


# Exact Quench Dynamics of the Floquet Quantum East Model at the Deterministic Point

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We study the nonequilibrium dynamics of the Floquet quantum East model (a Trotterized version of the kinetically constrained quantum East spin chain) at its “deterministic point,” where evolution is defined in terms of CNOT permutation gates. We solve exactly the thermalization dynamics for a broad class of initial product states by means of “space evolution.” We prove: (i) the entanglement of a block of spins grows at most at one-half the maximal speed allowed by locality (i.e., half the speed of dual-unitary circuits); (ii) if the block of spins is initially prepared in a classical configuration, speed of entanglement is a quarter of the maximum; (iii) thermalization to the infinite temperature state is reached exactly in a time that scales with the size of the block.

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**Introduction.**—Systems with constrained dynamics are of interest in many areas of nonequilibrium physics. Kinetically constrained models (KCMs) [1–3] provide a framework for explaining [4–6] the emergence of slow and heterogeneous dynamics in glasses [7–10], and their study has spurred the development of dynamical large deviation and trajectory ensemble methods [11–13]. Quantum constrained dynamics emerges naturally in systems such as Rydberg atoms under blockade conditions [14–17], leading to questions about slow thermalization and nonergodicity in the absence of disorder [18–31].

The simplest setting for implementing kinetic constraints is in lattice systems with discrete dynamics, such as cellular automata [32,33] or quantum circuits [34]. For such setups it has been possible to obtain many exact results that underpin our understanding of quantum dynamics, including on operator dynamics, information spreading, and thermalization (see, e.g., Refs. [35–66]). Quantum circuits are also vital for experimental simulation of quantum systems and quantum computation, having been used to demonstrate quantum advantage, perform randomized benchmarking, and to study nonequilibrium Floquet dynamics [67–77]. Here we consider this setting to characterize the dynamical effects of kinetic constraints by studying a circuit version of the quantum East model [78–80], itself a quantum generalization of the classical East model [2]. Using methods similar to those employed for dual-unitary circuits [53,61,61], we solve exactly the thermalization dynamics.

**Model setting.**—More specifically, we consider the nonequilibrium dynamics of the Floquet Quantum East model [80] at its deterministic point, which we refer to as the “deterministic Floquet quantum East” (DFQE) model. This system can be thought of as a brickwork quantum circuit, see Fig. 1, acting on a chain of  $2L$  qubits and with the local gate given by

$$U = \mathbb{1} \otimes \bar{P} + X \otimes P, \quad (1)$$

where  $P = \mathbb{1} - \bar{P} = (\mathbb{1} + Z)/2$  is the projector to the up state  $|1\rangle$  of the qubit (the down state is denoted by  $|0\rangle$ ) and

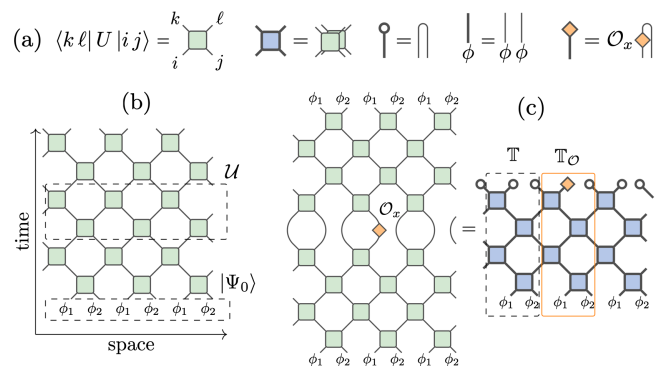


FIG. 1. Deterministic Floquet quantum East model. (a) Diagrammatic representation of the gate Eq. (1). Thick lines correspond to the folded representation of the forward and backward branches. (b) Time evolution of the state as a quantum circuit,  $|\Psi(t)\rangle = \mathcal{U}^t |\Psi_0\rangle$ . The dashed and solid-line boxes indicate the initial state  $|\Psi_0\rangle$  and the evolution operator  $\mathcal{U}$  for one time step. (c) One point function of a local operator (lhs) and its folded representation (rhs). The dashed and full boxes outline the space transfer matrices.

