THEORETICAL AND COMPUTATIONAL ASPECTS OF HILBERT - MAASS FORMS

SUJEET SINGH AND FREDRIK STRÖMBERG

Contents

1. Introduction

One of the key areas of modern number theory is that of modular forms. In the classical setting these correspond to holomorphic functions on the complex upper half-plane with certain transformation properties under subgroups of the modular group, $\mathrm{SL}_2(\mathbb{Z})$.

The theory of classical modular forms, associated geometry and representation theory played an important role in Wiles, Taylor et al.'s proof of Fermat's Last Theorem, through the so-called modularity theorem (formerly conjecture) for elliptic curves over $\mathbb{Q}$. To extend this conjecture to a totally real number field K it is necessary to introduce Hilbert modular forms, which can be represented by holomorphic functions on products of upper half-planes and with certain transformation properties under subgroups of the Hilbert modular group, $\mathrm{SL}_2(\mathcal{O}_K)$.

In this paper we extend this even further and study non-holomorphic Hilbert modular forms, also called Hilbert - Maass cusp forms.

One of the most striking applications of non-holomorphic Hilbert modular forms is towards the resolution of Hilbert's 11-th problem about quadratic forms in many variables. This, in turn, has far-reaching applications, for instance in quantum computing, where the strong approximation properties of certain quadratic forms can be used to design universal quantum gates [**?**].

Similar to the situation for classical Maass cusp forms, there are no explicit examples of general Hilbert - Maass cusp forms, apart from some special cases, including lifts from subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and one of the primary goals of the project presented here was to develop algorithms to find explicit numerical examples to obtain a better understanding of both individual Hilbert - Maass cusp forms as well as the arithmetic and analytic tools required to study them.

The Hilbert - Maass cusp forms are eigenfunctions of an invariant Laplace - Beltrami operator and their study is closely related to the spectral theory for the associated Hilbert modular variety.

For the sake of clarity and to avoid technical arguments and heavy notation we restrict the current paper to a discussion of the case of narrow class number one, and leave the general case to a forthcoming paper, which is currently in preparation. All algorithms used and discussed in this paper can be obtained from GitHub [**?**].

In the following sections we will first provide background and notations for Hilbert modular groups, followed by an overview of the most relevant definitions and fundamental results about Hilbert - Maass forms. Subsequently we develop an explicit analogue of the classical Hecke theory for Hilbert - Maass forms in the case of totally real number fields with narrow class number one.

2. Background and Notation

2.1. **Number Fields and embeddings.** Let $K/\mathbb{Q}$ be a totally real number field of degree $n > 1$ and class number $h(K) = 1$ with ring of integers $\mathcal{O}_K$ and embeddings $\sigma_i : K \hookrightarrow \mathbb{R}$, $i = 1, 2, \ldots, n$. The norm $\mathrm{N}(\alpha) = \prod \sigma(\alpha)$ and trace $\mathrm{Tr}(\alpha) = \sum \sigma(\alpha)$ over $\mathbb{Q}$ are defined as usual and an element $\alpha \in K$ is said to be toally positive, $\alpha \gg 0$, if $\sigma(\alpha) > 0$ for all embeddings σ.

To simplify notation we introuce the ring $\mathbb{C}_K = \mathbb{C} \otimes_{\mathbb{Q}} K$ viewed as an algebra over both K and $\mathbb{C}$ with the natural embeddings. It is then possible to define multiplication in $\mathbb{C}_K$ by considering pure tensors. For $\mathbf{z}, \mathbf{z}' \in \mathbb{C}_K$ with $\mathbf{z} = z \otimes \alpha$ and $\mathbf{z}' = z' \otimes \alpha'$ for some $z, z' \in \mathbb{C}$ and $\alpha, \alpha' \in K$ we define

$$\mathbf{z}\mathbf{z}' = zz' \otimes \alpha\alpha', \quad z'\mathbf{z} = \mathbf{z}z' = (z'z) \otimes \alpha, \quad \alpha'\mathbf{z} = \mathbf{z}\alpha' = z \otimes (\alpha'\alpha),$$

and then extend these operations to the whole of $\mathbb{C}_K$ by linearity. The embeddings are extended to $\sigma_i : \mathbb{C}_K \hookrightarrow \mathbb{C}$ by $\sigma_i(z \otimes \alpha) = z\sigma_i(\alpha)$ and norm, trace, real and imaginary parts are defined on $z \otimes \alpha \in \mathbb{C}_K$ by $\mathrm{N}(z \otimes \alpha) = z^n \mathrm{N}(\alpha)$, $\mathrm{Tr}(z \otimes \alpha) = z\mathrm{Tr}(\alpha)$, $\Re(z \otimes \alpha) = \Re(z) \otimes \alpha$ and $\Im(z \otimes \alpha) = \Im(z) \otimes \alpha$ and then extended linearly. For any $\mathbf{z} \in \mathbb{C}_K$ we set $z_j = \sigma_j(\mathbf{z})$ and define the sign $\mathrm{sign}(\mathbf{z}) = \frac{\mathbf{z}}{|\mathbf{z}|}$ where $\sigma_j(|\mathbf{z}|) = |\sigma_j(\mathbf{z})|$.

We also consider the subring $\mathbb{R}_K = \mathbb{R} \otimes_{\mathbb{Q}} K \subset \mathbb{C}_K$ and the generalized upper half-plane $\mathbb{H}_K = \{\mathbf{z} \in \mathbb{C}_K \mid \Im(\mathbf{z}) \gg 0\}$.

For explicit numerical computations we often make use of the identifications $\mathbb{C}_K \simeq \mathbb{C}^n$, $\mathbb{R}_K \simeq \mathbb{R}^n$ and $\mathbb{H}_K \simeq \mathbb{H}^n$, where $\mathbb{H} = \{x + iy \mid y > 0\}$ is the usual upper half-plane.

We write by $G = GL_2^+(\mathbb{R})^n$. We define the slash operator on any function on $\mathbb{H}^n$ by

$$f \mid \mathbf{g} = f\left(\frac{\mathbf{az}+\mathbf{b}}{\mathbf{cz}+\mathbf{d}}\right), \quad \mathbf{g} \in G.$$

For $\mathbf{x} \in \mathbb{R}^n$, we define the signature $sgn\ \mathbf{x} \in \{\pm 1\}^n$ by

$$(sgn\ x)_v = \frac{x_v}{|x_v|}.$$

2.2. **Hilbert modular groups.** It is possible to equip the generalised upper half-plane with a hyperbolic metric $d\mathbf{s}(\mathbf{z}) = \mathbf{y}^{-1}|d\mathbf{z}|$ and volume measure $d\boldsymbol{\mu}(\mathbf{z}) = \mathbf{y}^{-2}d\mathbf{x}d\mathbf{y}$ where $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{H}_K$. It is easy to verify that the corresponding isometries of $\mathbb{H}_K$ are given by elements or $\mathrm{PGL}^+(K) \simeq \mathrm{GL}^+(K)/\{\pm 1_2\}$ corresponding to matrices in $M_2(K)$ with totally positive determinants, acting on $\mathbb{H}_K$ via fractional linear transformations

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathbf{z} \mapsto (a\mathbf{z}+b)(c\mathbf{z}+d)^{-1}.$$

For the purpose of this paper we consider the Hilbert modular group to be $\Gamma_K = \mathrm{PSL}_2(\mathcal{O}_K)/\{\pm 1_2\}$ but stress that in other contexts, e.g. when studying Hilbert modular forms of non-zero weight it is more common to study e.g. $\mathrm{SL}_2(\mathcal{O}_K)$ or $\mathrm{GL}^+(\mathcal{O}_K)$. todo[inline]Mention companion groups and cusps / class number > 1

3. Hilbert Maass Forms

3.1. **Introduction and definition.** Recall that the Laplace–Beltrami operator acting on the hyperbolic upper half-plane $\mathbb{H}$ is an essentially self-adjoint, positive semi-definite, elliptic differential operator defined as

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

where $z = x + iy \in \mathbb{H}$. If we write $\Delta_j = \Delta \circ \sigma_j$ and and let $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_n)$, then for a twice-differentiable function $f : \mathbb{H}_K \to \mathbb{C}$ we define

$$\boldsymbol{\Delta} f = (\Delta_1 f, \ldots, \Delta_n f)$$

and say that f is an eigenform of $\boldsymbol{\Delta}$ with eigenvalue $\boldsymbol{\lambda} \in \mathbb{C}_K$ if

$$\boldsymbol{\Delta} f = \boldsymbol{\lambda} f,$$

i.e. if $\Delta_j f = \lambda_j f$ for all j. Since Δ_j is positive semi-definite we know that $\lambda_j \geq 0$ and we write $\boldsymbol{\lambda} = \frac{1}{4} + \mathbf{R}^2 = \mathbf{s}(1-\mathbf{s})$ as usual with $R_j \in i[0, 1/2] \cup [0, \infty)$ or $s_j \in [0, 1] \cup \{1/2 + i\mathbb{R}\}$.

Definition 1. A real-analytic function $f : \mathbb{H}_K \to \mathbb{C}$ is said to be a *Hilbert-Maass form* for the group Γ_K if

(1) $(f \mid A)(\mathbf{z}) = f(\mathbf{z})$ for all $A \in \Gamma_K$,
(2) $\boldsymbol{\Delta} f = \lambda f$,
(3) $\int_{\Gamma_K \backslash \mathbb{H}_K} |f|^2 d\boldsymbol{\mu} < 0$

In addition, f is said to be a cusp form if

4. $\lim f(A_\lambda \mathbf{z}) \to 0$ as $\Im \mathbf{z} \to \infty$ for all cusps $\lambda \in P_1(K)$.

