ON SYMMETRIES OF SPHERES IN UNIVALENT FOUNDATIONS

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Abstract. Working in univalent foundations, we investigate the symmetries of spheres, i.e., the types of the form $\mathbb{S}^n = \mathbb{S}^n$. The case of the circle has a slick answer: the symmetries of the circle form two copies of the circle. For higher-dimensional spheres, the type of symmetries has again two connected components, namely the components of the maps of degree plus or minus one. Each of the two components has $\mathbb{Z}/2\mathbb{Z}$ as fundamental group. For the latter result, we develop an EHP long exact sequence.

1. Introduction

Martin-Löf's dependent type theory [22] can serve as a basis for proof assistants and dependently typed programming languages. As pioneered by Voevodsky [32] as well as by Awodey and Warren [2], it allows for homotopy-theoretic semantics. Concretely, types can be interpreted as ∞-groupoids [18], and more generally, as objects in any Grothendieck $(\infty, 1)$-topos [25]. These models justify Voevodsky's *univalence axiom* and admit a range of *higher inductive types*. The field that embraces the view of types as spaces qua homotopy types is known as *homotopy type theory* (HoTT) and the setting as *univalent foundations* (UF) [30].

It turns out that various results that hold for spaces in standard homotopy theory can be stated and proved for types in homotopy type theory. The framework enforces all arguments to be purely axiomatic in nature, and the resulting subfield of homotopy type theory is sometimes called *synthetic homotopy theory*. Examples of results are the type-theoretic Seifert–van Kampen theorem [17], the Blakers–Massey connectivity theorem [16], and the construction of the Hopf fibration [30, Ch. 8.5].

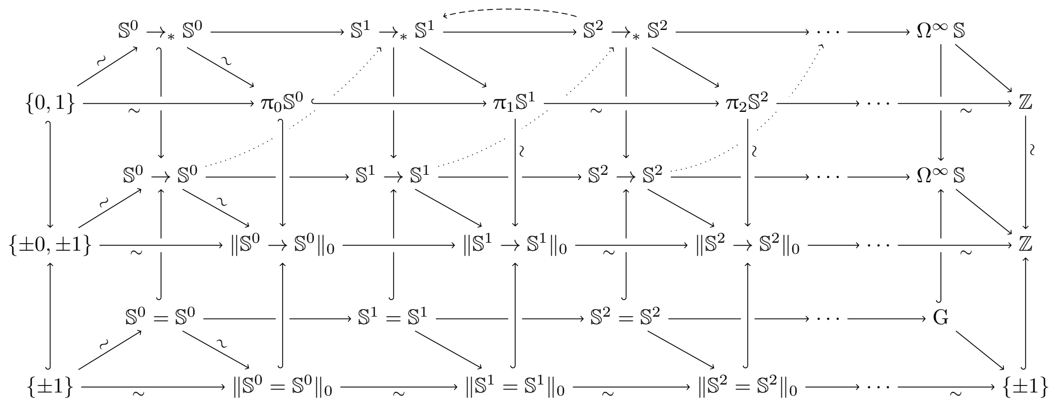
A central type of study in synthetic homotopy theory is the n-dimensional sphere type $\mathbb{S}^n$. The calculation of the fundamental group of the circle $\mathbb{S}^1$ was among the first results in the area [21], and the result was quickly extended to the n^{th} homotopy group of $\mathbb{S}^n$, i.e., to $\pi_n(\mathbb{S}^n)$ [20]. Further homotopy groups of higher spheres have been studied by Brunerie, showing that $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$ [4].

In this paper, we are interested in the type of symmetries, or self-equivalences, of the spheres $\mathbb{S}^n$. By univalence, this type can be written simply as $\mathbb{S}^n = \mathbb{S}^n$. Trivial cases occur for $n = -1$, where $\mathbb{S}^{-1} = \emptyset$, the empty type, so $(\mathbb{S}^{-1} = \mathbb{S}^{-1}) = \mathbf{1}$, and for $n = 0$, where $\mathbb{S}^0 = \mathbf{2}$, the type of booleans, so $(\mathbb{S}^0 = \mathbb{S}^0) = \mathbf{2} = (\mathbf{1} + \mathbf{1})$. For $n = 1$, a relatively simple calculation shows that $(\mathbb{S}^1 = \mathbb{S}^1) = (\mathbb{S}^1 + \mathbb{S}^1)$. For $n \geq 2$, a similarly elegant answer does not seem to be possible. Our main result for this case is that the type $(\mathbb{S}^n = \mathbb{S}^n)$ has two equivalent connected components, each with fundamental group $\mathbb{Z}/2\mathbb{Z}$. Perhaps surprisingly, this turns out to be easier to prove for $n \geq 3$ than for $n = 2$.

Our study is closely related to the calculation of homotopy groups mentioned above. Recall that the n^{th} homotopy group of $\mathbb{S}^n$ is, by definition, the set-truncation of the iterated loop space $\Omega^n(\mathbb{S}^n)$. The latter type is equivalent to $\mathbb{S}^n \to_* \mathbb{S}^n$, the type of *pointed* endofunctions on $\mathbb{S}^n$. In contrast, we study the types of self-equivalences of $\mathbb{S}^n$, not the pointed ones. Many of our arguments use techniques similar to the ones used in the calculation of higher homotopy groups, such as the Hopf fibration [30, Sec. 8.5], or Freudenthal's suspension theorem [30, Thm. 8.6.4]. For the characterisation of the fundamental group of the components of $\mathbb{S}^2 = \mathbb{S}^2$, Brunerie's [4] calculation of $\pi_4(\mathbb{S}^3)$ is of great use.

The type $\mathbb{S}^n = \mathbb{S}^n$ is the type of elements of a (higher) group, viz., the automorphism group of the n-sphere, traditionally denoted $\mathrm{G}(n+1)$. The classifying space $\mathrm{BG}(n+1)$ classifies spherical fibrations

FIGURE 1. Relating pointed endomaps, endomaps, and self-identifications of spheres.



with fiber $\mathbb{S}^n$, and these play an important role in the branch of geometric topology that deals with the homotopy theory of manifolds via surgery theory (see, e.g., Sullivan [28]). The homotopy type of the symmetries of the 2-sphere in classical topology has been determined by Hansen [15], and we discuss this in our conclusions. We refer to Smith [26] for a general survey of the homotopy theory of function types in classical topology.

Setting and assumptions. To be precise and to fix notations, we work inside a version of intuitionistic Martin-Löf type theory with Σ-, Π- and Id-types and with a cumulative hierarchy of universes, simply written $\mathcal{U}$, for which Voevodsky's univalence axiom holds. Our type theory corresponds to the one developed in the HoTT book [30], although we only assume the higher inductive types specified below. The basic concepts introduced in the first four chapters of [30] will mostly be used without further explanation.

We use the same notations as in [30], with the following exceptions. If $p : x = y$ and $q : y = z$ are paths, then we denote their composition by qp, or by $q \cdot p$. The (dependent) application of a function $f : \prod_{x:X} Y(x)$ on paths is denoted by $[f]$.

We assume a type $\mathbb{N}$ of natural numbers with its inductive property (cf. [30, Ch. 1.9]), from which is crafted a type $\mathbb{Z}$ of integers as in [5, `core/lib/types/Int.agda`]. One important property of $\mathbb{Z}$ is that it has decidable equality, as proved in the cited file.

A *pointed type* is a type A with an implicitly or explicitly given point $a_0 : A$. Given such a pointed type, we write $\Omega(A) :\equiv (a_0 = a_0)$ for the *loop space*, which is itself pointed at refl_{a_0}. The *iterated loop space* is given by $\Omega^0(A) :\equiv A$ and $\Omega^{n+1} :\equiv \Omega^n(\Omega(A))$. Note that Ω, and hence Ω^n, will be given the structure of *wild endofunctors* (see Example B.6), which means in particular that they can be applied to functions between pointed types. The universe of pointed types is denoted by $\mathcal{U}_*$, and the forgetful map $\mathcal{U}_* \to \mathcal{U}$ is a silent coercion: given a pointed type A, its underlying unpointed type is still written A. To avoid confusion, throughout this paper, the type $A = B$ will always denote the type of paths from A to B in $\mathcal{U}$ (that is when A and B are considered as unpointed types), while $A =_* B$ will denote the type of paths from A to B in $\mathcal{U}_*$ (that is, when A and B are considered as pointed types).

We further assume the following higher inductive types, referring to [30, Ch. 6] for the details: the circle (denoted by $\mathbb{S}^1$); propositional truncation (denoted by $\|A\|$); set truncation (denoted by $\|A\|_0$); suspension (denoted by ΣA); join (denoted by $A * B$); wedge sum (denoted by $A \vee B$). The latter three are defined as certain pushouts. For the circle we assume the usual definition with a base point $\bullet$ and a loop $\circlearrowleft$, although it can equivalently be described as the suspension of $\mathbf{2}$. (In fact, all the above higher inductive types can be constructed from pushouts alone, see [24].)

The spheres $\mathbb{S}^n$ are then defined by induction on $n : \mathbb{N}$ by $\mathbb{S}^n :\equiv \Sigma(\mathbb{S}^{n-1})$ for all $n \geq 2$. We also set $\mathbb{S}^{-1} :\equiv \emptyset$ and $\mathbb{S}^0 :\equiv \mathbf{2} :\equiv \mathbf{1} + \mathbf{1}$. Then we have $\mathbb{S}^n = \Sigma(\mathbb{S}^{n-1})$ for all $n \geq 0$ (for $n = 1$ this follows from

[30, Lem. 6.5.1]). Another basic fact about spheres is that $\mathbb{S}^n$ is $(n-1)$-connected, for all $n \geq -1$, [30, Cor. 8.2.2]. This implies that $\mathbb{S}^1$ is connected and that $\mathbb{S}^2$ is simply connected, i.e., $\mathbb{S}^2$ and all its path types are connected.

For pointed types we adopt the following conventions. Suspensions are pointed at N. If A and B are pointed types, then $A + B$ is pointed at $\mathrm{inl}(a_0)$, with a_0 the point of A. A similar convention is followed for pushouts, wedges and joins.

A *fiber sequence* (cf. [30, Def. 8.4.3]) is a sequence $F \xrightarrow{\iota}_* X \xrightarrow{p}_* Y$ where F is the fiber of p at the point y_0 of Y and ι is the first projection. The *connected component* of type A at point $a : A$ is $A_{(a)} :\equiv \sum_{x:A} \|a = x\|$.

Throughout this paper we treat univalence as transparent, in the sense that equivalences $f : A \simeq B$ will be treated as paths of type $A = B$ without any warning, and vice versa. (Possible universe level issues can be solved by cumulativity and will be disregarded.) We adopt a similar attitude towards function extensionality. We treat a homotopy $h : \prod_{x:A}(f(x) = g(x))$ as a path of type $f = g$. Conversely, any $p : f = g$ is treated as the induced homotopy, with $p(x)$ (or p_x) denoting the induced path of type $f(x) = g(x)$.

Contributions and overview of the paper. In Section 2 we show that the type $\mathbb{S}^1 = \mathbb{S}^1$ is equivalent to $\mathbb{S}^1 + \mathbb{S}^1$. To do so, we take a detour to univalent group theory and establish a far more general result: Gottlieb's theorem (new in UF). We get $(\mathbb{S}^1 = \mathbb{S}^1) \simeq (\mathbb{S}^1 + \mathbb{S}^1)$ by applying Gottlieb's theorem to the group of integers $\mathbb{Z}$. In Section 3 we deal with $\mathbb{S}^2 = \mathbb{S}^2$. This case is much more difficult than the previous. Using the Hopf fibration as defined in UF in [4], and two definitions of the degree function (a variation of [10]), we prove that the type $\mathbb{S}^2 = \mathbb{S}^2$ has exactly two connected components, equivalent to one another. In Section 4 we prove by induction on $n \geq 2$ that the type $\mathbb{S}^n = \mathbb{S}^n$ has exactly two connected components, equivalent to one another. Each induction step relies on Freudenthal's suspension theorem and on the result that suspension and negation commute (new in UF in 2020). In Section 5 we explain the 3-dimensional Fig. 1, the inventory of the types and maps between them studied so far, and discuss comparisons with other approaches. In Section 6 we prepare the study of the structure of the connected components of $\mathbb{S}^2 = \mathbb{S}^2$ by some results on the generalized Whitehead product (new in UF). We partly develop an EHP long exact sequence (new in UF). In Section 7 we show that the fundamental group of each component of $\mathbb{S}^n = \mathbb{S}^n$ is $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. Final remarks are made in Section 8. Appendix A contains proofs that are left out or only sketched. Appendix B provides the basics of wild categories (including $\mathcal{U}$ and $\mathcal{U}_*$), wild functors (including Ω and Σ), wild adjunctions ($\Sigma \dashv \Omega$), and wild monoids.

2. Symmetries of the circle

In this section, we will prove the following result.

Theorem 2.1. *There is an equivalence*

$$(\mathbb{S}^1 = \mathbb{S}^1) \simeq (\mathbb{S}^1 + \mathbb{S}^1).$$

We will obtain the result as a consequence of Theorem 2.2. In order to state the theorem and prove it, we need a bit of group theory in univalent foundations. For any group G, a delooping of G is a connected pointed 1-type (groupoid) A such that $G = \Omega A$ as groups. Such a delooping always exist, and two deloopings are always equal as pointed types [6]. We usually write BG for such a delooping, with the point denoted by s_G. Given groups G and H, the function Ω is an equivalence from $BG \to_* BH$ to the type of group homomorphisms from G to H. Moreover, this equivalence retricts to an equivalence between $BG \simeq_* BH$ and the type of group isomorphisms from G to H. Recall that for a group G, there is a homomorphism $G \to \mathrm{Aut}(G)$ from G to the group of group automorphisms of G, that sends an element $g : G$ to the conjugation $x \mapsto gxg^{-1}$. The kernel of this homomorphism is called the center of G and is written $Z(G)$. The image of this homomorphism is called the group of inner automorphisms of G and is written $\mathrm{Inn}(G)$. The quotient $\mathrm{Out}(G) :\equiv \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is a set whose elements are called the outer automorphisms of G. It is naturally pointed at the class of the identity automorphism.

Theorem 2.2. *Let G be a group. There is a fiber sequence of the form:*

$$BZ(G) \xrightarrow{\iota}_* (BG = BG) \xrightarrow{p}_* \mathrm{Out}(G)$$

where $BG = BG$ is pointed at refl_{BG}.

This theorem is known in classical homotopy theory, and is attributed to Gottlieb [14].

Proof of Theorem 2.2. For any type A, by pointing $(A = A)$ at refl_A and $\|A = A\|_0$ at $|\mathrm{refl}_A|_0$, there is, by definition of the connected component $(A = A)_{(\mathrm{refl}_A)}$, a fiber sequence

$$(A = A)_{(\mathrm{refl}_A)} \xrightarrow{\iota}_* (A = A) \xrightarrow{|\,|_0}_* \|A = A\|_0.$$

When G is a group, we can apply this fact to the (unpointed) type BG. We will get the result stated in Theorem 2.2 if we can exhibit pointed equivalences $BZ(G) \simeq_* (BG = BG)_{(\mathrm{refl}_{BG})}$ and $\|BG = BG\|_0 \simeq_* \mathrm{Out}\, G$. The former is the content of Lemma 2.3, and the latter that of Lemma 2.4. □

Lemma 2.3. *For any group G there is a pointed equivalence $BZ(G) \simeq_* (BG = BG)_{(\mathrm{refl}_{BG})}$.*

Sketch of proof. Define $z_G : (BG = BG)_{(\mathrm{refl}_{BG})} \to_* BG$ as the restriction of the evaluation $(BG = BG) \to BG$ that maps $x : BG = BG$ to $x(s_G) : BG$. The associated group homomorphism $\Omega\,(z_G)$ is injective and has image $Z(G)$. □

To prove the next lemma, we need to make observations about subgroups and quotients in univalent foundations. The curious reader can refer to [3, Sec. 5.2 and 5.3]. Given a group G and a subgroup H, the inclusion $H \subseteq G$ corresponds to a pointed map $i_H : BH \to_* BG$ (pointed by a path $(i_H)_0$) whose fibers are all sets. The fiber $i_H^{-1}(s_G)$ can be identified with the set G/H of H-cosets, in such a way that the element $(s_H, (i_H)_0)$ corresponds to the class of the neutral element of G. Subsequently, every fiber is merely equivalent to G/H. Moreover, the pointed map $a_H : BG \to_* \mathcal{U}_{(G/H)}$ mapping $y : BG$ to $i_H^{-1}(y)$ is such that the homorphism $\Omega(a_H)$ is precisely the action of G on G/H by multiplication. Now, when $f : G \to G'$ is any group homomorphism, with corresponding pointed map $Bf : BG \to_* BG'$, we point $(Bf)^{-1}(s_{G'})$ at $(s_G, (Bf)_0)$, where $(Bf)_0$ is the path pointing Bf. One can then prove that $B\,\mathrm{im}(f)$ is equivalent to $\sum_{y':BG'} \|(Bf)^{-1}(y')\|_0$, pointed at $(s_{G'}, |(s_G, (Bf)_0)|_0)$, under which $i_{\mathrm{im}(f)}$ identifies with the first projection. In particular, there is a pointed equivalence $G'/\,\mathrm{im}(f) \simeq_* \|(Bf)^{-1}(s_{G'})\|_0$.

Lemma 2.4. *For any group G there is a pointed equivalence $\|BG = BG\|_0 \simeq_* \mathrm{Out}(G)$.*

Sketch of proof. Recall that $\mathrm{Out}(G) \equiv \mathrm{Aut}(G)/\,\mathrm{im}(\mathrm{inn})$ for the morphism of groups $\mathrm{inn} : G \to \mathrm{Aut}(G)$ that maps an element $g \in G$ to the inner automorphism $x \mapsto gxg^{-1}$. Apply the observation above to inn and recognize that the fiber of $B\mathrm{inn}$ is equivalent to $BG = BG$. □

Theorem 2.2 has a simpler statement whenever there is a section (right inverse) of $p : (BG = BG) \to_* \mathrm{Out}(G)$.

