominates that of $e^{Q(z)}$, while the reverse is true when $m < d_Q$.

Suppose first that $m \geq d_Q$. Then $m \geq 2$ by the hypothesis on the genus of f .

Let θ be as in Lemma 4.4. Then there exists $\theta_0 \in \{\theta, \theta + \pi\}$ and $A > 0$ such that for $z_0 = re^{i\theta_0}$ with $r > 1$, we have

$$\log |z_0^n e^{Q(z_0)}| \leq \begin{cases} Ar^m, & \text{if } m \text{ is even,} \\ Ar^{m-1}, & \text{if } m \text{ is odd.} \end{cases}$$

To see that this holds when m is odd, note that $\operatorname{Re}(Q(re^{i\theta_0}))$ is a polynomial in r of degree at most m . If this degree equals m , then $\theta_0 \in \{\theta, \theta + \pi\}$ can be chosen to make the leading coefficient of $\operatorname{Re}(Q(re^{i\theta_0}))$ negative, so in this case $\log |z_0^n e^{Q(z_0)}| < 0$ for large r .

Next consider $\log |P(z_0)|$. Let C and T_0 be as in Lemma 4.4 and write $\alpha = 1/T_0$. By (4.4) and (4.5),

$$\begin{aligned} \log |P(z_0)| &= \sum_{k=1}^{\infty} \log \left| E\left(\frac{re^{i\theta_0}}{a_k}, m\right) \right| \\ &\leq \sum_{|a_k| < \alpha r} \log \left| E\left(\frac{re^{i\theta_0}}{a_k}, m\right) \right| \leq \begin{cases} -Cr^m \sum_{|a_k| < \alpha r} |a_k|^{-m}, & \text{if } m \text{ is even,} \\ -Cr^{m-1} \sum_{|a_k| < \alpha r} |a_k|^{-(m-1)}, & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

Since these two sums diverge as $r \rightarrow \infty$, it follows from the estimates above that

$$\log |f(re^{i\theta_0})| = \log |z_0^n e^{Q(z_0)}| + \log |P(z_0)| \rightarrow -\infty \quad \text{as } r \rightarrow \infty,$$

as required.

Now suppose instead that $d_Q \geq m + 1$. We first observe that

$$(4.7) \quad \log |P(z)| = o(|z|^{m+1}) \quad \text{as } z \rightarrow \infty.$$

The proof of (4.7) depends only on the fact that $\sum |a_k|^{-(m+1)} < \infty$, and not on the fact that these zeros are real. Indeed, this convergence implies (see [8, p.17 and Lemma 1.4]) that $n(r, P)$ is of at most order $m + 1$ convergence class; that is,

$$\int_0^\infty \frac{n(t, P)}{t^{m+2}} dt < \infty, \quad \text{so } n(r, P) = o(r^{m+1}) \text{ as } r \rightarrow \infty.$$

The estimate (4.7) then follows from [8, Theorem 1.11] (see in particular the antepenultimate sentence of its proof) or alternatively from [14, p. 233, (3.3)].

Since Q is a polynomial, there exist $\theta \in [0, 2\pi]$ and $c = c(\theta) > 0$ such that

$$|e^{Q(re^{i\theta})}| < \exp(-cr^{d_Q}), \quad \text{for sufficiently large } r > 0.$$

In fact an estimate of this type holds for all θ in a union of open subintervals of $[0, 2\pi]$ of total length π . We conclude, using (4.7) and the assumption that $d_Q \geq m + 1$, that for such θ we have

$$|f(re^{i\theta})| \leq r^n \exp((-c + o(1))r^{m+1}) \rightarrow 0 \text{ as } r \rightarrow \infty,$$

as required.

The proof of part (b) also involves two cases.

First suppose that $m \geq 2$. Then we can simply apply Lemma 4.3 to deduce that 0 is a deficient value of f .