The space of Hilbert-Maass forms for Γ_K is denoted by $M(\Gamma_K)$ and the subspace of Maass cusp forms by $S(\Gamma_K)$.

3.2. **Fourier expansions.** Since we assume that the class number of K is one it is known that there is only one Γ_K inequivalent cusp, denoted by ∞. The stabiliser of this cusp in Γ_K is given by

$$\Gamma_{K\ \infty} = \left\{ \left(\begin{smallmatrix} \varepsilon & \alpha \\ 0 & \varepsilon^{-1} \end{smallmatrix} \right) : \varepsilon \in \mathcal{O}_K^{\times}, \alpha \in \mathcal{O}_K \right\} \simeq (\mathcal{O}_K^{\times})^2 \ltimes \mathcal{O}_K.$$

The subgroup of translations is usually identified with the ring of integers $\mathcal{O}_K$, which we view as a lattice in $\mathbb{R}^n$ together with the trace form

$$B(\alpha, \beta) = \mathrm{Tr}(\alpha\beta).$$

The dual lattice of $\mathcal{O}_K$ with respect to this bilinear form is the inverse different

$$\mathfrak{d}^{-1} = \mathcal{O}_K^{\#} = \{\alpha \in \mathbb{Q} \otimes \mathcal{O}_K : B(\alpha, \beta) \in \mathbb{Z}\ \forall \beta \in \mathcal{O}_K\}.$$

By the general theory of Fourier series in $\mathbb{R}^n$ and component-wise separation of variables in (2) it follows that a Hilbert-Maass cusp form f with eigenvalue $\lambda = \frac{1}{4} + \mathbf{R}^2$ has a Fourier-Bessel expansion of the form

$$f(\mathbf{z}) = \sum_{0 \neq \mathbf{v} \in \mathfrak{d}^{-1}} a(\mathbf{v}) \kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) e\left(\mathrm{Tr}(\mathbf{x}\mathbf{v})\right),$$

where

$$\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) = \sqrt{\mathrm{N}\mathbf{y}}\mathrm{N}K_{i\mathbf{R}}(2\pi|\mathbf{v}|\mathbf{y}) = \prod_{j=1}^{n} \sqrt{y_j} K_{iR_j}(2\pi|v_j|y_j)$$

and $K_{iR}(2\pi|v|y)$ is a standard K-Bessel function. It follows that

$$f(\mathbf{z}) = \sum_{0 \neq \mathbf{v} \in \mathfrak{d}^{-1}} a(\mathbf{v}) \kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) e\left(\mathrm{Tr}(\mathbf{x}\mathbf{v})\right).$$

For holomorphic Hilbert modular forms it is known that the Fourier series can be restricted to totally positive indices. This was first proven for $\mathbb{Q}(\sqrt{5})$ by Gï¿œtzky and in general by Gundlach, although it is often referred to as the Gï¿œtzky - Koecher principle. Unfortunately the same argument does not apply here and the invariance under $(\mathcal{O}_K^{\times})^2$ only implies that

$$a(\varepsilon^2 \mathbf{v}) = a(\mathbf{v}) \quad \text{for all } \varepsilon \in \mathcal{O}_K^{\times}.$$

This condition can either be used to aid in computations by reducing the number of coefficients necessary, or as a way of estimating the accuracy of computed coefficients. It is known the Fourier coefficients of Maass cusp forms for the modular group $\mathrm{PSL}_2(\mathbb{Z})$ satisfy $a(n) = O(|n|^{1/2})$ as $|n| \to \infty$. Estimates of this form were already known to Maass [8] (see also e.g. [4, p.\{} 585-587] or [7, Thm.\{} 3.2]).

Theorem 2. *Let $f \in S(\Gamma_K)$ and let $a(\mathbf{v})$ be the $\mathbf{v}$th Fourier coefficient of f. Then*

$$|a(\mathbf{v})| \leq C\sqrt{\mathrm{N}(|\mathbf{v}|)},$$

where C is a constant depending on f.

Proof. The proof is analogous to [7, Thm. 3.2] and follows from Parseval's identity in $\mathbb{R}^n$

$$\sum_{\mathbf{v} \in \mathfrak{d}^{-1}} |c(\mathbf{v}, y)|^2 = \frac{1}{V} \int_{\mathcal{O}_K \backslash \mathbb{R}^n} |f(\mathbf{z})|^2 dx_1 \cdots dx_n$$

together with the estimate

$$\int_{|s|}^{\infty} K_{s-1/2}^2(y)y^{-1}dy \gg \frac{e^{-\pi|s|}}{|s|}$$

applied to each variable separately. □

4. Hecke Theory for Hilbert - Maass forms

As an explicit description of the Hecke theory for non-holomorphic Hilbert modular forms does not appear in the literature we will provide a self-contained brief exposition, focussing on the simplest case. For the remainder of this section we will assume that the narrow class number of K, $h^+(K) = 1$ as this is the case which most closely resembles the theory for modular and Maass forms on $\mathrm{SL}_2(\mathbb{Z})$.

Since factorisation of integers into prime numbers in $\mathbb{Z}$ generalises to factorisation of integral ideals into prime ideals in $\mathcal{O}_K$ it is clear that the theory of multiplicative functions on $\mathcal{O}_K$ becomes particularly simple when this is a unique factorisation domain, i.e. if the class number $h(K) = 1$. The further restriction to $h^+(K) = 1$ ensures that for every integral ideal $\mathcal{N} \subseteq \mathcal{O}_K$ there exists a totally positive element $\mathbf{n} \in \mathcal{N}$ such that $\mathcal{N} = \mathbf{n}\mathcal{O}_K$ and we therefore obtain a theory resembling that of $\mathrm{SL}_2(\mathbb{Z})$ where Hecke operators are usually indexed by positive integers.

For an integral ideal $\mathcal{N} = \mathbf{n}\mathcal{O}_K$ we define the set

$$\Delta(\mathcal{N}) = \{g \in \mathrm{GL}_2(\mathcal{O}_K) \mid \mathbf{n}^{-1}\det(g) \in \mathcal{O}_K^{\times,+}\},$$

where $\mathcal{O}_K^{\times,+}$ is the set of totally positive units. Let us write by

$$\mathcal{Z} := \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \mid \varepsilon \in \mathcal{O}_K^{\times} \right\}$$

the center of $GL_2^+(\mathcal{O}_K)$. The set $\Delta(\mathcal{N})$ admits both a right and left action by $\mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)$ and we have the following.

Lemma 3. *The set $\Delta(\mathcal{N})$ decomposes into a finite number of left cosets under the action of $\mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)$*

$$\Delta(\mathcal{N}) = \bigsqcup_i \mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)\beta_i$$

and a complete set of representatives is given by

$$X(\mathcal{N}) = \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ 0 & \mathbf{d} \end{pmatrix} \mid \mathbf{a}, \mathbf{d} \in \mathcal{O}_K/\mathcal{O}_K^{\times}, \mathbf{b} \in \mathcal{O}_K,\ \mathbf{d} \gg 0, \mathbf{ad} = \mathbf{n},\ \mathbf{b} \in \mathcal{O}_K/\mathbf{d}\mathcal{O}_K \right\}.$$

Proof. Let $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta(\mathcal{N})$ such that $ad - bc = n\varepsilon$, for some $\varepsilon \in \mathcal{O}_K^{\times,+}$. If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then we have

$$\gamma\beta = \begin{pmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{pmatrix} = \begin{pmatrix} a_1, & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

We will show that there exist $\gamma \in SL_2(\mathcal{O}_K)$ such that $Ca + Dc = 0$. Let $a\mathcal{O}_K + c\mathcal{O}_K = t\mathcal{O}_K$. The solution space of $Ca + Dc = 0$ is of the form $C = \frac{-\lambda c}{t}$ and $D = \frac{\lambda a}{t}$ for some $\lambda \in \mathcal{O}_K$. To extend the pair (C, D) to be an element of $\mathrm{SL}_2(\mathcal{O}_K)$, we need that $C\mathcal{O}_K + D\mathcal{O}_K = \mathcal{O}_K$. This implies $\lambda \in \mathcal{O}_K^{\times}$. We take $C = -\frac{c}{t}$ and

$D = \frac{a}{t}$ and hence there exist $A, B \in \mathcal{O}_K$ such that $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$ and $\gamma\beta = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$, $a_1 d_1 = n\varepsilon$. We check that $d_1 = Cb + Dc = \frac{n\varepsilon}{t}$ and $a_1 = t$. Since $h^+(K) = 1$ there exist $\varepsilon_1 \in \mathcal{O}_K^\times$ such that $\varepsilon_1^2 = \varepsilon$. Let $\xi = \begin{pmatrix} \varepsilon_1^{-1} & 0 \\ 0 & \varepsilon_1^{-1} \end{pmatrix} \in \mathcal{Z}$. Now $\xi\gamma\beta = \begin{pmatrix} a_1' & b_1' \\ 0 & d_1' \end{pmatrix}$, where $a_1' d_1' = n$. Also we check that $d_1' = \frac{n\varepsilon}{t\varepsilon_1}$ and $a_1' = \frac{t}{\varepsilon}$. We choose the sign of t ($sgn(t) = sgn(\varepsilon_1)$) in such a way that $d_1' \gg 0$. Now suppose $\beta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $\beta' = \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$ are $\mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)$ equivalent. This implies that $\beta'\beta^{-1} = \begin{pmatrix} 1 & (b-b')/d \\ 0 & 1 \end{pmatrix} \in \mathcal{Z}SL_2(\mathcal{O}_K)$. This holds if and only if $b - b' \in d\mathcal{O}_K$. Therefore we have proved that the representative of γ for the action of $\mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)$ is uniquely determined. □

Definition 4. For an integral ideal $\mathcal{N}$ we define the Hecke operator of index $\mathcal{N}$, $T_{\mathcal{N}} : M(\Gamma_K) \to M(\Gamma_K)$ by

$$T_{\mathcal{N}}(f) = \frac{1}{\sqrt{N(\mathcal{N})}} \sum_{\beta \in \mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)\backslash\Delta(\mathcal{N})} f \mid \beta,$$

for $f \in M(\Gamma_K)$, where the sum is taken over any complete set of left-coset representatives for $\mathcal{Z}\mathrm{SL}_2(\mathcal{O}_K)\backslash\Delta(\mathcal{N})$.

