



Isotropic and numerical equivalence for Chow groups and Morava K-theories

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Abstract

In this paper we prove the conjecture claiming that, over a flexible field, *isotropic Chow groups* coincide with *numerical Chow groups* (with \mathbb{F}_p -coefficients). This shows that Isotropic Chow motives coincide with Numerical Chow motives. In particular, homs between such objects are finite groups and \otimes has no zero-divisors. It provides a large supply of new points for the Balmer spectrum of the Voevodsky motivic category. We also prove the Morava K-theory version of the above result, which permits to construct plenty of new points for the Balmer spectrum of the Morel-Voevodsky \mathbb{A}^1 -stable homotopy category. This substantially improves our understanding of the mentioned spectra whose description is a major open problem.

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1 Introduction

The idea of *isotropic realisation* is to provide a local handy version for an algebro-geometric object, a version whose complexity will be similar to that of a topological object (and so, much simpler). In the first approximation, such local versions should be parametrized by *points* of the algebro-geometric world, that is, by all possible extensions of the base field. A closer look though reveals that a prime number p should be chosen and only an equivalence class of an extension under certain p -equivalence relation matters. In the motivic case, the idea is to annihilate the motives of p -anisotropic varieties (that is, varieties which have no closed points of degree prime to p). This idea appeared first in the work of Bachmann [2], who considered the $\otimes - \Delta$ -subcategory generated by motives of quadrics and applied this method successfully to the study of the Picard group of the Voevodsky category. With the latter purpose in mind, the *motivic category of a field extension* $DM(E/k; \mathbb{F}_p)$ was

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introduced in [25, Sect. 4] and studied extensively in [26]. The trivial extension k/k case of it is the *isotropic motivic category* $DM(k/k; \mathbb{F}_p)$ - [26, Definition 2.4]. It is the Verdier localisation of the “global” Voevodsky category $DM(k; \mathbb{F}_p)$ with respect to the localising subcategory generated by motives of p -anisotropic varieties. Combining the global to local localisation functor with the restriction via field extension we get natural functors from $DM(k)$ to $DM(E/E; \mathbb{F}_p)$, for various field extensions E/k . The next important observation is that, for the target isotropic category to be “handy”, the field E should be chosen carefully. Fortunately, there is a large class of *flexible fields* - [26, Definition 1.1], which are fit for the role. These are purely transcendental extensions of an infinite transcendence degree of some other fields. Any field E can be embedded into its *flexible closure* $\tilde{E} := E(\mathbb{P}^\infty)$ and the respective motivic restriction functor is conservative. We get the family of *isotropic realisations*:

$$\psi_{p,E} : DM(k) \rightarrow DM(\tilde{E}/\tilde{E}; \mathbb{F}_p),$$

where p is prime and E/k runs over all extensions of the base field. These functors are tensor-triangulated and so, the kernels $\alpha_{p,E} := \text{Ker}(\psi_{p,E})$ are $\otimes - \Delta$ -ideals of the Voevodsky category. It appears that some of these realisations will be equivalent in a sense. Namely, one can introduce an equivalence relation \sim on the set of extensions - see Sect. 5.2, such that the respective kernels $\alpha_{p,E}$ coincide if and only if the extensions are equivalent - see Theorem 1.3.

As in the global case, the isotropic motivic category $DM(k/k; \mathbb{F}_p)$ possesses the natural weight structure in the sense of Bondarko [6] whose heart is the category of *isotropic Chow motives* $\text{Chow}(k/k; \mathbb{F}_p)$ - see Proposition 5.7. This consists of direct summands of the isotropic motives of smooth projective varieties. Many questions about the isotropic motivic category can be reduced to questions about the heart. In this article we will be able to answer various such crucial questions.

The Homs in the category of isotropic Chow motives are given by *isotropic Chow groups* Ch_{iso}^* . The latter is the quotient of the usual Chow groups modulo *anisotropic classes*, that is, elements coming from p -anisotropic varieties. On $\text{Ch}^* = \text{CH}^*/p$ of a smooth projective variety X there is the *degree pairing* $\text{Ch}^*(X) \times \text{Ch}^*(X) \rightarrow \mathbb{F}_p$. Modding out the kernel of it, we get the *numerical version* Ch_{Num}^* . By obvious reasons, anisotropic classes are numerically trivial and so, we get the natural surjection $\text{Ch}_{iso}^* \rightarrow \text{Ch}_{Num}^*$ of oriented cohomology theories. Many important properties of the isotropic motivic categories and isotropic realisations depend on the following conjecture - [26, Conjecture 4.7].

Conjecture 1.1 *If k is flexible, then $\text{Ch}_{iso}^* = \text{Ch}_{Num}^*$.*

In [26], this conjecture was proven for varieties of dimension ≤ 5 , for cycles of dimension ≤ 2 , and for divisors. In [27], it was extended to varieties of dimension $\leq 2p$ and cycles of dimension $< p$ (as well as to few other cases). Finally, we can prove the entire Conjecture - see Theorem 4.12.

Theorem 1.2 *The conjecture 1.1 is true.*

An immediate consequence of it is that *isotropic Chow groups* are finite groups - see Corollary 4.13. Another one is that the category of *isotropic Chow motives* is equivalent to the category of *numerical Chow motives* - see Corollary 5.4. The latter tensor additive category is semi-simple and has no \otimes -zero-divisors. The weight complex functor considerations then imply that the zero ideal (0) in the isotropic motivic category of a flexible field is prime $\otimes - \Delta$ -ideal. Hence, the kernels $\alpha_{p,E}$ of our isotropic realisations $\psi_{p,E}$ provide points of the Balmer spectrum of the Voevodsky category - see Theorem 5.13:

Theorem 1.3 *The $\otimes - \Delta$ -ideal $\alpha_{p,E}$ is prime and so, defines a point of the Balmer spectrum $\mathrm{Spc}(DM(k)^c)$ of the Voevodsky category. Two such points $\alpha_{p,E}$ and $\alpha_{q,F}$ are equal if and only if $p = q$ and $E/k \stackrel{p}{\sim} F/k$.*

We get many new points of the Balmer spectrum, complementing the “classical” ones (provided by the topological realisation). This substantially improves our understanding of the spectrum. Balmer and Gallauer managed to completely describe the spectra of the subcategories of Tate and Artin-Tate motives in the Voevodsky category in [5] and [10] for special fields, but very little was known about $\mathrm{Spc}(DM(k)^c)$ itself, in general. Our results show, in particular, that this Balmer spectrum is pretty large. For example, the cardinality of $\mathrm{Spc}(DM(\mathbb{R})^c)$ is $2^{2^{\aleph_0}}$ - see Example 5.14.

A similar techniques can be applied to the study of the Balmer spectrum of Morel-Voevodsky stable homotopy category $SH(k)^c$. Here one needs to generalise the notion of *anisotropy*. In Sect. 2 below the notion of A -anisotropy for any (small) oriented theory A^* is introduced. A smooth projective variety is A -anisotropic, if the push-forward $\pi_* : A_*(X) \rightarrow A_*(\mathrm{Spec}(k))$ to the point is zero. In the case of $A^* = \mathrm{Ch}^* = \mathrm{CH}^*/p$ it gives the good old p -anisotropy. Then, in the same way as for Chow groups, one may introduce the *isotropic* version A_{iso}^* of the theory, with the natural surjection to the numerical one A_{Num}^* . The theories of interest are Morava K -theories $K(m)^*$ and closely related theories $P(m)^*$ - see Sect. 3.2. The latter is the *free* theory (in the sense of Levine-Morel - [14]) obtained from the Brown-Peterson theory BP^* by change of coefficients $P(m)^* := BP^* \otimes_{BP} P(m)$, where $P(m) = BP/I(m)$ and $I(m) = (v_0, \dots, v_{m-1})$ is the invariant ideal of Landweber. In the case $m = \infty$, we get $P(\infty)^* = K(\infty)^* = \mathrm{Ch}^*$. Theorems 4.7 and 4.17 generalise Theorem 1.2 to the case of an arbitrary m :

Theorem 1.4 *Let k be flexible. Then, for any $1 \leq m \leq \infty$, we have: $P(m)_{iso}^* = P(m)_{Num}^*$ and $K(m)_{iso}^* = K(m)_{Num}^*$.*

This is the main result of the paper, which implies all the rest. The broadening of view from Chow groups to $P(m)$ -theories was crucial in the solution of the Conjecture 1.1. The right set up here is provided by a theory Q^* which is a quotient of the BP -theory by any invariant ideal, and this is the generality in which I prove the Main Theorem 1.4. In the proof of the original cases in [26, Theorem 4.11] and [27, Theorem 2.3], the key step was to represent our numerically trivial cycle by the class of a regular connected subvariety and then annihilate its’ Chern classes numerically. This was achieved by a sophisticated surgery-type techniques involving blow-ups,

deformations, as well as Steenrod and Adams operations, and was quite non-trivial. Now it is replaced by the claim that any numerically trivial element of Q^* can be lifted to an element of Ω^* , so that all the Landweber-Novikov operations of it are Q -numerically trivial - see Proposition 4.3. The latter statement is proven by a short transparent argument involving the multiplicative projector defining the BP -theory. It allows to reduce to the case of a numerically trivial class of a regular embedding $[Y \rightarrow X]$, such that the normal bundle of it has the same Q -Chern classes as some bundle of the same dimension coming from X . Passing to the flag variety, one may assume that the class of our normal bundle in K_0 is a sum of classes of line bundles, all defined on X . Finally, one employs the deformation to the normal cone in a new unusual role. This famous construction, among other things, also makes a class into a complete intersection (in a situation like ours). Thus, we may reduce to the case of a numerically trivial complete intersection, which is straightforward - see Proposition 4.5.

Theorem 1.4, in particular, shows that the *isotropic category of $K(m)$ -Chow motives* $Chow_{iso}^{K(m)}(k)$ coincides with its' numerical counterpart $Chow_{Num}^{K(m)}(k)$, which is semi-simple and has no \otimes -zero-divisors - see Theorems 5.1 and 5.3, which plays a key role in the study of isotropic realisations of $SH^c(k)$.

To any prime p , any $1 \leq m \leq \infty$ and a choice of a field extension E/k we may assign the isotropic realisation

$$\psi_{(p,m),E} : SH(k)^c \rightarrow SH_{(p,m)}(\tilde{E}/\tilde{E})^c,$$

where the target is an *isotropic stable homotopic category* (over a flexible field). This is done in our paper with Du [8]. This category is obtained from $SH(\tilde{E})^c$ by moding out the $\Sigma_{\mathbb{P}^1}^\infty$ -spectra of $K(m)$ -anisotropic varieties as well as objects whose MGL -motive is annihilated by some power of v_m (where we set $v_\infty = 1$) - see Sect. 5.3.

Let $\alpha_{(p,m),E} = \text{Ker}(\psi_{(p,m),E})$. One may introduce the $K(m)$ -equivalence relation $(\sim^{(p,m)})$ on the set of extensions - see Sect. 5.3. Theorem 1.4 provides the crucial ingredient, which permits to establish that the zero ideal (0) in a (flexible) isotropic stable homotopic category is prime, and hence, the same holds for $\alpha_{(p,m),E}$. This is done in [8] - see Theorem 5.17:

Theorem 1.5 ([8]) (1) $\alpha_{(p,m),E}$ provide points of the Balmer spectrum of Morel-Voevodsky category $SH(k)^c$.

- (2) $\alpha_{(p,m),E} = \alpha_{(q,n),F}$ if and only if $p = q$, $m = n$ and $E/k \sim^{(p,m)} F/k$.
 (3) $\alpha_{(p,\infty),E}$ is the image of the point $\alpha_{p,E}$ under the natural map of spectra $\text{Spc}(DM(k)^c) \rightarrow \text{Spc}(SH(k)^c)$.

Thus, we get plenty of new “isotropic” points of the Balmer spectrum of Morel-Voevodsky category. These points are analogous to and complement the “classical” Morava-points $\alpha_{(p,m),Top}$ given by the topological realisation. The structure of $\text{Spc}(SH(k)^c)$ was studied, in particular, by Balmer and Heller-Ormsby in [4] and [12], who have shown that this spectrum surjects to the spectra of Grothendieck-Witt and Milnor-Witt rings of k . There are few other results, in particular, comparison with equivariant homotopic spectra. But the question of the description of $\text{Spc}(SH(k)^c)$

remained widely open, and the above isotropic points provide by far the largest known structured piece of it (note that there is a huge number of equivalence classes of extensions, for a general field).

2 Isotropic equivalence

Everywhere in this article, the ground field k will be of characteristic zero.

Let A^* be an oriented cohomology theory with localisation in the sense of [24, Definition 2.1] (which is the standard axioms of Levine-Morel [14, Definition 1.1.2] plus the excision axiom (*EXCI*)). We can introduce the notion of A -anisotropic varieties.

Definition 2.1 Let $X \xrightarrow{\pi} \text{Spec}(k)$ be a smooth projective variety. We say that X is A -anisotropic, if the map $\pi_* : A_*(X) \rightarrow A_*(\text{Spec}(k)) = A$ is zero.