Next suppose that $m \leq 1$. Then $d_Q \geq 2$ by our hypothesis about the genus, so $d_Q \geq m + 1$. Now we argue as in the proof of (4.7) to deduce that

$$n(r, f) = o(r^{m+1}) \text{ as } r \rightarrow \infty, \quad \text{so } N(r) = o(r^2) \text{ as } r \rightarrow \infty.$$

Since $d_Q \geq 2$, the function f has order at least 2 and, moreover,

$$\liminf_{r \rightarrow \infty} \frac{T(r)}{r^2} > 0.$$

We deduce that $N(r) = o(T(r))$ as $r \rightarrow \infty$, and hence

$$(4.8) \quad \delta(0, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r)}{T(r)} = 1.$$

Therefore 0 is again a deficient value, in this case with defect 1.

This completes the proof of Theorem 4.1. □

We conclude this section by mentioning another family of transcendental entire functions for which we can show that $m(r) \rightarrow 0$ as $r \rightarrow \infty$, and hence that property (1.3) does not hold.

Theorem 4.5. *Let f be a transcendental entire function of infinite order, with zeros a_n , $n \in \mathbb{N}$, such that*

$$(4.9) \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^\xi} < \infty,$$

for some $\xi \in (0, \infty)$. Then $\delta(0, f) = 1$, so $m(r) \rightarrow 0$ as $r \rightarrow \infty$, and hence f does not satisfy (1.3).

Proof. It follows from the argument used to prove (4.7) that $N(r)$ has finite order. On the other hand f has infinite order so we can write

$$f(z) = z^n e^{h(z)} P(z),$$

where $n \geq 0$, h is a transcendental entire function, and P is the Hadamard product associated with f , which has finite order. It follows that f has infinite lower order; see [10, Proof of Theorem 3], for example. Hence (4.8) holds, so $\delta(0, f) = 1$ and hence $m(r) \rightarrow 0$ as $r \rightarrow \infty$ by the reasoning following the statement of Theorem 4.1. $\square$

In view of Theorems 4.1 and 4.5, it is natural to ask whether it is the case that (1.3) fails to hold if f is *any* transcendental entire function of infinite order with only real zeros. For a general transcendental entire function f of infinite order with its zeros lying on a finite number of rays from 0, Miles [13, Theorem 1] has proved that there is a set $E \subset [0, \infty)$ of zero logarithmic density such that

$$\lim_{r \rightarrow \infty, r \notin E} \frac{N(r)}{T(r)} = 0, \quad \text{so} \quad \lim_{r \rightarrow \infty, r \notin E} m(r) = 0.$$

On its own, however, this property is not sufficient to show that (1.3) fails to hold.

5. FUNCTIONS WITH ORDER IN THE INTERVAL $[1/2, 2]$

We have seen that if f is any transcendental entire function of order less than $1/2$, then (1.3) holds, and if f is a finite order function with real zeros of genus at least 2 (which includes the case that $\rho(f) > 2$), then (1.3) does not hold. In this section, we consider entire functions with order in the interval $[1/2, 2]$.

We start by giving examples of real transcendental entire functions with all their zeros on the positive real axis, all of genus 0 and having all possible orders in the interval $[1/2, 1]$. For each order in this interval, we give one example that does satisfy (1.3) and another example that does not.

We can then adapt these examples to construct examples of functions of genus 1 with only real zeros. Indeed, for real transcendental entire functions f with only positive zeros, of genus 0 and order ρ say, the function $g(z) = f(z^2)$ is real with all real zeros, of genus 1 and having order 2ρ , and (1.3) holds for g if and only if it holds for f . Therefore, for each possible order in $[1, 2]$, we obtain one example of this type that does satisfy (1.3) and another example that does not.

First, for order $1/2$ we have the following examples mentioned in [16, Example 8.4]:

- property (1.3) does not hold for the function $f(z) = \cos \sqrt{z}$,
- property (1.3) holds for the functions $g(z) = 2z \cos \sqrt{z}$.

