EHRHART THEORY OF SPANNING LATTICE POLYTOPES

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ABSTRACT. The key object in the Ehrhart theory of lattice polytopes is the numerator polynomial of the rational generating series of the Ehrhart polynomial, called h^*-polynomial. In this paper we prove a new result on the vanishing of its coefficients. As a consequence, we get that $h_i^* = 0$ implies $h_{i+1}^* = 0$ if the lattice points of the lattice polytope affinely span the ambient lattice. This generalizes a recent result in algebraic geometry due to Blekherman, Smith, and Velasco, and implies a polyhedral consequence of the Eisenbud–Goto conjecture. We also discuss how this study is motivated by unimodality questions and how it relates to decomposition results on lattice polytopes of given degree. The proof methods involve a novel combination of successive modifications of half-open triangulations and considerations of number-theoretic step functions.

1. INTRODUCTION

1.1. **Basics of Ehrhart theory.** The study of Ehrhart polynomials of lattice polytopes is an active area of research at the intersection of discrete geometry, geometry of numbers, enumerative combinatorics, and combinatorial commutative algebra. We refer to [Bec16, Bra16, Bre15] for three recent survey articles, as well as to the book [BR07]. In order to describe our main result, let us recall the basic notions of Ehrhart theory. We denote by *lattice point* any element in $\mathbb{Z}^d$. A *lattice polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many lattice points, i. e., $P = \mathrm{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $\mathbf{v}_i \in \mathbb{Z}^d$. To a lattice polytope P, one associates its *Ehrhart function* which counts lattice points in integral multiples of P, i. e., $\mathrm{ehr}_P(k) = \left| kP \cap \mathbb{Z}^d \right|$. This is a polynomial function (see [Ehr62]), called the *Ehrhart polynomial* of P. Its generating function is known to be a rational function (see [Sta80])

$$\sum_{k \geq 0} \mathrm{ehr}_P(k) t^k = \frac{h_P^*(t)}{(1-t)^{d+1}}$$

where $h_P^*(t) \in \mathbb{Z}_{\geq 0}[t]$ is a polynomial of degree $s \in \{0, \ldots, d\}$, denoted the h^*-*polynomial* (or δ-polynomial) of P. Its coefficient vector $(h_0^*, \ldots, h_d^*)$, or $(h_0^*(P), \ldots, h_d^*(P))$ if we want to emphasize that these are the coefficients of the h^*-polynomial of P, is the h^*-*vector* (or δ-vector) of P. The number $\deg(P) \coloneqq s$ is called the *degree* of P. For future reference, let us give the basic properties of the h^*-vector of a d-dimensional lattice polytope P of degree s:

$$(1) \qquad\qquad\qquad h_0^* = 1,$$

$$(2) \qquad\qquad\qquad h_1^* = \left| P \cap \mathbb{Z}^d \right| - d - 1,$$

$$(3) \qquad\qquad\qquad h_d^* = \left| P^\circ \cap \mathbb{Z}^d \right|,$$

$$(4) \qquad\qquad d + 1 - s = \min\left\{ k \in \mathbb{Z}_{>0} \colon (kP)^\circ \cap \mathbb{Z}^d \neq \emptyset \right\},$$

$$(5) \qquad\qquad\qquad \sum_{i=0}^{s} h_i^* = \mathrm{Vol}_{\mathbb{Z}}(P),$$

where $\mathrm{Vol}_{\mathbb{Z}}(P)$ denotes the *normalized volume* of P, i. e., it equals $d!$ times the usual Euclidean volume of P, and P° denotes the relative interior of P, i. e., the topological interior of P in its affine span.

1.2. **Spanning lattice polytopes.** Let us explain what we mean by "spanning" in the title.

Definition 1.1. A d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$ is called *spanning* if any lattice point in $\mathbb{Z}^d$ is an affine integer combination of the lattice points in P. Equivalently, P is spanning if any lattice point in $\mathbb{Z}^{d+1}$ is a linear integer combination of the lattice points in $\{1\} \times P$.