Using the representatives from Lemma 3 we obtain the explicit expression

$$T_{\mathcal{N}}(f) := \frac{1}{\sqrt{\mathrm{N}(\mathcal{N})}} \sum_{\substack{\mathbf{ad}=\mathbf{n} \\ \mathbf{d}\gg 0}} \sum_{\mathbf{b}\,(\mathrm{mod}*\,\mathbf{d})} f\left(\frac{\mathbf{az}+\mathbf{b}}{\mathbf{d}}\right), \tag{4.1}$$

where the inner sum is over a set of residue classes modulo $\mathbf{d}\mathcal{O}_K$ and we recall that $\mathbf{b} \equiv \mathbf{b}' \,(\mathrm{mod})^*\, \mathbf{d}$ if and only if $\mathbf{b} - \mathbf{b}' \in \mathbf{d}\mathcal{O}_K$.

Theorem 5. *Suppose that $f \in M(\Gamma_K)$ has Fourier expansion*

$$f(\mathbf{z}) = \sum_{\mathbf{v}\in\mathfrak{d}^{-1}} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) e\left(\mathrm{Tr}(\mathbf{xv})\right),$$

and

$$T_{\mathcal{N}}(f)(\mathbf{z}) = \sum_{\mathbf{v}\in\mathfrak{d}^{-1}} b(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) e\left(\mathrm{Tr}(\mathbf{xv})\right).$$

Then

$$b(\mathbf{v}) = \sum_{\substack{\mathbf{d}\gg 0 \\ \mathbf{d}\mid\mathbf{n},\, \frac{\mathbf{v}}{\mathbf{d}}\in\mathfrak{d}^{-1}}} a\left(\frac{\mathbf{vn}}{\mathbf{d}^2}\right).$$

Proof. A direct calculation shows that

$$\sum_{\mathbf{b}\,(\mathrm{mod}\,\mathbf{d})} f\left(\frac{\mathbf{az}+\mathbf{b}}{\mathbf{d}}\right) = \sum_{\mathbf{v}\in\mathfrak{d}^{-1}} a(\mathbf{v})\kappa_{i\mathbf{R}}\left(\mathbf{v}, \frac{\mathbf{ay}}{\mathbf{d}}\right) e\left(\mathrm{Tr}\left(\frac{\mathbf{axv}}{\mathbf{d}}\right)\right)$$
$$\times \sum_{\mathbf{b}\,(\mathrm{mod}\,\mathbf{d})} e\left(\mathrm{Tr}\left(\frac{\mathbf{bv}}{\mathbf{d}}\right)\right)$$

and it is easy to see that the following orthogonality relation holds

$$\sum_{\mathbf{b}\,(\mathrm{mod}\,\mathbf{d})} e\left(\mathrm{Tr}\left(\frac{\mathbf{bv}}{\mathbf{d}}\right)\right) = \begin{cases} N(\mathbf{d}) & \text{if } \frac{\mathbf{v}}{\mathbf{d}} \in \mathfrak{d}^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

See also Lemma 14 below for a more general analogue. Using the explicit expression in (4.1) we now find that

$$\begin{aligned} T_{\mathcal{N}}(f)(\mathbf{z}) &= \frac{1}{\sqrt{N(\mathbf{n})}} \sum_{\substack{\mathbf{ad}=\mathbf{n} \\ \mathbf{d}\gg 0}} \mathrm{N}(\mathbf{d}) \sum_{\mathbf{v}\in\mathbf{d}\mathfrak{d}^{-1}} a(\mathbf{v})\kappa_{i\mathbf{R}}\left(\mathbf{v}, \frac{\mathbf{ay}}{\mathbf{d}}\right) e\left(\mathrm{Tr}\left(\frac{\mathbf{axv}}{\mathbf{d}}\right)\right) \\ &= \frac{1}{\sqrt{N(\mathbf{n})}} \sum_{\substack{\mathbf{ad}=\mathbf{n} \\ \mathbf{d}\gg 0}} \mathrm{N}(\mathbf{d}) \sum_{\mathbf{v}\in\mathfrak{d}^{-1}} a(\mathbf{dv})\kappa_{i\mathbf{R}}\left(\mathbf{dv}, \frac{\mathbf{ay}}{\mathbf{d}}\right) e\left(\mathrm{Tr}\left(\mathbf{axv}\right)\right) \\ &= \sum_{\substack{a|\mathbf{n} \\ a\gg 0}} \sum_{\mathbf{v}\in\mathfrak{d}^{-1}} a\left(\frac{\mathbf{nv}}{\mathbf{a}}\right) \kappa_{i\mathbf{R}}\left(\mathbf{av}, \mathbf{y}\right) e\left(\mathrm{Tr}\left(\mathbf{axv}\right)\right). \end{aligned}$$

The conclusion now follows from the uniqueness of Fourier expansions. □

It is easy to see from the above theorem that $T_{\mathcal{N}}$ leaves the space of cusp forms invariant and we can also deduce the following analogue of the classical multiplicativity.

Corollary 6. *If $\mathcal{M} = \mathbf{m}\mathcal{O}_K$, and $\mathcal{N} = \mathbf{n}\mathcal{O}_K$ with $\mathbf{n}, \mathbf{m} \gg 0$ then*

$$T_{\mathcal{M}}T_{\mathcal{N}} = \sum_{\substack{d|gcd(\mathfrak{m},\mathfrak{n}) \\ d\gg 0}} T_{\frac{\mathfrak{m}\mathfrak{n}}{d^2}\mathcal{O}_K} \tag{4.2}$$

In particular, if $\mathcal{M} + \mathcal{N} = \mathcal{O}_K$ then

$$T_{\mathcal{M}}T_{\mathcal{N}} = T_{\mathcal{M}\mathcal{N}}.$$

Furthermore, if $\mathcal{P}$ is a prime ideal and $v \geq 1$ then

$$T_{\mathcal{P}^{v+1}} = T_{\mathcal{P}}T_{\mathcal{P}^v} - T_{\mathcal{P}^{v-1}}.$$

Proof. This result follows immediately from Theorem 5 by comparing the action on the Fourier coefficients of left- and right-hand side. □

Another important property of the Hecke operators is that they commute with all Laplace operators, in other words, for all ideals $\mathcal{N}$ and $i = 1, \ldots, n$

$$T_{\mathcal{N}}\Delta_i = \Delta_i T_{\mathcal{N}}$$

Let $M(\Gamma_K, \boldsymbol{\lambda})$ be the finite-dimensional space of Hilbert Maass forms with eigenvalue $\boldsymbol{\lambda}$ for $\boldsymbol{\Delta}$ and $S(\Gamma_K, \boldsymbol{\lambda})$ the subspace of cusp forms. It is clear that the Hecke operators acts on both $M(\Gamma_K, \boldsymbol{\lambda})$ and $S(\Gamma_K, \boldsymbol{\lambda})$. The latter becomes a finite dimensional Hilbert space equipped with the Petersson inner product

$$\langle f, g\rangle = \int_{\Gamma_K\backslash\mathbb{H}_K} f\overline{g}d\boldsymbol{\mu}$$

Theorem 7. *If $f, g \in S_k(\Gamma_K, \boldsymbol{\lambda})$ then*

$$\langle T_{\mathcal{N}}(f), g\rangle = \langle f, T_{\mathcal{N}}(g)\rangle.$$

More precisely $T_{\mathcal{N}}$ is Hermitian with respect to the Petersson inner product.

We use above theorem to get an orthogonal basis for $S_k(\Gamma, \boldsymbol{\lambda})$ consisting of simultaneous eigenvectors (eigenform) for all the Hecke operators $T_{\mathcal{N}}$.