Corollary 2.5. *If we have a section $s : \mathrm{Out}(G) \to_* (BG = BG)$ of the map $p : (BG = BG) \to_* \mathrm{Out}(G)$ in Theorem 2.2, then*

$$(BG = BG) \simeq (BZ(G) \times \mathrm{Out}(G)).$$

Proof. We know that $(BG = BG) \simeq \sum_{\varphi:\mathrm{Out}(G)} p^{-1}(\varphi)$. From Theorem 2.2, we know that the fiber $p^{-1}(\varphi)$ is equivalent to $BZ(G)$ when φ is the class of $\mathrm{id}_G : \mathrm{Aut}(G)$. So it suffices to prove that all fibers are equivalent to each other. However, since p is a set-truncation and we have a section s of it, the fiber $p^{-1}(\varphi)$ is simply the connected component of $BG = BG$ at $s(\varphi)$. There is an obvious equivalence from $(BG = BG)_{(\mathrm{refl}_G)}$ to $(BG = BG)_{(s(\varphi))}$, namely $\psi \mapsto \psi \circ s(\varphi)$. □

Let us come back to the study of $(\mathbb{S}^1 = \mathbb{S}^1)$, and define an element of it that is not in the connected component of $\mathrm{refl}_{\mathbb{S}^1}$. Under univalence, this is equivalent to defining an equivalence $\mathbb{S}^1 \simeq \mathbb{S}^1$ that is not merely equal to $\mathrm{id}_{\mathbb{S}^1}$. Let $-\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ be the function defined by circle induction as $-\mathrm{id}_{\mathbb{S}^1} :\equiv \mathrm{ind}(\bullet, ⟲^{-1})$. In other words, $-\mathrm{id}_{\mathbb{S}^1}$ is the (propositionally) unique function $\mathbb{S}^1 \to \mathbb{S}^1$ such that $-\mathrm{id}_{\mathbb{S}^1}(\bullet) \equiv \bullet$ and $[-\mathrm{id}_{\mathbb{S}^1}](⟲) = ⟲^{-1}$. It is an equivalence because it is its own inverse. Indeed, we can construct a proof of $-\mathrm{id}_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$ by function extensionality and $\mathbb{S}^1$-induction: since $\mathrm{refl}_\bullet$ is an element of $(-\mathrm{id}_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1})(\bullet) = \bullet$, we only need to provide an element of $\mathrm{refl}_\bullet =^T_{⟲} \mathrm{refl}_\bullet$ where T is the type family $x \mapsto (-\mathrm{id}_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1})(x) = x$. But the transport in the type family T over $⟲$ is given by $p \mapsto ⟲ \cdot p \cdot [-\mathrm{id}_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1}](⟲)^{-1}$. Expanding the expression as $⟲ \cdot p \cdot ⟲^{-1}$, we find that $\mathrm{trp}^T_{⟲}(\mathrm{refl}_\bullet) = \mathrm{refl}_\bullet$ by simple path algebra, as we wanted.

Now, we shall prove that:

$$\mathrm{id}_{\mathbb{S}^1} \neq -\mathrm{id}_{\mathbb{S}^1}. \tag{1}$$

In order to do so, consider the evaluation fiber sequence:

$$(\mathbb{S}^1 \to_* \mathbb{S}^1) \to (\mathbb{S}^1 \to \mathbb{S}^1) \xrightarrow{\mathrm{ev}_\bullet} \mathbb{S}^1 \tag{2}$$

Here, all the fibers can be identified via a function

$$f : \prod_{x:\mathbb{S}^1} (\bullet = \bullet) \simeq (x = x) \tag{3}$$

with $f(\bullet) \equiv \mathrm{id}_{\bullet=\bullet}$ and $[f](⟲) : (⟲ - ⟲^{-1} = \mathrm{id}_{\bullet=\bullet})$ is the reflexivity path of $\mathrm{id}_{\bullet=\bullet}$ transported using commutativity in $\bullet = \bullet$ and path algebra. Because $(\bullet = \bullet)$ is equivalent to $\mathbb{Z}$, it follows that the sequence gives an equivalence

$$(\mathbb{S}^1 \to \mathbb{S}^1) \simeq \left(\sum_{x:\mathbb{S}^1} x = x\right) \simeq \left(\sum_{x:\mathbb{S}^1} \mathbb{Z}\right) \simeq (\mathbb{S}^1 \times \mathbb{Z}), \tag{4}$$

where $\mathrm{id}_{\mathbb{S}^1}$ is sent to $(\bullet, 1)$ while $-\mathrm{id}_{\mathbb{S}^1}$ is sent to $(\bullet, -1)$. These elements of $\mathbb{S}^1 \times \mathbb{Z}$ belong to different connected components, so $\mathrm{id}_{\mathbb{S}^1}$ and $-\mathrm{id}_{\mathbb{S}^1}$ as well.

We can finally prove Theorem 2.1.

Proof of Theorem 2.1. The classifying type $B\mathbb{Z}$ of the group $\mathbb{Z}$ of integers is equivalent to the circle $\mathbb{S}^1$. Because $\mathbb{Z}$ is abelian, $BZ(\mathbb{Z})$ is equivalent to $\mathbb{S}^1$ itself, and $\mathrm{Inn}(\mathbb{Z})$ is trivial. In particular, $\mathrm{Out}(\mathbb{Z})$ is the set underlying the group $\mathrm{Aut}(\mathbb{Z})$. But $\mathbb{Z}$ has exactly two automorphisms, namely the identity and $k \mapsto -k$. To apply Corollary 2.5 for the desired result, we give a section of the map $p : (\mathbb{S}^1 \simeq \mathbb{S}^1) \to_* \mathrm{Out}(\mathbb{Z})$. Because we want the section to be pointed, we need to send the identity to $\mathrm{id}_{\mathbb{S}^1}$ and $k \mapsto -k$ to $-\mathrm{id}_{\mathbb{S}^1}$.[1] □

Note that Theorem 2.1 also gives that the equivalence (4) restricts to the equivalence $(\mathbb{S}^1 \simeq \mathbb{S}^1) \simeq (\mathbb{S}^1 \times \{\pm 1\})$ of the corresponding subtypes.

Remark 2.6. Although we used Theorem 2.2 only to prove Theorem 2.1 here, Gottlieb's result has other consequences in univalent group theory, which are worth mentioning. For example, we can apply Corollary 2.5 to the group $\mathfrak{S}_n$ of permutations of n elements. By definition, its classifying type $B\mathfrak{S}_n$ is equivalent to the connected component $\mathcal{U}_{(\mathbf{n})}$ of the standard set $\mathbf{n}$ with n elements in the universe $\mathcal{U}$. A surprising fact of group theory is that $\mathrm{Out}(\mathfrak{S}_n)$ is always a singleton except for $n = 6$ for which it is a 2-element set. For $n \geq 3, n \neq 6$, both the center of $\mathfrak{S}_n$ and the set of outer automorphisms are trivial, so we get that $\mathcal{U}_{(\mathbf{n})} = \mathcal{U}_{(\mathbf{n})}$ is contractible. For $n = 6$, we get that $\mathcal{U}_{(\mathbf{6})} = \mathcal{U}_{(\mathbf{6})}$ is a set with two elements. In layman's terms, there is an invertible uniform way to associate to each 6-elements set another 6-elements set, and this mapping is drastically different from the identity. This mapping can actually be described in more details in terms of graph factorizations:

[1] To obtain a section from this reasoning, we need first an actual bijection between $\mathrm{Aut}(\mathbb{Z})$ and a set with two elements. To construct such a bijection, the decidability of equality on $\mathbb{Z}$ is crucial: For any automorphism, we need to decide if its image on 1 is 1 or -1.

for a 6-element set X, consider the complete graph on X and then craft the set of sets of perfect matchings not sharing any edge; it just happens that this resulting set also has 6-elements.

3. Symmetries of the 2-sphere

In this section, we will prove that the canonical inclusion

$$(\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})} + (\mathbb{S}^2 = \mathbb{S}^2)_{(-\mathrm{id}_{\mathbb{S}^2})} \to (\mathbb{S}^2 = \mathbb{S}^2)$$

is an equivalence, i.e., $\mathbb{S}^2 = \mathbb{S}^2$ has exactly two components, one containing the identity, and one corresponding to the equivalence $-\mathrm{id}_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{S}^2$, which is defined by $\mathbb{S}^2$-induction as the function such that $-\mathrm{id}_{\mathbb{S}^2}(\mathrm{N}) \equiv \mathrm{S}$, and $-\mathrm{id}_{\mathbb{S}^2}(\mathrm{S}) \equiv \mathrm{N}$ and $[-\mathrm{id}_{\mathbb{S}^2}](\mathrm{mrd}(x)) = (\mathrm{mrd}(x))^{-1}$ for all $x : \mathbb{S}^1$.

Lemma 3.1. *The function $-\mathrm{id}_{\mathbb{S}^2}$ is self-inverse and thus an equivalence.*

Sketch of proof. More generally, the same holds for the reflection $-\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ on any suspension, and an element of the type $\prod_{z:\Sigma X}\big(z = (-\mathrm{id}_{\Sigma X} \circ -\mathrm{id}_{\Sigma X})(z)\big)$ is easily constructed by induction. □

The plan now is as follows:

- First, we give a direct proof that $\mathrm{id}_{\mathbb{S}^2}$ and $-\mathrm{id}_{\mathbb{S}^2}$ are not in the same connected component;
- then, we give two definitions of the degree of a self-map $\mathbb{S}^2 \to \mathbb{S}^2$, from which it follows that every self-equivalence is either in the connected component of $\mathrm{id}_{\mathbb{S}^2}$ or in the connected component of $-\mathrm{id}_{\mathbb{S}^2}$;
- finally, we prove that the connected components of $\mathrm{id}_{\mathbb{S}^2}$ and $-\mathrm{id}_{\mathbb{S}^2}$ are equivalent to each other.

Notice that the last step is less ambitious than in the case of $\mathbb{S}^1$, where the two connected components were proven equivalent to each other but also each equivalent to $\mathbb{S}^1$ itself. We shall see in Section 7 that the connected components of $\mathrm{id}_{\mathbb{S}^2}$ and $-\mathrm{id}_{\mathbb{S}^2}$ are not equivalent to $\mathbb{S}^2$ itself. And indeed, the proof in the case of $\mathbb{S}^1$ relied heavily on two facts: $\mathbb{S}^1$ is 1-truncated and $\mathbb{S}^1$ is the classifying type of an abelian group. In other words, the homotopy structure of $\mathbb{S}^1$ is very well understood. This is not the case for $\mathbb{S}^2$: for example, it is certainly not 2-truncated ([4]), and is expected to be provably not n-truncated for any n.

The main tool for this section is the Hopf family, as defined by Brunerie in [4], to get an analogue in HoTT of the Hopf fibration in topology. We define, uniformly in $x : \mathbb{S}^1$, the function $\iota_x : \mathbb{S}^1 \to \mathbb{S}^1$ by $\mathbb{S}^1$-induction (giving the usual H-space structure on $\mathbb{S}^1$), putting $\iota_x(\bullet) \equiv x$ and $[\iota_x](\circlearrowleft) = f_x(\circlearrowleft)$. Here, $f : \prod_{x:\mathbb{S}^1}(\bullet = \bullet) \simeq (x = x)$ is the dependent function defined in (3). Clearly, $\iota_\bullet = \mathrm{id}_{\mathbb{S}^1}$ and hence, since $\mathbb{S}^1$ is connected, every ι_x is merely equal to $\mathrm{id}_{\mathbb{S}^1}$ and thus an equivalence. Recalling the transparency of univalence, we view $\iota_x : \mathbb{S}^1 = \mathbb{S}^1$ as a path. Note also that ι_x is the element of $(\mathbb{S}^1 = \mathbb{S}^1)_{(\mathrm{id}_{\mathbb{S}^1})}$ that corresponds to $x : \mathbb{S}^1$ under the evaluation equivalence $(\mathbb{S}^1 = \mathbb{S}^1)_{(\mathrm{id}_{\mathbb{S}^1})} \simeq \mathbb{S}^1$ exhibited in Section 2. Now define the type family $\mathcal{H} : \mathbb{S}^2 \to \mathcal{U}$ by $\mathbb{S}^2$-induction as the family:

$$\mathcal{H}(\mathrm{N}) \equiv \mathbb{S}^1, \quad \mathcal{H}(\mathrm{S}) \equiv \mathbb{S}^1, \quad \text{and} \quad [\mathcal{H}](\mathrm{mrd}(x)) = \iota_x \text{ for all } x : \mathbb{S}^1$$

Following Brunerie's exposition, we consider the map

$$\tau : \Omega\,\mathbb{S}^2 \to_* \mathbb{S}^1,\ p \mapsto [\mathcal{H}](p)(\bullet),\ \text{ pointed by } \mathrm{refl}_\bullet : \tau(\mathrm{refl}_\mathrm{N}) = \bullet. \tag{5}$$

This map is the key to getting the second homotopy group of the sphere (cf. [30, Sec. 8.4 and 8.5]). For now, recall from Example B.6 that there is a pointed map

$$\eta_{\mathbb{S}^1} : \mathbb{S}^1 \to_* \Omega\,\mathbb{S}^2, \quad x \mapsto \mathrm{mrd}(\bullet)^{-1} \cdot \mathrm{mrd}(x)$$

where the pointing path $\eta_0 : \eta_{\mathbb{S}^1}(\bullet) = \mathrm{refl}_N$ is given by path algebra.

Lemma 3.2. *The map τ is a retraction of $\eta_{\mathbb{S}^1}$, meaning that there is an element of $\tau \circ \eta_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$ as pointed functions.*

We remark that this is also an instance of a general fact about left-invertible H-spaces [9, Prop. 2.19]. The following is proved by circle induction.

Lemma 3.3. *There is an element of the type* $\eta_{\mathbb{S}^1}(-)^{-1} = \eta_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1}$.

Lemma 3.4. *The proposition* $\mathrm{id}_{\mathbb{S}^2} \neq -\mathrm{id}_{\mathbb{S}^2}$ *holds.*

Proof. Suppose $p : \mathrm{id}_{\mathbb{S}^2} = -\mathrm{id}_{\mathbb{S}^2}$ and derive a contradiction. Through function extensionality, it produces paths

$$p(\mathrm{N}) : \mathrm{N} = \mathrm{S} \quad \text{and} \quad p(\mathrm{S}) : \mathrm{S} = \mathrm{N},$$

and for all $x : \mathbb{S}^1$ a path over, $[p](\mathrm{mrd}(x)) : p(\mathrm{N}) =^T_{\mathrm{mrd}(x)} p(\mathrm{S})$, where $T : \mathbb{S}^2 \to \mathcal{U}$ is the type family $T(a) :\equiv (\mathrm{id}_{\mathbb{S}^2}(a) = -\mathrm{id}_{\mathbb{S}^2}(a))$. Because $\mathbb{S}^2$ is simply connected and we are targeting the empty type $\varnothing$, which is a proposition, we might as well assume paths of types $p(\mathrm{N}) = \mathrm{mrd}(\bullet)$ and $p(\mathrm{S}) = \mathrm{mrd}(\bullet)^{-1}$. Transporting $[p](\mathrm{mrd}(x))$ over these two paths, we get a path of type $\mathrm{mrd}(\bullet) =^T_{\mathrm{mrd}(x)} \mathrm{mrd}(\bullet)^{-1}$. Transport over $\mathrm{mrd}(x)$ in the type family T is the function $q \mapsto \mathrm{mrd}(x)^{-1} q\, \mathrm{mrd}(x)^{-1}$, so we get a path

$$\mathrm{mrd}(x)^{-1}\, \mathrm{mrd}(\bullet)\, \mathrm{mrd}(x)^{-1} = \mathrm{mrd}(\bullet)^{-1} \quad \text{for all } x : \mathbb{S}^1$$

Equivalently, this is a path $\eta_{\mathbb{S}^1}(-)^{-1} = \eta_{\mathbb{S}^1}$. Compose with the path from Lemma 3.3 to get a path $\eta_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1} = \eta_{\mathbb{S}^1}$. Using Lemma 3.2, we conclude that $-\mathrm{id}_{\mathbb{S}^1} = \tau \circ \eta_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1} = \tau \circ \eta_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$, which we already know to be absurd. □

This proves that $\mathrm{id}_{\mathbb{S}^2}$ and $-\mathrm{id}_{\mathbb{S}^2}$ belong to different connected components. We proceed to the second step of the road map: every equivalence in $\mathbb{S}^2 \simeq \mathbb{S}^2$ is either in the component of $\mathrm{id}_{\mathbb{S}^2}$ or in the component of $-\mathrm{id}_{\mathbb{S}^2}$. To this end we construct two, ultimately equal, degree functions $d, d' : (\mathbb{S}^2 \to \mathbb{S}^2) \to \mathbb{Z}$:

(i) The first, d, is directly seen to be a morphism of wild monoids, where the operations are given by composition and multiplication, respectively. In particular, it maps equivalences to invertible elements in $\mathbb{Z}$, that is 1 or -1.
(ii) The second, d', is more easily seen to be 'weakly injective', i.e., $d(f) = d(g)$ implies $\|f = g\|$. From Lemma 3.4 we then get that the degree of $-\mathrm{id}_{\mathbb{S}^2}$ is -1, and the degree induces equivalences $\|\mathbb{S}^2 \to \mathbb{S}^2\|_0 \simeq \mathbb{Z}$ and $\|\mathbb{S}^2 = \mathbb{S}^2\|_0 \simeq \{\pm 1\}$.

To define the degree, we recall from [30, Cor. 8.5.2] that the second homotopy group of $\mathbb{S}^2$ is $\mathbb{Z}$. Indeed, the second homotopy group $\pi_2(\mathbb{S}^2)$ is defined as the set-truncation $\|\Omega^2\, \mathbb{S}^2\|_0$, and [30, Sect. 8.4 and 8.5] proves that the map $\Omega\, \tau : \Omega^2\, \mathbb{S}^2 \to \Omega\, \mathbb{S}^1$ induces an isomorphism $\|\Omega\, \tau\|_0 : \pi_2(\mathbb{S}^2) \to \pi_1(\mathbb{S}^1)$ on the set-truncations, with $\|\Omega\, \eta\|_0$ as the inverse. Indeed, by the Freudenthal suspension theorem [30, Thm. 8.6.4], η is 0-connected, but it has a retraction, so it (and τ) are 1-equivalences. (A 1-equivalence is a map that induces an equivalence on 1-truncations.) Composing with the isomorphism $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ [30, Cor. 8.1.11], we get a group isomorphism $\zeta : \pi_2(\mathbb{S}^2) \to \mathbb{Z}$. Our first definition of the degree of a pointed map is then as the image of 1 through the induced homomorphism on second homotopy groups, transported back and forth by ζ:

$$d(f) :\equiv \left(\zeta \circ \pi_2(f) \circ \zeta^{-1}\right)(1) : \mathbb{Z} \qquad \text{for any } f : \mathbb{S}^2 \to_* \mathbb{S}^2.$$

Note that the degree of a map is *a priori* defined only when the map is pointed. However, the following lemma shows that the degree is independent of the choice of such a path.

Lemma 3.5. *Let X and Y be types, and $x : X$ and $y : Y$ be points. If Y is n-connected, then the map which forgets the pointing paths,* $\mathrm{pr}_1 : ((X, x) \to_* (Y, y)) \to (X \to Y)$, *is $(n-1)$-connected.*

Proof. Let $f : X \to Y$. The fiber $\mathrm{pr}_1^{-1}(f)$ is equivalent to $f(x) = y$. Now apply [30, Thm. 7.3.12] and use the assumption. □

In particular, if we have two paths $f_0, f_0' : f(\mathrm{N}) = \mathrm{N}$ pointing the map $f : \mathbb{S}^2 \to \mathbb{S}^2$, then the proposition $d(f, f_0) = d(f, f_0')$ holds, since $d(f, -) : (f(\mathrm{N}) = \mathrm{N}) \to \mathbb{Z}$ is a map from a connected type to a set, hence constant.