Example 1.2. Any 1- and 2-dimensional lattice polytope is spanning. For any $k \geq 2$, the Reeve-simplex is not spanning:

$$\mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + k\mathbf{e}_3)$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{Z}^3$ denotes the standard basis. We remark that $h_P^*(t) = 1 + (k-1)t^2$.

Spanning is a very mild condition for a lattice polytope (for instance, it is weaker than "very ample", cf. [Bru13]). In fact, *any* lattice polytope is associated to a spanning lattice polytope by a change of the ambient lattice (replace $\mathbb{Z}^d$ by the lattice affinely spanned by $P \cap \mathbb{Z}^d$). Especially in toric geometry it is natural to pass from the ambient lattice to the spanning lattice, e.g., for fake weighted projective spaces (see [Con02, Kas09]) or in the study of A-discriminants (see [Est10, Ito15]).

1.3. **Our main result.** In this work we initiate the study of Ehrhart polynomials of spanning lattice polytopes. The main goal of this paper is the following lower bound theorem on their h^*-vectors. It is a direct consequence of a new, general Ehrhart-theoretic result (Theorem 4.7), which applies to arbitrary lattice polytopes, and can be found in Section 4.2.

Theorem 1.3. *The h^*-vector of a spanning polytope P satisfies $h_i^* \geq 1$ for all $i = 0, \ldots, \deg(P)$.*

Recall that an h^*-vector satisfying the conclusion of Theorem 1.3 is said to have *no internal zeros* (see, for instance, [Sta89]). We remark that the h^*-vector of any lattice polytope with interior lattice points has no internal zeros by Hibi's lower bound theorem (see [Hib94]).

Example 1.4. The converse of Theorem 1.3 is not true. In dimension $d \geq 3$, there are non-spanning lattice polytopes whose h^*-vectors have no internal zeros. For instance, the lattice polytope $P := \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_3, 2\mathbf{e}_1 + 4\mathbf{e}_2 + \mathbf{e}_3)$ is not spanning and satisfies $h_P^*(t) = 1 + t + 2t^2$.

1.4. **Motivation from unimodality questions.** Let us explain why one should view Theorem 1.3 as an example of a positive result in the quest for unimodality results for h^*-vectors of lattice polytopes. We refer to the survey [Bra16] for motivation and background.

We recall that a lattice polytope P is *IDP* (with respect to $\mathbb{Z}^d$) if for $k \in \mathbb{Z}_{\geq 1}$ any lattice point $\mathbf{m} \in (kP) \cap \mathbb{Z}^d$ can be written as $\mathbf{m} = \mathbf{m}_1 + \cdots + \mathbf{m}_k$ for $\mathbf{m}_1, \ldots, \mathbf{m}_k \in P \cap \mathbb{Z}^d$. IDP stands for "integer decomposition property", a condition also referred to as being *integrally-closed*. One of the main open questions about IDP lattice polytopes (see [Sta89, OH06, SVL13]) is whether their h^*-vectors are *unimodal*, i.e., their coefficients satisfy $h_0^* \leq h_1^* \leq \cdots \leq h_i^* \geq h_{i+1}^* \geq \cdots \geq h_s^*$ for some $i \in \{0, \ldots, s\}$. Theorem 1.3 is a modest analogue of this conjecture. Clearly, IDP implies spanning, and unimodality implies no internal zeros.

Example 1.5. The 5-dimensional lattice simplex with vertices

$$\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_4, \mathbf{v} := 5(\mathbf{e}_1 + \ldots + \mathbf{e}_4) + 8\mathbf{e}_5$$

is spanning with h^*-vector $(1, 1, 2, 1, 2, 1)$, i.e., not unimodal. We have used the software `polymake` (see [GJ00]) to compute the h^*-vector and the lattice points contained in the simplex which are exactly the vertices and the point $\mathbf{w} := 2(\mathbf{e}_1 + \ldots + \mathbf{e}_4) + 3\mathbf{e}_5$. As $\mathbf{e}_5 = 2\mathbf{v} - 5\mathbf{w}$, it follows that the simplex is spanning.