4.1. **Reflection Operators.** In this section we define the reflection operators which will be used to characterize any eigenform f by the coefficients at totally positive element in $\mathfrak{d}^{-1}$. For any v, where $1 \leq v \leq n$, and function f on $\mathbb{H}^n$, we define $K_v(f)$ by

$$(K_v f)(z) = f(z'), \quad \text{where} \;\; z'_w = z_w \text{ for } w \neq v \text{ and } z'_v = -\overline{z_v}.$$

For any $\mathbf{g} \in G$, we check that

$$(K_v f) \mid \mathbf{g} = K_v(f \mid e_v \mathbf{g} e_v), \text{ where } e_v = \begin{pmatrix} (-1)_v & 0 \\ 0 & 1 \end{pmatrix} \in G$$

Here $(-1)_v$ is just the notation which denotes that if $w \neq v$ then the entry is 1 and the vth component is -1. Let $\epsilon \in \mathcal{O}_K^\times$ be such that sgn $\epsilon = (-1)_v$ and let $\beta = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \in G$. Now for each v we define the reflection operator Z_v on $S_k(\Gamma_K, \boldsymbol{\lambda})$ by

$$Z_v(f) = K_v(f \mid \beta e_v)$$

It is easy to verify that $Z_v^2 = I$. Therefore the eigenvalues of Z_v will be ± 1. We also check that the Hecke operators $T_{\mathcal{N}}$ commute with the reflection operators Z_v. Therefore it is sensible to break up $S_k(\Gamma_K, \lambda)$ into simultaneous eigenspaces of Z_v.for all v. Suppose f is an eigenform for Z_v with eigenvalue $t_v \in \{\pm 1\}$. Then we have $Z_v(f) = t_v f$. Comparing the Fourier coefficients we get

$$a(\mathbf{v}) = t_v a(\epsilon_v \mathbf{v}), \tag{4.3}$$

where $\epsilon_v \in \mathcal{O}_K^\times$ such that sgn $\epsilon_v = (-1)_v$. We utilize the equation above to calculate coefficients based on coefficients at totally positive elements.

Proposition 8. *Let $\delta \gg 0$ be such that $\mathfrak{d}^{-1} = \delta \mathcal{O}_K$. Let f be a non-zero eigenform in $S(\Gamma_K, \boldsymbol{\lambda})$ with Fourier series expansion*

$$f(\mathbf{z}) = \sum_{\mathbf{v} \in \mathfrak{d}^{-1}} a(\mathbf{v}) \kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y}) e\left(\text{Tr}(\mathbf{x}\mathbf{v})\right),$$

If $T_{\mathcal{N}}(f) = \gamma(\mathcal{N}) f$ with $\mathcal{N} = \mathfrak{n}\mathcal{O}$ then

$$a(\delta \mathfrak{n}) = \gamma(\mathcal{N}) a(\delta)$$

and

$$a(\delta) \neq 0.$$

Proof. (a). We use Theorem 5 to compare the coefficient indexed by δ in $T_{\mathcal{N}}(f) = \gamma(\mathcal{N}) f$

$$a(\delta \mathfrak{n}) = \gamma(\mathcal{N}) a(\delta).$$

(b). We write any $v \in \mathfrak{d}^{-1}$ by $v = \delta \mathfrak{v}$, where $\mathfrak{v} \in \mathcal{O}_K$. Now if $v \gg 0$, we consider the operator $T_{\mathcal{V}}$ with $\mathcal{V} = \mathfrak{v}\mathcal{O}_K$. So we will have $a(\delta \mathfrak{v}) = \gamma(\mathcal{V}) a(\delta)$, where $\gamma(\mathcal{V})$ is the eigenvalue of f for $T_{\mathcal{V}}$. Thus, if $a(\delta) = 0$ then $a(\mathbf{v}) = 0$ for all $v \gg 0$ and using (4.3) we conclude that $a(\mathbf{v}) = 0$ for all v. Hence $f = 0$. □

We say that f is a normalized eigenform if $a(\delta) = 1$. Using Corollary 6 and the above proposition, and properties of reflection operators we get the following corollary.

Corollary 9. *Let $f \in S_k(\Gamma_K, \boldsymbol{\lambda})$ be a normalized eigenform. Let $\mathfrak{m}, \mathfrak{n} \in \mathcal{O}_K$ such that $\mathfrak{m}\mathcal{O}_K + \mathfrak{n}\mathcal{O}_K = \mathcal{O}_K$. Then we have*

$$a(\delta\mathfrak{m}\mathfrak{n}) = a(\delta\mathfrak{m})a(\delta\mathfrak{n}).$$

Suppose $\mathfrak{p}$ is a prime element of $\mathcal{O}_K$. Then

$$a(\delta\mathfrak{p}^{m+1}) = a(\delta\mathfrak{p})a(\delta\mathfrak{p}^m) - a(\delta\mathfrak{p}^{m-1}), \quad m \geq 1.$$

Remark 10. *If we assume that $h^+(K) > 1$ it is essential to use the adelic language to formulate a complete Hecke theory.*

Example 11. *We use Corollary 9 to get some explicit relations between the coefficients which will be used in the computations later. Suppose $K = \mathbb{Q}(\sqrt{2})$. Let $f \in S(\Gamma_K, \boldsymbol{\lambda})$ is an eigenform with Fourier series expansion*

$$f(\mathbf{z}) = \sum_{0 \neq \mathbf{v} \in \mathfrak{d}^{-1}} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y})e\left(\operatorname{Tr}(\mathbf{x}\mathbf{v})\right).$$

Let $\delta = \frac{1+\sqrt{2}}{2\sqrt{2}}$. We check that gcd $(2-\sqrt{2}, 3-\sqrt{2}) = 1$ and both are prime elements. So by Corollary 9, we have

$$a((8 - 5\sqrt{2})\delta) - a((2 - \sqrt{2})\delta)a((3 - \sqrt{2})\delta) = 0.$$

$$a((2 - \sqrt{2})^2\delta) - a((2 - \sqrt{2})\delta)^2 + 1 = 0.$$

If we index the coefficients by lattice points then the above two relations are written as

$$a(3, -2) - a(1, 0)a(2, 1) = 0$$

$$a(2, -2) - a(1, 0)^2 + 1 = 0$$

4.2. **The automorphy method.** The automorphy (or Hejhal's) method was first introduced in by Hejhal to compute Maass cusp forms for Hecke triangle groups [5, 6] and was later extended by the second author to compute Maass cusp forms for congrence and non-congruence subgroups and Maass waveforms with non-zero weight [11].

The key idea is to approximate a Maass cusp form with a finite Fourier series and use Fourier inversion and invariance under group transformations (automorphy) to construct a linear system of equations that can be solved to obtain approximations for Fourier coefficients.

Assume that $f \in S(\Gamma_K)$ has Fourier expansion as above

$$f(\mathbf{z}) = \sum_{0 \neq \mathbf{v} \in \mathfrak{d}^{-1}} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y})e\left(\operatorname{Tr}(\mathbf{x}\mathbf{v})\right).$$

Lemma 12. *Let $\varepsilon = 10^{-D}$ and $Y > 0$. Then there exists $M_0 = M_0(Y, D)$ such that*

$$f(\mathbf{z}) = \sum_{0 \neq \mathbf{v} \in \mathfrak{d}^{-1}, \mathbf{v} \in B} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v}, \mathbf{y})e\left(\operatorname{Tr}(\mathbf{x}\mathbf{v})\right) + [[\varepsilon]]$$

for all $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ with $\|\mathbf{y}\|_1 > Y$ for all j, $[[\varepsilon]]$ denotes an error of magnitude at most ε and $B = [-M_0, M_0]^n \subset \mathbb{R}^n$.

Proof. Using the integral representation

$$K_{iR}(y) = \int_0^\infty e^{-y\cosh t}\cosh(iRt)dt$$

and monotonicity it follows directly that for $0 \le R < y$:

$$|K_{iR}(y)| \le K_0(y) < K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}.$$

Note that for the purpose of computing K_{iR} it is useful to consider more refined bounds taking into account the oscillatory behaviour (cf. e.g. [2]). It follows that

$$|\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{y})| \le 2^{-n}\mathrm{N}(|\mathbf{v}|)^{-1/2}e^{-2\pi\mathrm{Tr}(|\mathbf{v}|\mathbf{y})}$$

for any $\mathbf{v} \in \mathfrak{d}^{-1}$ and $\mathbf{y} \in \mathbb{R}_K^+$. Let $C > 0$ be as in Thm. (2) then

$$|a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{y})| \le C \cdot 2^{-n}e^{-2\pi\mathrm{Tr}(|\mathbf{v}|\mathbf{y})} = C' \cdot e^{-2\pi\mathrm{Tr}(|\mathbf{v}|\mathbf{y})}$$

and the tail can be estimated by

$$\begin{aligned}\mathcal{E}(Y,M_0) = \left|\sum_{0\ne\mathbf{v}\in\mathfrak{d}^{-1},\mathbf{v}\notin B} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{y})e\left(\mathrm{Tr}(\mathbf{x}\mathbf{v})\right)\right| &\le C'\sum_{\mathbf{v}\in\mathfrak{d}^{-1},\mathbf{v}\notin B} e^{-2\pi\mathrm{Tr}(|\mathbf{v}|\mathbf{y})}\\ &= C'\sum_{\mathbf{v}\in\mathfrak{d}^{-1},\mathbf{v}\notin B} e^{-2\pi(|v_1|y_1+\cdots+|v_n|y_n)}\\ &= C''\prod_j\sum_{v_j\ge b} e^{-2\pi v_j y_j}\\ &= C''\prod_j e^{-2\pi M_0 y_j}\frac{1}{1-e^{-2\pi y_j}}\\ &= C''\prod_j e^{-2\pi M_0 y_j}e^{2\pi y_j}\frac{1}{e^{2\pi y_j}-1}\\ &\le C'' e^{-2\pi n(M_0-1)Y}\frac{1}{2\pi Y^n}\end{aligned}$$