Both these functions are real and have their zeros on the positive real axis, and are of order $1/2$ and genus 0.

Next, we give functions of genus 0 and all orders in $(1/2, 1)$ for which (1.3) does *not* hold. Consider the family of functions

$$(5.1) \quad f_\sigma(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\sigma}\right),$$

where $\sigma = 1/\rho$ and $\frac{1}{2} < \rho < 1$. Then f_σ is a real transcendental entire function of genus 0 and order ρ , with zeros at n^σ , $n \in \mathbb{N}$. Hardy [7] showed that

$$f_\sigma(z) \sim \frac{2}{\sqrt{2\pi z}} \sin(\pi z^\rho) \exp(\pi \cot(\pi \rho) z^\rho),$$

as z tends to ∞ within a domain that contains the positive real axis. In particular, since $\cot(\pi \rho) < 0$ for $\frac{1}{2} < \rho < 1$, we have $m(r, f_\sigma) \rightarrow 0$ as $r \rightarrow \infty$, so (1.3) fails for these functions.

For an example of a real transcendental entire function f of order 1 and genus 0 with only positive zeros for which (1.3) fails, we note that Lindelöf [12] showed that the functions

$$f_\alpha(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n(\log n)^\alpha}\right), \quad \text{where } 1 < \alpha < 2,$$

satisfy

$$f_\alpha(z) = \exp\left(\frac{1+o(1)}{1-\alpha} z(\log(-z))^{1-\alpha}\right),$$

as z tends to ∞ within any set of the form $\{z : |\arg z| < \pi - \delta\}$, $\delta > 0$. It is easy to deduce from this estimate that each such f_α , $1 < \alpha < 2$, is bounded on all rays of the form $\{z : \arg z = \theta\}$, where $\theta \in (0, \pi/2)$. Hence $m(r, f_\alpha)$ is bounded, so (1.3) fails for these functions.

The zeros of all the functions considered so far in this section are distributed very evenly on the positive real axis. With a more uneven distribution of positive zeros, we can construct real transcendental entire functions given by infinite products for which (1.3) *does* hold.

In the proof of the following result we show that (1.3) holds for the constructed function f by again using the fact that (1.3) is equivalent to the existence of $R = R(f) > 0$ such that

$$(5.2) \quad \tilde{m}(r) = \max_{0 \leq s \leq r} m(s) > r, \quad \text{for } r \geq R;$$

see Lemma 3.1.

Theorem 5.1. *Given any ρ , $0 < \rho \leq 1$, we can construct a function of the form*

$$(5.3) \quad f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)^{m_k},$$

where $a_k > 0$ and $m_k \in \mathbb{N}$ for $k \in \mathbb{N}$, such that f has order ρ and genus 0, and (1.3) holds.

Proof. We first assume that $0 < \rho < 1$. We shall construct the sequence of zeros (a_k) to be strictly increasing, with multiplicity (m_k) , and have several other properties. Let $n(r)$ be the number of such zeros in $\{z : |z| \leq r\}$, counted according to multiplicity. One condition we require is that the sequence (m_k) is chosen in such a way that

$$n(a_k) = \sum_{j=1}^k m_j = [a_k^\rho], \quad k \in \mathbb{N},$$

where $[.]$ denotes the integer part function. This ensures that the infinite product in (5.3) is convergent and f has order ρ by [8, Theorem 1.11 and Lemma 1.4].