From the viewpoint of commutative algebra, it was already evident that IDP implies no internal zeros. Theorem 1.3 provides a new combinatorial proof of this fact. Indeed, the Ehrhart ring associated to an IDP polytope P (cf. [BG09, Section 4]) is standard graded and Cohen–Macaulay, so its quotient modulo a linear system of parameters yields a standard graded Artinian algebra whose Hilbert series equals $h_P^*(t)$, which clearly has no internal zeros. Let us remark that for spanning lattice polytopes it is unclear whether such an algebraic proof exists, the difficulty being that the Ehrhart ring of non-IDP lattice polytopes is not standard graded.

Another conjecture of interest is Oda's question whether every smooth lattice polytope is IDP [Gub12]. Here, a lattice polytope is *smooth* if the primitive edge directions at each vertex form a lattice basis. As smooth polytopes are spanning, Theorem 1.3 shows that the condition of having no internal zeros cannot be used to distinguish between smoothness and IDP.

The methods of the proof of Theorem 1.3 combine modifications of half-open triangulations and considerations of number-theoretical step functions. We hope that these methods will also be fruitful to prove stronger inequalities on the coefficients of h^*-polynomials. Let us remark that Schepers and van Langenhoven (see [SVL13]) suggested that a successive change of lattice triangulations should be essential in achieving new unimodality results in Ehrhart theory. In this sense, our results and methods could be seen as a first implementation of their proposed approach.

1.5. **Organization of the paper.** In Section 2 we explain how Theorem 1.3 implies a consequence of the Eisenbud–Goto conjecture from commutative algebra in this polyhedral setting and give some combinatorial consequences. Theorem 1.3 can be seen as a generalization of a recent result on the vanishing of the second coefficient of the h^*-polynomial (see [BSV16]). This observation and applications to decomposition results of lattice polytopes of given degree are discussed in Section 3. In Section 4 we recall the language of half-open decompositions and describe how Theorem 1.3 follows from Theorem 4.7, a general result in Ehrhart theory. Section 5 contains the proof of Theorem 4.7.

2. Application 1: Polyhedral Eisenbud–Goto

One of the original motivations of the present work is a connection with the famous Eisenbud–Goto conjecture from commutative algebra, which we explain in this section. For the algebraic concepts used in this chapter, we refer the reader to the monographs by Eisenbud [Eis95] or Brodmann and Sharp [BS13]. Let us recall the statement of the conjecture:

Conjecture 2.1 (Eisenbud–Goto conjecture [EG84])**.** *Let $\Bbbk$ be a field and let $S = \Bbbk[X_1, \ldots, X_n]$ be a polynomial ring with the standard grading, and let $I \subseteq S$ be a homogeneous prime ideal. Then*

$$(6) \qquad \operatorname{reg}(S/I) \leq \deg(S/I) - \operatorname{codim}(S/I)$$

Here, $\operatorname{reg}(S/I)$ denotes the *(Castelnuovo–Mumford) regularity* of S/I, which is defined as

$$\operatorname{reg}(S/I) \coloneqq \sup\{i + j : i, j \in \mathbb{N}_0, H^i_{\mathfrak{m}}(S/I)_j \neq 0\},$$

where $\mathfrak{m} = (X_1, \ldots, X_n)$ is the maximal homogeneous ideal of S and $H^i_{\mathfrak{m}}(S/I)_j$ denotes the j-th homogeneous component of the i-th local cohomology module of S/I with support in $\mathfrak{m}$. Further, the *degree* of S/I, denoted by $\deg(S/I)$, can be defined as $(\dim(S/I) - 1)!$ times the leading coefficient of the Hilbert polynomial of S/I. Moreover, $\operatorname{codim}(S/I) = \dim_{\Bbbk}(S/I)_1 - \dim(S/I)$ is the codimension of S/I (inside its linear hull).