It is clear that $|\mathcal{E}(Y,M_0)| < \varepsilon\cdot C/(2^{n+1}\pi)$ if M_0 is chosen such that $-2\pi n(M_0-1) < log(Y^n\varepsilon)$ and $2\pi M_0 Y > R$, in other words, if

$$M_0 > 1-\frac{1}{2\pi nY}\ln(Y^n\varepsilon) = 1-\frac{1}{2\pi nY}[n\ln Y - D\ln 10] = 1+\frac{D}{2\pi nY}\ln 10-\frac{1}{2\pi Y}\ln Y.$$

and $M_0 > \frac{R}{2\pi Y}$. Thus, $-2\pi n(M_0-1) < log(Y^n E\varepsilon)$

$$\begin{aligned}M_0 &> 1-\frac{1}{2\pi nY}\ln(Y^n\varepsilon) = 1-\frac{1}{2\pi nY}[n\ln Y - D\ln 10 + \ln(C) - (n+1)\ln 2 - \ln\pi]\\ &= \frac{D}{2\pi nY}\ln 10 + \frac{1}{2\pi nY}[-\ln C + (n+1)\ln 2 + \ln\pi] + 1 - \frac{1}{2\pi Y}\ln Y.\end{aligned}$$

□

Example 13. If $n = 2$, $Y = 0.3$ and we want $D = 16$ digits precision we can choose $M_0 = 13$.

4.3. **Fourier inversion in lattices.** By an integral lattice L in $\mathbb{R}^n$ we will always mean a finitely generated $\mathbb{Z}$-module of full rank with a symmetric non-degenerate bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $B(L, L) \subset \mathbb{Z}$. The dual lattice of L is defined as

$$L^{\#} = \{\mathbf{x} \in \mathbb{R}^n : B(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}, \quad \forall \mathbf{y} \in L\}.$$

It is clear that $L \subseteq L^{\#}$ and it is easy to show that $L^{\#}/L$ is a finite abelian group. Furthermore, if $M \supseteq L$ is a, not necessarily integral, superlattice of finite index over L then $M^{\#} \subseteq L^{\#}$ and for a fixed $\mathbf{v} \in L^{\#}$ the function $B(\mathbf{v}, \cdot) : M/L \to \mathbb{Q}/\mathbb{Z}$ is well-defined and non-degenerate unless $\mathbf{v} \in M^{\#}$.

The following Lemma and Corollary are fundamental to the theory of finite Fourier series on lattices and while well-known for $\mathbb{Z}^n$ with the Euclidean inner product we have not been able to find the following general formulation in the literature.

Lemma 14. *Let L be an integral lattice in $\mathbb{R}^n$ and M a superlattice. If $v \in L^{\#}$ then*

$$\frac{1}{[M : L]} \sum_{x \in M/L} e\left(B(v, x)\right) = \begin{cases} 1, & \text{if } v \in M^{\#}, \\ 0, & \text{if } v \notin M^{\#}. \end{cases}$$

Proof. For any $x_0 \in M$ we have

$$e\left(B(v, x_0)\right) \sum_{x \in M/L} e\left(B(v, x)\right) = \sum_{x \in M/L} e\left(B(v, x + x_0)\right) = \sum_{x \in M/L} e\left(B(v, x)\right).$$

It follows that either $\sum_{x \in M/L} e\left(B(v, x)\right) = 0$ or $e\left(B(v, x_0)\right) = 1$ for all $x_0 \in M$, in other words, $v \in M^{\#}$. □

Note that the above is simply a generalisation of the well-known orthogonality relation for complex exponentials. As an illustration in the simplest possible setting we consider the the following situation, which is used for the automorphy method in e.g. [6, 10, 5].

Example 15. Consider the integral lattice $L = \mathbb{Z}$ together with the bilinear form $B(x, y) = xy$. Then $L^{\#} = L$ and if we fix $v = a \in L^{\#}$, choose an integer Q and set $M = \frac{1}{2Q} L$ then then $M^{\#} = 2QL^{\#} = 2Q\mathbb{Z}$ and

$$\frac{1}{[M : L]} \sum_{x \in M/L} e\left(B(v, x)\right) = \frac{1}{2Q} \sum_{x \in \frac{1}{2Q}\mathbb{Z}/\mathbb{Z}} e\left(vx\right) = \frac{1}{2Q} \sum_{b \mod Q} e\left(\frac{ab}{2Q}\right) = \begin{cases} 1, & 2Q \mid a, \\ 0, & \text{else.} \end{cases}$$

To be able to apply Lemma 14 to finite Fourier inversion we would like the right-hand side to be 1 exactly when $a = 0$. To achieve this in the example above we simply need to take $2Q > a$. For a general lattice this is less straight-forward but the immediate generalisation leads to the following

Corollary 16. *Let $L \subset \mathbb{R}^n$ be an integral lattice. For any bounded subset $V \subseteq L^{\#}$ it is possible to find a superlattice $M \supset L$ of finite index such that if $\mathbf{v} \in V$ then*

$$\frac{1}{[M : L]} \sum_{\mathbf{x} \in M/L} e\left(B(\mathbf{v}, \mathbf{x})\right) = \begin{cases} 1, & \mathbf{v} = 0, \\ 0, & \mathbf{v} \neq 0. \end{cases}$$

Proof. Consider $M = \lambda^{-1} L$ for some $\lambda > 1$. Then $M^{\#} = \lambda L^{\#}$ and we can choose λ sufficiently large such that $M^{\#} \cap V = \{0\}$. □

The subsets we will be concerned with here are of the form $[-M_0, M_0]^n$ and it is posible to choose the superlattice of the form $M = \frac{1}{Q}L$ with

$$Q > M_0 \left\| B_L^T \right\|_\infty , \tag{4.4}$$

where B_L is the basis matrix for L, that is, $L = B_L\mathbb{Z}^n$. This can either be shown by using the analog of [3, Cor. 2] with $\mathfrak{o}$ replaced by $L^{\#}$ and $\mathfrak{n}$ by $M^{\#}$, or directly in terms of lattices and basis matrices.

Let $f \in S(\Gamma_K)$ have Fourier series expansion as above and let $\epsilon > 0$. For $Y > 0$ let $M_0 = M_0(\varepsilon, Y)$ be as in Lemma 12 and consider the finite Fourier series

$$\hat{f}(\mathbf{z}) = \sum_{0\neq\mathbf{v}\in\mathfrak{d}^{-1}, \|\mathbf{v}\|_\infty \le M_0} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{y})e\left(\mathrm{Tr}(\mathbf{x}\mathbf{v})\right).$$

Choose a Q such that $M = Q^{-1}L$ satisfy the conclusion of Corollary 16 for the bounded set $\{v \in L^{\#} : \|\mathbf{v}\|_\infty \le 2M_0\}$. Then, for any $\mathbf{w} \in L^{\#}$ with $\|\mathbf{w}\|_\infty \le M_0$ we have

$$\begin{aligned}
&\frac{1}{[M:L]}\sum_{\mathbf{x}\in M/L} \hat{f}(\mathbf{x}+iY)e(-B(\mathbf{w},\mathbf{x})) \\
&= \frac{1}{[M:L]}\sum_{0\neq\mathbf{v}\in\mathfrak{d}^{-1}, \|\mathbf{v}\|_\infty \le M_0} a(\mathbf{v})\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{Y})\sum_{\mathbf{x}\in M/L} e\left(B(\mathbf{x},\mathbf{v}-\mathbf{w})\right) \\
&= a(\mathbf{w})\kappa_{i\mathbf{R}}(\mathbf{w},\mathbf{Y}).
\end{aligned}$$

For $\mathbf{z} \in \mathbb{H}_K$ we let $\mathbf{z}^* \in \Gamma_K\backslash\mathbb{H}_K$ be the pullback of $\mathbf{z}$ into the fundamental domain of Γ_K. Since f is invariant under $\Gamma_K$we know that $f(\mathbf{z}) = f(\mathbf{z}^*)$ and if Y is small enough that so that $\mathbf{z} = \mathbf{x} + i\mathbf{Y}$ has pullback $\mathbf{z}^* = \mathbf{x}^* + i\mathbf{y}^*_{\mathbf{x}}$ with $\mathbf{y}^*_{\mathbf{x}} \gg \mathbf{Y}$ then

$$\hat{f}(\mathbf{z}^*) = \hat{f}(\mathbf{z}) + [[\epsilon]]$$

and hence

$$\begin{aligned}
\frac{1}{[M:L]}\sum_{\mathbf{x}\in M/L}\hat{f}(\mathbf{x}+iY)e(-B(\mathbf{w},\mathbf{x})) + [[\epsilon]] &= \frac{1}{[M:L]}\sum_{\mathbf{x}\in M/L}\hat{f}(\mathbf{z}^*)e(-B(\mathbf{w},\mathbf{x})) \\
&= \sum_{0\neq\mathbf{v}\in\mathfrak{d}^{-1}, \|\mathbf{v}\|_\infty \le M_0} a(\mathbf{v})V(\mathbf{w},\mathbf{v}),
\end{aligned}$$

where

$$V(\mathbf{w},\mathbf{v}) = \frac{1}{[M:L]}\sum_{\mathbf{x}\in M/L}\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{y}^*_{\mathbf{x}})e(B(\mathbf{v},\mathbf{x}^*) - B(\mathbf{w},\mathbf{x})).$$