Observation 2.2 Let X be a smooth projective variety over k . Then it is A -anisotropic if and only if the A^* -numerical ring $A_{\text{Num}}^*(X)$ of X is zero.

Proof It is sufficient to recall that the numerical pairing $A^*(X) \times A^*(X) \xrightarrow{\langle \cdot, \cdot \rangle} A$ is defined by $\langle x, y \rangle := \pi_*(x \cdot y)$. \square

Observation 2.3 Let $f : X \rightarrow Y$ be a morphism of smooth projective varieties. Then if Y is A -anisotropic, then so is X .

Proof Since f is proper, $(\pi_X)_* = (\pi_Y)_* \circ f_*$. \square

Remark 2.4 (1) Since every smooth projective variety contains a point $\text{Spec}(E)$ of finite degree $n = [E : k]$ and for the respective projection, $\pi_* \circ \pi^* = \cdot n$, it follows that A -anisotropic varieties may exist only if the theory A^* is n -torsion: $n \cdot A = 0$, for some natural n .

(2) When $A^* = \text{CH}^*/n$ is the Chow groups modulo n , our notion of A -anisotropy coincides with the notion of n -anisotropy: X is CH^*/n -anisotropic if and only if the degrees of all closed points of X are divisible by n - see [26, Definition 2.14]. \triangle

Now we can introduce *isotropic equivalence* on A^* .

Definition 2.5 Let X be a smooth projective variety and $x \in A^*(X)$. We say that x is anisotropic, if $x = f_*(y)$ for some $f : Y \rightarrow X$ and A -anisotropic (smooth projective) Y .

We have:

Proposition 2.6 Anisotropic classes form an ideal which is stable under pull-backs and push-forwards, and so, under A^* -correspondences.

Proof The fact that anisotropic classes are stable under push-forwards is obvious from definition. As for pull-backs, if x belongs to the image of f_* , where $f : Q \rightarrow X$ and Q is A -anisotropic, then it also belongs to the image of π_* , where $\pi : X \times Q \rightarrow X$ is the projection and $X \times Q$ is still anisotropic. But π is transversal to any $g : Y \rightarrow X$. Thus, $g^*(x)$ belongs to the image of $\tilde{\pi}_*$, where $\tilde{\pi} : Y \times Q \rightarrow Y$, by the axiom (A2) - see [14, Definition 1.1.2], and $Y \times Q$ is anisotropic too. Also, clearly, anisotropic classes form an ideal under external products. The remaining properties follow. \square

We get the *isotropic version* of the theory A^* .

Definition 2.7 Let X be smooth projective over k . Define

$$A_{iso}^*(X) := A^*(X) / (\text{anisotropic classes}).$$

Using [26, Example 4.1] (cf. [11, Example 4.6]) this can be extended to an oriented cohomology theory (with localisation) on \mathbf{Sm}_k as A_Γ^* , where Γ consists of all classes from A -anisotropic (smooth projective) varieties.

If $f : Y \rightarrow X$ is any morphism of smooth projective varieties and $y \in A^*(Y)$, $x \in A^*(X)$, then $\langle y, f^*(x) \rangle = \langle f_*(y), x \rangle \in A$. Thus, any *anisotropic* class is *numerically trivial* (by Observation 2.2) and we have a surjection $A_{iso}^* \rightarrow A_{Num}^*$ of oriented cohomology theories.

3 Brown-Peterson theory and Morava K-theories

3.1 Multiplicative operations

To any oriented cohomology theory A^* one can assign a formal group law (A, F_A) , where $A = A^*(\text{Spec}(k))$ and F_A describes the Chern class of the tensor product of two line bundles in terms of the Chern classes of the factors: $c_1^A(L \otimes M) = F_A(c_1^A(L), c_1^A(M))$.

If A^* and B^* are two oriented cohomology theories on \mathbf{Sm}_k , a *multiplicative operation* $G : A^* \rightarrow B^*$ is a collection of ring homomorphisms $A^*(X) \rightarrow B^*(X)$, for all smooth X/k , commuting with pull-back maps. The difference with the *morphisms of theories* is that the above maps don't have to commute with push-forwards.

To any multiplicative operation $G : A^* \rightarrow B^*$ one can assign the morphism

$$(\varphi_G, \gamma_G) : (A, F_A) \rightarrow (B, F_B)$$

of the respective formal group laws, where $\varphi_G : A \rightarrow B$ is the action of G on $\text{Spec}(k)$ and $\gamma_G(x) \in B[[x]]x$ describes, how G acts on the Chern classes of line bundles:

$$G(c_1^A(L)) = \gamma_G(c_1^B(L)).$$

These data satisfies:

$$\varphi_G(F_A)(\gamma_G(x), \gamma_G(y)) = \gamma_G(F_B(x, y)).$$

An oriented cohomology theory is *free*, if it is obtained from algebraic cobordism of Levine-Morel [14] by change of coefficients: $A^*(X) = \Omega^*(X) \otimes_{\mathbb{L}} A$. Such theories are in 1-to-1 correspondence with formal group laws: $A^* \leftrightarrow \mathbb{L} \rightarrow A$. By [24, Theorem 6.9], the multiplicative operations $G : A^* \rightarrow B^*$ between free theories are in 1-to-1 correspondence with the morphisms of the respective FGLs.

We say that a multiplicative operation $G : A^* \rightarrow B^*$ is of *invertible type*, if the leading coefficient $\gamma'(0)$ is invertible in B . Such a power series is then invertible with respect to the composition of series. We say that G is *stable*, if this coefficient is equal to 1.

In the case when the source theory is the algebraic cobordism of Levine-Morel Ω^* , a multiplicative operation G of invertible type is completely determined by γ_G (due to universality of the respective FGL), which can be any power series with an invertible coefficient at x .

The structure on the algebraic cobordism is provided by the Landweber-Novikov operations. The *total Landweber-Novikov operation*

$$S_{L-N}^{Tot} : \Omega^* \rightarrow \Omega^*[\bar{b}],$$

where $\bar{b} = b_i, i \geq 1$ and $\dim(b_i) = i$ is given by the homomorphism (φ, γ) of formal group laws, where the (stable) change of parameter $\gamma(x) = x + b_1x^2 + b_2x^3 + \dots$ has formal variables as coefficients - see [14, Example 4.1.25]. This operation is universal among *stable multiplicative operations*. Namely, any such operation G can be completed uniquely to a commutative square

$$\begin{array}{ccc} \Omega^* & \xrightarrow{S_{L-N}^{Tot}} & \Omega^*[\bar{b}] \\ \theta_A \downarrow & & \downarrow \psi \\ A^* & \xrightarrow{G} & B^* \end{array}$$

where θ_A is the canonical morphism of theories (reflecting the universality of algebraic cobordism - [14, Theorem 1.2.6]) and ψ extends a similar canonical morphism θ_B by sending the formal variables b_i to the coefficients of the power series $\gamma_G(x) \in B[[x]]x$.

Let $G : A^* \rightarrow B^*$ be a multiplicative operation of invertible type between *free* theories. Let X be a smooth projective variety and $\text{Td}_G(X) \in B^*(X)$ be the *Todd genus* of X corresponding to the operation G - see [17, Definition 2.5.2]. In other words,

$$\text{Td}_G(X) = \prod_j \left(\frac{x}{\gamma_G(x)} \right) (\lambda_j),$$

where λ_j runs over all *B-roots* of the Tangent bundle T_X (recall, that *B-roots* are “formal elements”, elementary symmetric functions of which are *B-Chern* classes of our bundle - see [24, 2.3] and [14, 4.1.8]; below we will also call by this name any collection of elements of degree 1 in $B^*(X)$ satisfying the same property, if one

exists). $\text{Td}_G(X)$ is an invertible element in $B^*(X)$. Then the Riemann-Roch theorem for multiplicative operations (of invertible type) - Panin [17, Theorem 2.5.4] claims that the transformation $G' : A^* \rightarrow B^*$, defined on a given variety Y by $G \cdot \text{Td}_G(Y)$, commutes with (proper) push-forwards.

3.2 p -typical theories

The theories we are interested in are related to a choice of a prime number p and can be produced out of the algebraic cobordism Ω^* of Levine-Morel [14]. Algebraic cobordism is universal among all oriented cohomology theories. For $\mathbb{Z}_{(p)}$ -localised theories, this role is played by the Brown-Peterson theory BP^* [7]. This theory is obtained from $\Omega^*_{\mathbb{Z}_{(p)}}$ by the same multiplicative projector ρ as in topology (see [28, I.3] and [1]). Namely, $\rho : \Omega^*_{\mathbb{Z}_{(p)}} \rightarrow \Omega^*_{\mathbb{Z}_{(p)}}$ is a multiplicative operation, which satisfies: $\rho \circ \rho = \rho$ and is determined by the property that $\rho([\mathbb{P}^n]) = [\mathbb{P}^n]$, if n is a power of p minus one, and is zero, otherwise. In particular, (by the formula of Mischenko) the logarithm of this theory is given by:

$$\log_{BP}(x) = \sum_{i \geq 0} \frac{[\mathbb{P}^{p^i-1}]}{p^i} x^{p^i}.$$

By [24, Proposition 4.9(2)], BP^* is obtained from algebraic cobordism by change of coefficients: $BP^*(X) = \Omega^*(X) \otimes_{\mathbb{L}} BP$, that is, it is a *free theory*. Here $\mathbb{L} \rightarrow BP$ is the universal p -typical formal group law. A torsion-free FGL over $\mathbb{Z}_{(p)}$ is called *p-typical*, if its logarithm has only terms of degree a power of p , for the general case see [19, Definition 2.1.22]. We call an oriented cohomology theory A^* over $\mathbb{Z}_{(p)}$ *p-typical* if the respective FGL is. The coefficient ring of the Brown-Peterson theory has the form $BP = \mathbb{Z}_{(p)}[v_i; i \geq 1]$, where $\dim(v_i) = p^i - 1$. It is traditional to denote $v_0 = p$. The Brown-Peterson theory is universal among *p-typical* oriented theories. Namely, for any such theory A^* , there exists a unique morphism of theories $\theta_A : BP^* \rightarrow A^*$ - see [19, 4.1.12, A2.1.25] and [24, Theorem 6.9].

The *Total Landweber-Novikov operation*

$$S_{BP}^{Tot} : BP^* \rightarrow BP^*[\bar{t}],$$

where $\bar{t} = t_i, i \geq 1, \dim(t_i) = p^i - 1$, is a stable multiplicative operation given by the morphism (φ, γ) of formal group laws with $\gamma^{-1}(x) = \sum_{i \geq 0} t_i x^{p^i}$ and $t_0 = 1$ (note the formal summation!). This operation is universal among stable multiplicative operations between p -typical cohomology theories. That is, any such operation G extends uniquely to a commutative square:

$$\begin{array}{ccc} BP^* & \xrightarrow{S_{BP}^{Tot}} & BP^*[\bar{t}] \\ \theta_A \downarrow & & \downarrow \psi \\ A^* & \xrightarrow{G} & B^* \end{array} \quad (1)$$

where θ_A is the canonical morphism of theories (classifying p -typical theories) and ψ is an extension of θ_B , mapping t_i to $d_i \in B$, where $\gamma_G^{-1}(x) = \sum_{i \geq 0} d_i x^{p^i}$ - see [19, Lemma A2.1.26].

We have an algebraic version $K(p, m)$ of Morava K-theory, which is an oriented cohomology theory in the sense of [24, Definition 2.1] obtained from BP -theory by the change of coefficients: $K(p, m)^* = BP^* \otimes_{BP} K(p, m)$, where $BP \rightarrow K(p, m)$ is a p -typical formal group law of level m . That is, for some choice of generators of BP , $K(p, m) = \mathbb{F}_p[v_m, v_m^{-1}]$, where $v_i \mapsto 0$, for $i \neq m$. Following the topological tradition, we will also denote this theory simply as $K(m)^*$.

By the result of Landweber [13, Theorem 2.7], the prime ideals of BP invariant under Landweber-Novikov operations are exactly the ideals $I(m) = (v_0, v_1, \dots, v_{m-1})$, for $0 \leq m \leq \infty$. Consider the free theory $P(m)^*$ obtained from BP^* by change of coefficients $P(m)^*(X) = BP^*(X) \otimes_{BP} P(m)$, where $P(m) = BP/I(m)$. In particular, $P(0)^* = BP^*$, $P(1)^*$ is the mod- p version $\overline{BP}^* = BP^*/p$ and $P(\infty)^* = \text{Ch}^* = \text{CH}^*/p$ is Chow groups modulo p . For $m \geq 1$, these theories are p -torsion. In particular, it makes sense to speak about *isotropic equivalence* on $P(m)^*$.