Very recently, McCullough and Peeva [MP16] found counterexamples to this conjecture. However, we are going to show that a certain consequence of it is nevertheless true. Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope, and let $\Bbbk$ be an algebraically closed field of characteristic 0. We denote by $\Bbbk[P]$ the *toric ring* generated by the lattice points in P, i.e., the subalgebra of $\Bbbk[Y_0, \ldots, Y_d]$ generated by the monomials

$$Y_0 \cdot \prod_i Y_i^{v_i} \qquad \text{with } \mathbf{v} = (v_1, \ldots, v_d) \in P \cap \mathbb{Z}^d.$$

The algebraic invariants on the right-hand side of (6) have a combinatorial interpretation for $S/I = \Bbbk[P]$:

$$\deg(\Bbbk[P]) = \mathrm{Vol}_{\Gamma_P}(P)$$
$$\mathrm{codim}(\Bbbk[P]) = |P \cap \mathbb{Z}^d| - (d+1).$$

Here, Vol_{Γ_P} is the volume form normalized with respect to the affine lattice generated by the lattice points in P. In particular, if P is spanning, then this simply equals $\mathrm{Vol}_{\mathbb{Z}}(P)$.

The regularity of $\Bbbk[P]$ does not have a direct combinatorial interpretation. However, if P is spanning and $\mathbf{v} = (v_0, \dots, v_d)$ is an interior lattice point of the cone C over P, then $H_{\mathfrak{m}}^{d+1}(\Bbbk[P])_{-v_0} \neq 0$, cf. [SS90, Theorem 5.6] or [Kat15, Proposition 4.2]. Thus, if we let $r \in \mathbb{Z}_{>0}$ be the minimal value for the first coordinate of an interior lattice point in C, i.e., the minimal number such that the multiple rP of P has an interior lattice point, then it holds that $\mathrm{reg}(\Bbbk[P]) \geq d+1-r$.

In conclusion, the following proposition is a consequence of Conjecture 2.1:

Proposition 2.2. *Let $P \subseteq \mathbb{R}^d$ be a d-dimensional spanning lattice polytope. Then the following holds:*

$$(7) \qquad\qquad |P \cap \mathbb{Z}^d| \leq \mathrm{Vol}_{\mathbb{Z}}(P) + \min\{k \in \mathbb{Z}_{>0}\colon (kP)^\circ \cap \mathbb{Z}^d \neq \emptyset\}$$

or equivalently,

$$(8) \qquad\qquad h_1^* + \deg(P) \leq \mathrm{Vol}_{\mathbb{Z}}(P).$$

As the Eisenbud–Goto conjecture has been disproven in [MP16], we show that this inequality is also a consequence of our main result Theorem 1.3. Let us remark that (8) is sharp for every value of $\deg(P)$, as can be seen by considering the lattice simplices $\mathrm{conv}(\mathbf{e}_1, \dots, \mathbf{e}_d, -\mathbf{e}_1 - \dots - \mathbf{e}_d)$.

Proof. Equations (7) and (8) are equivalent by the properties (2) and (4) of h^*-vectors. By properties (1) and (5), we can reformulate (8) as

$$\deg(P) \leq 1 + \sum_{i=2}^{\deg(P)} h_i^*(P).$$

This equation holds as $h_i^*(P) \geq 1$ for $2 \leq i \leq \deg(P)$ by Theorem 1.3. $\qquad\square$

Example 2.3. In dimension 5 there exists a non-spanning lattice simplex with binomial h^*-polynomial $1 + t^3$ (see, for instance, [HHN11, end of Section 2] or [HT09, paragraph below Lemma 1.3]). Hence, the left side in Equation (7) equals 6, while the right side equals $2 + 3 = 5$. This shows that the spanning assumption cannot be dropped in Proposition 2.2.

Proposition 2.2 has an immediate combinatorial consequence. Let us recall that two polytopes in $\mathbb{R}^d$ are *affinely equivalent* if they are mapped onto each other by an affine-linear automorphism of $\mathbb{R}^d$. Moreover, we say that two affinely equivalent lattice polytopes in $\mathbb{R}^d$ are *unimodularly equivalent* if such an affine-linear automorphism maps $\mathbb{Z}^d$ to $\mathbb{Z}^d$. In fixed dimension there are only finitely many lattice polytopes of bounded volume up to unimodular equivalence (see [LZ91]). Batyrev showed more generally that there are only finitely many lattice polytopes (of arbitrary dimension) of given degree and of bounded volume up to unimodular equivalence and lattice pyramid constructions (see [Bat06]). Here, $P \subseteq \mathbb{R}^d$ is a *lattice pyramid* if P is unimodularly equivalent to $\mathrm{conv}(\{\mathbf{0}\}, \{1\} \times P')$ for some lattice polytope $P' \subseteq \mathbb{R}^{d-1}$. We recall that h^*-vectors of lattice polytopes are invariant under lattice pyramid constructions (see, for instance, [BR07, Theorem 2.4]).