Proposition 3.6. *The degree function d is a morphism of wild monoids from $\mathbb{S}^2 \to \mathbb{S}^2$ (Remark B.13) to the multiplicative monoid $\mathbb{Z}$.*

Sketch of proof. Clearly, $d(\mathrm{id}_{\mathbb{S}^2}) = 1$. From Lemma 3.5 we may assume $f, g : \mathbb{S}^2 \to \mathbb{S}^2$ are pointed. By functoriality of the second fundamental group we have $\pi_2(g \circ f) = \pi_2(g) \circ \pi_2(f)$, from which we readily conclude $d(g \circ f) = d(g)d(f)$. □

Remark 3.7. As in [10], one can also put a group structure on $\mathbb{S}^2 \to_* \mathbb{S}^2$ such that the degree function becomes a group morphism onto $\mathbb{Z}$ with its additive structure. Together with Proposition 3.6, the degree function on pointed maps then becomes a wild ring morphism. In general, pointed self-maps $\Sigma X \to_* \Sigma X$ only form a wild *near-ring*.

Corollary 3.8. *The degree of a self-equivalence of the sphere $\mathbb{S}^2$ is either 1 or -1.*

Sketch of proof. For a self-equivalence $f : \mathbb{S}^2 \simeq \mathbb{S}^2$ we have $1 = d(\mathrm{id}_{\mathbb{S}^2}) = d(f \circ f^{-1}) = d(f)d(f^{-1})$, so $d(f)$ and $d(f^{-1})$ are multiplicatively inverse integers, hence ± 1. □

To prove that the degree map is an injection on connected components, we will define another map $\bar{d} : (\mathbb{S}^2 \to_* \mathbb{S}^2) \to \mathbb{Z}$, which is easily proven an injection on connected components, and then we will prove that $d = \bar{d}$.

Recall from Example B.6 that for each pointed type A, there is a map $\eta_A : A \to_* \Omega\,\Sigma\,A$. These maps are such that the following function is an equivalence for each $A, B : \mathcal{U}_*$ (see Remark B.8):

$$\Phi_{A,B} :\equiv \Omega - \circ\, \eta_A : (\Sigma\,A \to_* B) \simeq (A \to_* \Omega\,B) \tag{6}$$

There is now an equivalence $\gamma : (\mathbb{S}^2 \to_* \mathbb{S}^2) \simeq \Omega^2\,\mathbb{S}^2$ defined as the composition:

$$(\mathbb{S}^2 \to_* \mathbb{S}^2) \overset{\Phi_{\mathbb{S}^1,\mathbb{S}^2}}{\simeq} (\mathbb{S}^1 \to_* \Omega\,\mathbb{S}^2) \overset{\Phi_{\mathbf{2},\Omega\,\mathbb{S}^2}}{\simeq} (\mathbf{2} \to_* \Omega^2\,\mathbb{S}^2) \overset{\mathrm{ev}_1}{\simeq} \Omega^2\,\mathbb{S}^2,$$

using that $\mathbb{S}^1 \simeq \Sigma\,\mathbf{2}$, and with ev_1 evaluating its argument at the non-base point of $\mathbf{2}$. We now define $\bar{d}$ as the composition

$$(\mathbb{S}^2 \to_* \mathbb{S}^2) \xrightarrow{\gamma} \Omega^2\,\mathbb{S}^2 \xrightarrow{|-|_0} \pi_2(\mathbb{S}^2) \xrightarrow{\zeta} \mathbb{Z}.$$

We now show that d and $\bar{d}$ coincide, allowing us to compute the degree through $\bar{d}$ when necessary.

Proposition 3.9. *The equation $d = \bar{d}$ holds.*

Proof. Taking the definitions of d and $\bar{d}$ into account, we have to prove $\pi_2(f)(\zeta^{-1}(1)) = |\gamma(f)|_0$ for all f. Unfolding definitions, we have to prove that the outer diagram commutes in the following:

$$\begin{tikzcd}
(\mathbb{S}^2 \to_* \mathbb{S}^2) \arrow[rrr, "\Phi_{\mathbb{S}^1,\mathbb{S}^2}"] \arrow[d, "\Omega"'] \arrow[drr, phantom, "①"] & & & (\mathbb{S}^1 \to_* \Omega\,\mathbb{S}^2) \arrow[ddll, "\Omega"] \arrow[ddd, "\Phi_{\mathbf{2},\Omega\,\mathbb{S}^2}"] \arrow[ddl, phantom, "③"] \\
(\Omega\,\mathbb{S}^2 \to_* \Omega\,\mathbb{S}^2) \arrow[urrr, "- \circ \eta_{\mathbb{S}^1}"'] \arrow[d, "\Omega"'] \arrow[dr, phantom, "②"] & & & \\
 & (\Omega\,\mathbb{S}^1 \to_* \Omega^2\,\mathbb{S}^2) \arrow[drr, "- \circ \eta_{\mathbf{2}}"'] & & \\
(\Omega^2\,\mathbb{S}^2 \to_* \Omega^2\,\mathbb{S}^2) \arrow[ur, "- \circ \Omega(\eta_{\mathbb{S}^1})"'] \arrow[drrr, "\mathrm{ev}_\ell"] \arrow[d, "\|-\|_0"'] \arrow[rr, phantom, "④"] & & & (\mathbf{2} \to_* \Omega^2\,\mathbb{S}^2) \arrow[d, "\mathrm{ev}_1"] \\
(\pi_2(\mathbb{S}^2) \to \pi_2(\mathbb{S}^2)) \arrow[ddrr, "\mathrm{ev}_{\zeta^{-1}(1)}"'] \arrow[rr, phantom, "⑤"] & & & \Omega^2\,\mathbb{S}^2 \arrow[ddl, "|-|_0"] \\
 & & & \\
 & & \pi_2(\mathbb{S}^2) &
\end{tikzcd}$$

We will do that by proving that each of the small inner diagrams denoted ①, …, ⑤ commute, for elementary reasons. Triangles ① and ③ commute as instances of (6). The commutativity of square ② simply expresses the functoriality of Ω. In order to prove that ④ and ⑤ commute, we first need to define ℓ in ev_ℓ: we set $\ell :\equiv \Omega(\eta_{\mathbb{S}^1})(\circlearrowleft)$. Then it is almost immediate that ④ commutes because under the equivalence $\mathbb{S}^1 = \Sigma\,\mathbf{2}$, we have $\circlearrowleft = \eta_{\mathbf{2}}(1)$. Now the commutativity of ⑤ will follow from the functoriality of $\|-\|_0$ once we have shown $|\ell|_0 = \zeta^{-1}(1)$. By definition of ζ, this is equivalent to showing that $\Omega(\tau)(\ell) = \circlearrowleft$ in $\Omega\,\mathbb{S}^1$. However, we have seen in Lemma 3.2 that τ is a retraction of $\eta_{\mathbb{S}^1}$. Hence it follows that:

$$\Omega(\tau)(\ell) = \Omega(\tau)(\Omega(\eta_{\mathbb{S}^1})(\circlearrowleft)) = \circlearrowleft. \qquad \square$$

Corollary 3.10. *The degree map $d : (\mathbb{S}^2 \to \mathbb{S}^2) \to \mathbb{Z}$ is 0-connected. Hence any self-equivalence of $\mathbb{S}^2$ is in the connected component of either $\mathrm{id}_{\mathbb{S}^2}$ or $-\mathrm{id}_{\mathbb{S}^2}$, and the canonical inclusion*

$$(\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})} + (\mathbb{S}^2 = \mathbb{S}^2)_{(-\mathrm{id}_{\mathbb{S}^2})} \to (\mathbb{S}^2 = \mathbb{S}^2)$$

is an equivalence.

Proof. The previous result shows that $d(f) = \bar{d}(f) \equiv \zeta(|\gamma(f)|_0)$ with γ, ζ equivalences. As $|-|_0$ is 0-connected, so is d. From Corollary 3.8 we know any self-equivalence has degree ± 1, and since $-\mathrm{id}_{\mathbb{S}^2}$ is in a different component than $\mathrm{id}_{\mathbb{S}^2}$ by Lemma 3.4, we get that $d(-\mathrm{id}_{\mathbb{S}^2}) = -1$. $\square$

Next, we show that the two components $(\mathbb{S}^2 = \mathbb{S}^2)_{(\mathrm{id}_{\mathbb{S}^2})}$ and $(\mathbb{S}^2 = \mathbb{S}^2)_{(-\mathrm{id}_{\mathbb{S}^2})}$ are equivalent. In fact, this has little to do with $\mathbb{S}^2$ itself, and one can state a more general result.

Proposition 3.11. *Let A be a type with a point $a : A$ and a loop $p : a = a$. Then:*

$$(a = a)_{(\mathrm{refl}_a)} \simeq (a = a)_{(p)}$$

Proof. Define $f : (a = a) \to (a = a)$ by mapping q to qp. Then f is an equivalence with pseudo-inverse given by mapping r to rp^{-1}. Moreover, we have $f(\mathrm{refl}_a) = p$.

The equivalence f then restricts to an equivalence between the connected component of refl_a and the connected component of p. $\square$

Remark 3.12. Another way to state this result is that for any ∞-group G with an element g, we get an equivalence $G_{(e)} \simeq G_{(g)}$ by mapping h to hg. Indeed, the point a in A has an ∞-group of symmetries, whose elements form the type $a = a$.

Of course, a similar result holds generally for wild groups.

We shall show in 7 that $\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2}) \simeq \mathbb{Z}/2\mathbb{Z}$. From Proposition 3.11 we then also get $\pi_1(\mathbb{S}^2 = \mathbb{S}^2, -\mathrm{id}_{\mathbb{S}^2}) \simeq \mathbb{Z}/2\mathbb{Z}$.

4. Symmetries of the n-sphere

Having discussed the cases $n = 1$ and $n = 2$ in some detail, we finally wish to establish the result that $\mathbb{S}^n = \mathbb{S}^n$ has two connected components, for all $n \geq 1$. The main ideas for the proof are already contained in the result that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ [20] and the definition of degrees by Buchholtz and Favonia [10], but the argument that we need does not seem to have been written out informally in detail so far, though it has been formalized.

An important tool in this section is the wild adjunction $\Sigma \dashv \Omega$ from Proposition B.9. We recall that the suspension Σ acts like a wild functor, see Example B.6(ii), on morphisms: For $f : A \to B$, define $\Sigma(f) : \Sigma(A) \to \Sigma(B)$ by $\mathrm{N}_{\Sigma(A)} \mapsto \mathrm{N}_{\Sigma(B)}$, $\mathrm{S}_{\Sigma(A)} \mapsto \mathrm{S}_{\Sigma(B)}$, and $\mathrm{mrd}_{\Sigma(A)}(a) \mapsto \mathrm{mrd}_{\Sigma(B)}(f(a))$. The map $\Sigma(f)$ is pointed by the reflexivity path.

A core result is the following, to be proved in Section 4.1:

Theorem 4.1. *For all natural numbers $n \geq 1$, the wild monoid morphism,*

$$(\mathbb{S}^n \to_* \mathbb{S}^n) \xrightarrow{\Sigma} (\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1})$$

is a 0-equivalence.

Recall that an n-equivalence is a map that becomes an equivalence after n-truncation, cf. [13, Sec. 2]. In Section 4.2 we shall see how Theorem 4.1 implies that Σ induces an isomorphism $\|\mathbb{S}^n \to \mathbb{S}^n\|_0 \to \|\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}\|_0$ of monoids. Hence, by induction, $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$ and $(\mathbb{Z}, \times)$ are isomorphic as monoids, and so $\mathbb{S}^n = \mathbb{S}^n$ has two connected components. In Section 4.3 we give one concrete symmetry in each of the components.

4.1. **The suspension morphism is 0-connected.** Recall the equivalence $\Phi_{A,B}$ and the unit η_A in Section 3, (6) and Proposition B.9. Taking ΣB for B in (6) we get:

$$\Phi_{A,\Sigma B} :\equiv \Omega - \circ \eta_A : (\Sigma A \to_* \Sigma B) \simeq (A \to_* \Omega \Sigma B)$$

We will use the following naturality properties.

Lemma 4.2. *For any pointed map $f : A \to_* B$ we have:*

$$(\Phi_{A,\Sigma B} \circ \Sigma)(f) \equiv \Omega(\Sigma(f)) \circ \eta_A = \eta_B \circ f$$

Proof. By the naturality witness nat_η from Remark B.8. ☐

Lemma 4.3. *Let X be a pointed type and $f : A \to_* B$ a pointed function. Then for any $g : \Sigma X \to_* A$ we have:*

$$\Phi_{X,B}(f \circ g) = \Omega(f) \circ \Phi_{X,A}(g)$$

Proof. This is the last part of Remark B.8. ☐

Definition 4.4. Let A be a pointed type. Recall that $\mathbb{S}^0 :\equiv \mathbf{2}$ is pointed at $\mathrm{inl}(*)$ and that $(\mathbb{S}^0 \to_* A) \simeq A$ by the equivalence $\phi^0_A(f : \mathbb{S}^0 \to_* A) :\equiv f(\mathrm{inr}(*))$. For $n \geq 1$, define equivalences $\phi^n_A : (\Sigma \mathbb{S}^{n-1} \to_* A) \to \Omega^n A$ by induction: $\phi^n_A :\equiv \phi^{n-1}_{\Omega A} \circ \Phi_{\mathbb{S}^{n-1},A}$.

Lemma 4.5. *For all $n \geq 0$ and $f : A \to_* B$ and $g : \mathbb{S}^n \to_* A$ we have:*

$$\phi^n_B(f \circ g) = \Omega^n(f)(\phi^n_A(g))$$

Proof. By induction on $n : \mathbb{N}$. The base case $n \equiv 0$ is trivial. The step from n to $n+1$ is an application of Lemma 4.3, using that $\mathbb{S}^{n+1} \equiv \Sigma \mathbb{S}^n$ and $\Omega^{n+1}(f) \equiv \Omega^n(\Omega(f))$. ☐

The above lemmas allow us to formulate a connection between the maps Σ in Theorem 4.1 and the unit $\eta_{\mathbb{S}^n}$ from Proposition B.9. The following diagram commutes by Lemma 4.2 (top triangle) and Lemma 4.5 (bottom quadrangle). This proves Corollary 4.6.

$$\begin{tikzcd}
(\mathbb{S}^n \to_* \mathbb{S}^n) \arrow[r, "\Sigma"] \arrow[dr, "\eta_{\mathbb{S}^n} \circ -"'] \arrow[dd, "\phi^n_{\mathbb{S}^n}"'] & (\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1}) \arrow[d, "\Phi_{\mathbb{S}^n,\mathbb{S}^{n+1}}"] \\
 & (\mathbb{S}^n \to_* \Omega \Sigma \mathbb{S}^n) \arrow[d, "\phi^n_{\Omega\mathbb{S}^{n+1}}"] \\
\Omega^n(\mathbb{S}^n) \arrow[r, "\Omega^n(\eta_{\mathbb{S}^n})"] & \Omega^{n+1}(\mathbb{S}^{n+1})
\end{tikzcd}$$

Corollary 4.6. $\phi^{n+1}_{\mathbb{S}^{n+1}} \circ \Sigma \equiv \phi^n_{\Omega\mathbb{S}^{n+1}} \circ \Phi_{\mathbb{S}^n,\mathbb{S}^{n+1}} \circ \Sigma = \Omega^n(\eta_{\mathbb{S}^n}) \circ \phi^n_{\mathbb{S}^n}$

Since the ϕ's in Corollary 4.6 are equivalences we can transport knowledge about $\Omega^n(\eta_{\mathbb{S}^n})$ to Σ, using the following result:

Theorem 4.7 (Freudenthal suspension theorem [30, Thm. 8.6.4])**.** *If X is n-connected and pointed, with $n \geq 0$, then the map $\eta_X : X \to_* \Omega \Sigma X$ is $2n$-connected. (Note $\eta_X \equiv \sigma_X$ in [30].)*

Taking $X \equiv \mathbb{S}^n$ and using that $\mathbb{S}^n$ is $(n-1)$-connected [30, Cor. 8.2.2], we get in particular the following instantiation:

Corollary 4.8. *The map $\eta_{\mathbb{S}^n} : \mathbb{S}^n \to \Omega(\mathbb{S}^{n+1})$ is a $2(n-1)$-connected for all $n \geq 1$,*

To make use of this, we show how connectedness of functions interacts with loop spaces (proof in appendix):

Lemma 4.9. *Let A and B be types and $f : A \to B$ be a k-connected function ($k \geq -1$). For all $a_1, a_2 : A$, the function $[f] : a_1 = a_2 \to f(a_1) = f(a_2)$ is $(k-1)$-connected.*

Iterating this n times combined with Corollary 4.8 we get:

Corollary 4.10. *The map*

$$\Omega^n(\eta_{\mathbb{S}^n}) : \Omega^n(\mathbb{S}^n) \to \Omega^{n+1}(\mathbb{S}^{n+1})$$

is $(n-2)$*-connected for all* $n \geq 1$*.*

We are now ready to make good on our promise made above.

Proof of Theorem 4.1. Let $n \geq 1$. We have to prove that $\Sigma : (\mathbb{S}^n \to_* \mathbb{S}^n) \to (\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1})$ is a 0-equivalence. It is a wild monoid morphism by functoriality of Σ, see Example B.6(ii), since identity and composition of the monoid structures are just given by the identity function and function composition.

We now show that Σ above is a 0-equivalence. By Corollary 4.10 we have that $\Omega^n(\eta_{\mathbb{S}^n})$ is $(n-2)$-connected. For $n \geq 2$, it is then directly a 0-equivalence. For $n = 1$, only have that $\eta_{\mathbb{S}^1}$ is 0-connected, so induces a surjection on fundamental groups. But $\eta_{\mathbb{S}^1}$ has a retraction τ by Lemma 3.2, so it also induces an injection, hence a bijection, on fundamental groups, so $\Omega(\eta_{\mathbb{S}^1})$ is also a 0-equivalence.

It follows by Corollary 4.6 that also Σ is a 0-equivalence, since the ϕ's there are equivalences. $\square$

4.2. **Connected components of $\mathbb{S}^n = \mathbb{S}^n$.** Theorem 4.1 implies that, for all $n \geq 1$, the map $\|\Sigma\|_0 : \|\mathbb{S}^n \to_* \mathbb{S}^n\|_0 \to \|\mathbb{S}^{n+1} \to_* \mathbb{S}^{n+1}\|_0$ is an isomorphism of monoids. Two more, smaller steps are needed to be able to determine the number of components of $\mathbb{S}^n \simeq \mathbb{S}^n$. One is to remove the base points of this monoid morphism. The other is to consider equivalences rather than just maps $\mathbb{S}^n \to \mathbb{S}^n$.

Lemma 3.5 allows us to remove the point in the (co)domain of Σ:

Lemma 4.11. *For all natural numbers* $n \geq 1$*, the wild monoid morphism* $\Sigma : (\mathbb{S}^n \to \mathbb{S}^n) \to (\mathbb{S}^{n+1} \to \mathbb{S}^{n+1})$ *is a* 0*-equivalence.*

Proof. We have $\mathrm{pr}_1 \circ \Sigma_* = \Sigma \circ \mathrm{pr}_1$, with Σ_* the suspension morphism for pointed maps. Since every $\mathbb{S}^n$ (for $n \geq 1$) is 0-connected, we get that pr_1 is 0-connected by Lemma 3.5. It follows that Σ is a 0-equivalence. $\square$

Corollary 4.12. *For all natural numbers* $n \geq 1$*, the map*

$$\|\Sigma\|_0 : \|\mathbb{S}^n \to \mathbb{S}^n\|_0 \to \|\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}\|_0 \tag{7}$$

is an isomorphism of monoids.