Due to the rapid decay of the K-Bessel functions the term with $\mathbf{y} = \mathbf{Y}$ is dominating and if we let

$$\tilde{V}(\mathbf{w},\mathbf{v}) = V(\mathbf{w},\mathbf{v}) - \delta_{\mathbf{v}=\mathbf{w}}\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{Y})$$

we obtain a well-conditioned homogeneous linear system of equations

$$\sum_{0\neq\mathbf{v}\in\mathfrak{d}^{-1}, \|\mathbf{v}\|_\infty < M_0} a(\mathbf{v})\tilde{V}(\mathbf{w},\mathbf{v}) = [[\epsilon]]. \tag{4.5}$$

Now, if we are given a spectral parameter $\mathbf{R}$ corresponding to a Hilbert-Maass cusp form f and choose a suitable $Y > 0$ we can solve (4.5) to obtain an approximation of the Fourier coefficients $a(\mathbf{v}) := a(\mathbf{v}; \mathbf{R}, Y)$ up to an error of magnitude $\epsilon \times \|\tilde{V}^{-1}\|_\infty$, where we observe that $\|\tilde{V}^{-1}\|_\infty$ is not too large provided that $\kappa_{i\mathbf{R}}(\mathbf{v},\mathbf{Y})$ is not too small for $\|\mathbf{v}\|_\infty < M_0$. The main issue is of course that we don't know the spectrum apriori so we need simultaneouly locate eigenvalues and compute Fourier coefficients.

4.4. **Normalization.** We know from Proposition **??** that if the dimension of $S(\Gamma_K, \mathbf{R})$ is non-zero then there exist a forms with Fourier coefficient $c(\delta) \neq 0$, where $\mathfrak{d}^{-1} = \delta\mathcal{O}_K$. We solve the system of equation 4.5 by putting $c(\delta) = 1$. Observe the the solution space of the system of of equation for a true eigenvalue $\mathbf{R}$ will be unique Hecke eigenform if the $\dim(\mathrm{S}(\Gamma_\mathrm{K}, \mathbf{R}) = 1$ and if the $\dim \mathrm{S}(\Gamma_\mathrm{K}, \mathbf{R}) > 1$ then the solution will be a linear combination of Hecke eigenforms.

4.5. **Locating eigenvalues.** It is worth noting that all methods mentioned here are heuristic and in order to provably locate eigenvalues it is necessary to use alternative methods similar to those of [1] or more recently [9].

To locate eigenvalues of Hilbert-Maass forms we use the fact that the Fourier coefficients satisfy certain non-trivial relations, either implied by automorphy or arithmeticity (i.e. Hecke operators) to construct a function $h : \mathbb{R}^n \to \mathbb{R}$ which is continous in the spectral parameter $\mathbf{R}$ within a certain region and is approximately zero when $\mathbf{R}$ corresponds to a true eigenvalue. We use the Broyden's method to find the minimas or zeros of this functions and treat these zeros as *tentative* eigenvalues and use a range of other heuristic tests to determine if they are true eigenvalues or not. We will outline some possible choices but there are many alternatives.

(1) The first observation is that as long as the value of Y is chosen such that the pullback $\mathbf{z}^*$ of $\mathbf{x} + i\mathbf{Y}$ has imaginary part $\mathbf{y}^* \gg \mathbf{Y}$ then the solution $a(\mathbf{v}; \mathbf{R}; Y)$ to (4.5) is independent of Y (up to an error of magnitude ϵ). In the same way as in the one dimensional case, e.g. [6, 10, 5] we can choose two suitable values of Y, say Y_1 and Y_2 and use these to compute two sets of coefficients $a_1(\mathbf{v}) = a(\mathbf{v}; \mathbf{R}, Y_1)$ and $a_2(\mathbf{v}) = a(\mathbf{v}; \mathbf{R}, Y_2)$ and for a fixed index $\mathbf{v}$ we write down a function $h : \mathbb{R}^n \to \mathbb{R}$ by
$$h_{Y_1,Y_2,\mathbf{v}}(\mathbf{R}) = a(\mathbf{v}; \mathbf{R}, Y_1) - a(\mathbf{v}; \mathbf{R}, Y_2).$$
(2) We observed earlier that $a(\varepsilon^2\mathbf{v}) = a(\mathbf{v})$ for any unit ε and index $\mathbf{v}$ and the corresponding function is
$$h_{\varepsilon,\mathbf{v}}(\mathbf{R}) = a(\varepsilon^2\mathbf{v}; \mathbf{R}; Y) - a(\mathbf{v}; \mathbf{R}; Y).$$
(3) We use Hecke relations from Corollary 9 to construct the following two functions
$$h_{\delta,\mathfrak{m},\mathfrak{n}}(\mathbf{R}) = a(\delta\mathfrak{m}\mathfrak{n}, \mathbf{R}) - a(\delta\mathfrak{m}, \mathbf{R})a(\delta\mathfrak{n}, \mathbf{R}),$$
$$h_{\delta,\mathfrak{p}}(\mathbf{R}) = a(\delta\mathfrak{p}^{m+1}, \mathbf{R}) - a(\delta\mathfrak{p}, \mathbf{R})a(\delta\mathfrak{p}^m, \mathbf{R}) + a(\delta\mathfrak{p}^{m-1}, \mathbf{R}),$$
where $\mathfrak{m}, \mathfrak{n}, \mathfrak{p} \in \mathcal{O}_K$, $\mathfrak{d}^{-1} = \delta\mathcal{O}_K$, $\mathfrak{m}\mathcal{O}_K + \mathfrak{n}\mathcal{O}_K = \mathcal{O}_K$ and $\mathfrak{p}$ is prime ideal $\mathcal{O}_K$.

In this article we mainly focused on locating eigenvalues of real quadratic field with narrow class number 1. In the above all h are functions from $\mathbb{R}^2 \to \mathbb{C}$ but Broyden method is applicable to a function from $\mathbb{R}^2 \to \mathbb{R}^2$. We define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by taking the real and imaginary part of $h(\mathbf{R})$. We also note that Broyden's method converge to a zero of the function when the initial point is close to that zero.

Algorithm:

(1) We construct objects in the database for different $\mathbf{R}$ by putting $c(\delta) = 1$ in 4.5 separated by a very small distance 0.01 with $M_0 = 6$.

(2) Run through the constructed objects and sort out the objects whose coefficient satisfy
$$|a(\varepsilon^2\mathbf{v},\mathbf{R}) - a(\mathbf{v},\mathbf{R})| < \epsilon$$
for a fixed $\mathbf{v}$ and $\varepsilon \in \mathcal{O}_K^\times$.
(3) On a sorted objects we apply the Broyden's method to get the zero of
$$h_{\varepsilon,\mathbf{v}}(\mathbf{R}) = a(\varepsilon^2\mathbf{v},\mathbf{R}) - a(\mathbf{v},\mathbf{R}).$$
(4) The zero of $h_{\varepsilon,\mathbf{v}}(\mathbf{R})$ is a *tentative* eigenvalue. We test if it is a true eigenvalue by testing the accuracy of $a(\epsilon_1\mathbf{v_1}) - a(\mathbf{v}_1) = 0$, where the pair $\epsilon_1, \mathbf{v_1}$ is different than what we used in step (3).
(5) We check if the solution is a Hecke eigenform by testing the accuracy of certain Hecke relations.
(6) We make the eigenvalues precise in this steps. We apply Broyden method on Hecke relation function $h_{\delta,\mathfrak{m},\mathfrak{n}}(\mathbf{R})$ with initial point as eigenvalue and $M_0 = 17$.

5. Computational Results

The primary aim of this article has been to develop a robust and efficient algorithm for computing Hilbert-Maass forms on the full modular group $\mathrm{SL}_2(\mathcal{O}_K)$, where the narrow class number of K is 1. Our main results include the following:

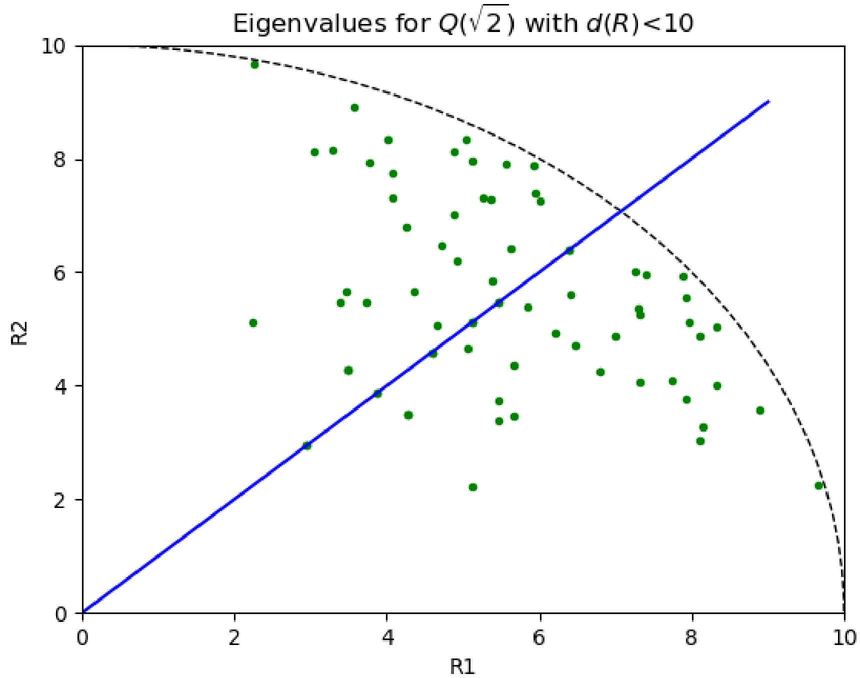
(1) Extensive list of eigenvalues for the field $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5})$.
(2) One Eigenvalue for $K = \mathbb{Q}(\sqrt{3})$ whose class number is 1 and narrow class number is 2.