There exist (non-spanning) lattice polytopes of normalized volume 2 for each degree, none of them being a lattice pyramid of the other (see [HHN11, HT09]). Such a situation cannot happen for spanning lattice polytopes, since by Equation (8) a bound on the normalized volume also implies a bound on the degree.

Corollary 2.4. *There are only finitely many spanning lattice polytopes of given normalized volume (and arbitrary dimension) up to unimodular equivalence and lattice pyramid constructions.*

Remark 2.5. While the generalization in [Nil08] of Batyrev's result might suggest this, we remark that it is not enough to fix h_1^* and the degree of a spanning lattice polytope in order to bound its volume. To see this, we consider the three-dimensional lattice polytope P with vertices

$$\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + a\mathbf{e}_3$$

with $a \in \mathbb{Z}_{\geq 2}$. Then P is spanning of (normalized) volume $a + 1$ where the only lattice points in P are its vertices, so, $h_1^* = 1$ and $s = 2$.

3. APPLICATION 2: ON THE VANISHING OF h^*-COEFFICIENTS

3.1. **Passing to spanning lattice polytopes.** Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope (with respect to $\mathbb{Z}^d$). Let us denote by Γ_P the affine sublattice in $\mathbb{Z}^d$ generated by $P \cap \mathbb{Z}^d$, i.e., the set of all integral affine combinations of $P \cap \mathbb{Z}^d$. We define the *spanning polytope $\tilde{P}$ associated to P* as the lattice polytope given by the vertices of P with respect to the lattice Γ_P.

Let us say that two lattice polytopes P, P' are *lattice-point equivalent* if there is an affine-linear automorphism of $\mathbb{R}^d$ mapping P to P' such that the lattice points in P map bijectively to the lattice points in P'. In particular P and $\tilde{P}$ are lattice-point equivalent. Clearly, unimodularly equivalent implies lattice-point equivalent implies affinely equivalent, however, none of the converses is generally true. As $\text{Vol}_{\mathbb{Z}}(\tilde{P}) \leq \text{Vol}_{\mathbb{Z}}(P)$, Corollary 2.4 has the following Corollary 3.1 as an immediate consequence for lattice polytopes that are not necessarily spanning. For this, we call P a *lattice-point pyramid* if there is a facet of P that contains all lattice points of P except for one. Note that lattice pyramids are lattice-point pyramids, but not vice versa.

Corollary 3.1. *There are only finitely many lattice polytopes of given normalized volume (and arbitrary dimension) up to lattice-point equivalence and lattice-point pyramid constructions.*

We remark that this corollary can be also obtained from [NP15, Corollary 3.9].

3.2. **Bounding the degree of the spanning lattice polytope.** As h_1^* equals the number of lattice points minus dimension minus one, we get $h_1^*(\tilde{P}) = h_1^*(P)$. For $i \geq 2$, it holds $h_i^*(\tilde{P}) \leq h_i^*(P)$. This follows from the description of h_i^* as the number of lattice points in half-open parallelepipeds, see Equation (9) in Section 4.1. In particular, $\deg(\tilde{P}) \leq \deg(P)$.

The previous considerations show that Theorem 1.3 has the following corollary.

Corollary 3.2. *If P is a lattice polytope with $h_i^*(P) = 0$, then $\deg(\tilde{P}) \leq i - 1$.*

In other words, the first zero in the h^*-vector of P bounds the degree of its spanning polytope.