The second step is to consider equivalences rather than just maps $\mathbb{S}^n \to \mathbb{S}^n$. The following result implies that $\|\mathbb{S}^n = \mathbb{S}^n\|_0$ is the group of invertible elements of the monoid $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$. The proof is in the appendix.

Lemma 4.13. *Let* A *be a type. Then* $\|A \simeq A\|_0$ *is equivalent to the set of invertible elements in the monoid* $\|A \to A\|_0$*.*

We now see that there are two connected components of symmetries of spheres:

Theorem 4.14. *For any* $n \geq 1$*, we have an equivalence of types*

$$\|\mathbb{S}^n = \mathbb{S}^n\|_0 \simeq \mathbf{2}$$

Proof. For $n = 1$ and $n = 2$, we have established this result in the previous sections (see Theorem 2.1 and Corollary 3.10). For higher n, it follows by induction on n with the help of Corollary 4.12 that the monoids $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$ have exactly two invertible elements. Then, Lemma 4.13 allows us to conclude. $\square$

4.3. **Concrete symmetries of $\mathbb{S}^n$.** As in the cases of $\mathbb{S}^1$ and $\mathbb{S}^2$, we want to construct one concrete element for each of the two connected components of symmetries of $\mathbb{S}^n$ that were established in Theorem 4.14. As before, we take $\mathrm{id}_{\mathbb{S}^n}$ in one component. For any type A, define $-\mathrm{id}_{\Sigma(A)}$ by $\mathrm{N}_{\Sigma(A)} \mapsto \mathrm{S}_{\Sigma(A)}$, $\mathrm{S}_{\Sigma(A)} \mapsto \mathrm{N}_{\Sigma(A)}$, and $\mathrm{mrd}_{\Sigma(A)}(a) \mapsto \mathrm{mrd}_{\Sigma(A)}(a)^{-1}$. Clearly, being self-inverse, $-\mathrm{id}_{\Sigma(A)}$ is a symmetry of $\Sigma(A)$. What is less obvious is that $-\mathrm{id}_{\mathbb{S}^n}$ is indeed in the other component of $\mathbb{S}^n = \mathbb{S}^n$. Preparing for a proof of $\mathrm{id}_{\mathbb{S}^n} \neq -\mathrm{id}_{\mathbb{S}^n}$ by induction on $n \geq 1$, we show that suspension and "negation" commute.

Lemma 4.15. *For any type A, the two functions $\Sigma(-\mathrm{id}_{\Sigma(A)})$ and $-\mathrm{id}_{\Sigma(\Sigma(A))}$ of type $\Sigma(\Sigma(A)) \to \Sigma(\Sigma(A))$ are equal.*

Proof. For brevity we abbreviate $N :\equiv \mathrm{N}_{\Sigma\,\Sigma\,A}$ and $S :\equiv \mathrm{S}_{\Sigma\,\Sigma\,A}$. We construct $h(x)$ of type $T(x) :\equiv (\Sigma(-\mathrm{id}_{\Sigma\,A})(x) = -\mathrm{id}_{\Sigma\,\Sigma\,A}(x))$ by induction on $x : \Sigma\,\Sigma\,A$ setting $h(N) :\equiv \mathrm{mrd}(\mathrm{N}_{\Sigma\,A}) : (N = S)$ and $h(S) :\equiv (\mathrm{mrd}(\mathrm{S}_{\Sigma\,A}))^{-1} : (S = N)$. To complete the definition of h (and the proof of the lemma) we need to define, for all $y : \Sigma\,A$, a higher path $[h](\mathrm{mrd}(y))$ whose type is $h(N) =^{T}_{\mathrm{mrd}(y)} h(S)$. By [30, Lem. 2.11.3], and using abbreviations $m_N :\equiv \mathrm{mrd}(\mathrm{N}_{\Sigma\,A})$ and $m_S :\equiv \mathrm{mrd}(\mathrm{S}_{\Sigma\,A})$, the latter type is equivalent to

$$U(y) :\equiv (m_S^{-1} \cdot \mathrm{mrd}(-\mathrm{id}_{\Sigma\,A}(y)) = \mathrm{mrd}(y)^{-1} \cdot m_N).$$

We construct $g(y)$ of type $U(y)$ by induction on $y : \Sigma\,A$. The type $U(\mathrm{N}_{\Sigma\,A})$ is, after simplification (normalising), $m_S^{-1} \cdot m_S = m_N^{-1} \cdot m_N$. For $g(\mathrm{N}_{\Sigma\,A})$ we take the path $c(m_S, m_N)$, where c is defined by double path induction, setting $c(\mathrm{refl}, \mathrm{refl}) \equiv \mathrm{refl}_{\mathrm{refl}}$. Similarly, $U(\mathrm{S}_{\Sigma\,A}) :\equiv (m_S^{-1} \cdot m_N = m_S^{-1} \cdot m_N)$, and we take $g(\mathrm{S}_{\Sigma\,A}) :\equiv \mathrm{refl}_{m_S^{-1} \cdot m_N}$. To complete the definition of g we need to define, for all $z : A$, a higher path $[g](\mathrm{mrd}(z))$ whose type is $c(m_S, m_N) =^{U}_{\mathrm{mrd}(z)} \mathrm{refl}_{m_S^{-1} \cdot m_N}$. In general, transport of an arbitrary $c : U(y)$ along $p : y = y'$ in $\Sigma\,A$ yields a $c' : U(y')$ given by

$$[- \cdot m_N]\big([{}^{-1} \circ \mathrm{mrd}](p)\big) \cdot c \cdot \Big(\big[m_S^{-1} \cdot -\big]\big([\mathrm{mrd} \circ -\mathrm{id}_{\Sigma\,A}](p)\big)\Big)^{-1}$$

This follows again from [30, Lem. 2.11.3], or by path induction on p.

We now instantiate this transport with $p \equiv \mathrm{mrd}(z)$ for $z : A$, and abbreviate $\beta :\equiv [\mathrm{mrd}](\mathrm{mrd}(z)) : m_N = m_S$. By unfolding the definition of $-\mathrm{id}_{\Sigma\,A}$ and path algebra we get $[\mathrm{mrd}]([-\mathrm{id}_{\Sigma\,A}](\mathrm{mrd}(z))) = \beta^{-1}$. Here $[\mathrm{mrd}] \equiv [\mathrm{mrd}_{\Sigma\,\Sigma\,A}]$ and $\mathrm{mrd}(z) \equiv \mathrm{mrd}_{\Sigma\,A}(z)$. Further calculations show that to define $[g](\mathrm{mrd}(z))$ it suffices to find an element of

$$\big[(- \cdot m_N) \circ {}^{-1}\big](\beta) \cdot c(m_S, m_N) \cdot \big[m_S^{-1} \cdot -\big](\beta) = \mathrm{refl}_{m_S^{-1} \cdot m_N}$$

of type $m_S^{-1} \cdot m_N = m_S^{-1} \cdot m_N$. In other words, we should fill the following diagram of 2-paths:

$$\begin{tikzcd}
m_N^{-1} \cdot m_N \arrow[d, Rightarrow, "{\big[(- \cdot m_N) \circ {}^{-1}\big](\beta)}"'] & m_S^{-1} \cdot m_S \arrow[l, Rightarrow, "{c(m_S, m_N)}"'] \\
m_S^{-1} \cdot m_N & m_S^{-1} \cdot m_N \arrow[l, Rightarrow, "\mathrm{refl}_{m_S^{-1} \cdot m_N}"] \arrow[u, Rightarrow, "{\big[m_S^{-1} \cdot -\big](\beta)}"']
\end{tikzcd}$$

The easiest way to fill the above diagram is to abstract completely from suspension types to a type T with points $N, S : T$, paths $m_N, m_S : N = S$, and a 2-path $\beta : m_N = m_S$. One starts by doing path induction on m_N, reducing the task to the case $S \equiv N$ and $m_N \equiv \mathrm{refl}_N$, and arbitrary $m_S : N = N$ and $\beta : m_N = m_S$. One can then do path induction on β, reducing the task to the case $m_S \equiv m_N$ and $\beta \equiv \mathrm{refl}_{m_N}$. But now, as $m_N \equiv \mathrm{refl}_N$, all paths appearing in the diagram, including $c(m_S, m_N)$, are reflexivity paths. Hence we can conclude by simple path algebra. $\square$

A formal proof of this lemma is available in cubical Agda [7].

Corollary 4.16. *For any $n \geq 1$, we have $\mathrm{id}_{\mathbb{S}^n} \neq -\mathrm{id}_{\mathbb{S}^n}$, and any symmetry of $\mathbb{S}^n$ is merely equal to either $\mathrm{id}_{\mathbb{S}^n}$ or $-\mathrm{id}_{\mathbb{S}^n}$.*

Proof. We know the statement for $n \equiv 1$ (Eq. (1)) and $n \equiv 2$ (Lemma 3.4) from the previous sections. The rest is done by induction on n, so we assume $\mathrm{id}_{\mathbb{S}^n} \neq -\mathrm{id}_{\mathbb{S}^n}$. By Corollary 4.12, we have $\Sigma(\mathrm{id}_{\mathbb{S}^n}) \neq \Sigma(-\mathrm{id}_{\mathbb{S}^n})$. As $\mathrm{id}_{\mathbb{S}^{n+1}} = \Sigma(\mathrm{id}_{\mathbb{S}^n})$ trivially holds and we further have $-\mathrm{id}_{\mathbb{S}^{n+1}} = \Sigma(-\mathrm{id}_{\mathbb{S}^n})$ by Lemma 4.15, the claimed inequality follows. Therefore, the two symmetries lie in different components, and any symmetry lies in one of the two components given by Theorem 4.14. □

5. Summary and comparison

In Fig. 1 we depict some relationships between the types studied so far. In the back, we see the types of pointed maps $\mathbb{S}^n \to_* \mathbb{S}^n$ on the top, the types of maps $\mathbb{S}^n \to \mathbb{S}^n$ in the middle, and the types of identifications $\mathbb{S}^n = \mathbb{S}^n$ on the bottom, each related by suspension as we go left-to-right. In the front, we see the set truncations thereof, with the set truncation maps going back-to-front. Additionally, we see on the front left concrete monoids that are equivalent to the types involving the 0-sphere. (The map $-0 : \mathbb{S}^0 \to \mathbb{S}^0$ is the constant map at the non-base point; this becomes identified with 0 after one suspension.)

On the right, we see the sequential colimits. (Sequential colimits can be defined using pushouts and coproducts over $\mathbb{N}$ or as a HIT as in [27, Sec. 3].) The dotted arrows are lifts of the top back squares since we form the suspension of a pointed map by first forgetting the pointedness. The top two sequences in the back thus sit cofinally inside the zigzagging sequence, and hence they have the same colimit, which we identify with the elements of the sphere spectrum, $\Omega^\infty \mathbb{S}$. (The sphere spectrum $\mathbb{S}$ is the spectrification of the prespectrum of spheres, i.e., $\Sigma^\infty \mathbb{S}^0$, where Σ^∞ maps a pointed type to the corresponding suspension spectrum, and Ω^∞ is the right adjoint thereof, which maps a spectrum to its underlying infinite loop type. See [31, Sec. 5.3] for more on spectra in HoTT.) The sequential colimit of the self-identification groups $\mathrm{G}(n+1) \equiv (\mathbb{S}^n = \mathbb{S}^n)$ is the group of units of the sphere spectrum, $\mathrm{G} \equiv \mathrm{GL}_1(\mathbb{S}) = (\mathbb{S} = \mathbb{S})$.

The diagram commutes, with the exception of the dashed arrow. This takes a pointed map $f : \mathbb{S}^2 \to_* \mathbb{S}^2$ to the composite $\tau \circ \Omega(f) \circ \eta_{\mathbb{S}^1} : \mathbb{S}^1 \to_* \mathbb{S}^1$, where $\tau : \Omega\, \mathbb{S}^2 \to_* \mathbb{S}^1$ comes from the H-space structure on the circle, cf. Eq. (5). This is a retraction of the suspension operation, because if $f \equiv \Sigma(g)$, then

$$\tau \circ \Omega\, \Sigma(g) \circ \eta_{\mathbb{S}^1} = \tau \circ \eta_{\mathbb{S}^1} \circ g = g$$

by naturality of η and Lemma 3.2. Relative to the equivalences $(\mathbb{S}^1 \to_* \mathbb{S}^1) \simeq \Omega\, \mathbb{S}^1$ and $(\mathbb{S}^2 \to_* \mathbb{S}^2) \simeq \Omega^2\, \mathbb{S}^2$, the dashed map can be identified with $\Omega(\tau) : \Omega^2\, \mathbb{S}^2 \to_* \Omega\, \mathbb{S}^1$, which is a 0-equivalence.

The homotopy groups of $\Omega^\infty \mathbb{S}$ are of course the stable homotopy groups of the spheres, π_k^{s}. Since the type of units G embeds into $\Omega^\infty \mathbb{S}$, it has the same homotopy groups, except for the connected components:

$$\pi_k(\mathrm{G}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{for } k = 0, \\ \pi_k^{\mathrm{s}}, & \text{otherwise.} \end{cases}$$

Though not shown in Fig. 1, the types of pointed identifications $\mathbb{S}^n =_* \mathbb{S}^n$ are equivalent to pullbacks of the vertical cospans in the back. We have embeddings $(\mathbb{S}^n =_* \mathbb{S}^n) \hookrightarrow \Omega^n\, \mathbb{S}^n$ onto the subtypes $\Omega^n_{\pm 1}\mathbb{S}^n$ corresponding to the generators $\{\pm 1\}$ in $\pi_n \mathbb{S}^n \simeq \mathbb{Z}$. Hence the homotopy groups of $\mathbb{S}^n =_* \mathbb{S}^n$ are the usual (unstable) homotopy groups of spheres, except at degree 0:

$$\pi_k(\mathbb{S}^n =_* \mathbb{S}^n, \pm\mathrm{id}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{for } k = 0, \\ \pi_{n+k}\mathbb{S}^n, & \text{otherwise.} \end{cases}$$

Our construction of degree functions establishing the above picture follows in most respects the approach outlined in [10, Sec. 5] and formalized in [5]. The difference is that we focus on the (wild) monoid structures given by composition instead of the group structures given either by the cogroup structures on the spheres or by transport from the wild groups $\Omega^n\, \mathbb{S}^n$. We thus give a direct proof of Corollary 4.16 that is interesting its own right, even though one could also conclude that $d(-\mathrm{id}_{\mathbb{S}^n}) = -1$ from the facts that $-\mathrm{id}_{\mathbb{S}^n}$ is the additive inverse of $\mathrm{id}_{\mathbb{S}^n}$ and that the degree is a 0-equivalence sending $\mathrm{id}_{\mathbb{S}^n}$ to 1.

6. Interlude on Whitehead products

Before focusing on the components of $\mathbb{S}^n = \mathbb{S}^n$, we need a few general results on Whitehead products. Recall from [30, Chapter 6.8] the join $*$ and the wedge $\vee$, higher inductive operations on types that can be constructed using pushouts. The proof of the following lemma is in the appendix.

Lemma 6.1. *Let A, B, and X be pointed types. We have the following equivalence: $(A \to_* (B \to_* \Omega X)) \simeq (A * B \to_* X)$.*

For the rest of this section we fix two pointed types A and B, and we denote a_0 and b_0 the base points of A and B. We repeat here the definition of the generalized Whitehead product from [4, Sec. 3.3].

Definition 6.2. Define the map $W = W_{A,B} : A * B \to \Sigma A \vee \Sigma B$ by $W(\mathrm{inl}(a)) :\equiv \mathrm{inr}(\mathrm{N}_{\Sigma B})$, $W(\mathrm{inr}(b)) :\equiv \mathrm{inl}(\mathrm{N}_{\Sigma A})$, and

$$[W](\mathrm{glue}(a,b)) := [\mathrm{inl}](\eta_A(a)) \cdot (\mathrm{glue}(*))^{-1} \cdot [\mathrm{inr}](\eta_B(b))$$

(η from B.9, $\mathrm{glue}(*) : \mathrm{inl}(\mathrm{N}_{\Sigma A}) = \mathrm{inr}(\mathrm{N}_{\Sigma B})$). The map $W_{A,B}$ is pointed by the path $(\mathrm{glue}(*))^{-1} : W(\mathrm{inl}(a_0)) \equiv \mathrm{inr}(\mathrm{N}) = \mathrm{inl}(\mathrm{N})$.

The map W makes a pushout square with the wedge inclusion i:

$$\begin{tikzcd} A * B \arrow[r, "W"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \Sigma A \vee \Sigma B \arrow[d, "i"] \\ 1 \arrow[r] & \Sigma A \times \Sigma B \end{tikzcd} \tag{8}$$

The fact that (8) is a pushout square is deduced using the 3×3-lemma in [4, Prop. 3.3.2], but plays no role in our arguments below.

We now also fix another pointed type (X, x_0).

Definition 6.3. The generalized Whitehead product of $\alpha : \Sigma A \to_* X$ and $\beta : \Sigma B \to_* X$ is the composition

$$[\alpha, \beta] :\equiv (\alpha \vee \beta) \circ W_{A,B} : A * B \to_* X.$$

Here $\alpha \vee \beta :\equiv \mathrm{ind}(\alpha, \beta, \beta_0^{-1}\alpha_0)$ by $\vee$-induction, with α_0, β_0 the pointing paths, so $\beta_0^{-1}\alpha_0 : \alpha(\mathrm{N}_{\Sigma A}) = \beta(\mathrm{N}_{\Sigma B})$.

Remark 6.4. If $A \equiv \mathbb{S}^p$ and $B \equiv \mathbb{S}^q$, then $A * B \simeq \mathbb{S}^{p+q+1}$ [4, Prop. 1.8.8], so one can obtain a map (using the same denotation)

$$[-,-] : \pi_{p+1}(X) \times \pi_{q+1}(X) \to \pi_{p+q+1}(X),$$

which is the usual Whitehead product on homotopy groups. Explicitly, given $a : \pi_{p+1}(X)$ and $b : \pi_{q+1}(X)$, one wants to define the element $[a, b]$ of the set $\pi_{p+q+1}(X)$. As we are targeting a set, one can as well assume $a \equiv |\alpha|_0$ and $b \equiv |\beta|_0$ for $\alpha : \Omega^{p+1}(X)$ and $\beta : \Omega^{q+1}(X)$. Using the equivalences

$$\phi_X^{p+1} : (\Sigma\, \mathbb{S}^p \to_* X) \simeq \Omega^{p+1}(X), \quad \phi_X^{q+1} : (\Sigma\, \mathbb{S}^q \to_* X) \simeq \Omega^{q+1}(X),$$

one gets pointed maps $(\phi_X^{p+1})^{-1}(\alpha) : \Sigma\, \mathbb{S}^p \to_* X$ and $(\phi_X^{q+1})^{-1}(\beta) : \Sigma\, \mathbb{S}^q \to_* X$. The element $[a, b] : \pi_{p+q+1}(X)$ is then defined as $|[(\phi_X^{p+1})^{-1}(\alpha), (\phi_X^{q+1})^{-1}(\beta)]|_0$ (where the bracket follows Definition 6.3).