We will also use the theory $P\{m\}^* = P(m)^*[v_m^{-1}]$ obtained from $P(m)^*$ by inverting v_m . There are the following natural morphisms among our p -typical theories:

$$BP^* \rightarrow P(m)^* \rightarrow P\{m\}^* \rightarrow K(m)^*.$$

4 The main theorem

Our main result compares isotropic and numerical properties of the $P(m)^*$ theory. Namely, we will show that, for $1 \leq m \leq \infty$, the theories $P(m)_{iso}^*$ and $P(m)_{Num}^*$ coincide. This plays the crucial role in applications below, and implies the respective facts about Chow groups and Morava K-theories.

4.1 The strategy of the proof

Let me start by explaining the strategy of the proof. The argument applies to any free theory Q^* obtained from BP^* by moding out a non-zero ideal J of BP invariant under BP -Landweber-Novikov operations.

Let X be a smooth projective variety over a flexible field. We need to show that any *numerically trivial* class u in $Q^*(X)$ is *Q-anisotropic*.

The first step is to lift u to an element of the algebraic cobordism of Levine-Morel $\Omega^*(X)$ (via the natural maps $\Omega^*(X) \rightarrow BP^*(X) \rightarrow Q^*(X)$) in such a way that the action of all Landweber-Novikov operations on it produces elements which are numerically trivial in $Q^*(X)$ (as u itself). This is achieved by applying the multiplicative projector defining the BP -theory to an arbitrary lifting and uses the fact that the BP -Landweber-Novikov operations respect Q -numerical equivalence. This is done in Proposition 4.3 below, see also Proposition 4.1. Thus, we may assume that u is represented by the class of a projective map $[Y \xrightarrow{y} X]$ from some smooth variety Y of dimension d . The Q -numerical triviality of the action of Landweber-Novikov operations on $[Y \xrightarrow{y} X]$ means that any polynomial in Chern classes of T_Y is Q -numerically trivial on X .

The second step is to substitute u by a class of a regular embedding $[Y \xrightarrow{y} X]$, with the normal bundle Q -equivalent to a pull-back of some element from $K_0(X)$ (where we call two elements of K_0 to be Q -equivalent, if they have the same Q -Chern classes). By twisting T_Y by a sufficiently high p -primary power of a very ample line bundle (which is a Q -equivalence), we may assume that T_Y is generated by global sections and defines an embedding $Y \xrightarrow{f} Gr(d, N)$ into some Grassmannian. Then

$$v = [Y \xrightarrow{(y,f)} X \times Gr(d, N)]$$

is a class of a regular embedding, whose projection to X is u and which is still Q -numerically trivial, since the restriction of $Q^*(X \times Gr(d, N))$ to Y as an algebra over $Q^*(X)$ is generated exactly by the Chern classes of T_Y , polynomials in which are Q -numerically trivial by Step 1. It is sufficient to prove the claim for v . Moreover, since T_Y is Q -equivalent to the restriction of the tautological vector bundle from the Grassmannian, the normal bundle of our embedding has the needed property.

With a bit of extra work, we may reduce to the case of a regular embedding $u = [Y \xrightarrow{y} X]$, whose normal bundle is Q -equivalent to the restriction of the sum $[L_1] + \dots + [L_n]$ of classes of line bundles from X , where $n = \text{codim}(u)$.

The third step is to use the deformation to the normal cone construction and reduce to the case of a complete intersection. Here we replace X by $W = Bl_{X \times \mathbb{P}^1}(Y \times 0)$ and u by the pull-back of $-[Y \times 0]$. This class is still Q -numerically trivial and its push-forward to X is $-u$. But, at the same time, it is equal to the product of 1-st Q -Chern classes of line bundles $\xi(\xi +_Q a_1) \cdot \dots \cdot (\xi +_Q a_n)$, where $\xi = c_1^Q(O(1))$ and $a_i = c_1^Q(L_i)$. The key here is the fact that the Q -roots of the normal bundle $N_{Y \times 0 \subset X \times \mathbb{P}^1}$ are defined on X and there is a zero among them. By twisting by a sufficiently high p -primary power of an ample line bundle, we may assume that our line bundles are very ample and so, our class is a Q -numerically trivial complete intersection.

Finally, on the fourth step we prove that any numerically trivial complete intersection is anisotropic. This result holds for any *free* theory A^* and is proven in Proposition 4.5. This finishes the proof of the Main Theorem.

4.2 The proof

Let $Q = BP/J$, where $J \subset BP$ is a non-zero ideal invariant under Landweber-Novikov operations. Consider the *free* theory $Q^* = BP^* \otimes_{BP} Q$. Let Q_{iso}^* and Q_{Num}^* be the *isotropic* and the *numerical* versions of the theory (the latter one is obtained from Q^* by moding-out the kernel of the numerical pairing on Q^* - see [26, Definition 4.3]). We are going to show that $Q_{iso}^* = Q_{Num}^*$. The case where J is the Landweber ideal $I(m)$ will give us the statement for $P(m)$ then.

Proposition 4.1 *Let $J \subset BP$ be an invariant ideal, $Q = BP/J$ and Q^* be the free theory given by: $Q^*(X) = BP^*(X) \otimes_{BP} Q$. Then the Landweber-Novikov operations descend to Q_{Num}^* .*

Proof Since the ideal J is invariant, the Total BP -Landweber-Novikov operation descends to the Q^* -theory:

$$S_Q^{Tot} : Q^* \rightarrow Q^*[\bar{t}]$$

We need to show that Landweber-Novikov operations preserve the ideal of numerically-trivial classes.

Our operation can be extended to a multiplicative endomorphism of the $Q^*[\bar{t}]$ -theory

$$G = \tilde{S}_Q^{Tot} : Q^*[\bar{t}] \rightarrow Q^*[\bar{t}],$$

by mapping t_i to t_i . In other words, for the respective morphism (φ_G, γ_G) of the FGLs, φ_G acts on Q as $\varphi_{S_Q^{Tot}}$ and maps t_i to t_i , while $\gamma_G^{-1}(x) = \sum_{i \geq 0} t_i x^{p^i}$, with $t_0 = 1$. Recall, that according to [24, Theorem 6.9], multiplicative operations between *free* theories are in 1-to-1 correspondence with the morphisms of the respective formal group laws. Since the constant term of $\varphi_{S_Q^{Tot}}$ is the identity map on Q , and the terms at higher monomials in \bar{t} decrease the dimension of elements, it follows that $\varphi_G : Q[\bar{t}] \rightarrow Q[\bar{t}]$ is invertible. Also, the power series γ_G is invertible (with respect to the composition). Then the pair (φ_H, γ_H) given by: $\varphi_H = \varphi_G^{-1}$ and $\gamma_H(x) = \varphi_G^{-1}(\gamma_G^{-1}(x))$ defines an endomorphism of the FGL $(Q[\bar{t}], F_Q)$ inverse to (φ_G, γ_G) . Recall, that the composition of morphisms of FGLs is defined as: $(\varphi_H \circ \varphi_G, \gamma_H \circ \gamma_G(x)) = (\varphi_H \circ \varphi_G, \varphi_H(\gamma_G(\gamma_H(x))))$. Hence, the operation G is invertible. In particular, it is surjective.

Let $X \xrightarrow{\pi} \text{Spec}(k)$ be a smooth projective variety and $\text{Td}_G(X) \in Q^*(X)[\bar{t}]$ be the *Todd genus* of X corresponding to our operation G - see Sect. 3.1 and [17, Definition 2.5.2]. This is an invertible element in $Q^*(X)[\bar{t}]$. Recall, that the Riemann-Roch theorem for multiplicative operations - Panin [17, Theorem 2.5.4] claims that the transformation $G' : Q^*[\bar{t}] \rightarrow Q^*[\bar{t}]$ defined on a given variety Y by $G \cdot \text{Td}_G(Y)$ commutes with (proper) push-forwards.

Let $x \in Q^*(X)[\bar{t}]$ be a numerically trivial element. Since $G = \tilde{S}_Q^{Tot}$ is surjective and $\text{Td}_G(X)$ is invertible, for any $z \in Q^*(X)[\bar{t}]$, there exists $y \in Q^*(X)[\bar{t}]$, such that $z = \tilde{S}_Q^{Tot}(y) \cdot \text{Td}_G(X)$. Then

$$\pi_*(\tilde{S}_Q^{Tot}(x) \cdot z) = \pi_*(\tilde{S}_Q^{Tot}(x) \tilde{S}_Q^{Tot}(y) \cdot \text{Td}_G(X)) = \pi_*(\tilde{S}_Q^{Tot}(x \cdot y) \cdot \text{Td}_G(X)).$$

By the Riemann-Roch theorem [17, Theorem 2.5.4], using the fact that $\text{Td}_G(\text{Spec}(k)) = 1$, this is equal to

$$\tilde{S}_Q^{Tot}(\pi_*(x \cdot y)) \cdot \text{Td}_G(\text{Spec}(k)) = \tilde{S}_Q^{Tot}(\pi_*(x \cdot y)).$$

So, it is zero, since x was numerically trivial. Hence, $\tilde{S}_Q^{Tot}(x)$ is numerically trivial too. \square

Remark 4.2 This statement applies, among others, to the theories $Q^* = BP^* \otimes_{BP} BP/I(m)^r$, for any $r \in \mathbb{N}$ and $0 \leq m \leq \infty$. In particular, to all theories $P(m)^*$ (including $P(0)^* = BP^*$). \triangle

The following result is one of the key steps of the construction. It permits to substitute the non-trivial and elaborate considerations of [26, Theorem 4.11] and [27] by a clean and transparent multiplicative projector argument.

Proposition 4.3 *Let $J \subset BP$ be a non-zero invariant ideal and $Q^* = BP^* \otimes_{BP} BP/J$. Let X/k be a smooth projective variety and $x \in Q^*(X)$ be a numerically trivial class. Then it may be represented by $\tilde{x} \in \Omega^*(X)$, such that the projection of $S_{L-N}^{Tot}(\tilde{x})$ to Q^* is numerically trivial.*

Proof Let $\rho : \Omega_{\mathbb{Z}(p)}^* \rightarrow \Omega_{\mathbb{Z}(p)}^*$ be the multiplicative projector defining the theory BP^* . It corresponds to the endomorphism $(\varphi_\rho, \gamma_\rho)$ of the $(\mathbb{Z}(p)$ -localised) universal FGL - see [24, Theorem 6.9]. It can be decomposed as $\Omega_{\mathbb{Z}(p)}^* \xrightarrow{\mu} BP^* \xrightarrow{\eta} \Omega_{\mathbb{Z}(p)}^*$, where μ is the canonical morphism of theories (so, the respective morphism of FGLs has the form (pr, x)) and η is the multiplicative operation corresponding to the morphism (em, γ_ρ) , where φ_ρ decomposes as $\mathbb{L}_{\mathbb{Z}(p)} \xrightarrow{pr} BP \xrightarrow{em} \mathbb{L}_{\mathbb{Z}(p)}$ - see [24, the proof of Prop. 4.9(2)]. Moreover $\mu \circ \eta = id_{BP^*}$. Thus, BP^* can be realised as a quotient of $\Omega_{\mathbb{Z}(p)}^*$, or as a direct summand of the *reorientation* (as in Panin-Smirnov [18]) $(\Omega_{\mathbb{Z}(p)}^{\gamma_\rho})^*$ of this theory. Consider the composition:

$$BP^* \xrightarrow{\eta} \Omega_{\mathbb{Z}(p)}^* \xrightarrow{S_{L-N}^{Tot}} \Omega_{\mathbb{Z}(p)}^*[\bar{b}] \xrightarrow{\mu[\bar{b}]} BP^*[\bar{b}].$$

It is a multiplicative operation $\delta : BP^* \rightarrow BP^*[\bar{b}]$ between p -typical theories and so, by (1), is a specialization of $S_{BP}^{Tot} : BP^* \rightarrow BP^*[\bar{t}]$. That is, there exists a morphism of theories $\varepsilon : BP^*[\bar{t}] \rightarrow BP^*[\bar{b}]$ (which is automatically BP^* -linear), such that $\delta = \varepsilon \circ S_{BP}^{Tot}$. In other words, individual δ -operations (coefficients of δ at particular monomials in \bar{b}) are BP -linear combinations of the individual BP -Landweber-Novikov operations $S_{BP}^{t^r}$. In turn, it is easy to see (from universality of the cobordism operation S_{L-N}^{Tot}) that S_{BP}^{Tot} is a specialization of δ (and so, these operations are equivalent), but we will not use it.