5.1. **Eigenvalues for $\mathbb{Q}(\sqrt{2})$.** In table 5.1 we give the eigenvalues for Hilbert Maassform for $\mathbb{Q}(\sqrt{2})$ with $d(\mathbf{R}) = \sqrt{R_1^2 + R_2^2} < 10$ and in Fig. 5.1 we give the picture of their distribution in $R_1 - R_2$ plane. To obtain these eigenvalues we have put $a(\delta) = a(1,1) = 1$ in our algorithm for finding the eigenvalues. Here $\delta = \frac{1+\sqrt{2}}{2\sqrt{2}}$. To make the eigenvalues precise we have applied Broyden method on the Hecke relation
$$\begin{aligned} h_{\delta,\mathfrak{m},\mathfrak{n}}(\mathbf{R}) &= a((2-\sqrt{2})^2\delta,\mathbf{R}) - a((2-\sqrt{2})\delta,\mathbf{R})^2 + 1 \\ &= a((2,-2),\mathbf{R}) - a((1,0),\mathbf{R})^2 + 1. \end{aligned}$$
From (4.3) we know that $a(\epsilon_v\mathbf{v}) = t_v a(\mathbf{v})$, where $t_v \in \{1,-1\}$ is the eigenvalue of the reflection operator Z_v. Therefore if we put $\mathbf{v} = \delta$ then we get the following relation
$$a(0,-1) = \pm a(1,1) = \pm 1, \quad a(0,1) = \pm a(1,1) = \pm 1.$$
We measure the accuracy of above two relations in table 5.1 denoted by
$$A_1 = ||a(0,-1)| - 1| \qquad A_2 = ||a(0,1)| - 1|$$

Eigenvalues	$Z_{v_1}(f)$	$Z_{v_2}(f)$	A_1	A_2
(2.94644197057517, 2.94644197057590)	-	-	8.8E-13	1.0E-12
(3.87932871354483, 3.87932871354844)	-	-	2.0E-11	1.0E-11
(4.27438019881703, 3.48836224262859)	+	-	1.0E-11	1.1E-11
(3.48836224263069, 4.27438019881912)	-	+	5.0E-12	4.3E-12
(5.12163961866447, 2.23423570738229)	+	-	1.4E-11	1.3E-11
(5.46508286150025, 3.39346897067854)	-	-	1.0E-11	9.9E-12
(4.59042962923236, 4.59042962923230)	+	+	6.0E-11	4.8E-11

(5.46161670252051, 3.73542725949592)	-	+	4.4E-12	2.0E-12
(5.66104208632875, 3.47695155477744)	+	+	2.9E-11	2.9E-11
(5.06085806549068, 4.65976815171535)	-	+	5.0E-11	5.3E-11
(5.65780725055356, 4.35919904030488)	+	-	1.1E-11	1.1E-11
(5.12632439882674, 5.12632439882759)	-	-	2.1E-11	2.2E-11
(5.46201508044348, 5.46201508045897)	+	+	1.9E-10	1.8E-10
(6.21282006328438, 4.92772644054263)	+	+	1.4E-8	8.0E-8
(5.84122247681663, 5.39266054483805)	-	+	7.1E-11	4.9E-11
(6.46571992406758, 4.72046555875545)	-	-	1.6E-12	2.1E-12
(6.78429191271241, 4.25975922391160)	-	+	2.3E-11	6.5E-11
(7.31202098937811, 4.07336856494287)	+	-	1.6E-10	1.4E-10
(6.41541797236412, 5.61833900465313)	-	+	2.7E-11	2.6E-11
(7.00191462014148, 4.87474146214415)	-	+	1.6E-10	1.4E-11
(8.11405927120330, 3.04617456025092)	-	+	3.3E-10	4.9E-10
(7.73864329691478, 4.07907141835314)	+	+	4.1E-10	4.1E-10
(7.91986732150452, 3.76493165378960)	-	+	1.0E-10	3.9E-11
(8.14361808340775, 3.27891498664495)	+	-	3.7E-9	4.2E-9
(7.32256762533282, 5.26329241029498)	-	-	1.7E-10	2.1E-10
(6.39195860202002, 6.39195860202030)	+	+	2.7E-11	2.5E-11
(7.29516242878050, 5.35885878392577)	+	+	5.4E-12	5.5E-12
(8.33147125719617, 4.00372675101176)	-	-	2.5E-10	4.1E-9
(7.26070634689921, 6.00298919945343)	-	-	2.6E-9	4.6E-10
(8.10963385595443, 4.87293559898530)	+	-	8.3E-9	8.2E-9
(5.12486731338244, 7.96801591737712)	+	-	9.9E-9	5.7E-9
(7.40482588031344, 5.94791794712751)	+	+	3.2E-10	9.7E-11
(8.89573788860815, 3.56505186533443)	+	-	6.8E-9	2.6E-9
(7.91609668483091, 5.55547029432387)	-	-	1.3E-10	3.0E-10
(8.32992873691812, 5.04016720386581)	+	+	3.0E-9	7.6E-9
(7.89031373086060, 5.91824294473287)	+	-	6.2E-10	1.1E-10

Table 5.1



In the above picture it is interesting to observe that the eiegenvalues are scattered in the middle and there no eigenvalue when we move toward the R_1 or R_2 axis.

In the following table we compare the coefficient of Hilbert Maass form corresponding to $\mathbf{R} = (2.94644197057517, 2.94644197057590)$ at two different y. In Table 5.3, we measure the accuracy of different Hecke relation on the Fourier coefficients of the object for the eigenvalues $(4.27438019881703, 3.48836224262859)$ with $M = 17$ and $y = 0.31$.

$\mathbf{R} = (2.94644197057517, 2.94644197057590)$			
Index	coefficients at $y_1 = 0.31$	Coefficients at $y_2 = 0.29$	Difference
c(-2, -2)	0.450207392017623	0.450207392025016	8.7E-12
c(-2, -1)	0.347283781931002	0.347283781931164	1.1E-12
c(-2, 0)	-0.662085312387992	-0.662085312418177	3.5E-11
c(-2, 1)	0.347283781932313	0.347283781972468	4.6E-11
c(-2, 2)	0.450207392016154	0.450207392049946	3.9E-11
c(-1, -2)	-1.20424556964650	-1.20424556967296	3.1E-11
c(-1, -1)	1.00000000000036	1.00000000007000	8.0E-11
c(-1, 0)	1.20424556964772	1.20424556969050	4.9E-11
c(-1, 1)	0.999999999997163	1.00000000001836	2.5E-11
c(-1, 2)	-1.20424556964851	-1.20424556971471	7.6E-11
c(0, -2)	-0.450207392016980	-0.450207392045077	3.2E-11
c(0, -1)	-0.999999999999121	-1.00000000005203	6.1E-11
c(0, 0)	0.000000000000000	0.000000000000000	0.0E-14

c(0, 1)	-0.999999999998984	-1.00000000003666	4.3E-11
c(0, 2)	-0.450207392016765	-0.450207392018310	1.8E-12
c(1, -2)	-1.20424556964837	-1.20424556969316	5.1E-11
c(1, -1)	0.999999999997649	1.00000000006775	8.1E-11
c(1, 0)	1.20424556964768	1.20424556972147	8.5E-11
c(1, 1)	1.00000000000000	1.00000000000000	0.0E-14
c(1, 2)	-1.20424556964658	-1.20424556971723	8.2E-11
c(2, -2)	0.450207392015957	0.450207392028644	1.5E-11
c(2, -1)	0.347283781932022	0.347283781917048	1.7E-11
c(2, 0)	-0.662085312387989	-0.662085312417507	3.4E-11
c(2, 1)	0.347283781931334	0.347283781970199	4.5E-11
c(2, 2)	0.450207392017814	0.450207392051174	3.9E-11