Fix now a pointed map $\beta : \Sigma B \to_* X$ with pointing path $\beta_0 : \beta(\mathrm{N}) = x_0$, and consider the fiber sequence of evaluation at β:

$$(\Sigma B \to_* X)_{(\beta)} \xrightarrow{\iota}_* (\Sigma B \to X)_{(\beta)} \xrightarrow{\mathrm{ev}_\beta}_* X.$$

Figure 2. Long fiber sequence of evaluation.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{n+1}\left((\Sigma B \to X)_{(\beta)}\right) \arrow[r, "(-1)^{n+1}\,\Omega^{n+1}(\mathrm{ev}_\beta)"] & \Omega^{n+1} X \arrow[dll, rounded corners, "(-1)^n\,\Omega^n\,(\kappa_\beta)"'] \\
\Omega^n\left((\Sigma B \to X)_{(\beta)}\right) \arrow[r, "(-1)^n\,\Omega^n(\iota)"] & \Omega^n\left((\Sigma B \to X)_{(\beta)}\right) \arrow[r] & \cdots
\end{tikzcd}$$

(Taking connected components is not necessary, but allows us to emphasize the base points of the various function types.) This fiber sequence induces a long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(\Sigma B \to X, \beta) \arrow[r, "\pi_{n+1}(\mathrm{ev}_\beta)"] & \pi_{n+1}(X) \arrow[dll, rounded corners, "\partial^n_\beta"] \\
\pi_n(\Sigma B \to_* X, \beta) \arrow[r, "\pi_n(\iota)"'] & \pi_n(\Sigma B \to X, \beta) \arrow[r] & \cdots
\end{tikzcd}$$

The construction presented in [30, Ch. 8.4] of this exact sequence is as follows. Consider the map $\kappa_\beta : \Omega\, X \to (\Sigma\, B \to_* X)_{(\beta)}$ that associates to a loop α the function β pointed by the path $\alpha \cdot \beta_0$. Then the long fiber sequence is shown equivalent to the one in Fig. 2. It is then shown that the set-truncation of this sequence is a long exact sequence of sets, and the very last paragraph of the proof of [30, Thm. 8.4.6] proceeds to replace the truncations of the form $\|-\Omega^n(h)\|_0$ (which are group antimorphisms) by $\pi_n(h)$ (which are actual group morphisms) for $n \geq 1$. In particular, the boundary map ∂^n_β in that long exact sequence can be taken to be $\pi_n(\kappa_\beta)$.

However, we wish to express ∂^n_β at $n \geq 0$ in terms of the Whitehead product. In order to do so, we use the equivalences ϕ^n_A from Definition 4.4 and reason about the map $\delta^n_\beta : (\Sigma\, \mathbb{S}^n \to_* X) \to (\mathbb{S}^n \to_* (\Sigma\, B \to_* X)_{(\beta)})$ defined by

$$\delta^n_\beta :\equiv \left(\phi^n_{(\Sigma\, B \to_* X)_{(\beta)}}\right)^{-1} \circ \Omega^n(\kappa_\beta) \circ \phi^{n+1}_X$$

so that in particular $\partial^n_\beta = \|\phi^n_{(\Sigma\, B \to_* X)_{(\beta)}} \circ \delta^n_\beta \circ (\phi^{n+1}_X)^{-1}\|_0$. Notice that this function δ^n_β has a simple expression through the use of Lemma 4.5:

$$\begin{aligned}
\delta^n_\beta &\equiv \left(\phi^n_{(\Sigma\, B \to_* X)_{(\beta)}}\right)^{-1} \circ \Omega^n(\kappa_\beta) \circ \phi^{n+1}_X \\
&= \left(\phi^n_{(\Sigma\, B \to_* X)_{(\beta)}}\right)^{-1} \circ \Omega^n(\kappa_\beta) \circ \phi^n_X \circ \Phi_{\mathbb{S}^n, X} \\
&= \kappa_\beta \circ (\Phi_{\mathbb{S}^n, X}(-))
\end{aligned}$$

In other words, for every $\alpha : \Sigma\, \mathbb{S}^n \to_* X$ and $x : \mathbb{S}^n$, the pointed map $\delta^n_\beta(\alpha)(x)$ is just β as an unpointed function, but is pointed by the path:

$$(\Omega(\alpha) \circ \eta_{\mathbb{S}^n})(x) \cdot \beta_0 : \beta(\mathrm{N}) = x_0$$

To express δ^n_β in terms of generalized Whitehead products, we are going to construct a commuting square of the following form:

$$\begin{tikzcd}
(\Sigma\, \mathbb{S}^n \to_* X) \arrow[r, "\delta^n_\beta"] \arrow[d, "\rho^n_\beta"'] & (\mathbb{S}^n \to_* (\Sigma\, B \to_* X)_{(\beta)}) \arrow[d, "\xi^n_\beta \circ -", "\wr"'] \\
(\mathbb{S}^n * B \to_* X) & (\mathbb{S}^n \to_* (\Sigma\, B \to_* X)_{(0)}) \arrow[l, "\sim", "\varphi^n"']
\end{tikzcd}$$

where the maps ξ^n_β, ρ^n_β and φ^n are to be defined, and the element of $\Sigma\, B \to_* X$ denoted 0 is the constant map at x_0.

Since nothing hinges on having a sphere $\mathbb{S}^n$, let us generalize and construct a commuting diagram for any pointed connected type A:

$$\begin{tikzcd} (\Sigma\, A \to_* X) \arrow[r, "\delta_\beta"] \arrow[d, "\rho_\beta"'] & (A \to_* (\Sigma\, B \to_* X)_{(\beta)}) \arrow[d, "\wr", "\xi_\beta \circ -"'] \\ (A * B \to_* X) & (A \to_* (\Sigma\, B \to_* X)_{(0)}) \arrow[l, "\sim", "\varphi"'] \end{tikzcd} \tag{9}$$

where δ_β is defined on α as follows: for $a : A$, $\delta_\beta(\alpha)(a) \equiv \beta$ as an unpointed function, pointed by the path:

$$(\delta_\beta(\alpha)(a))_0 :\equiv (\Omega(\alpha) \circ \eta_A)(a) \cdot \beta_0 = \alpha_0 \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} : \beta(\mathrm{N}) = x_0$$

Here we write $\nu_{\alpha,\beta} :\equiv \alpha_0^{-1} \cdot \beta_0 : \beta(\mathrm{N}) = \alpha(\mathrm{N})$ for short, where $\alpha_0 : \alpha(\mathrm{N}) = x_0$ is the path pointing α. Notice that the map $\delta_\beta(\alpha)$ is indeed a pointed map: the element $\delta_\beta(\alpha)(a_0)$ is the map β pointed by the path $\alpha_0 \cdot [\alpha](\eta_A(a_0)) \cdot \nu_{\alpha,\beta}$; using the fact that $\eta_A(a_0) = \mathrm{refl}_\mathrm{N}$, one finds a path $(\delta_\beta(\alpha)(a_0))_0 = \beta_0$, providing an element of $\delta_\beta(\alpha)(a_0) = \beta$ as pointed functions.

Next define $\rho_\beta :\equiv [-, \beta]$ as the Whitehead product, or explicitly:

$$\begin{aligned} \rho_\beta(\alpha)\,(\mathrm{inl}(a)) &\equiv \beta(\mathrm{N}) \\ \rho_\beta(\alpha)\,(\mathrm{inr}(b)) &\equiv \alpha(\mathrm{N}) \\ [\rho_\beta(\alpha)](\mathrm{glue}(a,b)) &= [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} \cdot [\beta](\eta_B(b)) \end{aligned}$$

The map $\rho_\beta(\alpha)$ is pointed by the path $\beta_0 : \rho_\beta(\alpha)(\mathrm{inl}(a)) \equiv \beta(\mathrm{N}) = x_0$.

Let us now describe ξ_β. Since $\Sigma\, B \to_* X$ is equivalent to $B \to_* \Omega\, X$, and because $\Omega\, X$ is a wild group, so is $\Sigma\, B \to_* X$. Its unit is the map $0 \equiv (- \mapsto x_0)$ already described. The multiplication of two elements γ and γ' is defined by induction:

$$\begin{aligned} (\gamma' {+} \gamma)(\mathrm{N}) &:\equiv \gamma(\mathrm{N}) \\ (\gamma' {+} \gamma)(\mathrm{S}) &:\equiv \gamma'(\mathrm{S}) \\ [\gamma' {+} \gamma](\mathrm{mrd}(b)) &:= [\gamma'](\mathrm{mrd}(b)) \cdot \nu_{\gamma',\gamma} \cdot [\gamma](\eta_B(b)) \end{aligned}$$

The map $\gamma' {+} \gamma$ is pointed by the path $\gamma_0 : \gamma(\mathrm{N}) = x_0$ pointing γ itself. The inverse $-\gamma$ of an element γ is given by $\gamma \circ (-\mathrm{id}_{\Sigma\, B})$ (where $-\mathrm{id}_{\Sigma\, B}$ is pointed by $\mathrm{mrd}(b_0)^{-1}$). Then, there is an equivalence $(\Sigma\, B \to_* X) \simeq (\Sigma\, B \to_* X)$ that maps γ to $-\beta {+} \gamma$, cf. Proposition 3.11. This equivalence sends the connected component at β to the connected component at 0, hence providing the pointed equivalence ξ_β. Explicitly:

$$\begin{aligned} \xi_\beta(\gamma)(\mathrm{N}) &\equiv \gamma(\mathrm{N}) \\ \xi_\beta(\gamma)(\mathrm{S}) &\equiv \beta(\mathrm{N}) \\ [\xi_\beta(\gamma)](\mathrm{mrd}\, b) &= [\beta](\eta_B(b))^{-1} \cdot \nu_{\beta,\gamma} \cdot [\gamma](\eta_B(b)) \end{aligned}$$

Notice that $\xi_\beta(\gamma)$ is pointed by the path γ_0 that points γ.

Let us now define φ from diagram (9). Since A is connected, the inclusion of $A \to_* (\Sigma\, B \to_* X)_{(0)}$ in $A \to_* (\Sigma\, B \to_* X)$ is an equivalence. Now, use the equivalence between $\Sigma\, B \to_* X$ and $B \to \Omega\, X$ before simply applying Lemma 6.1. Unfolding definition, we see that the composition of equivalences

$$\begin{tikzcd} (A \to_* (\Sigma\, B \to_* X)_{(0)}) \arrow[r, hook] & (A \to_* (\Sigma\, B \to_* X)) \arrow[dl, hook, "\Phi_{B,X} \circ -"'] \\ (A \to_* (B \to_* \Omega\, X)) \arrow[r, "f \mapsto \bar{f}"'] & (A * B \to_* X) \end{tikzcd}$$

can be identified with the function φ defined by induction as follows:

$$\begin{aligned}
\varphi(h)(\mathrm{inl}(a)) &:\equiv x_0 \\
\varphi(h)(\mathrm{inr}(b)) &:\equiv x_0 \\
[\varphi(h)](\mathrm{glue}(a,b)) &:= (h(a))_0 \cdot [h(a)](\eta_B(b)) \cdot (h(a))_0^{-1}
\end{aligned}$$

The map $\varphi(h)$ is pointed by the reflexivity path at x_0.

With the preliminaries out of the way, let us show that ρ_β can be identified with $\psi_\beta :\equiv \varphi\circ(\xi_\beta\circ -)\circ\delta_\beta$. Let $\alpha : \Sigma A \to_* X$. Unfolding the above definitions, let us first examine $h :\equiv \xi_\beta \circ (\delta_\beta(\alpha)) : A \to_* (\Sigma B \to_* X)_{(0)}$:

$$\begin{aligned}
h(a)(\mathrm{N}) &\equiv h(a)(\mathrm{S}) \equiv \beta(\mathrm{N}) \\
[h(a)](\mathrm{mrd}(b)) &= [\beta](\eta_B(b))^{-1} \cdot \beta_0^{-1} \cdot (\delta_\beta(\alpha)(a))_0 \cdot [\beta](\eta_B(b)) \\
&= [\beta](\eta_B(b))^{-1} \cdot \nu_{\beta,\alpha} \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} \cdot [\beta](\eta_B(b)) \\
(h(a))_0 &= (\delta_\beta(\alpha)(a))_0 \\
&= \alpha_0 \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta}
\end{aligned}$$

Finally, we can insert this into the definition of φ to obtain the function $g :\equiv \psi_\beta(\alpha) = \varphi(h) : A * B \to_* X$:

$$\begin{aligned}
g(\mathrm{inl}(a)) &\equiv x_0 \\
g(\mathrm{inr}(b)) &\equiv x_0 \\
[g](\mathrm{glue}(a,b)) &= (h(a))_0 \cdot [h(a)](\eta_B(b)) \cdot (h(a))_0^{-1} \\
&= (h(a))_0 \cdot [h(a)](\mathrm{mrd}(b_0)^{-1} \cdot \mathrm{mrd}\, b) \cdot (h(a))_0^{-1} \\
&= (h(a))_0 \cdot [h(a)](\mathrm{mrd}\, b_0)^{-1} \cdot [h(a)](\mathrm{mrd}(b)) \\
&\qquad \cdot (h(a))_0^{-1} \\
&= \big(\alpha_0 \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta}\big) \cdot \big(\nu_{\beta,\alpha} \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta}\big)^{-1} \\
&\qquad \cdot \big([\beta](\eta_B(b))^{-1} \cdot \nu_{\beta,\alpha} \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} \\
&\qquad \cdot [\beta](\eta_B(b))\big) \cdot \big(\alpha_0 \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta}\big)^{-1} \\
&= \beta_0 \cdot [\beta](\eta_B(b))^{-1} \cdot \nu_{\beta,\alpha} \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} \\
&\qquad \cdot [\beta](\eta_B(b)) \cdot \nu_{\beta,\alpha} \cdot [\alpha](\eta_A(a))^{-1} \cdot \alpha_0^{-1}
\end{aligned}$$

The path g_0 pointing g is, by definition of φ, the reflexivity path at x_0. It only remains to construct a path $H : \rho_\beta(\alpha) = g$ as mere functions such that the type $H(\mathrm{inl}(a_0)) = \beta_0$ has an element. We proceed by induction on an element of the join:

$$\begin{aligned}
H(\mathrm{inl}(a)) &:\equiv \alpha_0 \cdot [\alpha](\eta_A(a)) \cdot \nu_{\alpha,\beta} : \beta(\mathrm{N}) = x_0 \\
H(\mathrm{inr}(b)) &:\equiv \beta_0 \cdot [\beta](\eta_B(b))^{-1} \cdot \nu_{\beta,\alpha} : \alpha(\mathrm{N}) = x_0
\end{aligned}$$

Finally, we must produce an element of

$$\prod_{a:A}\prod_{b:B}\Big(H(\mathrm{inl}(a)) =^{z\mapsto\rho_\beta(\alpha)(z)=g(z)}_{\mathrm{glue}(a,b)} H(\mathrm{inr}(b))\Big) \tag{10}$$

which corresponds to filling the square in Fig. 3. We fill this as indicated. This proves that $\rho_\beta(\alpha)$ and g are equal as mere functions. We must still check that $H(\mathrm{inl}(a_0)) = \beta_0$. This follows directly from $[\alpha](\eta_A(a_0)) = \mathrm{refl}$.

Specializing again to the case where $B \equiv \mathbb{S}^q$ is a sphere, we have proved the following:

$$
\begin{tikzcd}
\beta(\mathrm{N}) \arrow[r, "\nu_{\alpha,\beta}"] \arrow[ddd, "{[\beta](\eta_B(b))}"] \arrow[dddrrr, equal, dashed] & \alpha(\mathrm{N}) \arrow[r, "{[\alpha](\eta_A(a))}"] \arrow[ddrr, equal, dashed] & \alpha(\mathrm{N}) \arrow[r, "\alpha_0"] \arrow[dr, equal, dashed] & x_0 \arrow[d, "\alpha_0^{-1}"] \\
 & & & \alpha(\mathrm{N}) \arrow[d, "{[\alpha](\eta_A(a))^{-1}}"] \\
 & & & \alpha(\mathrm{N}) \arrow[d, "\nu_{\beta,\alpha}"] \\
\beta(\mathrm{N}) \arrow[ddd, "\nu_{\alpha,\beta}"] \arrow[drrr, equal, dashed] & & & \beta(\mathrm{N}) \arrow[d, "{[\beta](\eta_B(b))}"] \\
 & & & \beta(\mathrm{N}) \arrow[d, "\nu_{\alpha,\beta}"] \\
 & & & \alpha(\mathrm{N}) \arrow[d, "{[\alpha](\eta_A(a))}"] \\
\alpha(\mathrm{N}) \arrow[ddd, "{[\alpha](\eta_A(a))}"] \arrow[urrr, equal, dashed] & & & \alpha(\mathrm{N}) \arrow[d, "\nu_{\beta,\alpha}"] \\
 & & & \beta(\mathrm{N}) \arrow[d, "{[\beta](\eta_B(b))^{-1}}"] \\
 & & & \beta(\mathrm{N}) \arrow[d, "\beta_0"] \\
\alpha(\mathrm{N}) \arrow[r, "\nu_{\beta,\alpha}"'] \arrow[uuurrr, equal, dashed] & \beta(\mathrm{N}) \arrow[r, "{[\beta](\eta_B(b))^{-1}}"'] \arrow[uurr, equal, dashed] & \beta(\mathrm{N}) \arrow[r, "\beta_0"'] \arrow[ur, equal, dashed] & x_0
\end{tikzcd}
$$

FIGURE 3. The square corresponding to (10).

Theorem 6.5. *For any $\beta : \mathbb{S}^{q+1} \to_* X$, there is a long exact sequence*

$$
\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(\mathbb{S}^{q+1} \to X, \beta) \arrow[r] & \pi_{n+1}(X) \arrow[dll, "\partial_\beta^{n,q}"'] \\
\pi_{n+q+1}(X) \arrow[r] & \pi_n(\mathbb{S}^{q+1} \to X, \beta) \arrow[r] & \cdots
\end{tikzcd}
$$

where the connecting homomorphisms are Whitehead products $\partial_\beta^{n,q} = [-, |\phi_X^{q+1}(\beta)|_0]$ (where the bracket refers to the one defined in Remark 6.4).