Let $\bar{x} \in \Omega_{\mathbb{Z}(p)}^*$ be any lifting of $x \in Q^*$, and $\tilde{x} = \rho(\bar{x}) \in \Omega_{\mathbb{Z}(p)}^*$. Then

$$\mu[\bar{b}] \circ S_{L-N}^{Tot}(\tilde{x}) = \mu[\bar{b}] \circ S_{L-N}^{Tot} \circ \eta \circ \mu(\bar{x}) = \delta \circ \mu(\bar{x}).$$

We know that the projection of $\mu(\bar{x})$ to Q^* is numerically trivial and that δ is a specialization of S_{BP}^{Tot} . From Proposition 4.1, the projection of $\delta \circ \mu(\bar{x})$ to Q^* is numerically trivial. This means that the projection of $S_{L-N}^{Tot}(\tilde{x})$ to Q^* is numerically trivial. It remains to recall that $\mu \circ \eta = id_{BP^*}$. Hence, the projection of \tilde{x} to Q^* is equal to that of $\mu \circ \eta \circ \mu(\bar{x}) = \mu(\bar{x})$ and so, coincides with x . Finally, since J is non-zero and invariant, it contains p^r , for some r (note that, by the results of Landweber [13], the radical of this ideal is some Landweber ideal $I(m)$, for $m > 0$). Then multiplying \tilde{x} by an appropriate integer $q \equiv 1 \pmod{p^r}$, we may assume that the class \tilde{x} is integral, and so, represented by some $[Y \xrightarrow{y} X]$ with Y -smooth projective. \square

Remark 4.4 It applies, in particular, to the theories $P(m)$, for any $1 \leq m \leq \infty$. When $J = 0$, that is, $Q^* = BP^*$, we may still represent x by $\tilde{x} \in \Omega_{\mathbb{Z}(p)}^*(X)$ with the result of Landweber-Novikov operations on it BP -numerically trivial, but it will not be represented by an integral class, in general. \triangle

Finally, we will need the following result showing that numerically trivial complete intersections are anisotropic (over a flexible field).

Proposition 4.5 *Let A^* be a free oriented cohomology theory, k be flexible and X/k be smooth projective variety. Let $u = \prod_{i=1}^n x_i \in A^*(X)$, where $x_i = c_1^A(L_i)$, for very ample line bundles L_i on X . Suppose, u is numerically trivial. Then it is anisotropic.*

Proof Let Z be a smooth projective variety and $L = O(D)$ a very ample line bundle on it. The linear system $|D|$ of effective divisors linearly equivalent to D is parametrized by a projective space \mathbb{P}^N . These divisors are hyperplane sections of X in the projective embedding given by L . Consider the subvariety $Y \subset Z \times \mathbb{P}^N$ consisting of pairs $\{(z, H) | z \in H\}$. We have natural projections $Z \xleftarrow{\pi} Y \xrightarrow{\beta} \mathbb{P}^N$, where π is a \mathbb{P}^{N-1} -bundle (note that, for a given z , the condition $z \in H$ gives a single linear equation on H). Let $\xi = c_1^A(O(1))$ on it. Let η be the generic point of \mathbb{P}^N , and Y_η be the generic fiber of the projection β . It is the *generic* representative of the linear system $|D|$ defined over the purely transcendental extension $F = k(\mathbb{P}^N)$ of the base field. By the projective bundle axiom (PB) [14, Def. 1.1.2] and by excision (EXCI) and [11, Def. 2.6, Cor. 2.13], we have surjections:

$$A^*(Z)[\xi] \twoheadrightarrow A^*(Y) \twoheadrightarrow A^*(Y_\eta).$$

Here $O(1)$ on Y is the restriction of the bundle $O(1)$ on \mathbb{P}^N . Thus, positive powers of ξ are supported in positive codimension on \mathbb{P}^N and so, restrict to zero in $A^*(Y_\eta)$.

Thus, we obtain the surjection $A^*(Z) = A^*(Z_F) \xrightarrow{j^*} A^*(Y_\eta)$ for the embedding $j : Y_\eta \rightarrow Z_F$ of the generic section.

Let $L_i = O(D_i)$ and $\mathbb{P}^{N_i} = |D_i|$ be the respective ample linear systems. Let $\mathbb{P} = \prod_i \mathbb{P}^{N_i}$, η be the generic point of \mathbb{P} and $j : W \hookrightarrow X_E$ be the intersection of the generic representatives of our linear systems. It is defined over a purely transcendental extension $E = k(\mathbb{P})$ of k . Applying the above argument inductively, we obtain that the restriction $A^*(X) = A^*(X_E) \xrightarrow{j^*} A^*(W)$ is surjective. Since $u_E = [W \rightarrow X_E]^A$ is numerically trivial, this shows that all elements of $A^*(W)$ are numerically trivial. Indeed, the (zero) pairing $\langle u_E, - \rangle : A^*(X_E) \rightarrow A$ is just the composition

$$A^*(X_E) \xrightarrow{j^*} A^*(W) \xrightarrow{pr_*} A^*(\text{Spec}(E)) = A.$$

This means that W is anisotropic. Thus, u becomes anisotropic over some purely transcendental extension $E = k(\mathbb{P})$. Since k is *flexible*, that is, $k = k_0(t_1, t_2, \dots)$ and the variety X and the class u are defined on some finite level: over some $k_n = k_0(t_1, \dots, t_n)$, there is an isomorphism of field extensions k/k_n with E/k_n , which identifies X with X_E and u with u_E - see [26, Proposition 1.3]. It follows that u is anisotropic already over k . \square

Remark 4.6 Note that here we don't have any restrictions on the free theory A^* . In particular, we see that if A^* is not torsion, then complete intersections can't be numerically trivial (since there are no non-zero anisotropic classes, in this case). The same result holds for any quotients of free theories. \triangle

Now, we are ready to prove our Main Theorem.

Theorem 4.7 *Let k be flexible and $1 \leq m \leq \infty$. Then $P(m)_{iso}^* = P(m)_{Num}^*$.*

Again, we will prove a more general version.

Theorem 4.8 *Let k be flexible, $J \subset BP$ be a non-zero invariant ideal and $Q^* = BP^* \otimes_{BP} BP/J$. Then $Q_{iso}^* = Q_{Num}^*$.*

Proof We need to show that all numerically trivial classes in Q^* are anisotropic. Suppose, X/k is smooth projective and $x \in Q^*(X)$ is numerically trivial. By Proposition 4.3, we may represent x by the class $u = [Y \xrightarrow{y} X] \in \Omega^*(X)$, such that $S_{L-N}^{Tot}(u) \in \Omega^*(X)[\bar{b}]$ is numerically trivial in $Q^*(X)$.

The operation S_{L-N}^{Tot} commutes with pull-backs. The value of it on u is given by

$$S_{L-N}^{Tot}(u) = y_*(\text{Td}(Y)) \cdot \text{Td}(X)^{-1},$$

where $\text{Td}(Z) = \text{Td}_{S_{L-N}^{Tot}}(Z)$ is equal to $\prod_j \left(\frac{x}{\gamma(x)} \right) (\lambda_j)$, where λ_j runs over Ω^* -roots of the tangent bundle T_Z of Z , and $\gamma = \gamma_{S_{L-N}^{Tot}}$ is the “generic” stable power series $x + b_1x^2 + b_2x^3 + \dots$ - see Sect. 3.1, [17, Theorem 2.5.3] and [14, Example 4.1.25]. There is the homological version S_{Tot}^{L-N} of it, which on X is equal to $S_{L-N}^{Tot} \cdot \text{Td}(X)$. The value of it on u is equal to

$$S_{Tot}^{L-N}(u) = y_*(\text{Td}(Y)).$$

Due to the Riemann-Roch theorem for multiplicative operations - Panin [17, Theorem 2.5.4], S_{Tot}^{L-N} commutes with (proper) push-forwards (which is obvious from the above presentation). Since $S_{L-N}^{Tot}(u)$ is Q^* -numerically trivial, so is $S_{Tot}^{L-N}(u) = S_{L-N}^{Tot}(u) \cdot \text{Td}(X)$.

The individual homological Landweber-Novikov operations $S_{b^*}^{L-N}(u)$ are just $(y_*$ of the) polynomials in various Chern classes $c_s^\Omega(-T_Y)$ of the minus tangent bundle of Y . More precisely, $S_{b^*}^{L-N}(u)$ is the coefficient at the monomial $b^{\vec{r}}$ in $y_*(\text{Td}(Y))$. Note that, this way, we get all possible polynomials in the Chern classes of $-T_Y$, or which is the same, all possible polynomials in the Chern classes of T_Y . Since all these classes are Q^* -numerically trivial on X , arbitrary polynomials in the Chern classes of the tangent bundle of Y are Q^* -numerically trivial on X .

Suppose, $\dim(Y) = d$. Let L be a very ample line bundle on Y . Then, for sufficiently large k , $T_Y \otimes L^{p^k}$ is generated by global sections and defines a regular embedding $f: Y \rightarrow Gr(d, N)$ into the Grassmanian of d -dimensional subspaces of some

vector space, such that $T_Y \otimes L^{p^k} = f^*(Tav)$ is the pull-back of the tautological vector bundle.

Thus, our class $u \in \Omega^*(X)$ is the push-forward of the class of the regular embedding $g = (y, f)$:

$$v = [Y \xrightarrow{g} X \times Gr(d, N)] \in \Omega^*(X \times Gr(d, N)).$$

Note that $Q^*(X \times Gr(d, N))$ as a $Q^*(X)$ -algebra is generated by the Chern classes $c_i^Q(Tav)$ (see [9, Example 14.6.6]) and the pairing $\langle v^Q, - \rangle : Q^*(X \times Gr(d, N)) \rightarrow Q$ is given by

$$Q^*(X \times Gr(d, N)) \xrightarrow{g^*} Q^*(Y) \xrightarrow{pr^*} Q^*(Spec(k)) = Q. \quad (2)$$

By Lemma 4.9, for k sufficiently large in comparison to the $\dim(Y) = d$, the first Chern class $c_1^Q(L^{p^k})$ is zero (in other words, L^{p^k} is Q -equivalent to the trivial bundle O). As Q -equivalence is respected by the \otimes (as Chern classes of the tensor product are expressible in terms of Chern classes of factors), the Chern classes of the pull-back $f^*(Tav)$, for sufficiently large k , are just the Chern classes of T_Y . But we know that any polynomial in the latter Chern classes is Q^* -numerically trivial on X . Hence, for such k , the map (2) is zero, and so, the class $v \in \Omega^*(X \times Gr(d, N))$ is Q^* -numerically trivial.

Note that the normal bundle of g is Q -equivalent to $g^*([T_{X \times Gr(d, N)}] - [Tav])$. Thus, replacing X by $X \times Gr(d, N)$ and u by v , we reduce to the case, where $u = [Y \xrightarrow{y} X]$ is a class of a regular embedding and the normal bundle $N_{Y \subset X}$ is Q -equivalent to the pull-back $y^*([V] - [U])$, for some vector bundles V and U on X . By construction, $\dim(N_{Y \subset X}) = \dim([V] - [U])$. Since the normal bundle of $X = \mathbb{P}_X(O) \rightarrow \mathbb{P}_X(O \oplus U)$ is U , replacing X by $\mathbb{P}_X(O \oplus U)$ and $Y \xrightarrow{y} X$ by $Y \xrightarrow{y} X \rightarrow \mathbb{P}_X(O \oplus U)$ (note that Q^* -numerical triviality of these elements is equivalent, and the same applies to Q^* -anisotropy, as each can be obtained from the other by push-forwards), we can assume that $N_{Y \subset X}$ is Q -equivalent to the pull-back $y^*(V)$ of a vector bundle of the same dimension.

Let $\varepsilon : Fl_X(V) \rightarrow X$ be the variety of complete flags on V . Replacing X by $Fl_X(V)$ and Y by $\varepsilon^{-1}(Y)$, we may assume that $[V] = [L_1] + \dots + [L_n] \in K_0(X)$, for some line bundles L_i . Let $c_1^Q(L_i) = a_i$. Note that anisotropy of $[y]$ is equivalent to the anisotropy of $[\varepsilon^*(y)]$, since anisotropic classes are stable under pull-backs, push-forwards and multiplication by any elements by Proposition 2.6 (and $[y]$ can be obtained from $[\varepsilon^*(y)]$ by the latter two operations - recall, that ε is a consecutive projective bundle). Similarly, Q^* -numerical triviality of $[y]$ implies that of $[\varepsilon^*(y)]$.