We now describe how to choose (a_k) . First we estimate $m(r_k) = |f(r_k)|$, where $r_k = 3a_k$. We have

$$(5.4) \quad \left| \prod_{j=1}^k \left(1 - \frac{r_k}{a_j}\right)^{m_j} \right| \geq \prod_{j=1}^k 2^{m_j} = 2^{[a_k^\rho]},$$

and we shall choose the sequence (a_k) so large that

$$(5.5) \quad \left| \prod_{j=k+1}^{\infty} \left(1 - \frac{r_k}{a_j}\right)^{m_j} \right| \geq \frac{1}{2}.$$

This can be achieved by choosing (a_k) such that

$$(5.6) \quad \sum_{j=k+1}^{\infty} \frac{m_j r_k}{a_j} \leq \frac{1}{2}.$$

Since $m_k \leq a_k^\rho$ for $k \in \mathbb{N}$, we have

$$\sum_{j=k+1}^{\infty} \frac{m_j r_k}{a_j} \leq r_k \sum_{j=k+1}^{\infty} a_j^{\rho-1}.$$

If we take

$$(5.7) \quad a_0 = 1, \quad a_{k+1} = (12a_k^\rho)^{1/(1-\rho)}, \quad k = 0, 1, \dots,$$

then

$$a_j \geq (12a_k^\rho)^{(j-k)/(1-\rho)} a_k = T_k^{(j-k)/(1-\rho)} a_k, \quad \text{for } j \geq k+1,$$

say, where $T_k = 12a_k^\rho$. Hence

$$r_k \sum_{j=k+1}^{\infty} a_j^{\rho-1} \leq r_k a_k^{\rho-1} \sum_{j=k+1}^{\infty} \left(\frac{1}{T_k}\right)^{j-k} = \frac{3a_k^\rho}{T_k - 1} \leq \frac{1}{2},$$

which proves (5.6), and also shows that f has genus 0. On combining (5.4) and (5.5), we obtain

$$m(r_k) = |f(r_k)| \geq 2^{[a_k^\rho]-1}, \quad \text{for } k \geq 1.$$

Finally, we prove that (5.2) holds. Given r we choose k such that $r_k \leq r < r_{k+1}$. Then

$$\tilde{m}(r) \geq m(r_k) \geq 2^{[a_k^\rho]-1} > 3(12a_k^\rho)^{1/(1-\rho)} = 3a_{k+1} = r_{k+1} > r,$$

provided that k is sufficiently large, as required.

The proof when $\rho = 1$ is a modification of the argument above in which f remains of the form (5.3) but we take

$$n(a_k) = \sum_{j=1}^k m_j = [a_k^{1-\varepsilon_k}], \quad \text{where } \varepsilon_k = \frac{1}{k}, \quad \text{for } k \in \mathbb{N},$$

which implies that f has order 1, and we replace (5.7) by

$$a_0 = 1, \quad a_{k+1} = (12a_k)^{1/\varepsilon_{k+1}}, \quad k = 0, 1, \dots$$

Then $a_k \geq 12^k$ for $k \in \mathbb{N}$, so

$$a_j^{\varepsilon_j} \geq 12^{j-k} a_k \geq 4^{j-k} r_k, \quad \text{for } j \geq k+1,$$

where $r_k = 3a_k$ as before. It readily follows, by splitting the product as above, that

$$m(r_k) = |f(r_k)| \geq 2^{[a_k^{1-\varepsilon_k}] - 1}, \quad \text{for } k \geq 1.$$

Hence, for $r_k \leq r < r_{k+1}$ and k sufficiently large,

$$\tilde{m}(r) \geq m(r_k) \geq 2^{a_k^{1/2}} > 3(12a_k)^{1/\varepsilon_{k+1}} = 3a_{k+1} = r_{k+1} > r,$$

since $a_k \geq 12^k$ for $k \in \mathbb{N}$, and so (1.3) holds. $\square$

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SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM NG7 2RD, UK

E-mail address: `Dan.Nicks@nottingham.ac.uk`

SCHOOL OF MATHEMATICS AND STATISTICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES MK7 6AA, UK

E-mail address: `Phil.Rippon@open.ac.uk`

SCHOOL OF MATHEMATICS AND STATISTICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES MK7 6AA, UK

E-mail address: `Gwyneth.Stallard@open.ac.uk`