Table 5.2

$\mathbf{R} = (4.27438019881703, 3.48836224262859)$	
Hecke Coprime relations	Value
c(-1, 4)-c(-1, 0)c(-1, -3)	3.0E-12
c(1, -4)-c(1, 0)c(-1, -3)	1.6E-12
c(1, -4)-c(1, 3)c(-1, 0)	2.5E-12
c(-1, 4)-c(1, 3)c(1, 0)	1.6E-12
c(1, 4)-c(-1, -2)c(-2, -1)	5.6E-12
c(3, -2)-c(-1, 0)c(-2, -1)	3.0E-12
c(-3, 2)-c(1, 0)c(-2, -1)	1.5E-12
c(-1, -4)-c(1, 2)c(-2, -1)	2.1E-12
c(-1, 4)-c(-1, -2)c(-2, 1)	1.7E-11
c(1, -4)-c(1, 2)c(-2, 1)	1.6E-11
c(3, 0)-c(0, -3)c(-1, -2)	2.3E-11
c(-3, 0)-c(0, 3)c(-1, -2)	2.2E-11
c(1, -4)-c(2, -1)c(-1, -2)	1.7E-11
c(-1, -4)-c(2, 1)c(-1, -2)	1.2E-12
c(-3, 2)-c(2, 1)c(-1, 0)	3.1E-12
c(-3, 0)-c(1, 2)c(0, -3)	2.2E-11
c(3, 0)-c(1, 2)c(0, 3)	2.3E-11
c(3, -2)-c(2, 1)c(1, 0)	1.7E-12
c(-1, 4)-c(2, -1)c(1, 2)	1.7E-11
c(1, 4)-c(2, 1)c(1, 2)	4.9E-12
c(3, 2)-c(-1, -2)c(-1, -3)	1.6E-12
c(-3, -2)-c(1, 2)c(-1, -3)	1.9E-12
c(-3, 2)-c(-1, 3)c(-1, -2)	1.4E-11
c(3, -2)-c(1, -3)c(-1, -2)	1.3E-11
c(-3, -2)-c(1, 3)c(-1, -2)	6.3E-13
c(3, -2)-c(1, 2)c(-1, 3)	1.3E-11
c(-3, 2)-c(1, 2)c(1, -3)	1.4E-11
c(3, 2)-c(1, 3)c(1, 2)	1.2E-12

c(3, 0)-c(-1, 0)c(-3, -3)	8.6E-12
c(-3, 0)-c(1, 0)c(-3, -3)	8.2E-12
c(-3, 0)-c(3, 3)c(-1, 0)	4.6E-12
c(3, 0)-c(3, 3)c(1, 0)	2.1E-12
c(4, -2)-c(0, -2)c(-1, -3)	2.0E-11
c(-4, 2)-c(0, 2)c(-1, -3)	2.0E-11
c(-4, 2)-c(1, 3)c(0, -2)	2.1E-11
c(4, -2)-c(1, 3)c(0, 2)	2.1E-11
c(4, 2)-c(-2, -1)c(-2, -2)	8.4E-12
c(4, -2)-c(-2, 1)c(-2, -2)	2.7E-12
c(-4, 2)-c(2, -1)c(-2, -2)	2.7E-12
c(-4, -2)-c(2, 1)c(-2, -2)	8.9E-12
c(-4, -2)-c(2, 2)c(-2, -1)	6.9E-12
c(-4, 2)-c(2, 2)c(-2, 1)	3.0E-12
c(4, -2)-c(2, 2)c(2, -1)	3.5E-12
c(4, 2)-c(2, 2)c(2, 1)	8.3E-12
c(1, 4)-c(-1, 0)c(-3, -5)	1.6E-11
c(-1, -4)-c(1, 0)c(-3, -5)	1.9E-11
c(-1, -4)-c(-1, 3)c(-3, -4)	5.3E-11
c(1, 4)-c(1, -3)c(-3, -4)	4.8E-11
c(-1, -4)-c(3, 5)c(-1, 0)	1.9E-11
c(1, 4)-c(3, 4)c(-1, 3)	4.4E-11
c(-1, -4)-c(3, 4)c(1, -3)	4.9E-11
c(1, 4)-c(3, 5)c(1, 0)	1.3E-11
c(3, 2)-c(-2, 1)c(-3, -4)	2.8E-11
c(-3, -2)-c(2, -1)c(-3, -4)	2.8E-11
c(-3, -2)-c(3, 4)c(-2, 1)	2.5E-11
c(3, 2)-c(3, 4)c(2, -1)	2.5E-11
c(-3, 2)-c(-1, 2)c(-3, -5)	3.9E-12
c(3, -2)-c(1, -2)c(-3, -5)	6.2E-12
c(3, -2)-c(3, 5)c(-1, 2)	3.1E-12
c(-3, 2)-c(3, 5)c(1, -2)	3.0E-12
c(3, 0)-c(-3, 3)c(-3, -4)	3.8E-11
c(-3, 0)-c(3, -3)c(-3, -4)	3.8E-11
c(-3, 0)-c(3, 4)c(-3, 3)	3.5E-11
c(3, 0)-c(3, 4)c(3, -3)	3.3E-11

TABLE 5.3

5.2. **Eigenvalues for $\mathbb{Q}(\sqrt{5})$.** In table 5.4 we give the eigenvalues for Hilbert Maassform for $\mathbb{Q}(\sqrt{5})$ with $d(\mathbf{R}) < 10$. To obtain these eigenvalues we have put $a(\delta) = a(1, -1) = 1$ in our algorithm for finding the eigenvalues, where $\delta = \frac{-1+\sqrt{5}}{2\sqrt{5}}$. We know that $a(\epsilon_v \mathbf{v}) = t_v a(\mathbf{v})$, where $t_v \in \{1, -1\}$ is the eigenvalue of the reflection operator Z_v. We put $\mathbf{v} = \delta$ then we get the following relation

$$a(1, -2) = \pm a(1, -1) = \pm 1, \quad a(-1, 2) = \pm a(1, -1) = \pm 1.$$

We measure the accuracy of above two relations in table 5.4 denoted by

$$A_1 = ||a(1,-2)| - 1| \qquad A_2 = ||a(-1,2)| - 1|$$

Eigenvalues. **Not precise**	$Z_{v_1}(f)$	$Z_{v_2}(f)$	A_1	A_2
(5.82272405351904, 2.23131291693068)	+	+	3.2E-3	3.0E-3
(4.97740834415463, 3.79543977278699)	-	+	1.9E-4	8.4E-4
(5.55700872827857, 3.70247210734705)	+	-	8.1E-5	5.5E-5
(4.89378533293492, 4.89377736229831)	-	-	1.3E-5	1.3E-5
(6.50233753972765, 3.43328482949946)	-	-	8.6E-3	8.7E-3
(6.60248584146319, 4.59809464666161)	-	+	1.0E-3	1.2E-3
(7.56251335652632, 2.94032983170630)	+	-	2.8E-2	1.1E-2
(5.88126826097356, 5.61420301691631)	+	-	7.9E-4	1.2E-3
(6.69005230712794, 5.06205137741998)	-	-	5.3E-3	7.8E-3
(7.87824237484104, 3.50706079669053)	+	+	3.1E-3	3.2E-3
(7.25170082398076, 4.74677588728822)	+	-	9.3E-3	4.3E-3
(6.96267505167020, 6.96503288419134)	-	-	1.3E-3	1.5E-3

TABLE 5.4

5.3. **Eigenvalues for $\mathbb{Q}(\sqrt{3})$.** Note that in this case we do not have a totally positive generator of $\mathfrak{d}^{-1}$. We put $a(\delta) = a(0,-1) = 1$ in our algorithm to find the any eigenvalue, where $\delta = -\frac{1}{2\sqrt{3}}$. We do not have Hecke relations to use in this case. We have used the unit relations function $h_{\varepsilon,\mathbf{v}}(\mathbf{R}) = a(\varepsilon^2\mathbf{v},\mathbf{R}) - a(\mathbf{v},\mathbf{R})$. in our algorithm to find this eigenvalue. The eigenvalue we discovered is

$$\mathbf{R} = ()$$

In the following table we measure the accuracy of certain unit relations

6. Further Work

References

[1] A. R. Booker, A. Strömbergsson, and A. Venkatesh. Effective computation of Maass cusp forms. *Int. Math. Res. Not.*, pages Art. ID 71281, 34, 2006.

[2] Andrew R. Booker, Andreas Strömbergsson, and Holger Then. Bounds and algorithms for the k-bessel function of imaginary order. *LMS Journal of Computation and Mathematics*, 16:78–108, 2013.

[3] Mikoł aj Fraczyk, Gergely Harcos, and Péter Maga. Counting bounded elements of a number field. *Int. Math. Res. Not. IMRN*, (1):373–390, 2022.

[4] D. A. Hejhal. *The Selberg trace formula for* PSL(2, $\mathbf{R}$). *Vol. 2*, volume 1001 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[5] Dennis A. Hejhal. Eigenvalues of the Laplacian for Hecke triangle groups. *Mem. Amer. Math. Soc.*, 97(469):vi+165, 1992.

[6] Dennis A. Hejhal. On the calculation of Maass cusp forms. In *Hyperbolic geometry and applications in quantum chaos and cosmology*, volume 397 of *London Math. Soc. Lecture Note Ser.*, pages 175–185. Cambridge Univ. Press, Cambridge, 2012.

[7] H. Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2002.

[8] H. Maass. Ü eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121:141–183, 1949.

[9] Andrei Seymour-Howell. Rigorous computation of Maass cusp forms of squarefree level. *Res. Number Theory*, 8(4):Paper No. 83, 15, 2022.

[10] F. Strömberg. Maass waveforms on $(\Gamma_0(N), \chi)$ (computational aspects). In *Hyperbolic geometry and applications in quantum chaos and cosmology*, volume 397 of *London Math. Soc. Lecture Note Ser.*, pages 187–228. Cambridge Univ. Press, Cambridge, 2012.
[11] Fredrik Strömberg. Computation of Maass waveforms with nontrivial multiplier systems. *Math. Comp.*, 77(264):2375–2416, 2008.

University of Nottingham
Email address: sujeet.singh@nottingham.ac.uk

University of Nottingham
Email address: fredrik.stromberg@nottingham.ac.uk