Let's apply the deformation to the normal cone construction. Consider $W = Bl_{X \times \mathbb{P}^1}(Y \times 0)$ with the projection $\pi : W \rightarrow X \times \mathbb{P}^1$. The class $\bar{u} = [Y \times 1] = [Y \times 0] \in Q^*(X \times \mathbb{P}^1)$ is still numerically trivial (since $[Y] \in Q^*(X)$ is). Hence, so is $\pi^*(\bar{u})$. Since the normal bundle $N_{Y \times 0 \subset X \times \mathbb{P}^1}$ is $O \oplus N_{Y \subset X}$, which has Q -roots $0, a_1, \dots, a_n$, all defined on X , by [23, Proposition 5.27],

$$-\pi^*(\bar{u}) = \xi(\xi + Q a_1) \cdot \dots \cdot (\xi + Q a_n) - 0 \cdot a_1 \cdot \dots \cdot a_n = \xi(\xi + Q a_1) \cdot \dots \cdot (\xi + Q a_n),$$

where $\xi = c_1^Q(O(1))$ - see Lemma 4.10 for details. Thus, it is a product of first Chern classes of certain line bundles. Note that any line bundle L is a ratio of some very ample line bundles $K \otimes M^{-1} = (K \otimes M^{p^k-1}) \otimes M^{-p^k}$. Thus, tensoring by M^{p^k} , for a very ample line bundle M on W , which is a Q -equivalence for sufficiently large k , and using the fact that the tensor product of very ample line bundles is very ample, we may assume that the Q^* -divisor classes $\xi, (\xi +_Q a_1), \dots, (\xi +_Q a_n)$ are represented by very ample divisors on W . Hence, $-\pi^*(\bar{u})$ is a Q^* -numerically trivial complete intersection. By Proposition 4.5, $\pi^*(\bar{u})$ is anisotropic. But the push-forward of this class to $Q^*(X)$ is equal to u (recall that $\bar{u} = [Y \times 1]$ and this cycle is away from the center of the blow-up). Thus, u is anisotropic. The Theorem is proven. \square

Lemma 4.9 *Let L be a line bundle on Y . Then, for k sufficiently large in comparison to the $\dim(Y) = d$, the Q -Chern class $c_1^Q(L^{p^k})$ is zero.*

Proof Indeed, $c_1^Q(L^{p^k}) = [p^k]_Q(c_1^Q(L))$, but since Q^* is p^r -torsion, for some r (as J is invariant and non-zero), the Q -formal multiplication $[p^r]_Q(x)$ has no linear term, and so, $[p^{rm}]_Q(x)$ has no terms of degree less than 2^m , so it is sufficient to take $k > r \cdot \log_2(d)$ (recall, that the first Chern class is supported in positive co-dimension, and so, the $(d+1)$ -st power of it is zero). \square

Lemma 4.10 *Let $B \xrightarrow{g} A$ be a regular embedding of co-dimension d , where $N_{B \subset A} = N$ is Q -equivalent to $g^*(M)$ for some vector bundle M on A . Let $\pi : Bl_A(B) \rightarrow A$ be the blow-up of A at B and $\tilde{M} = \pi^*(M)$. Then $\pi^*([B]) = c_d^Q(\tilde{M}) - c_d^Q(\tilde{M} \otimes O(1)) \in Q^*(Bl_A(B))$.*

Proof Applying [23, Proposition 5.27] to the cartesian square and blow-up diagram:

$$\begin{array}{ccc} A & \xleftarrow{g} & B \\ \uparrow g & & \parallel \\ B & \xlongequal{\quad} & B \end{array} \quad \begin{array}{ccc} A & \xleftarrow{\pi} & Bl_A(B) \\ \uparrow g & & \uparrow j \\ B & \xleftarrow{\varepsilon} & \mathbb{P}_B(N) \end{array},$$

taking into account that (in the notations of loc. cit.) $\mathcal{M} = N$, we get:

$$0 = \pi^* g_* + j_* \left(\frac{c_d^Q(N \otimes O(1)) - c_d^Q(N)}{c_1^Q(O(-1))} \cdot \varepsilon^* \right).$$

Since the Chern classes of N are j^* of the Chern classes of \tilde{M} , using the projection formula and the fact that $j_*(1) = c_1^Q(O(-1))$, we obtain:

$$\pi^*([B]) = c_d^Q(\tilde{M}) - c_d^Q(\tilde{M} \otimes O(1)). \quad \square$$

Remark 4.11 Since free oriented cohomology theories are stable under purely transcendental field extensions, the above Theorem shows that if k is an arbitrary field (of

characteristic zero) and $\tilde{k} = k(t_1, t_2, \dots)$ is its *flexible closure*, then, for any smooth projective X/k and any theory Q^* as above,

$$Q_{Num}^*(X) = Q_{Num}^*(X_{\tilde{k}}) = Q_{iso}^*(X_{\tilde{k}}).$$

For $m = \infty$, $P(\infty)^* = \text{Ch}^* = \text{CH}^*/p$ is Chow groups modulo p , and we obtain:

Theorem 4.12 *Let k be flexible. Then isotropic Chow groups $\text{Ch}_{k/k}^*$ coincide with numerical Chow groups Ch_{Num}^* .*

This settles [26, Conjecture 4.7].

Corollary 4.13 *Let k be flexible. Then isotropic Chow groups $\text{Ch}_{k/k}^*(X)$ of varieties are finite groups.*

Proof Let X be a smooth projective variety over k . It is defined over some finitely generated subfield $l \subset k$, that is, $X = \bar{X}_k$, for some \bar{X}/l . Moreover, any algebraic cycle on X and any rational (mod p) equivalence among such cycles is defined over some finitely generated subfield $l \subset F \subset k$. Thus, $\text{Ch}^*(X)$ is the colimit of $\text{Ch}^*(\bar{X}_F)$, where F runs over all such intermediate finitely generated subfields. Since $\text{char}(k) = 0$, any such F has a complex embedding: $F \rightarrow \mathbb{C}$, defining a map $\text{Ch}^*(\bar{X}_F) \rightarrow H^{2*}(\bar{X}(\mathbb{C}); \mathbb{F}_p)$. Since $H^*(\bar{X}(\mathbb{C}); \mathbb{F}_p) = H_{et}^*(\bar{X}_{\mathbb{C}}; \mathbb{F}_p)$ and the latter group is equal to $H_{et}^*(\bar{X}_{\bar{F}}; \mathbb{F}_p)$ by the smooth base change theorem [15, Chapter VI, Cor. 4.3], the above map is just the restriction $\text{Ch}^*(\bar{X}_F) \rightarrow H_{et}^{2*}(\bar{X}_{\bar{F}}; \mathbb{F}_p) = H_{et}^{2*}(\bar{X}_{\bar{l}}; \mathbb{F}_p)$. In particular, it doesn't depend on the choice of the complex embedding. The same smooth base change theorem shows that for any embedding $F \subset E$ of our finitely generated subfields, the maps agree and provide the topological realisation map $\text{Ch}^*(X) \rightarrow H_{et}^{2*}(\bar{X}_{\bar{l}}; \mathbb{F}_p)$.

Since numerical pairing is defined on the level of the topological realisation, the numerical Chow groups $\text{Ch}_{Num}^*(X)$ are sub-quotients of the singular cohomology $H^*(\bar{X}(\mathbb{C}); \mathbb{F}_p)$ of the topological realisation of \bar{X} . The latter groups are finite. Due to Theorem 4.12, the same is true for isotropic Chow groups (as long as k is flexible). \square

Theorem 4.7 permits to identify the isotropic and numerical versions for the Morava K -theory as well. For this we will need the following result.

Proposition 4.14 *Any numerically trivial $K(m)$ -class can be lifted to a numerically trivial $P\{m\}$ -class.*

Proof Let $CK(m)^*$ be the free theory with the coefficient ring $CK(m) = \mathbb{F}_p[v_m]$ (the connective Morava K -theory). Denote as $y \mapsto \bar{y}$ the standard projection $P(m)^* \rightarrow CK(m)^*$. Let $\pi : X \rightarrow \text{Spec}(k)$ be a smooth projective variety. Multiplying by some power of v_m (an invertible element), we may assume that our numerically trivial class is lifted to $\bar{x} \in CK(m)^*(X)$ which is the restriction of some $x \in P(m)^*(X)$. I claim that:

Claim 4.15 *Modulo multiplication by some power of v_m , x can be chosen in such a way that $S_{P(m)}^{Tot}(x) \in CK(m)^*[t](X)$ is numerically trivial.*

Proof Let $S_{P(m)}^{\bar{r}} = \bar{S}_{P(m)}^{\bar{r}}$ be the individual $P(m)$ -Landweber-Novikov operation (the coefficient of $S_{P(m)}^{Tot}$ at the respective monomial). This operation has degree $|\bar{r}| = \sum_i ir_i$ (equal to that of the monomial). We will show by a decreasing induction on d that we may choose x so that, for any \bar{r} and $y \in P(m)^*(X)$ with $|\bar{r}| + \text{codim}(y) \geq d$, we have $\pi_*(\bar{S}_{P(m)}^{\bar{r}}(x) \cdot \bar{y}) = 0$. Clearly, for a given choice of x , the statement is true for $d > \dim(x)$, since the dimension of the element $\bar{S}_{P(m)}^{\bar{r}}(x) \cdot \bar{y}$ is negative, in this case. On the other hand, the case $d = 0$ gives the claim above, because the map $P(m)^* \rightarrow CK(m)^*$ is surjective and $P(m)^*(X)$ is generated as a $P(m)$ -module by elements of non-negative codimension (since it is so for Ω^*).

Suppose, the lifting x satisfies the statement for $d + 1$. Consider the set

$$R = R(x) = \{|\bar{r}| \mid \exists y, \text{ such that } |\bar{r}| + \text{codim}(y) = d \text{ and } \pi_*(\bar{S}_{P(m)}^{\bar{r}}(x) \cdot \bar{y}) \neq 0\}.$$

We say that $\bar{s} \leq \bar{r}$, if $s_i \leq r_i$, for every i (in other words, the monomial $\bar{t}^{\bar{s}}$ is a divisor of $\bar{t}^{\bar{r}}$). Let

$$S = S(x) = \{|\bar{r}| \text{ such that } |\bar{r}| \leq d \text{ and } \forall \bar{s} \leq \bar{r}, \bar{s} \notin R(x)\}.$$

Let $\bar{r} \in R$ be a *minimal* element of R in the sense of the mentioned partial order. By definition of R , there exists $y \in P(m)^*(X)$, such that $|\bar{r}| + \text{codim}(y) = d$ and $\pi_*(\bar{S}_{P(m)}^{\bar{r}}(x) \cdot \bar{y}) = \lambda \cdot v_m^a$, where $\lambda \in \mathbb{F}_p$ is invertible. Let $u := \pi_*(x \cdot y) \in P(m)$. Note that since x satisfies the statement for $d + 1$ and by the definition of R , we have:

Fact A: *For any \bar{s} and any $z \in P(m)^*(X)$, $\pi_*(\bar{S}_{P(m)}^{\bar{s}}(x) \cdot \bar{z}) \in CK(m)$ is divisible by v_m^a . Moreover, if $\bar{s} \notin R(x)$, then it is divisible by v_m^{a+1} .*

Indeed, the dimension of any such non-zero element must be \geq , respectively $> \dim(x) - d = \dim(v_m^a)$. Since $S_{P(m)}^{Tot}$ is multiplicative and $S_{Tot}^{P(m)}$, which on X is equal to $S_{P(m)}^{Tot} \cdot \text{Td}(X)$, commutes with push-forwards, by the Riemann-Roch theorem (for multiplicative operations) - Panin [17, Theorem 2.5.4], for any \bar{q} , we have:

$$\overline{S_{P(m)}^{\bar{q}}(u)} = \overline{S_{\bar{q}}^{P(m)}(u)} = \sum_{\bar{q}_1 + \bar{q}_2 + \bar{q}_3 = \bar{q}} \pi_*(\text{Td}(X)^{\bar{q}_1} \cdot \overline{S_{P(m)}^{\bar{q}_2}(x)} \cdot \overline{S_{P(m)}^{\bar{q}_3}(y)}) \quad (3)$$

(here $\text{Td}(X)^{\bar{q}_1}$ is the \bar{q}_1 -component of $\text{Td}(X) = \text{Td}_{S_{P(m)}^{Tot}}(X)$). Combining it with the Fact A, we get:

Fact B: *$\overline{S_{P(m)}^{\bar{q}}(u)}$ is divisible by v_m^a , for any \bar{q} . Moreover, if $\bar{q} \in S(x)$, then it is divisible by v_m^{a+1} .*

For the second claim, we just need to observe that if $\bar{q} \in S$, then $\bar{q}_2 \notin R$.

Consider

$$x' := v_m^a \cdot x - \lambda^{-1} \cdot u \cdot \bar{S}_{P(m)}^{\bar{r}}(x).$$

Note that $\bar{u} = 0$ (as \bar{x} is $CK(m)$ -numerically trivial) and so, $\bar{x}' = \bar{x} \cdot v_m^a$. I claim that x' still satisfies the statement for $d + 1$, and moreover, $S(x') \supset S(x) \cup \{\bar{r}\}$.