7. Exploring the components of $\mathbb{S}^n = \mathbb{S}^n$

Having established that there are exactly two connected components of $(\mathbb{S}^n = \mathbb{S}^n)$, we want to examine the structure of each of these components. The first observation we make is simple:

Proposition 7.1. *For all $n \geq 1$, the two connected components of $(\mathbb{S}^n = \mathbb{S}^n)$ are equivalent.*

Proof. For $n = 1$ and $n = 2$, this statement is given by the main result of Section 2 and by Proposition 3.11, respectively. For $n \geq 3$, it follows from Proposition 3.11 and Corollary 4.16. □

Proposition 7.1 means that we can restrict ourselves to the connected component of id (or refl) in $(\mathbb{S}^n = \mathbb{S}^n)$. From now on, we use id as the implicit base point of $(\mathbb{S}^n = \mathbb{S}^n)$. The rest of this subsection is devoted to calculating the fundamental group of this type, which also allows us to see that the equivalence $(\mathbb{S}^1 = \mathbb{S}^1) = (\mathbb{S}^1 + \mathbb{S}^1)$ does not generalize for $n > 1$.

The proof of the following lemma is in the appendix.

Lemma 7.2. *For any $n \geq 1$, there is a group isomorphism:*

$$\xi_n : \pi_1(\mathbb{S}^n \to_* \mathbb{S}^n, \mathrm{id}) \simeq \pi_{n+1}(\mathbb{S}^n)$$

Remark 7.3. More generally, since $\mathbb{S}^n \to_* \mathbb{S}^n$ is a wild group, given any $\alpha, \beta : \mathbb{S}^n \to_* \mathbb{S}^n$, there is an equivalence between the corresponding components

$$(\mathbb{S}^n \to_* \mathbb{S}^n)_{(\alpha)} \simeq (\mathbb{S}^n \to_* \mathbb{S}^n)_{(\beta)}.$$

This is given by $\gamma \mapsto \gamma - \alpha + \beta$, where we write the group operation additively, cf. Proposition 3.11. This induces, for any $k \geq 1$, particular group isomorphisms of types

$$\pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, \alpha) \simeq \pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, \beta).$$

We have now every tool needed to prove the result we alluded to in Section 3.

Theorem 7.4. *The fundamental group of the type of symmetries of $\mathbb{S}^2$ is the cyclic group of order 2. That is, the following type has an element:*

$$\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}) = \mathbb{Z}/2\mathbb{Z}$$

where both sides of the equality are considered as groups.

Sketch of proof. Using the long exact sequence of Theorem 6.5 with $X \equiv \mathbb{S}^2$, $n = q = 1$ and $\beta \equiv \mathrm{id}_{\mathbb{S}^2}$, we get in particular an exact sequence:

$$\pi_2(\mathbb{S}^2) \xrightarrow{[-,i_2]} \pi_3(\mathbb{S}^2) \xrightarrow{\kappa} \pi_1(\mathbb{S}^2 \to \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2}) \to 0$$

where i_2 generates the group $\pi_2(\mathbb{S}^2)$ (equivalent to $\mathbb{Z}$). This shows that:

$$\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}) \simeq \pi_1(\mathbb{S}^2 \to \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2}) \simeq \pi_3(\mathbb{S}^2)/\langle [i_2, i_2] \rangle$$

The element $[i_2, i_2]$ is precisely that studied by Brunerie in its thesis: It generates a subgroup of index 2 in the infinite cyclic group $\pi_3(\mathbb{S}^2)$. □

The result generalizes to higher spheres. However, as it can be expected from the use of Brunerie's number in the case of the sphere $\mathbb{S}^2$, the results goes through for $n \geq 3$ for different and actually simpler reasons, as shown now.

Theorem 7.5. *For $n \geq 3$, and any function $f : \mathbb{S}^n \to \mathbb{S}^n$, the fundamental group of the connected component of f in the type $\mathbb{S}^n \to \mathbb{S}^n$ is the cyclic group of order 2:*

$$\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) = \mathbb{Z}/2\mathbb{Z}$$

as groups.

Sketch of proof. Because $\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) = \mathbb{Z}/2\mathbb{Z}$ is a proposition and $\mathbb{S}^n$ is connected, we can consider f to be pointed. By considering the fiber sequence associated with the evaluation $\mathrm{ev}_* : \mathbb{S}^n \to \mathbb{S}^n \to_* \mathbb{S}^n$ at the north pole, we get a long exact sequence that allows to establish $\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) \simeq \pi_1(\mathbb{S}^n \to_* \mathbb{S}^n, f)$. By Remark 7.3 and the wild adjunction $\Sigma \dashv \Omega$, the latter is equivalent to $\pi_{n+1}(\mathbb{S}^n)$, that is known to be $\mathbb{Z}/2\mathbb{Z}$. □

Theorem 7.6. *For $n \geq 3$, the fundamental group of the type of symmetries of $\mathbb{S}^n$ is the cyclic group of order 2:*

$$\pi_1(\mathbb{S}^n = \mathbb{S}^n, \mathrm{id}) = \mathbb{Z}/2\mathbb{Z}$$

as groups.

Proof. This is a direct corollary of Theorem 7.5, applied with $f :\equiv \mathrm{id}_{\mathbb{S}^n}$. □

Let us summarize the results of the current section, putting Theorems 4.14, 7.4 and 7.6 and Proposition 7.1 together:

Theorem 7.7. *For $n \geq 2$, the type of symmetries of $\mathbb{S}^n$ has two connected components. The two components are equivalent and both have fundamental group $\mathbb{Z}/2\mathbb{Z}$.* □

Remark 7.8. By Theorems 7.4 and 7.6, for $n \geq 2$, the group $\pi_1(\mathbb{S}^n = \mathbb{S}^n, \mathrm{id})$ is non-trivial. At the same time, $\pi_1(\mathbb{S}^n)$ is trivial by [30], and by 0-connectedness of $\mathbb{S}^n$, the group $\pi_1(\mathbb{S}^n + \mathbb{S}^n, x)$ is trivial for any x. Therefore, in contrast to the result for the case $n \equiv 1$ in Section 2, we have

$$(\mathbb{S}^n = \mathbb{S}^n) \neq (\mathbb{S}^n + \mathbb{S}^n).$$

8. Conclusion

We have shown that the two connected components of $(\mathbb{S}^n = \mathbb{S}^n)$ are equivalent and have fundamental group $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. The only result the authors are aware of that gives the complete homotopy type of these components is a result proved in classical topology about the topological 2-sphere in [15, Sec. 5]. Write S^2 for the topological 2-sphere, and $\mathrm{M}(S^2, S^2)$ for the space of continuous maps from S^2 to S^2 with the uniform topology:

Theorem 8.1. *The connected component of the identity map in $\mathrm{M}(S^2, S^2)$ is homotopy equivalent to $\mathrm{SO}(3) \times \Omega$, where Ω is the universal covering space of the connected component of the constant loop in $\Omega^2(S^2)$.*

This result can be stated as well in homotopy type theory, since $\mathrm{SO}(3) \simeq \mathbb{R}\mathrm{P}^3$, and the real projective 3-space $\mathbb{R}\mathrm{P}^3$ can be defined as in [11]. To prove the result, however, it might be necessary to define the classifying type $\mathrm{BSO}(3)$, which is itself another open problem. (This problem is closely related to that of defining the classifying type $\mathrm{BSU}(2)$, where $\mathrm{SU}(2) \simeq \mathbb{S}^3$.) We leave the further investigations for future work.

Our proof of Theorem 6.5 was inspired by the similar result [19, Thm. 2.7], the λ-*component EHP sequence*. As shown there, if we take $\beta \equiv \mathrm{id}_{\mathbb{S}^{q+1}}$, we obtain an approximation to the classical EHP sequence [34], valid in the range $n \leq 3q+1$. It would be interesting to construct this in homotopy type theory, as well as, of course, more modern refinements such as the various EHP spectral sequences for each prime p, see [23, Sec. 1.5] for a discussion in the classical setting. For $p = 2$, this is very much in reach using the James construction, while for odd p, it would need Toda's fibrations [29].

Something special happens for the spheres that happen to be H-spaces, namely $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$. Indeed, if X is an H-space such that, say, left-multiplication is invertible, meaning $\mu(x, -) : X \to X$ is invertible for all $x : X$, then we get an equivalence

$$(X \to X) \simeq (X \to_* X) \times X.$$

This equivalence uses left-multiplication to adjust any function f to a pointed function $x \mapsto \mu(f(x_0), -)^{-1}(f(x))$, together with the image of the base point, $f(x_0)$. Since this equivalence maps equivalences to pointed equivalences and vice versa, it restricts to an equivalence $(X = X) \simeq (X \simeq_* X) \times X$. Since $(\mathbb{S}^0 \simeq_* \mathbb{S}^0) \simeq \mathbf{1}$ and $(\mathbb{S}^1 \simeq_* \mathbb{S}^1) \simeq \mathbb{Z}/2$, we recover the equivalences

$$(\mathbb{S}^0 = \mathbb{S}^0) \simeq \mathbb{S}^0, \qquad (\mathbb{S}^1 = \mathbb{S}^1) \simeq \mathbb{Z}/2 \times \mathbb{S}^1.$$

From [12] we know that also $\mathbb{S}^3$ is an H-space, and further, that once we know that it's a loop space, then also $\mathbb{S}^7$ is an H-space. It also follows from our calculations that $\mathbb{S}^2$ is not an H-space, since all the components of $(\mathbb{S}^2 \to_* \mathbb{S}^2) \times \mathbb{S}^2$ are equivalent, but that's not the case for the components of $(\mathbb{S}^2 \to \mathbb{S}^2)$.

We have focused on the self-identification types $\mathbb{S}^n = \mathbb{S}^n$, but we could of course also look at the other components of the function type $\mathbb{S}^n \to \mathbb{S}^n$. For $n \geq 3$, we don't get anything new by the proof of Theorem 7.6, i.e., $\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) \simeq \pi_1(\mathbb{S}^n \to \mathbb{S}^n, \mathrm{id}) \simeq \mathbb{Z}/2\mathbb{Z}$, for any f. However, for $n = 2$, from the proof of Theorem 7.4, the fundamental group of a component of $\mathbb{S}^2 \to \mathbb{S}^2$ depends on the corresponding degree. Once we know that the Whitehead product is bilinear (possibly up to a sign), as in [1, Prop. 3.4], we can conclude that $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2, f) = \mathbb{Z}/2d(f)\mathbb{Z}$.

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Appendix A. Proofs

Lemma 2.3. *For any group G there is a pointed equivalence $BZ(G) \simeq_* (BG = BG)_{(\mathrm{refl}_{BG})}$.*

Proof. Notice that $(BG = BG)_{(\mathrm{refl}_{BG})}$ is a connected groupoid, pointed at $(\mathrm{refl}_{BG}, |\mathrm{refl}_{\mathrm{refl}_{BG}}|)$. So, to exhibit a pointed equivalence as in the statement, we construct a pointed map

$$z_G : (BG = BG)_{(\mathrm{refl}_{BG})} \to_* BG$$

such that $\Omega\,(z_G)$ is injective and has image $Z(G)$. Write $s : BG$ for the distinguished point of the delooping BG. The map z_G is defined to be the restriction to the connected component at refl_{BG} of the evaluation $\mathrm{ev}_s : (BG = BG) \to BG$ that sends an equality $x : BG = BG$ to the point $x(s) : BG$ (where x is seen as an equivalence $BG \simeq BG$). Note that z_G is pointed, trivially, by the path $\mathrm{refl}_s : s = \mathrm{ev}_s(\mathrm{refl}_{BG})$.

Note that $\mathrm{ev}_y : (BG = BG) \to BG$ can be defined in the same way for any $y : BG$ instead of s. Then we can prove, by double path-inductions,

$$[x](q) \cdot [\mathrm{ev}_s](p) = [\mathrm{ev}_y](p) \cdot q$$

for all $x : (BG = BG)$, all $p : \mathrm{refl}_{BG} = x$, all $y : BG$, and all $q : s = y$. (Indeed, the equation holds for $p \equiv \mathrm{refl}_{\mathrm{refl}_{BG}}$ and $q \equiv \mathrm{refl}_s$). In particular, the equation holds when $x \equiv \mathrm{refl}_{BG}$ and $y \equiv s$, so that we have: for all $p : \mathrm{refl}_{BG} = \mathrm{refl}_{BG}$ and all $g : s = s$, $g \cdot [\mathrm{ev}_s](p) = [\mathrm{ev}_s](p) \cdot g$. By restriction to the subtype $(BG = BG)_{(\mathrm{refl}_{BG})}$, we get: for all $h : \Omega\,(BG = BG)_{(\mathrm{refl}_{BG})}$, and all $g : G$, $g \cdot [z_G](h) = [z_G](h) \cdot g$. Because z_G is pointed by refl_s, path algebra shows that $\Omega\,(z_G) = [z_G]$. Hence, for any $h : \Omega\,(BG = BG)_{(\mathrm{refl}_{BG})}$, $\Omega\,(z_G)(h)$ lies in the center of G.

Conversely, we must show that any element of the center is in the image of by $\Omega\,(z_G)$. Take g in the center of G, and construct an element of $\Omega\,(BG = BG)_{(\mathrm{refl}_{BG})}$ as follows. Define $\hat{g} : \mathrm{refl}_{BG} = \mathrm{refl}_{BG}$ through univalence by giving an equality of type $\mathrm{id}_{BG} = \mathrm{id}_{BG}$, that is, under function extensionality, by giving a homotopy of type $\prod_{y:BG} y = y$. We will obtain such a homotopy by taking, for all $y : BG$, the first component of a center of contraction of $\sum_{q:y=y} \prod_{q':s=y}(q'g = qq')$. The contractibility of $\sum_{q:y=y} \prod_{q':s=y}(q'g = qq')$ is a proposition, so we can use the connectedness of BG and only prove it for $y \equiv s$. But because g commutes with all elements of G,

$$\sum_{q:G} \prod_{h:G} (hg = qh) \simeq \sum_{q:G} G \to (g = q) \simeq \sum_{q:G} (g = q) \simeq 1.$$

We obtain in that way $\hat{g} : \mathrm{refl}_{BG} = \mathrm{refl}_{BG}$ and we consider the element $(\hat{g}, !) : \Omega\,(BG = BG)_{(\mathrm{refl}_{BG})}$. Its image though $\Omega\,(z_G)$ is $\hat{g}(s)$ when $\hat{g}$ is seen as an homotopy $\prod_{y:BG} y = y$. And by definition, $\hat{g}(s)$ is the first component of the center of contraction of $\sum_{q:G} \prod_{h:G}(hg = qh)$. Tracking back the equivalences above, that is exactly g. We thus have proven that $\Omega\,(z_G)$ has image $Z(G)$.

Lastly, we have to show that $\Omega\,(z_G)$ is injective. Equivalently, we want to prove that all fibers of z_G are sets. This is a proposition, so it boils down to proving that the fiber at s is a set. This fiber is the type of elements $x : BG = BG$, merely equal to refl_{BG}, together with an equality from s to $[\mathrm{ev}_s](x)$. Through univalence, this is thus the type of pointed equivalences from BG to itself. But this is equivalent to the type of group automorphisms of G, which is a set. □

Lemma 2.4. *For any group G there is a pointed equivalence $\|BG = BG\|_0 \simeq_* \mathrm{Out}(G)$.*

Proof. Recall the morphism of groups $\mathrm{inn} : G \to \mathrm{Aut}(G)$ that maps an element $g \in G$ to the inner automorphism $x \mapsto gxg^{-1}$. By univalence, the delooping of $\mathrm{Aut}(G)$ can be described as the connected component of BG in $\mathcal{U}_*$. In particular, under this identification, inn is simply $\Omega(B\mathrm{inn})$

where $B\mathrm{inn} : BG \to_* \mathcal{U}_{*(BG)}$ is the function mapping each $y : BG$ to the type BG itself but pointed at y instead of the distinguished point s of BG. The path pointing $B\mathrm{inn}$ is simply refl_{BG}. Using the fact stated above the lemma, we find a pointed equivalence $\|B\mathrm{inn}^{-1}(BG)\|_0 \simeq \mathrm{Out}(G)$ by pointing the type on the left at $|(s, \mathrm{refl}_{BG})|_0$. But the fiber $B\mathrm{inn}^{-1}(BG)$ is by definition $\sum_{y:BG}(BG, y) =_* BG$, which is equivalent to $BG = BG$. Moreover, through this last equivalence, $\mathrm{refl}_{BG} : BG = BG$ corresponds to $(s, \mathrm{refl}_{BG}) : B\mathrm{inn}^{-1}(BG)$. Truncating the equivalence, we get a pointed equivalence $\|BG = BG\|_0 \simeq_* \|B\mathrm{inn}^{-1}(BG)\|_0$, and thus by composing with the first pointed equivalence, we get $\|BG = BG\|_0 \simeq_* \mathrm{Out}(G)$ as wanted. □

Lemma 3.1. *The function $-\mathrm{id}_{\mathbb{S}^2}$ is self-inverse and thus an equivalence.*

Proof. More generally, the same holds for the reflection $-\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ on any suspension, so we prove it in this generality. We produce by induction an element of the type $\prod_{z:\Sigma X} T(z)$, where $T(z) :\equiv (z = (-\mathrm{id}_{\Sigma X} \circ -\mathrm{id}_{\Sigma X})(z))$. By definition of $-\mathrm{id}_{\Sigma X}$, $T(\mathrm{N}) \equiv (\mathrm{N} = \mathrm{N})$ and $T(\mathrm{S}) \equiv (\mathrm{S} = \mathrm{S})$, so we take $\mathrm{refl}_\mathrm{N} : T(\mathrm{N})$ and $\mathrm{refl}_\mathrm{S} : T(\mathrm{S})$. To complete the induction, we need to provide an element of type $\prod_{x:X} \mathrm{refl}_\mathrm{N} =^T_{\mathrm{mrd}\, x} \mathrm{refl}_\mathrm{S}$. Transporting over a meridian in the family T is conjugation by the meridian: indeed, the transport over any path $p : x = x'$ in T is given by $q \mapsto [-\mathrm{id}_{\Sigma X} \circ -\mathrm{id}_{\Sigma X}](p) \cdot q \cdot p^{-1}$, and

$$\begin{aligned}
[-\mathrm{id}_{\Sigma X} \circ -\mathrm{id}_{\Sigma X}](\mathrm{mrd}(x)) &= [-\mathrm{id}_{\Sigma X}](\mathrm{mrd}(x)^{-1}) \\
&= \big([-\mathrm{id}_{\Sigma X}](\mathrm{mrd}(x))\big)^{-1} \\
&= \big(\mathrm{mrd}(x)^{-1}\big)^{-1} \\
&= \mathrm{mrd}(x).
\end{aligned}$$

Hence $\mathrm{refl}_\mathrm{N} =^T_{\mathrm{mrd}(x)} \mathrm{refl}_\mathrm{S}$ is equivalent to $\mathrm{mrd}(x)\, \mathrm{refl}_\mathrm{N}\, \mathrm{mrd}(x)^{-1} = \mathrm{refl}_\mathrm{S}$, which is indeed inhabited for any $x : \mathbb{S}^1$ by simple path algebra. □

Lemma 3.2. *The map τ is a retraction of $\eta_{\mathbb{S}^1}$, meaning that there is an element of $\tau \circ \eta_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$ as pointed functions.*

Proof. For all $z \in \mathbb{S}^1$, we can calculate:

$$\begin{aligned}
\tau(\eta_{\mathbb{S}^1}(z)) &= [\mathcal{H}](\mathrm{mrd}(\bullet)^{-1} \cdot \mathrm{mrd}(z))(\bullet) \\
&= (\iota_\bullet^{-1} \circ \iota_z)(\bullet) = \mathrm{id}_{\mathbb{S}^1}^{-1}(\iota_z(\bullet)) = z
\end{aligned}$$

It remains to verify that $[\tau](\eta_0)$ coincides with the path above instantiated at $z \equiv \bullet$. In fact, τ factors through an *adjoint* equivalence $\bar{\tau} : \|\Omega\, \mathbb{S}^2\|_1 \simeq_* \mathbb{S}^1$, with inverse $\bar{\eta}_{\mathbb{S}^1} = |-|_1 \circ \eta_{\mathbb{S}^1}$. The identification $\bar{\eta}_{\mathbb{S}^1} \circ \bar{\tau} = \mathrm{id}_{\|\Omega\, \mathbb{S}^2\|_1}$ defined by the encode-decode method is more easily seen to be pointed, but then the identification $\bar{\tau} \circ \bar{\eta}_{\mathbb{S}^1} = \tau \circ \eta_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$ is pointed as well. The full details of this argument are formalized in [5]. □

Lemma 3.3. *There is an element of the type $\eta_{\mathbb{S}^1}(-)^{-1} = \eta_{\mathbb{S}^1} \circ -\mathrm{id}_{\mathbb{S}^1}$.*

Proof. We construct the element κ by induction on $\mathbb{S}^1$ by first defining

$$\kappa_\bullet : (\mathrm{mrd}(\bullet)^{-1} \cdot \mathrm{mrd}(\bullet))^{-1} = \mathrm{mrd}(\bullet)^{-1} \cdot \mathrm{mrd}(\bullet)$$

by simple path algebra.