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$\{X, Y, Z\}$  are Pauli matrices [81]. The quantum circuit with local gate (1) was first studied in Ref. [82] (see also Ref. [83]) and is the quantum counterpart of the classical Floquet East model of Ref. [84]. The gate (1) deterministically implements the constraint that defines both the classical [2,3] and quantum [78–80] East models, where a site can flip only if its right neighbour is in the up state. The local gate (1) is part of the so called second hierarchy of generalized dual-unitary circuits (DU2s) introduced in Ref. [85]. In the jargon of quantum circuits, Eq. (1) is a CNOT (controlled NOT) gate [86], and as such it is a Clifford gate [87,88]. This implies that there exists a class of initial *stabilizer states* whose dynamics can be efficiently simulated classically. Our discussion, however, is not restricted to this class.

Following a standard quantum quench protocol [89], the system is prepared in an initial state  $|\Psi_0\rangle$ , which we take to be a product state in space, and then let to evolve unitarily as in Fig. 1(b). We characterize the ensuing dynamics using the so called *space evolution* approach (also known as folding algorithm) [90] (see also Refs. [91–95]). This can be used to characterize the evolution of general local observables [53,56,57,96], quantum information [58,60,97–102], and even spectral properties [44,46–48,51,52], but is most easily explained by considering the one-point function of an operator,  $\mathcal{O}_x$ , acting on a single qubit. We represent this via a tensor-network diagram and fold on the portions of the network representing forward and backward evolution, see Fig. 1(c), and contract the network horizontally (in space rather than time). Namely, we write the one-point function using the space transfer matrices  $\mathbb{T}$  and  $\mathbb{T}_{\mathcal{O}}$ , defined in Fig. 1(c), as

$$\langle \Psi(t) | \mathcal{O}_x | \Psi(t) \rangle = \text{tr}[\mathbb{T}_{\mathcal{O}} \mathbb{T}^{L-1}]. \quad (2)$$

Note that the transfer matrices appearing in this expressions act on the vertical folded lattice, i.e., they act on the Hilbert space  $\mathcal{H}_t = \mathbb{C}^{4^{2t}}$ , and for simplicity we assumed the initial state to be two-site shift invariant. This latter assumption is not necessary for our analysis and will be explicitly lifted in the second part of this work.

Because of unitarity and locality of the interactions the transfer matrix  $\mathbb{T}$  has a very simple spectrum: its only eigenvalues are 1 and 0 [57,99]. The transfer matrix itself is *not* rank one, since the eigenvalue zero has generically a nontrivial Jordan structure. However, the size of its Jordan blocks are bounded by  $2t$ , which implies that  $\mathbb{T}^{2t}$  is rank one. It can be written as  $\mathbb{T}^{2t} = |R\rangle\langle L|$ , where  $|R\rangle$  and  $\langle L|$  are the right and left fixed points of  $\mathbb{T}$  [103]. In other words, for  $L \geq 2t$  the one-point function of interest is fully specified the fixed points. In particular we have

$$\lim_{L \rightarrow \infty} \langle \Psi(t) | \mathcal{O}_x | \Psi(t) \rangle = \langle L | \mathbb{T}_{\mathcal{O}} | R \rangle. \quad (3)$$

This expression suggests an interesting physical interpretation of the fixed points: they are the mathematical objects

encoding the influence of the rest of the system on the subsystem where  $\mathcal{O}$  acts. For this reason they are also referred to as influence matrices [96].

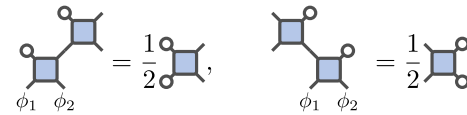
Equation (3) might seem to be a drastic simplification of Eq. (2), as it replaces a complicated matrix product with a matrix element between two fixed points. This form offers practical advantage only when the influence matrices can be computed efficiently, e.g., when they can be represented by matrix product states with low bond dimension. This is not possible in general: for generic systems and initial states, influence matrices have volume law entanglement in time [104]. Some systems, however, avoid this general rule. This includes a class of chaotic dual-unitary circuits [61] evolving from a family of compatible initial states [53], and evolution from compatible states in the Rule 54 quantum cellular automaton [56,57]. In fact, Ref. [105] argued that in the presence of integrability every low entanglement initial state should generate low entangled influence matrices. Here we show that also the nonintegrable DFQE model admits *solvable* initial states generating analytically tractable influence matrices. We find *three* distinct families of initial states with influence matrices in dimer-product form, i.e., entangling together only pairs of sites along time. We then use this result to study the exact quench dynamics of a block of spins when the rest of the system is prepared in a solvable state. We show that, regardless of the initial state, the block relaxes to the infinite temperature state in a finite number of time steps. Moreover, we provide an exact description of the full entanglement dynamics if the block is initially prepared in a solvable state.