Let $|\bar{s}| + \text{codim}(z) > d$, respectively, $|\bar{s}| + \text{codim}(z) = d$ and $\bar{s} \in S(x)$. Since $S_{P(m)}^{Tot}$ is a multiplicative operation and $S_{P(m)}^{Tot}(v_m) = v_m$ (note that v_i , for $i < m$, are zero in $P(m)$), we have: $\pi_*(\overline{S_{P(m)}^{\bar{s}}}(v_m^a \cdot x) \cdot \bar{z}) = v_m^a \cdot \pi_*(\overline{S_{P(m)}^{\bar{s}}}(x) \cdot \bar{z}) = 0$, as x satisfies the statement for $d + 1$, respectively, $\bar{s} \notin R(x)$. At the same time,

$$\pi_*(\overline{S_{P(m)}^{\bar{s}}}(u \cdot S_{P(m)}^{\bar{r}}(x)) \cdot \bar{z}) = \sum_{\bar{s}_1 + \bar{s}_2 = \bar{s}} \pi_*(\overline{S_{P(m)}^{\bar{s}_1}}(u) \cdot \overline{S_{P(m)}^{\bar{s}_2}} S_{P(m)}^{\bar{r}}(x) \cdot \bar{z})$$

is equal to $v_m^b \cdot \sum_{\bar{s}_1 + \bar{s}_2 = \bar{s}} \pi_*(\overline{S_{P(m)}^{\bar{s}_2}} S_{P(m)}^{\bar{r}}(x) \cdot \overline{w_{\bar{s}_1}})$, for some $\overline{w_{\bar{s}_1}} \in CK(m)^*(X)$, where $b = a$, respectively, $b = a + 1$, because $\overline{S_{P(m)}^{\bar{s}_1}}(u)$ is divisible by v_m^b , by Fact B (note that $\bar{s}_1 \in S(x)$, if \bar{s} does). Since $\overline{S_{P(m)}^{\bar{s}_2}} S_{P(m)}^{\bar{r}}(x)$ is a linear combination of Landweber-Novikov operations, by Fact A, our expression is divisible by v_m^{a+b} and so, is zero (since the dimension of the result is less, respectively, equal to that of v_m^{2a}). Thus, $\pi_*(\overline{S_{P(m)}^{\bar{s}}}(x') \cdot \bar{z}) = 0 \in CK(m)$. Moreover, in the second case, the same holds for all $\bar{s}' \leq \bar{s}$ (as these elements also belong to $S(x)$). Hence, x' satisfies the statement for $d + 1$ and $S(x) \subset S(x')$.

Finally, since \bar{r} is a minimal element of $R(x)$, $\overline{S_{P(m)}^{\bar{r}}}(u) = \lambda \cdot v_m^a$ (as the only non-zero term in (3) corresponds to $\bar{q}_2 = \bar{r}$), while $\overline{S_{P(m)}^{\bar{r}'}}(u)$ is divisible by v_m^{a+1} , for every other $\bar{r}' \leq \bar{r}$, because the dimension of this element is $> \dim(v_m^a)$. Hence, for $|\bar{r}| + \text{codim}(z) = d$, we have:

$$\begin{aligned} \pi_*(\overline{S_{P(m)}^{\bar{r}}}(\lambda^{-1} \cdot u \cdot S_{P(m)}^{\bar{r}}(x)) \cdot \bar{z}) &= \lambda^{-1} \sum_{\bar{r}_1 + \bar{r}_2 = \bar{r}} \pi_*(\overline{S_{P(m)}^{\bar{r}_1}}(u) \cdot \overline{S_{P(m)}^{\bar{r}_2}} S_{P(m)}^{\bar{r}}(x) \cdot \bar{z}) \\ &= v_m^a \cdot \pi_*(\overline{S_{P(m)}^{\bar{r}}}(x) \cdot \bar{z}) \\ &\quad + v_m^{a+1} \sum_{\bar{q}} \pi_*(\overline{S_{P(m)}^{\bar{q}}}(x) \cdot \overline{w_{\bar{q}}}), \end{aligned}$$

for some $\overline{w_{\bar{q}}} \in CK(m)^*(X)$. The latter summand is divisible by v_m^{2a+1} , by Fact A, and so, is zero, since the dimension of the result is equal to that of $\dim(v_m^{2a})$. The first summand is equal to $\pi_*(\overline{S_{P(m)}^{\bar{r}}}(v_m^a \cdot x) \cdot \bar{z})$. Hence, $\pi_*(\overline{S_{P(m)}^{\bar{r}}}(x') \cdot \bar{z}) = 0$ and consequently, $\bar{r} \notin R(x')$ and so, $\bar{r} \in S(x')$ (as all other $\bar{r}' \leq \bar{r}$ were contained in $S(x) \subset S(x')$). Thus, $S(x')$ contains $S(x) \cup \{\bar{r}\}$.

We managed to increase the cardinality of $S(x)$ while keeping the condition $d + 1$. After finitely many such steps $S(x)$ will contain all \bar{r} of degree $\leq d$, which will mean that x satisfies the condition for d . The induction step and the claim is proven. \square

So, we may assume that $\overline{S_{P(m)}^{Tot}}(x) \in CK(m)^*[\bar{t}](X)$ is numerically trivial. Consider the ideal $J \subset P(m) = BP/I(m)$ generated by $\pi_*(\overline{S_{P(m)}^{\bar{r}}}(x) \cdot z)$, for all \bar{r} and all $z \in P(m)^*(X)$. It follows from the Riemann-Roch theorem that this ideal is invariant

under $P(m)$ -Landweber-Novikov operations. On the other hand, we know that the projection of it to $CK(m)$ is zero. Hence, it doesn't contain powers of v_m . By the result of Landweber [13, Proposition 2.11], the ideal J must be zero. In particular, x is $P(m)$ -numerically trivial. Dividing by an appropriate power of v_m , we obtain a numerically trivial lifting of the original class from $K(m)^*$ to $P\{m\}^*$. \square

Let $\pi : X \rightarrow \text{Spec}(k)$ be a smooth projective variety and $I(X) = \text{Im}(\pi_* : BP_*(X) \rightarrow BP) \subset BP$. This is an invariant ideal in BP which is finitely generated, since $BP^*(X)$ is generated by elements of non-negative codimension by the result of Levine-Morel [14, Corollary 1.2.13]. By the result of Landweber [13, Theorem 2.7, Proposition 3.4], $\sqrt{I(X)}$ is either $I(n)$, for some $1 \leq n < \infty$, or the whole BP . It appears that this ideal contains the information on $P(m)$ and $K(m)$ -anisotropy of X .

Proposition 4.16 *The following conditions are equivalent:*

- (1) X is $P(m)$ -anisotropic;
- (2) X is $K(m)$ -anisotropic;
- (3) $\sqrt{I(X)} = I(n)$, for some $1 \leq n \leq m$.

Proof As $K(m)^*$ is obtained from $P(m)^*$ by change of coefficients, (1) \Rightarrow (2). The condition (2) implies that $I(X)$ doesn't contain powers of v_m , which implies (3). Finally, (3) implies that $I(X) \subset I(m) \Leftrightarrow$ (1). \square

Now we can prove the Morava-analogue of the Main Theorem.

Theorem 4.17 *Let k be flexible and $1 \leq m \leq \infty$. Then $K(m)_{iso}^* = K(m)_{Num}^*$.*

Proof We know that any $K(m)$ -anisotropic class is $K(m)$ -numerically trivial. Conversely, by Proposition 4.14 any $K(m)$ -numerically trivial class can be lifted to a $P\{m\}$ -numerically trivial one, which must be $P\{m\}$ -anisotropic by Theorem 4.7. Hence, the projection of it to $K(m)^*$ is $K(m)$ -anisotropic, since the notions of $K(m)$ and $P(m)$ -anisotropy are equivalent by Proposition 4.16. \square

5 Applications

5.1 Isotropic Chow motives

Let A^* be an oriented cohomology theory. The category of A -correspondences is some linearization of the category of smooth projective varieties. The objects of $Cor^A(k)$ are smooth projective varieties over k , while morphism are defined as $\text{Hom}_{Cor^A(k)}(X, Y) = A^{\dim(Y)}(X \times Y)$. Considering the pseudo-abelian (Karoubian) envelope of $Cor^A(k)$ (that is, adding kernels and cokernels of projectors) and formally inverting the Tate-motive $T\{1\}$ (which can be produced as a direct summand of the motive $M(\mathbb{P}^1) = T \oplus T\{1\}$ of a projective line), one obtains the category of A -Chow motives $Chow^A(k)$. This is a tensor additive category with $M(X) \oplus M(Y) := M(X \amalg Y)$ and $M(X) \otimes M(Y) := M(X \times Y)$. In the case $A^* = CH^*$ of Chow

groups, we get the classical category of Chow motives $Chow^{CH}(k) = Chow(k)$ of Grothendieck.

Any morphism of theories $A^* \xrightarrow{\mu} B^*$ (a homomorphism of rings commuting with both pull-backs and push-forwards) defines the natural functor $Chow^A(k) \rightarrow Chow^B(k)$.

Let A^* be a free theory and A_{iso}^* and A_{Num}^* be isotropic and numerical versions of it. We have a natural (surjective) morphism of theories: $A_{iso}^* \rightarrow A_{Num}^*$. Denote the respective Chow-motivic categories as $Chow_{iso}^A(k)$ and $Chow_{Num}^A(k)$, respectively. Our Main Theorem 4.7 and Theorem 4.17 imply:

Theorem 5.1 *Let k be flexible and $0 < m \leq \infty$. Then*

- (1) *The category of isotropic $P(m)$ -Chow motives $Chow_{iso}^{P(m)}(k)$ is equivalent to the category of numerical $P(m)$ -Chow motives $Chow_{Num}^{P(m)}(k)$.*
- (2) *The category of isotropic $K(m)$ -Chow motives $Chow_{iso}^{K(m)}(k)$ is equivalent to the category of numerical $K(m)$ -Chow motives $Chow_{Num}^{K(m)}(k)$.*

Proof Indeed, from Theorem 4.7 it follows that the projection $P(m)_{iso}^* \rightarrow P(m)_{Num}^*$ is, actually, an isomorphism of theories. Similarly, Theorem 4.17 implies that the projection $K(m)_{iso}^* \rightarrow K(m)_{Num}^*$ is an isomorphism of theories. \square

Proposition 5.2 *Let A^* be an oriented cohomology theory whose coefficient ring A is an integral domain. Then \otimes has no zero-divisors in $Chow_{Num}^A(k)$ (on objects or morphisms).*

Proof Let $\alpha_i \in A_{Num}^*(X_i)$, for $i = 1, 2$ be non-zero classes. Then $\alpha_1 \times \alpha_2 \in A_{Num}^*(X_1 \times X_2)$ is non-zero. Indeed, since, $\alpha_i \neq 0$ numerically, there exist $\beta_i \in A_{Num}^*(X_i)$, such that $\langle \alpha_i, \beta_i \rangle \neq 0 \in A$. But then $\langle \alpha_1 \times \alpha_2, \beta_1 \times \beta_2 \rangle = \langle \alpha_1, \beta_1 \rangle \cdot \langle \alpha_2, \beta_2 \rangle \neq 0 \in A$, since A is an integral domain. Thus, $\alpha_1 \times \alpha_2$ is numerically non-trivial. Non-zero objects U_i , $i = 1, 2$ of $Chow_{Num}^A(k)$ are given by non-trivial projectors $\rho_i \in A_{Num}^*(Y_i^{\times 2})$ in the motives of some varieties Y_i . Then $U_1 \otimes U_2$ is given by a non-zero projector $\rho_1 \times \rho_2$ in $A_{Num}^*((Y_1 \times Y_2)^{\times 2})$. \square

For $0 < m < \infty$, the coefficient ring of $P(m)^*$ is the polynomial ring $\mathbb{F}_p[v_m, v_{m+1}, \dots]$ in infinitely many variables over \mathbb{F}_p , while for $m = \infty$, it is \mathbb{F}_p . In any case, it is an integral domain. The same applies to the coefficient ring of Morava K-theory. Thus, combining Theorem 5.1 and Proposition 5.2, we obtain:

Theorem 5.3 *Let k be flexible, and $m > 0$. Then*

- (1) *The category $Chow_{iso}^{P(m)}(k)$ of isotropic $P(m)$ -Chow motives has no \otimes -zero-divisors (on objects or morphisms).*
- (2) *The category $Chow_{iso}^{K(m)}(k)$ of isotropic Morava-Chow motives has no \otimes -zero-divisors (on objects or morphisms).* \square

When $m = \infty$, the category $\text{Chow}_{iso}^{P(\infty)}(k)$ is the *isotropic Chow motivic category* $\text{Chow}(k/k; \mathbb{F}_p)$ of [26]. As a particular case of Theorems 5.1 and 5.3 we get:

Corollary 5.4 *Let k be flexible. Then the category $\text{Chow}(k/k; \mathbb{F}_p)$ of isotropic Chow motives - [26] is equivalent to the category $\text{Chow}_{Num}(k; \mathbb{F}_p)$ of numerical Chow motives (with \mathbb{F}_p -coefficients). In particular, the category $\text{Chow}(k/k; \mathbb{F}_p)$ has no \otimes -zero-divisors.*

Since Homs in the category $\text{Chow}(k/k; \mathbb{F}_p)$ are isotropic Chow groups, Corollary 4.13 gives:

Corollary 5.5 *Let k be flexible. Then Homs in the category $\text{Chow}(k/k; \mathbb{F}_p)$ are finite groups.*

The category $\text{Chow}(k/k; \mathbb{F}_p)$ is a full tensor additive subcategory of the compact part of the triangulated category $DM(k/k; \mathbb{F}_p)$ of *isotropic motives* - [26]. The above Corollary provides support for the following conjecture.

Conjecture 5.6 *Let k be flexible. Then Homs between compact objects of the isotropic motivic category $DM(k/k; \mathbb{F}_p)$ are finite groups.*

Another piece of evidence in support of this Conjecture is given by the calculation of isotropic motivic cohomology of a point (that is, Homs between Tate objects in $DM(k/k; \mathbb{F}_p)$). This was done for $p = 2$ in [26, Theorem 3.7].