It remains to find an element $\kappa_\circlearrowleft : \kappa_\bullet =_\circlearrowleft \kappa_\bullet$, which amounts to an element of

$$[\eta_{\mathbb{S}^1}](\circlearrowleft)^{-1} \cdot \kappa_\bullet = \kappa_\bullet \cdot \left[-^{-1}\right]([\eta_{\mathbb{S}^1}](\circlearrowleft))$$

Consider the following paths given by path algebra for each $x : \mathbb{S}^1$:

$$\begin{aligned}
\lambda_x &: (\mathrm{mrd}(x)^{-1} \cdot \mathrm{mrd}(\bullet))^{-1} = \eta_{\mathbb{S}^1}(x) \\
\mu_x &: \eta_{\mathbb{S}^1}(x)^{-1} = \mathrm{mrd}(x)^{-1}\, \mathrm{mrd}(\bullet)
\end{aligned}$$

In particular, the types $\lambda_\bullet = \kappa_\bullet$, $\mu_\bullet = \kappa_\bullet$, and $\lambda_\bullet = \mu_\bullet$ have elements. Moreover, by path induction on $p : \bullet = x$, one can construct an element

$$\xi_p : [\eta_{\mathbb{S}^1}](p)^{-1} \cdot \lambda_x = \mu_x \cdot \left[-^{-1}\right]([\eta_{\mathbb{S}^1}](p))$$

The element $\xi_{\circlearrowleft}$ gives the wanted $\kappa_{\circlearrowleft}$ by transport. □

Proposition 3.6. *The degree function d is a morphism of wild monoids from $\mathbb{S}^2 \to \mathbb{S}^2$ (Remark B.13) to the multiplicative monoid $\mathbb{Z}$.*

Proof. First, let us prove that $\mathrm{id}_{\mathbb{S}^2}$, pointed by refl_N, has degree 1. This is easy because $\pi_2(\mathrm{id}_{\mathbb{S}^2}) = \mathrm{id}_{\pi_2(\mathbb{S}^2)}$ so that $d(\mathrm{id}_{\mathbb{S}^2}) = \zeta(\zeta^{-1}(1)) = 1$.

Now, let us prove that $d(g \circ f) = d(g) \times d(f)$ for any $f, g : \mathbb{S}^2 \to_* \mathbb{S}^2$. This again comes mainly from the functoriality of π_2 ([30, after Lem. 7.3.3 and before Def. 8.4.2]), meaning that $\pi_2(g \circ f) = \pi_2(g) \circ \pi_2(f)$ holds. Hence:

$$\begin{aligned} d(g \circ f) &= \zeta\left(\pi_2(g \circ f)\left(\zeta^{-1}(1)\right)\right) \\ &= \zeta\left(\pi_2(g)\left(\pi_2(f)\left(\zeta^{-1}(1)\right)\right)\right) \\ &= \zeta\left(\pi_2(g)\left(\zeta^{-1}\left(\zeta\left(\pi_2(f)\left(\zeta^{-1}(1)\right)\right)\right)\right)\right) \\ &= \zeta\left(\pi_2(g)\left(\zeta^{-1}\left(d(f)\right)\right)\right) \end{aligned}$$

Here, we can use the fact that ζ is not just any equivalence but actually a group isomorphism. Because $\pi_2(g)$ is a homomorphism of groups, the composition $\zeta \circ \pi_2(g) \circ \zeta^{-1} : \mathbb{Z} \to \mathbb{Z}$ also is. Hence, for any $n : \mathbb{Z}$, one gets $(\zeta \circ \pi_2(g) \circ \zeta^{-1})(n) = (\zeta \circ \pi_2(g) \circ \zeta^{-1})(1) \times n$. We can then conclude:

$$d(g \circ f) = \zeta\left(\pi_2(g)\left(\zeta^{-1}(1)\right)\right) \times d(f) = d(g) \times d(f)$$ □

Corollary 3.8. *The degree of a self-equivalence of the sphere $\mathbb{S}^2$ is either 1 or -1.*

Proof. Given an equivalence $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$, pointed by $p : \varphi(\mathrm{N}) = \mathrm{N}$, any inverse ψ of φ is also pointed by the following path q:

$$\psi(\mathrm{N}) \overset{[\psi](p^{-1})}{=} \psi\varphi(\mathrm{N}) \overset{\alpha_\mathrm{N}}{=} \mathrm{N}$$

where $\alpha : \psi\varphi = \mathrm{id}$ is a witness of ψ being a left inverse for φ. In particular, $(\psi, q) \circ (\varphi, p)$ is an equivalence whose first component is equal to $\mathrm{id}_{\mathbb{S}^2}$. In determining the degree of this composite equivalence, Lemma 3.5 ensures that the path is irrelevant, and because $d(\mathrm{id}_{\mathbb{S}^2}, \mathrm{refl}_\mathrm{N}) = 1$, we can conclude that $d((\psi, q) \circ (\varphi, p)) = 1$ also holds. The previous result then proves that $d(\varphi, p)$ is a divisor of 1 in $\mathbb{Z}$, which is either 1 or -1 by decidability of the equality in $\mathbb{Z}$. □

Lemma 4.9. *Let A and B be types and $f : A \to B$ be a k-connected function ($k \geq -1$). For all $a_1, a_2 : A$, the function $[f] : a_1 = a_2 \to f(a_1) = f(a_2)$ is $(k-1)$-connected.*

Proof. Let $a_1, a_2 : A$, and $p : f(a_1) = f(a_2)$. We have to prove that $\|[f]^{-1}(p)\|_{k-1}$ is contractible. From [30, proof of Lem. 7.6.2] we get that the fiber $[f]^{-1}(p)$ is equivalent to the path type $(a_1, p) = (a_2, \mathrm{refl}_{f(a_2)})$ in the fiber $f^{-1}(f(a_2))$. Moreover, by [30, Thm. 7.3.12], $\|(a_1, p) = (a_2, \mathrm{refl}_{f(a_2)})\|_{k-1} \simeq |(a_1, p)|_k = |(a_2, \mathrm{refl}_{f(a_2)})|_k$. The latter path type is contractible if $\|f^{-1}(f(a_2))\|_k$ is contractible, which follows from the assumption of the lemma. □

Lemma 4.13. *Let A be a type. Then $\|A \simeq A\|_0$ is equivalent to the set of invertible elements in the monoid $\|A \to A\|_0$.*

Proof. For giving maps in both directions we use set truncation elimination. Being an equivalence and being invertible are propositions and we denote proofs if they exist simply by !. For each $(f, !) : A \simeq A$ we map $|(f, !)|_0$ to $(|f|_0, !)$; obviously, $|f|_0$ is invertible with inverse $|f^{-1}|_0$ if f is an equivalence. For the converse, if $x : \|A \to A\|_0$ is invertible, then we have the unique inverse x^{-1}. We may assume that $x \equiv |f|_0$ for some function $f : A \to A$ and $x^{-1} \equiv |g|_0$ for some $g : A \to A$. From the inverse law in the monoid $\|A \to A\|_0$, using [30, Thm. 7.3.12], one derives that both fg and gf

are merely equal to id_A. To prove the proposition that f is an equivalence, one can then assume actual witnesses of $fg = \mathrm{id}_A$ and $gf = \mathrm{id}_A$. Then g is a pseudo-inverse for f. Hence the map in the other direction is $(|f|_0, !) \mapsto |(f, !)|_0$. Clearly the maps in both directions are pseudo-inverses. □

Lemma 6.1. *Let A, B, and X be pointed types. We have the following equivalence: $(A \to_* (B \to_* \Omega\, X)) \simeq (A * B \to_* X)$.*

Proof. Let x_0 be the base point of X. Given $f : A \to_* (B \to_* \Omega\, X)$, construct $\bar{f} : A * B \to_* X$ by induction:

$$\begin{aligned} \bar{f}(\mathrm{inl}(a)) &:\equiv x_0 \quad \text{for } a : A \\ \bar{f}(\mathrm{inr}(b)) &:\equiv x_0 \quad \text{for } b : B \\ \left[\bar{f}\right](\mathrm{glue}(a,b)) &:= f(a)(b) \quad \text{for } a : A, b : B \end{aligned}$$

The map $\bar{f}$ is trivially pointed.

This construction $f \mapsto \bar{f}$ admits an inverse. Let a_0 and b_0 be the base points of A and B, respectively. Now, for any $a : A$ and $b : B$, one has an element $\tau_{a,b}$ of $\Omega(A * B)$ constructed as the following composition of paths:

$$\begin{tikzcd} \mathrm{inl}(a_0) \arrow[r, "\mathrm{glue}(a_0,b_0)"] & \mathrm{inr}(b_0) \arrow[r, "\mathrm{glue}(a,b_0)^{-1}"] & \mathrm{inl}(a) \arrow[r, "\mathrm{glue}(a,b)"] & \mathrm{inr}(b) \arrow[lll, bend left, "\mathrm{glue}(a_0,b)^{-1}"] \end{tikzcd} \tag{11}$$

Then, to any $g : A * B \to_* X$, one can map the function $\hat{g} : a \mapsto (b \mapsto \Omega(g)(\tau_{a,b}))$. The map $\hat{g}$ is a pointed, as $\tau_{a_0,b} = \tau_{a,b_0} = \mathrm{refl}_{\mathrm{inl}(a_0)}$ for all $a : A$ and $b : B$ by path algebra. The construction $g \mapsto \hat{g}$ provides an inverse to $f \mapsto \bar{f}$. A proof of this has been formalized in cubical Agda [8]. There, it's also checked that this equivalence arises from a wild adjunction, cf. Proposition B.10. □

Lemma 7.2. *For any $n \geq 1$, there is a group isomorphism:*

$$\xi_n : \pi_1(\mathbb{S}^n \to_* \mathbb{S}^n, \mathrm{id}) \simeq \pi_{n+1}(\mathbb{S}^n)$$

Proof. Recall the equivalence $\phi^n_{\mathbb{S}^n} : (\mathbb{S}^n \to_* \mathbb{S}^n) \to \Omega^n(\mathbb{S}^n)$, defined in Definition 4.4. Note that this is not a pointed equivalence. Indeed, $\phi^1_{\mathbb{S}^1}$ maps $(\mathrm{id}_{\mathbb{S}^1}, \mathrm{refl}_\bullet)$ to $\circlearrowleft : \Omega(\mathbb{S}^1)$ and from there, one can prove by induction that $\phi^n_{\mathbb{S}^n}$ maps $(\mathrm{id}_{\mathbb{S}^n}, \mathrm{refl}_\mathrm{N})$ to a point in $\Omega^n(\mathbb{S}^n)$ which is mapped to $\pm 1 : \mathbb{Z}$ by the set truncation $\Omega^n(\mathbb{S}^n) \to \pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$. However, the distinguished point of $\Omega^n(\mathbb{S}^n)$ is the iterated refl, which is sent to $0 : \mathbb{Z}$ by this set truncation.

Fortunately, there is an equivalence $\psi_n : \Omega^n(\mathbb{S}^n) \simeq \Omega^n(\mathbb{S}^n)$ defined as follows:

$$\psi_n(\alpha) :\equiv \phi^n_{\mathbb{S}^n}(\mathrm{id}_{\mathbb{S}^n}, \mathrm{refl}_\mathrm{N})^{-1} \cdot \alpha$$

This makes the composite $\psi_n \circ \phi^n_{\mathbb{S}^n}$ pointed by path algebra. The wanted equivalence is then:

$$\xi_n :\equiv \pi_1(\psi_n \circ \phi^n_{\mathbb{S}^n}) : \pi_1(\mathbb{S}^n \to_* \mathbb{S}^n) \simeq \pi_{n+1}(\mathbb{S}^n)$$

once we identify $\pi_{n+1}(\mathbb{S}^n)$ with $\|\Omega(\Omega^n(\mathbb{S}^n))\|_0$. □

Theorem 7.4. *The fundamental group of the type of symmetries of $\mathbb{S}^2$ is the cyclic group of order 2. That is, the following type has an element:*

$$\pi_1(\mathbb{S}^2 = \mathbb{S}^2, \mathrm{id}) = \mathbb{Z}/2\mathbb{Z}$$

where both sides of the equality are considered as groups.

Proof. We follow the proof (in classical topology) of [33]. Let us consider the long exact sequence given by Theorem 6.5 when specialized to $X \equiv \mathbb{S}^2$, $n = q = 1$ and $\beta \equiv (\mathrm{id}_{\mathbb{S}^2}, \mathrm{refl}_\mathrm{N}) : \mathbb{S}^2 \to_* \mathbb{S}^2$. Using $\phi^2_{\mathbb{S}^2}$ from Definition 4.4, the element in $\pi_2(\mathbb{S}^2)$ represented by $(\mathrm{id}_{\mathbb{S}^2}, \mathrm{refl}_\mathrm{N})$ is:

$$i_2 :\equiv |\Omega(\eta_{\mathbb{S}^1})(\circlearrowleft)|_0 \tag{12}$$

Recall that this element i_2 generates the group $\pi_2(\mathbb{S}^2)$, which is isomorphic to $\mathbb{Z}$.

We then have an exact sequence:

$$\pi_2(\mathbb{S}^2) \overset{[-,i_2]}{\to} \pi_3(\mathbb{S}^2) \overset{\kappa}{\to} \pi_1(\mathbb{S}^2 \to \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2}) \to \pi_1(\mathbb{S}^2) \tag{13}$$

We consider the type $\mathbb{S}^2 \to \mathbb{S}^2$ as pointed at the element $\mathrm{id}_{\mathbb{S}^2}$. Hence, we write only $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2)$ instead of $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2, \mathrm{id}_{\mathbb{S}^2})$. The sphere $\mathbb{S}^2$ is simply connected, that is $\pi_1(\mathbb{S}^2) = 0$. By exactness, it means that:

$$\pi_1(\mathbb{S}^2 \to \mathbb{S}^2) \simeq \mathrm{im}(\kappa) \simeq \pi_3(\mathbb{S}^2)/\ker(\kappa) \simeq \pi_3(\mathbb{S}^2)/\langle[i_2, i_2]\rangle \tag{14}$$

Recall from Lemma 7.2 that there is a group isomorphism $\xi_2 : \pi_3(\mathbb{S}^2) \simeq \mathbb{Z}$. Hence, one has:

$$\pi_1(\mathbb{S}^2 = \mathbb{S}^2) \simeq \pi_1(\mathbb{S}^2 \to \mathbb{S}^2) \simeq \mathbb{Z}/\xi_2([i_2, i_2])\mathbb{Z} \tag{15}$$

We now invoke the main result from [4] to conclude that $\pi_1(\mathbb{S}^2 = \mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$ (as groups). □

Theorem 7.5. *For $n \geq 3$, and any function $f : \mathbb{S}^n \to \mathbb{S}^n$, the fundamental group of the connected component of f in the type $\mathbb{S}^n \to \mathbb{S}^n$ is the cyclic group of order 2:*

$$\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) = \mathbb{Z}/2\mathbb{Z}$$

as groups.

Proof. As the group $\mathbb{Z}/2\mathbb{Z}$ has no non-trivial symmetries, the target type $\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) = \mathbb{Z}/2\mathbb{Z}$ is a proposition. So, as the sphere $\mathbb{S}^n$ is connected, we can suppose without loss of generality that f is pointed by a path $f_0 : f(N) = N$. In the rest of the proof, we make the abuse of writing f for both the pointed and unpointed map as the context allows to differentiate.

Consider the following fiber sequence:

$$(\mathbb{S}^n \to_* \mathbb{S}^n, f) \xrightarrow{\iota}_* (\mathbb{S}^n \to \mathbb{S}^n, f) \xrightarrow{\mathrm{ev}_*}_* \mathbb{S}^n$$

where $(\mathbb{S}^n \to_* \mathbb{S}^n, f)$ is the type $\mathbb{S}^n \to_* \mathbb{S}^n$ pointed at f, $(\mathbb{S}^n \to \mathbb{S}^n, f)$ is the type $\mathbb{S}^n \to \mathbb{S}^n$ pointed at f, ι is the map forgetting the pointing path, and ev_* is the evaluation at the point N of $\mathbb{S}^n$. By [30, Thm. 8.4.6], this induces a long exact sequence of groups:

$$\cdots \to \pi_{k+1}(\mathbb{S}^n) \to \pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, f) \to \pi_k(\mathbb{S}^n \to \mathbb{S}^n, f) \to \pi_k(\mathbb{S}^n) \to \ldots$$

Hence for every $1 \leq k < n-1$, this long sequence contains the following short exact sequence:

$$0 = \pi_{k+1}(\mathbb{S}^n) \to \pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, f) \to \pi_k(\mathbb{S}^n \to \mathbb{S}^n, f) \to \pi_k(\mathbb{S}^n) = 0$$

In other words, for every $1 \leq k < n-1$, one has

$$\pi_k(\mathbb{S}^n \to \mathbb{S}^n, f) \simeq \pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, f) \simeq \pi_k(\Omega^n \mathbb{S}^n) \simeq \pi_{n+k}(\mathbb{S}^n)$$

The group isomorphism in the middle is given by Remark 7.3 as follows: if we write r_n for the chosen point of $\Omega^n \mathbb{S}^n$ (that is the iterated refl on the north pole N of $\mathbb{S}^n$), then the remark allows to derive an isomorphism $\pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, f) \simeq \pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, (\phi^n_{\mathbb{S}^n})^{-1}(r_n))$, that can be composed with the isomorphism $\pi_k(\mathbb{S}^n \to_* \mathbb{S}^n, (\phi^n_{\mathbb{S}^n})^{-1}(r_n)) \simeq \pi_k(\Omega^n \mathbb{S}^n)$ induced by the equivalence $\phi^n_{\mathbb{S}^n} : (\mathbb{S}^n \to_* \mathbb{S}^n) \simeq \Omega^n \mathbb{S}^n$.