*Exact fixed points.*—We begin by observing that  $U$  in Eq. (1), see Fig. 1, obeys the local relations



$$\text{Diagram 1} = \text{Diagram 2}, \quad \text{Diagram 3} = \text{Diagram 4}, \quad (4)$$

which define DU2 circuits [85]. Relations (4) imply [85] that if the initial states of two neighboring sites fulfil



$$\text{Diagram 5a} = \frac{1}{2} \text{Diagram 5b}, \quad \text{Diagram 5c} = \frac{1}{2} \text{Diagram 5d}, \quad (5)$$

the fixed points of the transfer matrix  $\mathbb{T}$  are of the form given in Fig. 2(a). To see this consider  $\langle L_s | \mathbb{T}$ , with  $\langle L_s |$  given in Fig. 2(a). Starting from above we apply repeatedly the first of (4) until we remove the leftmost column of gates. We then proceed with removing the second column up to the gate applied on the initial state. The latter can be removed using the first of (5) while the numerical factors combine to give  $\langle L_s | \mathbb{T} = \langle L_s |$ . Analogously, using the right relations of (4) and (5) gives  $\mathbb{T} | R_s \rangle = | R_s \rangle$ . Here and in the following we add the subscript “s” to quantities computed for initial states fulfilling (5).

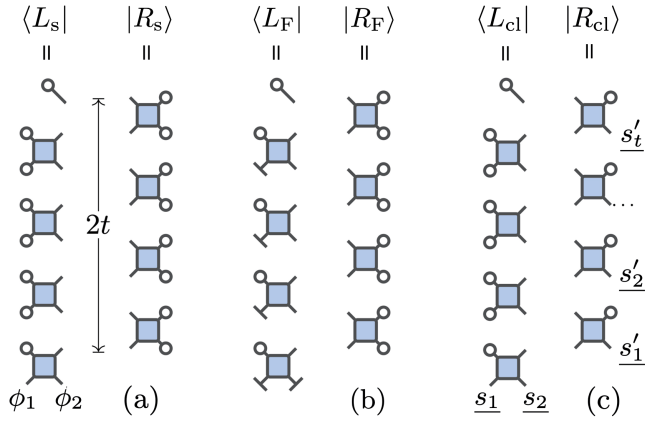


FIG. 2. Fixed points of space transfer matrices. (a) Initial states  $|\phi_{1,2}\rangle$  fulfilling Eq. (5). (b) Flat initial states,  $|\phi_{1,2}\rangle = |-\rangle$ . (c) Classical initial states, in the notation of Eq. (8), where  $s'_1 \equiv s_1 + s_2 \pmod{2}$ ,  $s'_2 = s_2$ , and  $s'_t \equiv s'_{t-1} + s'_{t-2} \pmod{2}$  for  $2 < t' \leq t$  (see [106] for details).

Expressing a general qubit state as  $|\phi\rangle = re^{i\alpha}|0\rangle + \sqrt{1-r^2}e^{i\beta}|1\rangle$ , a simple calculation reveals that all states  $|\phi_1\rangle \otimes |\phi_2\rangle$  fulfilling (5) can be parameterized as follows

$$\begin{aligned} 0 &= r_1 r_2 \sqrt{1-r_1^2} \sqrt{1-r_2^2} \cos(\alpha_1 - \beta_1) \cos(\alpha_2 - \beta_2), \\ 0 &= \left(r_1^2 - \frac{1}{2}\right) \left(r_2^2 - \frac{1}{2}\right), \end{aligned} \quad (6)$$

where the subscripts 1,2 refer to parameters of  $|\phi_{1,2}\rangle$ . The first equation is fulfilled if and only if the first of (5) holds while the second if and only if the second of (5) holds.