5.2 Isotropic realisations

In [26] the *isotropic realisations* of the Voevodsky triangulated category of motives $DM(k)$ were constructed (see also [27, Introduction]). Such realisations are parametrized by the choice of a prime number p and an equivalence class of extensions of the ground field.

Let k be an arbitrary field (of characteristic zero) and $\text{Ch}^* = \text{CH}^*/p$ be Chow groups modulo p . Define the following partial ordering on the set of field extensions E/k . Let $E = \text{colim}_\alpha E_\alpha$ and $F = \text{colim}_\beta F_\beta$, where $E_\alpha = k(Q_\alpha)$ and $F_\beta = k(P_\beta)$ are finitely generated extensions with smooth models Q_α and P_β . We say that $E/k \geq^p F/k$, if for any β , there exists α and a correspondence $Q_\alpha \rightsquigarrow P_\beta$ of degree prime to p (equivalently, the push-forward $\text{Ch}_*(Q_\alpha \times P_\beta) \rightarrow \text{Ch}_*(Q_\alpha)$ is surjective). We say that $E/k \stackrel{p}{\sim} F/k$, if $E/k \geq^p F/k$ and $F/k \geq^p E/k$. For a field E , denote as $\tilde{E} = E(\mathbb{P}^\infty)$ the *flexible closure* of E . Finally, for a field L , define the *isotropic motivic category* $DM(L/L; \mathbb{F}_p)$ as the Verdier localisation of $DM(L; \mathbb{F}_p)$ by the localising subcategory generated by motives of p -anisotropic ($= \text{Ch}^*$ -anisotropic) varieties over L - see [26, Def. 2.4, Rem. 2.8]. We get the family of *isotropic realisations*

$$\psi_{p,E} : DM(k) \longrightarrow DM(\tilde{E}/\tilde{E}; \mathbb{F}_p),$$

where p is an arbitrary prime and E runs over all equivalence classes of the $\overset{p}{\sim}$ relation above. Note that our realisations take values in isotropic motivic categories over flexible fields.

Corollary 5.4 permits to reveal the meaning of these functors. They provide the points of the Balmer spectrum $\mathrm{Spc}(DM(k)^c)$ - [3] of (the compact part of) the Voevodsky motivic category. Recall, that the points of the Balmer spectrum of a tensor triangulated category are prime $\otimes - \Delta$ -ed ideals in it. In our case, the ideal $\mathfrak{a}_{p,E} = \psi_{p,E}^{-1}(0)$ is the pre-image of the zero ideal. It is prime, since (the compact part of) the isotropic (flexible) motivic category $DM(\tilde{E}/\tilde{E}; \mathbb{F}_p)$ has no zero-divisors by Corollary 5.4 as we will prove below.

It was shown by Bondarko [6], that on $DM(k)^c$ there is a unique bounded non-degenerate weight structure whose heart is the category of Chow motives $\mathrm{Chow}(k)$. A similar structure exists on the category $DM(k/k; \mathbb{F}_p)^c$ of isotropic motives.

Proposition 5.7 *The compact part $DM(k/k; \mathbb{F}_p)^c$ of isotropic motivic category possesses a bounded non-degenerate weight structure whose heart is the category of isotropic Chow motives $\mathrm{Chow}(k/k; \mathbb{F}_p)$.*

Proof Recall from [26, Proposition 2.5] that the compact part $DM(k/k; \mathbb{F}_p)^c$ of the isotropic motivic category can be realized as a full thick subcategory of $DM(k; \mathbb{F}_p)$ which is an idempotent completion of the subcategory consisting of objects of the form $\tilde{\mathcal{X}}_Q \otimes U$, for all $U \in DM(k; \mathbb{F}_p)^c$, where $\tilde{\mathcal{X}}_Q$ is the reduced motive of the Čech simplicial scheme $\check{\mathrm{Cech}}(Q)$, where Q is the disjoint union of all smooth p -anisotropic varieties over k . Here $\tilde{\mathcal{X}}_Q$ is a tensor projector fitting the distinguished triangle

$$\mathcal{X}_Q \rightarrow T \rightarrow \tilde{\mathcal{X}}_Q \rightarrow \mathcal{X}_Q[1],$$

where \mathcal{X}_Q is the non-reduced motive of our Čech simplicial scheme (the complementary projector). The subcategory of isotropic Chow motives $\mathrm{Chow}(k/k; \mathbb{F}_p)$ is the idempotent completion of the subcategory consisting of objects $\tilde{\mathcal{X}}_Q \otimes U$, where U is a Chow motive. Since every compact Voevodsky motive is an extension of finitely many shifts of Chow motives, the same is true about isotropic motives. Following Bondarko, let's define the $\mathcal{D}_{\leq 0}$ as the idempotent completion of the subcategory of $DM(k/k; \mathbb{F}_p)$ consisting of all possible extensions (of finitely many) objects of the type $\tilde{U}[i]$, where \tilde{U} is an isotropic Chow motive and $i \leq 0$, and $\mathcal{D}_{\geq 0}$ as the idempotent completion of the subcategory of $DM(k/k; \mathbb{F}_p)$ consisting of all possible extensions (of finitely many) objects of the type $\tilde{U}[i]$, where $i \geq 0$. To show that this is indeed a weight structure, by [6, Theorem 4.3.2, Proposition 5.2.2], one only needs to check that $\mathrm{Hom}_{DM(k/k; \mathbb{F}_p)}(\tilde{U}, \tilde{V}[i]) = 0$, for any isotropic Chow motives \tilde{U} and \tilde{V} and any $i > 0$.

If $\tilde{U} = \tilde{\mathcal{X}}_Q \otimes U$ and $\tilde{V} = \tilde{\mathcal{X}}_Q \otimes V$, for (global) Chow motives U and V , then

$$\mathrm{Hom}_{DM(k/k; \mathbb{F}_p)}(\tilde{U}, \tilde{V}[i]) = \mathrm{Hom}_{DM(k; \mathbb{F}_p)}(\tilde{\mathcal{X}}_Q \otimes U, \tilde{\mathcal{X}}_Q \otimes V[i]).$$

As \otimes -projectors \mathcal{X}_Q and $\tilde{\mathcal{X}}_Q$ define a semi-orthogonal decomposition of the category $DM(k; \mathbb{F}_p)$ (by [22, Theorem 2.3.2]), this is equal to:

$$Hom_{DM(k; \mathbb{F}_p)}(T[-i], \tilde{\mathcal{X}}_Q \otimes U^\vee \otimes V) = H_{-i,0}^{\mathcal{M}}(\tilde{\mathcal{X}}_Q \otimes U^\vee \otimes V; \mathbb{F}_p).$$

But $U^\vee \otimes V$ is a direct summand in $M(Y)(j)[2j]$, for some smooth projective variety Y and some $j \in \mathbb{Z}$, while smooth simplicial schemes don't have motivic homology $H_{a,b}^{\mathcal{M}}$, for $a < 2b$. Thus, the group in question is zero and we obtain the needed weight structure. \square

A weight filtration presents an object X as an extension of its graded pieces X_i , where $X_i = C_i[i]$, for some objects C_i of the heart. The latter objects form a *weight complex* $t(X)$ - [6, Def. 2.2.1]

$$\dots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \dots$$

For a bounded weight structure, this complex will be finite. In contrast to the t -structure filtration, the filtration given by a weight structure is not canonical. In particular, the weight complex $t(X)$ is not uniquely defined. But, if for a given X , there exists a weight complex with zero differentials, then the respective terms C_i are invariants of X and can be considered as *weight cohomology* $H_w^i(X)$ of it. In particular, in this case, X is zero if and only if all $H_w^i(X)$ s are. The next statement shows that our case is exactly such.

Proposition 5.8 *Every object X of $DM(k/k; \mathbb{F}_p)^c$ possesses a weight filtration producing weight complex with zero differentials.*

Proof This is a consequence of the semi-simplicity of the category $Chow_{Num}(k, \mathbb{F}_p)$ - see Proposition 5.10, which follows from the following Lemma. \square

Lemma 5.9 *For any morphism $f : U \rightarrow V$ in $Chow_{Num}(k, \mathbb{F}_p)$, there are decompositions $U = W \oplus \tilde{U}$ and $V = W \oplus \tilde{V}$, such that $f = id_W \oplus 0$. In particular, the category $Chow_{Num}(k, \mathbb{F}_p)$ is semi-simple.*

Proof If f is non-zero, then the adjoint map $T \xrightarrow{\bar{f}} U^\vee \otimes V$ is non-zero too, which means that there exists $g : V \rightarrow U$, such that the composition $T \xrightarrow{\bar{f}} U^\vee \otimes V \xrightarrow{g} T$ is the identity. That means that the traces of $\alpha = g \circ f : U \rightarrow U$ and $\beta = f \circ g : V \rightarrow V$ are non-zero. Let us show that some powers of α and β are non-zero projectors. We may assume that U and V are motives of varieties.

Consider the topological realisation functor $TR : Chow(k; \mathbb{F}_p) \rightarrow D^b(\mathbb{F}_p)$ and denote $H_{Top}(Z) := H^*(TR(Z), \mathbb{F}_p)$. The latter is a finite-dimensional \mathbb{F}_p -vector space. We may identify $H_{Top}(U \times U)$ with $End_{\mathbb{F}_p}(H_{Top}(U))$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be TR of some liftings of α and β to non-numerical Chow groups. Since the degree pairing is defined on the level of the topological realisation, $\tilde{\alpha}$ and $\tilde{\beta}$, considered as \mathbb{F}_p -linear transformations, have non-zero traces. Define the filtration:

$H_{Top}(U) = F_0 \supset F_1 \supset F_2 \supset \dots$, where $F_k = \text{image}(\tilde{\alpha}^k)$. Since our space is finite-dimensional, there exists r , such that the embeddings $F_k \supset F_{k+1}$ are proper, for $k < r$, and are isomorphisms, for $k \geq r$. Since the ring $\text{End}_{\mathbb{F}_p}(H_{Top}(U))$ is finite, there exists N , such that $\tilde{\alpha}^N$ is a projector. Considered as a linear transformation, it should be a projection on F_r . Then $\text{tr}(\tilde{\alpha}^{N+1})$ coincides with $\text{tr}(\tilde{\alpha}|_{F_r})$ which is equal to $\text{tr}(\tilde{\alpha})$. Hence, $\deg(\tilde{\alpha} \cdot (\tilde{\alpha}^N)^\vee) = \text{tr}(\tilde{\alpha}^{N+1}) \neq 0$, and so, α^N is numerically non-trivial. Since numerical Chow groups are sub-quotients of topological cohomology (as the degree pairing factors through TR), it follows that $a = \alpha^N$ is a non-zero projector. Then so is $b = \beta^{2N}$, and maps f and $g' = g \circ \beta^{2N-1}$ identify (U, a) with (V, b) . Thus, there are decompositions $U = W' \oplus U'$ and $V = W' \oplus V'$ with non-zero W' , s.t. $f = \text{id}_{W'} \oplus f'$, for some $f' : U' \rightarrow V'$. Repeating these arguments with f' , f'' , ... and taking into account that the endomorphism rings of our objects are finite, we get the decomposition as required. \square

Proposition 5.10 *Let \mathcal{D} be a triangulated category with the non-degenerate bounded weight structure, whose heart \mathcal{C} is semi-simple. Then every object possesses a weight filtration, for which the weight complex has zero differentials.*

Proof Let X be an object of \mathcal{D} . Since our weight structure is bounded, X is supported on some segment $[m, n]$, that is, $X \in \mathcal{D}_{\geq m} \cap \mathcal{D}_{\leq n}$. In particular, we have an exact triangle

$$X_{>m}[-1] \rightarrow X_m \longrightarrow X \longrightarrow X_{>m}.$$

Here the differential $X_{m+1}[-1] \xrightarrow{d_{m+1}[m]} X_m$ of the (shifted) weight complex $t(X)$ factors as the composition $X_{m+1}[-1] \rightarrow X_{>m}[-1] \rightarrow X_m$. Since \mathcal{C} is semi-simple, we may present it as $(\text{id}_W \oplus 0)[m]$, for an appropriate $W \in \mathcal{C}$. Now we can split the triangle $W[m] \xrightarrow{\text{id}} W[m] \rightarrow 0 \rightarrow W[m+1]$ from the above one, and get another presentation

$$X'_{>m}[-1] \rightarrow X'_m \longrightarrow X \longrightarrow X'_{>m}.$$

with the zero differential $C'_{m+1} \xrightarrow{0} C'_m$ and $X'_{>m}$ supported on $[m+1, n]$. We conclude by induction on the support. \square

The weight cohomology is functorial.