In particular for $n \geq 3$, $k :\equiv 1$ enters the condition, and one gets:

$$\pi_1(\mathbb{S}^n \to \mathbb{S}^n, f) \simeq \pi_{n+1}(\mathbb{S}^n) \simeq \mathbb{Z}/2\mathbb{Z}$$

□

Appendix B. Wild categories

The types $\mathcal{U}_*$ and $\mathcal{U}$ certainly do not form categories in the usual sense (the intended types of morphisms $A \to_* B$ and $A \to B$ between two objects A, B are not necessarily sets), but some constructions on $\mathcal{U}_*$ and $\mathcal{U}$ are reminiscent of functors. This motivates the few following definitions.

Definition B.1. A *wild category in $\mathcal{U}$* is a dependent tuple $(\mathcal{C}, \rightsquigarrow, \circ, \mathrm{id}, \alpha, \iota)$ where:

$$
\begin{aligned}
\mathcal{C} &: \mathcal{U} \\
\rightsquigarrow &: \mathcal{C} \times \mathcal{C} \to \mathcal{U} \\
\circ &: \prod_{a,b,c:\mathcal{C}} b \rightsquigarrow c \times a \rightsquigarrow b \to a \rightsquigarrow c \quad (\text{notation } g \circ f :\equiv \circ(a,b,c)(g,f)) \\
\mathrm{id} &: \prod_{a:\mathcal{C}} a \rightsquigarrow a \quad (\text{notation } \mathrm{id}_a :\equiv \mathrm{id}(a)) \\
\alpha &: \prod_{a,b,c,d:\mathcal{C}} \prod_{f:a\rightsquigarrow b} \prod_{g:b\rightsquigarrow c} \prod_{h:c\rightsquigarrow d} h \circ (g \circ f) = (h \circ g) \circ f \\
\iota &: \prod_{a,b:\mathcal{C}} \prod_{f:a\rightsquigarrow b} (f \circ \mathrm{id}_a = f) \times (f = \mathrm{id}_b \circ f)
\end{aligned}
$$

One makes the abuse to denote a wild category by only its carrier type $\mathcal{C}$ when all the remaining data are clear from context.

Remark B.2. To be fully rigorous, one must say a word about the levels of universe one allows in the definition of wild categories. Assuming we have a cumulative hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \ldots$, we choose to consider locally small wild categories, by which are meant wild categories such that $\mathcal{C} : \mathcal{U}_{i+1}$ and $\rightsquigarrow : \mathcal{C} \times \mathcal{C} \to \mathcal{U}_i$ for some $i \geq 0$.

Example B.3. The type $\mathcal{U}_i$ together with function types, identity functions and the usual composition, is a wild category in $\mathcal{U}_{i+1}$, for any i. (We will henceforth ignore universe levels and just write $\mathcal{U}$.) The elements α and ι are given by function extensionality.

Similarly, the type $\mathcal{U}_*$ together with $\to_*$, identity functions pointed by refl paths, and composition of pointed functions, is a wild category. Again, the elements α and ι are given by function extensionality, completed by path algebra.

Definition B.4. Let $\mathcal{C}, \mathcal{D}$ be two wild categories. A *wild functor* from $\mathcal{C}$ to $\mathcal{D}$ is a dependent 4-tuple (F_0, F_1, c, u) where:

$$
\begin{aligned}
F_0 &: \mathcal{C} \to \mathcal{D}, \\
F_1 &: \prod_{a,b:\mathcal{C}} a \rightsquigarrow b \to F_0(a) \rightsquigarrow F_0(b) \\
c &: \prod_{a,b,c:\mathcal{C}} \prod_{f:a\rightsquigarrow b} \prod_{g:b\rightsquigarrow c} F_1(g \circ f) = F_1(g) \circ F_1(f) \\
u &: \prod_{a:\mathcal{C}} F_1(\mathrm{id}_a) = \mathrm{id}_{F_0(a)}
\end{aligned}
$$

As for non-wild functors, we usually write both F_0 and F_1 only as F. The only relevant fact about c and u is that they exist, even though their types are not propositions. Therefore, we will often (abusively) denote a wild functor by its first two components only.

Proposition B.5. *Given wild functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, there is a composite wild functor $G \circ F : \mathcal{C} \to \mathcal{E}$ with first components:*

$$
\begin{aligned}
(G \circ F)_0 &:\equiv G_0 \circ F_0, \\
(G \circ F)_1 &:\equiv (a,b) \mapsto G_1(F_0(a), F_0(b)) \circ F_1(a,b)
\end{aligned}
$$

Proof. Denote c_F, u_F and c_G, u_G the witnesses of functoriality for F and G respectively. Then one gets:

$$\begin{aligned} c_{G\circ F}(a,b,c)(f,g) &:\equiv c_G(F_0(a),F_0(b),F_0(c),F_1(f),F_1(g)) \\ &\quad \cdot [G_1](c_F(a,b,c,f,g)) \\ u_{G\circ F}(a) &:\equiv u_G(F_0(a)) \cdot [G_1](u_F(a)) \end{aligned}$$

□

Example B.6.

(i) There is a wild functor Ω from $\mathcal{U}_*$ to itself which maps a pointed type A (pointed at a) to $\Omega\, A :\equiv (a = a)$ (pointed at refl_a), and maps a pointed function $f : A \to_* B$ to the pointed function $\Omega(f) : \Omega\, A \to_* \Omega\, B$ defined as follows:

$$\Omega(f)(p) :\equiv f_0 \cdot [f](p) \cdot f_0^{-1}$$

where $f_0 : f(a) = b$ is the path pointing f. The map $\Omega(f)$ is itself pointed by a path obtained by using path algebra as follows:

$$\Omega(f)(\mathrm{refl}_a) \equiv f_0 \cdot [f](\mathrm{refl}_a) \cdot f_0^{-1} = f_0 \cdot \mathrm{refl}_{f(a)} \cdot f_0^{-1} = \mathrm{refl}_b\,.$$

A careful exposition of the witness created this way can be found in [5, `core/lib/types/LoopSpace.agda`]. It can also be found the witnesses c and u justifying that Ω is a wild functor.

(ii) There is a wild functor Σ from $\mathcal{U}$ to $\mathcal{U}_*$ which maps a type A to $\Sigma\, A$ (pointed at the pole N), and maps a function $f : A \to B$ to the pointed function $\Sigma(f) : \Sigma\, A \to_* \Sigma\, B$ defined by induction as follows:

$$\Sigma(f)(\mathrm{N}) :\equiv \mathrm{N}, \quad \Sigma(f)(\mathrm{S}) :\equiv \mathrm{N}, \quad [\Sigma(f)] \circ \mathrm{mrd} := \mathrm{mrd} \circ f.$$

Note that $\Sigma(f)$ is pointed by $\mathrm{refl}_{\mathrm{N}}$ as it maps N to N by definition. The witnesses u and c of Definition B.4 are defined through easy inductions on the suspension, and a careful exposition can be found in [5, `core/lib/types/Suspension.agda`].

(iii) The join operation $- * B$ from $\mathcal{U}_*$ to itself, defined in Section 6, has the structure of a wild functor, mapping $f : A \to_* A'$ to the pointed function $f * B : A * B \to_* A' * B$ defined by induction as follows:

$$\begin{aligned} (f * B)(\mathrm{inl}(a)) &:\equiv \mathrm{inl}(f(a)), \qquad (f * B)(\mathrm{inr}(b)) :\equiv \mathrm{inr}(b), \\ [f * B](\mathrm{glue}(a,b)) &:= \mathrm{glue}(f(a), b). \end{aligned}$$

Note that $f * B$ is pointed by $[\mathrm{inl}]\, f_0 : \mathrm{inl}(f(a_0)) = \mathrm{inl}(a_0')$, where a_0 and a_0' are the base points of A and A', and $f_0 : f(a_0) = a_0'$ witnesses that f is pointed.

A formalization of u and c of Definition B.4 in cubical Agda is in [8].

(iv) There is a wild functor $U : \mathcal{U}_* \to \mathcal{U}$ that maps a pointed type (A, a_0) to A and a pointed maps (f, f_0) to f. The witnesses c and u are both given by reflexivity. In practice, the application of U is left implicit: we write for example $\Sigma(X)$ for $\Sigma(U(X))$ when X is a pointed type. In particular, depending on the context, the wild functor Σ can be considered to have domain $\mathcal{U}_*$.

(v) There is a wild functor $\|-\|_0$ from $\mathcal{U}_*$ to $\mathcal{U}$ which maps a pointed type (A, a_0) to $\|A\|_0$ and a pointed map (f, f_0) to the map $\|f\|_0$ defined by induction as $|x|_0 \mapsto |f(x)|_0$. The witnesses c and u are defined by using well-known inhabitants of the following two types, respectively:

$$\begin{aligned} \|g\|_0 \circ \|f\|_0 \circ |-|_0 &= |-|_0 \circ (g \circ f), \\ \|\mathrm{id}\|_0 \circ |-|_0 &= |-|_0 \circ \mathrm{id} \end{aligned}$$

(vi) Building on the previous examples and Proposition B.5, there is for each $n : \mathbb{N}$, a wild functor π_n from $\mathcal{U}_*$ to $\mathcal{U}$ which acts on objects and maps as $\|\Omega^n(-)\|_0$. The witnesses c and u are given by successive transport and composition of the same witnesses for Ω and $\|-\|_0$, as explained in Proposition B.5.

Definition B.7. Let be given a wild functor L from $\mathcal{C}$ to $\mathcal{D}$, and a wild functor R from $\mathcal{D}$ to $\mathcal{C}$. A *wild adjunction* of type $L \dashv R$ consists of the data of two dependent functions, the unit and the counit:

$$\eta : \prod_{a:\mathcal{C}} a \rightsquigarrow R \circ L(a), \quad \epsilon : \prod_{b:\mathcal{D}} L \circ R(b) \rightsquigarrow b,$$

together with elements witnessing the naturality of the unit and the counit

$$\mathrm{nat}_\eta : \prod_{a,a':\mathcal{C}} \prod_{f:a \rightsquigarrow a'} (R \circ L)(f) \circ \eta(a) = \eta(a') \circ f,$$

$$\mathrm{nat}_\epsilon : \prod_{b,b':\mathcal{D}} \prod_{f:b \rightsquigarrow b'} f \circ \epsilon(b) = \epsilon(b') \circ (L \circ R)(f),$$

and elements witnessing the triangle identities

$$\mathrm{ltr}_{\epsilon,\eta} : \prod_{a:\mathcal{C}} \epsilon(L(a)) \circ L(\eta(a)) = \mathrm{id}_{L(a)},$$

$$\mathrm{rtr}_{\eta,\epsilon} : \prod_{b:\mathcal{D}} R(\epsilon(b)) \circ \eta(R(b)) = \mathrm{id}_{R(b)}.$$

Here again, even though the types of nat_η, nat_ϵ, $\mathrm{ltr}_{\eta,\epsilon}$ and $\mathrm{rtr}_{\eta,\epsilon}$ are not propositions, one only cares about their existence, and therefore one usually omits them when denoting a wild adjunctions.

Remark B.8. As carefully proven and formalized in Agda [5, `theorems/homotopy/PtdAdjoint.agda`], any such wild adjunction induces a dependent function Φ that maps elements $a : \mathcal{C}$ and $b : \mathcal{D}$ to an equivalence

$$\Phi_{a,b} : (L(a) \rightsquigarrow b) \simeq (a \rightsquigarrow R(b))$$

given by $f \mapsto R(f) \circ \eta(a)$, with inverse the function given by $g \mapsto \epsilon(b) \circ L(g)$.

This dependent function is natural in the following sense: for any $a, a' : \mathcal{C}$ and $b, b' : \mathcal{D}$, there are elements of $\Phi_{a,b}(-) \circ f = \Phi_{a',b}(- \circ L(f))$ for any $f : a' \rightsquigarrow a$ and $\Phi_{a,b'}(g \circ -) = R(g) \circ \Phi_{a,b}(-)$ for any $g : b \rightsquigarrow b'$.

(To be precise, this is proven in [5] only for wild functors L and R from $\mathcal{U}_*$ to $\mathcal{U}_*$. This is the only case we need for this paper, hence we rely on this proof.)

Proposition B.9. *There is a wild adjunction* $\Sigma \dashv \Omega$.

Proof. We refer to [5, `theorems/homotopy/SuspAdjointLoop.agda`] for a proper proof. However, we give the unit η and counit ϵ for convenience.

Let A be a pointed type with distinguished point $a_0 : A$. Then define

$$\eta_A :\equiv \eta(A) : A \to_* \Omega\,\Sigma\,A, \quad a \mapsto \mathrm{mrd}(a_0)^{-1} \cdot \mathrm{mrd}(a)$$

which is pointed by path algebra. And define $\epsilon(A) : \Sigma\,\Omega\,A \to_* A$ by induction by setting

$$\epsilon(A)(\mathrm{N}) :\equiv a_0, \quad \epsilon(A)(\mathrm{S}) :\equiv a_0, \quad [\epsilon(A)] \circ \mathrm{mrd} := \mathrm{id}_{\Omega\,A}. \qquad \square$$

Proposition B.10. *Given any pointed type* B*, there is a wild adjunction* $- * B \dashv (B \to_* \Omega\,-)$.

Here, the right adjoint is the composition of the loop functor Ω with the covariant hom-functor from $\mathcal{U}_*$ to itself.

Proof. We refer to [8] for a formalization in cubical Agda. Here we just give the unit η and the counit ϵ.

Let A be a pointed type with base point a_0. Then define

$$\eta(A) : A \to_* (B \to_* \Omega(A * B)), \quad a \mapsto (b \mapsto \tau_{a,b}),$$

where $\tau_{a,b}$ is defined in (11). And if X is a pointed type with base point x_0, we define $\epsilon(X) : (B \to_* \Omega\,X) * B \to_* X$ by induction by setting,

$$\epsilon(X)(\mathrm{inl}(f)) :\equiv x_0, \quad \epsilon(X)(\mathrm{inr}(b)) :\equiv x_0, \quad [\epsilon(X)](\mathrm{glue}(f,b)) := f(b). \qquad \square$$

Remark B.11. It is possible to obtain Proposition B.10 from a more general adjunction on pointed types involving the *smash product*, $- \wedge B \dashv (B \to_* -)$, where the smash product $A \wedge B$ is defined as the following pushout involving the wedge inclusion,

$$\begin{tikzcd} A \vee B \arrow[r, "i"] \arrow[d] & A \times B \arrow[d, "\mathrm{inr}"] \\ \mathbf{1} \arrow[r, "\mathrm{inl}"'] & A \wedge B. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This adjunction is described in details in [31, Sec. 4.3.3], and has been formalized in Lean.

To get Proposition B.10, we'd also need the natural equivalence $A * B \simeq \Sigma(A \wedge B)$, and then we could calculate

$$\begin{aligned}(A * B \to_* X) \simeq (\Sigma(A \wedge B) \to_* X) &\simeq (A \wedge B \to_* \Omega\, X) \\ &\simeq (A \to_* (B \to_* \Omega\, X)).\end{aligned}$$

However, it is considerably less work to establish Proposition B.10 directly, since both $\Omega\, X$ and $B \to_* \Omega\, X$ are *purely homogeneous types*, where a pointed type (X, x_0) with base point $x_0 : X$ is called purely homogeneous if $(X, x_0) =_{\mathcal{U}_*} (X, x)$ for all $x : X$. Indeed, in this case, two pointed maps $(A, a_0) \to_* (X, x_0)$ are equal as long as the underlying maps $A \to X$ are. This follows from Lemma B.12 below, using $e_x :\equiv \Omega(w(x))$ where w is a witness of pure homogeneity.

This notion of pure homogeneity is closely related to being an H-space. Indeed, Lemma B.12 also applies to left-invertible H-spaces, taking $e_x :\equiv \Omega(\mu(-, x))$, where $\mu : X \to X \to X$ is binary operation such that $\mu(-, x)$ is invertible for all $x : X$ and satisfying $\mu(x_0, -) = \mathrm{id}_X$.

The Hopf construction [30, Sec. 8.5.2] applies as well to any connected purely homogeneous type (X, x_0), since also the maps $w(-, y)$ will be invertible thanks to $w(-, x_0)$ being homotopic to the identity. In particular, this means that we can only expect $\mathbb{S}^n$ to be purely homogeneous for $n = 0, 1, 3, 7$.

Note that any pointed connected type (X, x_0) is *merely homogeneous* in the sense that we have $\|(X, x_0) =_{\mathcal{U}_*} (X, x)\|$ for all $x : X$, as witnessed by the identity.

Lemma B.12. *Let (X, x_0) be a pointed type and $e_x : (x_0 = x_0) \to (x = x)$ a family of equivalences parametrized by $x : X$. Let (A, a_0) also be a pointed type, $(f, f_0), (g, g_0)$ two pointed maps $(A, a_0) \to_* (X, x_0)$, and $h : f \sim g$ a homotopy. Then $(f, f_0) = (g, g_0)$.*

Proof. It suffices to give a homotopy $h' : f \sim g$ such that $h'(a_0) = g_0^{-1} f_0$. Define $p :\equiv e_{g(a_0)}^{-1}(g_0^{-1} f_0 h(a_0)^{-1}) : x_0 = x_0$ and $h'(a) :\equiv e_{g(a)}(p) h(a) : f(a) = g(a)$ for all $a : A$. Then indeed $h' : f \sim g$. Moreover, $h'(a_0) \equiv e_{g(a_0)}(e_{g(a_0)}^{-1}(g_0^{-1} f_0 h(a_0)^{-1})) h(a_0) = g_0^{-1} f_0$. □

Remark B.13. Define a *wild monoid* to be a pointed type $(M, 1)$ equipped with a function $\cdot : M \times M \to M$ and elements:

$$\alpha : \prod_{x,y,z:M} x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$\iota : \prod_{x:M} (x \cdot 1 = x) \times (x = 1 \cdot x)$$

If M is a set, then the types of α and ι become propositions and we are left with just a usual monoid. In particular, for any wild monoid M, the type $\|M\|_0$ has the induced monoid structure.

Note that any wild category $\mathcal{C}$, for any object $a : \mathcal{C}$, induces a *wild monoid* structure on $a \rightsquigarrow a$, pointed at id_a with the multiplication $\circ(a, a, a)$, and the witnesses $\alpha(a, a, a, a)$ and $\iota(a, a)$ coming from $\mathcal{C}$.

For example, $\mathbb{S}^n \to_* \mathbb{S}^n$ and $\mathbb{S}^n \to \mathbb{S}^n$, with composition of (pointed) maps, are wild monoids for each $n : \mathbb{N}$. Set truncation yields ordinary monoids $\|\mathbb{S}^n \to_* \mathbb{S}^n\|_0$ and $\|\mathbb{S}^n \to \mathbb{S}^n\|_0$.