A remarkable property of the DFQE model is that we do not need to fulfil both these conditions to have simple fixed points. If  $|\phi_{1,2}\rangle$  are both classical configurations  $|0\rangle$  or  $|1\rangle$ , only the first of (5) is fulfilled, however, the fixed points display the simple form of Fig. 2(b). Similarly, if both  $|\phi_{1,2}\rangle$  are the flat superpositions  $|\phi_1\rangle = |\phi_2\rangle = |-\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ , only the second of (5) holds but the fixed points take the simple form in Fig. 2(c). This is because the local gate fulfils the following relations:

$$\begin{aligned} \text{Diagram 1} &= \text{Diagram 2}, & \text{Diagram 3} &= \text{Diagram 4} \end{aligned} \quad (7)$$

The diagrams in equation (7) show tensor network identities. The first identity shows a blue square with legs connected to a horizontal line. The second identity shows a blue square with legs connected to a horizontal line labeled  $s_1 + s_2$ .

where we have introduced the diagrams and notation

$$\perp = |-\rangle \otimes_r |-\rangle, \quad \underline{s} = |s \bmod 2\rangle \otimes_r |s \bmod 2\rangle, \quad s \in \mathbb{N}_0, \quad (8)$$

and  $\otimes_r$  indicates that the tensor product is between the states of the same site in the forward and backward branches, see Fig. 1(a). The simplification mechanism is

very similar to the one discussed after Eq. (5) and we refer the reader to the Supplemental Material [106]. In the following we add the subscripts “cl” and “F” to quantities computed respectively for initial states that are classical configurations and flat superpositions of them [107].

*Subsystem dynamics.*—The exact expressions for the fixed points in Fig. 2 represent our first main result. To illustrate their power, we use them to determine the relaxation time of a block of  $2\ell$  qubits in a region denoted by  $A$ . In particular, we focus on the quench from the state  $|\Psi_0\rangle = (|\phi_1\rangle \otimes |\phi_2\rangle)^{\otimes L-\ell} \otimes |\Psi'\rangle$ , where  $|\phi_1\rangle \otimes |\phi_2\rangle$  are solvable states (5) and  $|\Psi'\rangle$  is arbitrary [108]. Using the exact expressions in Fig. 2(a) we find that the reduced density matrix at time  $t$  can be simplified as follows:

$$\rho_{A,\Psi'}(t) = \frac{1}{2^{2t}} \text{Diagram} \quad (9)$$

The diagram in equation (9) shows a large tensor network with a central region of blue squares and legs, and a boundary region labeled  $|\Psi'\rangle \otimes |\Psi'\rangle$ .

Note how Eq. (9) does not contain explicitly the environment ( $\bar{A}$ , the complement of  $A$ ): its effect is encoded in the boundaries of the time-evolution operator of  $A$ . In other words, the superoperator formed by two subsequent horizontal layers of the tensor network in Eq. (9),

$$C_x = \frac{1}{4} \text{Diagram} \quad (10)$$

The diagram in equation (10) shows a single layer of blue squares and legs, with a horizontal double-headed arrow labeled  $2x$  indicating the width of the region.

retains information on  $\bar{A}$  only through its depolarizing boundaries. In general, tracing out part of a unitarily evolving system gives rise to non-Markovian dissipative evolution on the subsystem [109]. In contrast, for the case of the DFQE model the evolution of the subsystem is Markovian and the superoperator  $C_x$  is a time-local quantum map. Equation (9) represents a drastic simplification: the dynamics of a block of  $2\ell$  spins can be fully determined by diagonalizing a  $4^{2\ell} \times 4^{2\ell}$  matrix, which can be done analytically for small  $\ell$  and numerically for larger values of  $\ell$ . Moreover, one can use Eq. (9) to show [106] that  $\rho_A(t) = 1/2^{2\ell}$  for any  $t \geq 2\ell$ , so that the subsystem reaches the maximal entropy state in a *finite* number of steps. This result contrasts with what happens in generic systems, where the presence of exponential corrections means that the stationary state is only reached exactly at infinite time. An analogous situation to the one here is found in