Corollary 5.11 *We have a conservative weight cohomology functor*

$$H_w^{Tot} = \oplus_i t^i \cdot H_w^i : DM(k/k; \mathbb{F}_p)^c \rightarrow \text{Chow}(k/k; \mathbb{F}_p)[t, t^{-1}].$$

Proof It is straightforward to check that a map $f : X \rightarrow Y$ between two objects with weight filtrations with zero differentials extends uniquely to maps of graded pieces. The conservativity is clear, as any object is an extension of graded pieces of it. \square

It is just the cohomology of the *weight complex functor*

$$t : DM(k/k; \mathbb{F}_p)^c \rightarrow K^b(Chow(k/k; \mathbb{F}_p))$$

- [6], which exists (and is triangulated) since, due to Proposition 5.8, the natural projection

$$K^b(Chow(k/k; \mathbb{F}_p)) \rightarrow K_w^b(Chow(k/k; \mathbb{F}_p))$$

is an equivalence of categories - see [6, Definition 3.1.6] and [6, Theorem 3.2.2(II)].

Moreover, the total weight cohomology functor respects the tensor product.

Theorem 5.12 (Kunneth formula) *The functor H_w^{Tot} respects the \otimes . In other words,*

$$H_w^k(X \otimes Y) \cong \bigoplus_{i+j=k} H_w^i(X) \otimes H_w^j(Y).$$

Proof This follows from the fact that the “global” *weight complex functor* is a tensor functor - see [2, Lemma 20], while the isotropic category is obtained from the global one by a tensor projector which commutes with the weight filtration and \otimes . \square

Applying Corollary 5.4, we get:

Theorem 5.13 *For any prime p and any field extension E/k , the $\otimes - \Delta$ -ideal*

$$\mathfrak{a}_{p,E} := \text{Ker}(\psi_{p,E} : DM(k)^c \rightarrow DM(\tilde{E}/\tilde{E}; \mathbb{F}_p)^c)$$

is prime and so, defines a point of the Balmer spectrum $\text{Spc}(DM(k)^c)$ of the Voevodsky category.

Two such points $\mathfrak{a}_{p,E}$ and $\mathfrak{a}_{q,F}$ are equal if and only if $p = q$ and $E/k \stackrel{p}{\sim} F/k$.

Proof This follows from the fact that the total weight cohomology is conservative and satisfies the Kunneth formula, while the \otimes has no zero-divisors on $Chow(k/k; \mathbb{F}_p)$ by Corollary 5.4.

Since $\text{char}(\mathfrak{a}_{p,E}) = p$, it is an invariant of a point.

Suppose, $E/k \stackrel{p}{\sim} F/k$. Let $U \in \mathfrak{a}_{p,E}$. Let $E = \text{colim}_\alpha E_\alpha$, where $E_\alpha = k(Q_\alpha)$ is a finitely generated extension with a smooth model Q_α . Since $U \in \mathfrak{a}_{p,E}$ and U is compact, there are finitely many p -anisotropic varieties R_i , $i = 1, \dots, n$ over \tilde{E} , such that $U_{\tilde{E}}$ is a direct summand of an extension of their motives. These varieties and all the morphisms involved should be defined over some finitely generated extension $\tilde{E}_\alpha = \tilde{k}(Q_\alpha)$ of \tilde{k} . Moreover, since $E/k \stackrel{p}{\sim} F/k$, these varieties remain p -anisotropic over $\tilde{F}(Q_\alpha)$. Hence, $U_{\tilde{F}(Q_\alpha)} = 0 \in DM(\tilde{F}(Q_\alpha)/\tilde{F}(Q_\alpha); \mathbb{F}_p)$. By [26, Prop. 2.9], then $U_{\tilde{F}} = 0 \in DM(\tilde{F}(Q_\alpha)/\tilde{F}; \mathbb{F}_p)$, where the latter is the *local motivic category* - see [26, Def. 2.3]. But since $Q_\alpha|_{\tilde{F}}$ is p -isotropic, the extensions $\tilde{F}(Q_\alpha)/\tilde{F}$ and \tilde{F}/\tilde{F} are p -equivalent and so, the respective local categories are identified - see loc. cit.. Thus, $U_{\tilde{F}} = 0 \in DM(\tilde{F}/\tilde{F}; \mathbb{F}_p)$. Hence $U \in \mathfrak{a}_{p,F}$. From symmetry, we get that $\mathfrak{a}_{p,E} = \mathfrak{a}_{p,F}$, if $E/k \stackrel{p}{\sim} F/k$. Conversely, if $E/k \stackrel{p}{\not\sim} F/k$, then either $Q_\alpha|_F$, for

some α , or $P_\beta|_E$, for some β , is p -anisotropic. W.l.o.g. may assume the former. Then $M(Q_\alpha) \in \mathfrak{a}_{p,F}$ (as Q_α stays anisotropic over \bar{F}), while $M(Q_\alpha) \notin \mathfrak{a}_{p,E}$, as $\psi_{p,E}$ passes through $DM(E)$ and a Tate-motive splits from $M(Q_\alpha)$ there. Hence, the points are different. \square

Thus we obtain a large supply of new points of the Balmer spectrum of $DM(k)^c$, which complement the points given by the topological realisation.

Example 5.14 If k is algebraically closed. then every variety of finite type over k has k -rational points. Hence, for every p , there is only one p -equivalence class of field extensions over k .

If $k = \mathbb{R}$ is real, then there is only one p -class, for odd p , and 2^c of such classes, for $p = 2$, where $c = \text{continuum}$. Indeed, there is a continuum of choices of the j -invariant for real curves of genus one without real points. If there is a correspondence of odd degree between two such curves, as there are no real points, the respective elliptic curves must be isogenous, and there are only countably many elliptic curves isogenous to a given one. Thus we have a set of real points free curves of genus one $\{X_i | i \in I\}$ parametrized by a set I of continuum cardinality, such that there are no correspondences of odd degree between X_i and X_j , for $i \neq j$, and the respective elliptic curves are not isogenous. Moreover, if Y is a product of finitely many X_i , where all i are different from a given j , then there is no odd degree correspondence from Y to X_j . Indeed, such a correspondence would give a point of odd degree on $(X_j)_{k(Y)}$, but this variety has also points of degree 2 (as we have complex points). Since it is a curve of genus one, by the (classical) Riemann-Roch theorem, it has a rational point then, that is, there is a rational map $Y \rightarrow X_j$, which must be dominant, since X_j has no real points. Then the respective map of Albanese varieties would have been dominant as well, which would mean that the map of one of the components $Alb(X_i) \rightarrow Alb(X_j)$ would be non-zero, which is impossible, since these elliptic curves are not isogenous. For any $J \subset I$, let us introduce the extension E_J , which is the composite of $k(X_j)$, for all $j \in J$. The above considerations show that, for different J s, these extensions are not 2-equivalent. Hence, we get, at least, 2^c different isotropic points of characteristic 2. On the other hand, the category $DM(\mathbb{R})$ has a continuum of compact objects, so the cardinality of $\text{Spc}(DM(\mathbb{R})^c)$ is no more than 2^c . Thus, the number of 2-equivalence classes of extensions E/\mathbb{R} is 2^c . Note, that the cardinality of the set of 2-equivalence classes of *finitely generated* extensions E/\mathbb{R} is only c . On our way, we established that the cardinality of $\text{Spc}(DM(\mathbb{R})^c)$ is exactly 2^c .

An extension of fields $l \rightarrow k$ defines the map of Balmer spectra $\text{Spc}(DM(k)^c) \rightarrow \text{Spc}(DM(l)^c)$ mapping the point $\mathfrak{a}_{p,E/k}$ to $\mathfrak{a}_{p,E/l}$.

Example 5.15 Let k be an arbitrary field (of characteristic zero) and \bar{E}/k and \bar{F}/k be two algebraically closed field extensions. Then $\bar{E}/k \stackrel{p}{\sim} \bar{F}/k$ and so, the points $\mathfrak{a}_{p,\bar{E}}$ and $\mathfrak{a}_{p,\bar{F}}$ coincide. Thus, we have the “algebraically closed isotropic point” (for every p) combining all such extensions. The respective p -equivalence class contains, in particular, all extensions E containing \bar{k} . Usually, it doesn’t contain any finitely-generated representatives. It is the highest class in the sense of the order $\stackrel{p}{\geq}$.

Remark 5.16 One can show that, at lest, for $p = 2$, the points $\mathfrak{a}_{p,E}$ and $\mathfrak{a}_{p,F}$, for non- p -equivalent finitely generated extensions E/k and F/k of a flexible field k , are not comparable in the sense of topology of $\mathrm{Spc}(DM(k)^c)$ (and the same should hold for other primes). Thus, the partial order $E/k \stackrel{p}{\geq} F/k$ provides a finer structure on the Balmer spectrum, a structure not detected by the topological specialisation of points.

△

5.3 Balmer spectrum of Morel-Voevodsky category

Let $SH(k)$ be the \mathbb{A}^1 -stable homotopy category of Morel-Voevodsky [16]. Theorem 5.3 permits to construct many new points of $\mathrm{Spc}(SH(k)^c)$. This is done in our paper with Du [8]. Here I will just list the main results.

Let p be a prime, n be a natural number, or ∞ , and $K(p, n) = K(n)$ be the Morava K -theory. Let $Q_{p,n}$ be the disjoint union of all $K(n)$ -anisotropic varieties over k , and $\mathfrak{X}_{(p,n)}$ be the $\Sigma_{\mathbb{P}^1}^\infty$ -spectrum of the respective Čech simplicial scheme $\check{Cech}(Q_{p,n})$. It is a \wedge -projector, and the complementary projector is $\Upsilon_{(p,n)} = \mathrm{Cone}(\mathfrak{X}_{(p,n)} \rightarrow \mathbb{1})$. Denote as $\widehat{SH}_{(p,n)}(k/k)$ the category $\Upsilon_{(p,n)} \wedge SH(k)$. It is naturally a full subcategory of $SH(k)$ equivalent to the Verdier localisation of $SH(k)$ by the localising subcategory generated by $K(n)$ -anisotropic varieties. The same projector can be applied to the category of modules over any E_∞ -spectrum. Applying it to MGL -modules we get the category $\widehat{MGL}_{(p,n)} - mod$ together with the natural functor

$$\widehat{M}^{MGL} : \widehat{SH}_{(p,n)}(k/k) \rightarrow \widehat{MGL}_{(p,n)} - mod.$$

Define the $K(p, n)$ -isotropic stable homotopy category $SH_{(p,n)}(k/k)$ as the Verdier localisation of $\widehat{SH}_{(p,n)}(k/k)$ by the localising subcategory generated by compact objects U , such that the action of v_n on $\widehat{M}^{MGL}(U)$ is nilpotent, where we set $v_\infty = 1$.

We can introduce the partial $K(p, n)$ -order on the set of extensions of k . Let $E = \mathrm{colim}_\alpha E_\alpha$ and $F = \mathrm{colim}_\beta F_\beta$, where $E_\alpha = k(Q_\alpha)$ and $F_\beta = k(P_\beta)$ are finitely-generated extensions with smooth models Q_α and P_β . We say that $E/k \stackrel{(p,n)}{\geq} F/k$, if for every β , there exists α , such that $P_\beta|_{E_\alpha}$ is $K(p, n)$ -isotropic ($\Leftrightarrow \pi_* : K(p, n)(Q_\alpha \times P_\beta) \twoheadrightarrow K(p, n)(Q_\alpha)$ is surjective). Then $E/k \stackrel{(p,n)}{\sim} F/k$ if $E/k \stackrel{(p,n)}{\geq} F/k$ and $F/k \stackrel{(p,n)}{\geq} E/k$.

We get a family of isotropic realisations

$$\psi_{(p,n),E} : SH(k) \rightarrow SH_{(p,n)}(\tilde{E}/\tilde{E}),$$

where p is a prime number, $1 \leq n \leq \infty$, and E/k runs over equivalence classes of the field extensions under the relation $\stackrel{(p,n)}{\sim}$ above.

For $n = \infty$, these are realisations closely related to $\psi_{p,E}$ of $DM(k)^c$ from the previous section. For not formally-real fields, these coincide with the categories studied by Tanania in [20] and [21].

Our Main Theorem 1.4 implies that these realisations provide points of the Balmer spectrum of $SH(k)^c$. Define $\mathfrak{a}_{(p,n),E} := \mathrm{Ker}(\psi_{(p,n),E})$.

- Theorem 5.17** ([8]) (1) *The $\otimes - \Delta$ -ideals $\mathfrak{a}_{(p,n),E}$ are prime and so, define points of $\mathrm{Spc}(SH(k)^c)$.*
- (2) *Two points $\mathfrak{a}_{(p,n),E}$ and $\mathfrak{a}_{(q,m),F}$ coincide if and only if $p = q$, $n = m$ and $E/k \stackrel{(p,n)}{\sim} F/k$.*
- (3) *The point $\mathfrak{a}_{(p,\infty),E}$ is the image of the point $\mathfrak{a}_{p,E}$ under the natural map $\mathrm{Spc}(DM(k)^c) \rightarrow \mathrm{Spc}(SH(k)^c)$.*

These points complement the classical topological points (of positive characteristic).

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