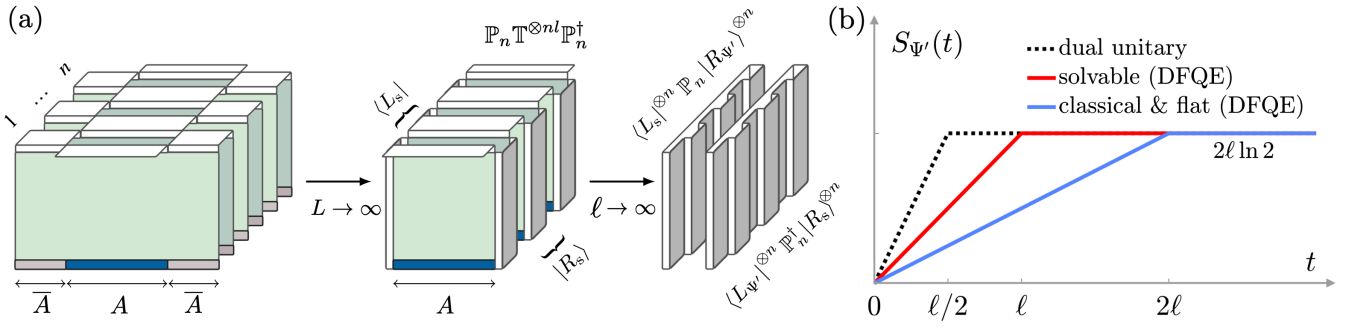


FIG. 3. Rényi entropies and entanglement dynamics. (a) Diagrammatic representation of  $\text{tr}[\rho_A^n]$  for  $n = 3$ . (b) Comparison of entanglement growth between the DFQE model (solid lines) and dual-unitary circuits (dashed line). For the DFQE model, the system  $A$  of length  $\ell \ll L$  is initially prepared in a solvable state  $|\Psi\rangle$  fulfilling (5) and (18) (red line), or one of (5) and one of (7) (blue line). For  $t > 2\ell$  all curves reach the infinite-temperature value  $2\ell \ln 2$ .

dual-unitary circuits [49], with an important difference: in the DFQE model the number of steps to approach stationarity is twice larger than for dual unitaries.

*Entanglement.*—Using the above properties we can compute the growth of entanglement from various homogeneous and inhomogeneous initial states. For concreteness we consider a system that is prepared in a solvable state everywhere, except for a finite subsystem  $A$  of length  $\ell = |A|$ . At some later time  $t$  the Rényi entanglement entropy between  $A$  and the rest is defined as

$$S_{\Psi'}^{(n)}(t) = \frac{1}{1-n} \text{tr}[\rho_{A,\Psi'}^n(t)], \quad (11)$$

where  $n$  is the Rényi index, and  $\rho_{A,\Psi'}(t)$  is given in Eq. (9). Within  $A$  the system is prepared in one of the classes of solvable states: those fulfilling both Eqs. (5), homogeneous flat states, or classical configurations.

The entanglement entropies show different scalings depending on the ratio between the subsystem size  $\ell$ , and time  $t$ . In particular, we expect a simple result in the limit  $t \rightarrow \infty$ , where the entropies saturate at a value extensive in  $\ell$  [see Fig. 3(b)]. In fact, the finite-time relaxation discussed above implies that for any initial state in  $A$  and  $t \geq 2\ell$  all Rényi entropies are the same,

$$S_{\Psi'}^{(n)}(t) \Big|_{t \geq 2\ell} = 2\ell \ln 2. \quad (12)$$

For times that are shorter than  $2\ell$ , there are no immediate simplifications at the level of the single reduced density matrix, but we need to consider the full trace in Eq. (11). Assuming for definiteness that inside of the subsystem  $A$  the initial state  $\Psi'$  is a product state and is invariant under lattice shifts by an even number of sites, the generalized purity can be compactly expressed in terms of the corresponding space transfer-matrix  $\mathbb{T}$  as

$$\text{tr}[\rho_{A,\Psi'}^n(t)] = \langle L_s | \otimes^n \mathbb{P}_n \mathbb{T}^{\otimes n|A|} | \mathbb{P}_n^\dagger | R_s \rangle^{\otimes n}, \quad (13)$$

where  $\mathbb{P}_n$  is an operator that implements the permutation of  $n$  copies, see Fig. 3(a). This expression suggests another conceptually simple regime: the “early time” regime where  $t$  is fixed and  $|A|$  is large. In this regime Eq. (13) is written in terms of a large power of a finite matrix expressible in terms of a fixed point: whenever  $t < \ell/2$  the powers of the transfer matrix factorize

$$\mathbb{T}^x|_{x > 2t} = |R_\Psi\rangle \langle L_\Psi|, \quad (14)$$

and (13) reduces to a product of two matrix elements,

$$\text{tr}[\rho_{A,\Psi'}^n(t)] = \langle L_s | \otimes^n \mathbb{P}_n | R_\Psi \rangle^{\otimes n} \langle L_\Psi | \otimes^n \mathbb{P}_n^\dagger | R_s \rangle^{\otimes n}. \quad (15)$$

With the specific form of fixed points, Fig. 2, we can evaluate these overlaps and obtain for the three classes [106]

$$\begin{aligned} S_s^{(n)}(t) \Big|_{2t < \ell} &= 2t \ln 2, \\ S_{\text{cl}}^{(n)}(t) \Big|_{2t < \ell} &= S_F^{(n)}(t) \Big|_{t < \ell} = t \ln 2. \end{aligned} \quad (16)$$

Interestingly, we see that the Rényi entanglement entropies with different  $n$  all grow with the same slope. Moreover, for classical configurations and their flat superposition this slope is reduced by 2: this is an explicit example of the initial-state dependence of the entanglement velocity. Note that for the flat superposition the range of validity of the early time expression is larger ( $t < \ell$  rather than  $2t < \ell$ ) due to the flat state being locally invariant under the dynamics.

The hardest regime to access is the intermediate-time regime  $\ell/2 < t < 2\ell$ . In this case Eq. (13) does not directly factorize, and cannot be determined only knowing the fixed points. Remarkably, and in contrast to other known solvable examples [56,58], Eq. (13) can be evaluated also in this regime for the three cases considered here. This leads to a complete description of the entanglement dynamics so far only attained for dual-unitary circuits [53,61]. In particular, when the subsystem is prepared in a classical configuration, or the flat state, the partition sum (13) evaluates to  $2^{(1-n)t}$  for



all  $t$  in the intermediate regime, i.e.,  $\ell/2 < t < 2\ell$ . This gives [106]

$$S_{\text{cl}}^{(n)}(t) = S_{\text{F}}^{(n)}(t) = \min(t, 2\ell) \ln 2. \quad (17)$$

Instead, if the subsystem is prepared in a solvable state that additionally satisfies

$$\begin{array}{c} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ \phi_1 \quad \phi_2 \end{array} = \begin{array}{c} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \text{---} \\ \phi_1 \quad \phi_2 \end{array}, \quad \text{---} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (18)$$

the stationary state is reached at  $t = \ell$  (rather than  $2\ell$ ), and for  $\ell/2 < t < \ell$  Eq. (13) gives  $2^{2t(1-n)}$ . This implies

$$S_s(t) = \min(2t, 2\ell) \ln 2. \quad (19)$$

An interesting question is whether this surprising result can be ascribed to a general property of the initial states. While the DFQE model is a Clifford circuit, the above is not a consequence of initial states being stabilizer states: even though classical configurations and the flat state are stabilizers, the solvable states fulfilling Eqs. (5) and (18) are generically not. As shown in [106], the exact results in Eqs. (17) and (19) rely on three different “microscopic mechanisms” of simplifications that combine properties of the initial states and the time evolution. The overall effect, however, is similar in the three cases: the microscopic simplifications decouple the entanglement production at the two boundaries between  $A$  and  $\bar{A}$ . This allows us to treat the problem as if it were always in the early time regime. Importantly, Eq. (18) is a necessary requirement for this to happen: finite-time numerics show that for solvable states that do not satisfy that condition the entanglement entropy deviates from (19) at intermediate times.

**Conclusions.**—We have solved exactly the entanglement dynamics of the deterministic Floquet quantum East model, a quantum circuit defined in terms of local CNOT gates that implement the same kinetic constraint as the East model [2,78]. To our knowledge, these are the first exact results for entangling dynamics in an interacting non-integrable circuit beyond those in the dual-unitary class. The simplicity of the DFQE model allows its dynamics to be solved in the large size limit for a broad class of initial product states that extends beyond Clifford stabilizers, exploiting the techniques of propagation in space. One can think of many avenues for future research. An immediate one is to characterize exactly operator spreading by extending the results of Ref. [110] on the butterfly velocity of DU2 circuits to determine the full profile of out-of-time-ordered correlators. Other directions include studying the effect of local measurements on entanglement at the level of quantum trajectories [62,63,111,112] and the

quantification of dynamical fluctuations as is done in the classical Floquet East [84].

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