Remark 3.3. For $i = 1$, Corollary 3.2 is even an equivalence. We give an elementary proof. Recall that a lattice polytope is an *empty* lattice simplex if $|P \cap \mathbb{Z}^d| = d + 1$, equivalently, $h_1^*(P) = 0$. Moreover, a lattice polytope P is a *unimodular simplex* if its vertices form an affine lattice basis. Equivalently, $\text{Vol}_{\mathbb{Z}}(P) = 1$, respectively, $\deg(P) = 0$. We observe that a spanning lattice polytope is an empty simplex if and only if it is a unimodular simplex. In particular, $h_1^*(P) = 0$ is equivalent to $\deg(\tilde{P}) = 0$.

For each $i \geq 2$, there exist empty lattice simplices P with $h_i^* = 1$ (see [HHN11, HT09]). Hence, the converse of Corollary 3.2 fails for $i \geq 2$.

3.3. **The vanishing criterion by Blekherman, Smith, and Velasco.** While Corollary 3.2 describes a necessary condition on the vanishing of h_i^*, it is a natural question how to strengthen it to get an equivalence also for $i \geq 2$. Recently such a criterion was proven for $i = 2$ (see [BSV16]). In order to describe this result, let us denote a lattice polytope $P \subseteq \mathbb{R}^d$ as *i-IDP* if any lattice point $\mathbf{m} \in (iP) \cap \mathbb{Z}^d$ can be written as $\mathbf{m} = \mathbf{m}_1 + \ldots + \mathbf{m}_i$ for $\mathbf{m}_1, \ldots, \mathbf{m}_i \in P \cap \mathbb{Z}^d$.

Proposition 3.4 ([BSV16, Proposition 6.6]). *A lattice polytope P satisfies $h_2^*(P) = 0$ if and only if $\deg(\tilde{P}) \leq 1$ and P is 2-IDP.*

This is a reformulation of [BSV16, Proposition 6.6] in our notation. The hard non-combinatorial part of their proof that relies on results from real and complex algebraic geometry is the statement $h_2^*(P) = 0$ implies $\deg(\tilde{P}) \leq 1$. This follows now from Corollary 3.2 for $i = 2$. The authors of [BSV16] communicated to us another purely combinatorial proof that relies on the classification of

lattice polytopes of degree one (see [BN07]). We remark that such a classification is not known for lattice polytopes of higher degree.

The sufficient condition on the vanishing of $h_2^*(P)$ in Proposition 3.4 easily generalizes.

Proposition 3.5. *If* $\deg(\tilde{P}) \leq i - 1$ *and* P *is* i-IDP, *then* $h_i^*(P) = 0$.

Proof. We show the contraposition, so assume $h_i^*(P) > 0$. Then there exists a lattice point of height i in some half-open parallelepiped of a given half-open triangulation of P, we refer to Section 4.1 for more details. As P is i-IDP, the lattice point is also contained in the sublattice Γ_P, hence, $h_i^*(\tilde{P}) > 0$, and thus $\deg(\tilde{P}) > i - 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.6. For $i \geq 3$ it is not true that $h_i^*(P) = 0$ implies that P is i-IDP. There exists a spanning (even very ample) lattice polytope $P' \subseteq \mathbb{R}^3$ with h^*-vector $h^*(P') = (1, 4, 5, 0)$ such that a lattice point in $2P'$ is not a sum of two lattice points in P' (see [Bru13, AGH+16]). This lattice polytope can be constructed as the Minkowski sum of the Reeve-simplex $R_4 := \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3) \subseteq \mathbb{R}^3$ and the edge $\mathrm{conv}(\mathbf{0}, \mathbf{e}_3) \subseteq \mathbb{R}^3$ (see [Oga13]). Therefore, the lattice pyramid $P \subseteq \mathbb{R}^4$ over P' is a 4-dimensional spanning lattice polytope of degree 2 that is not 3-IDP, as the lattice point $2\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3 \in 3P \cap \mathbb{Z}^4$ cannot be written as the sum of three lattice points in P.

3.4. **Generalizing results on the degree of lattice polytopes.** In [BN07, Nil08, HNP09] it was investigated how lattice polytopes of small degree can be decomposed into lower-dimensional lattice polytopes. This question is partly motivated by applications in algebraic geometry [DdP09, DN10, DHNP13, Ito15]. Corollary 3.2 allows to generalize these results up to a change of lattice. For this, let us recall that $P \subseteq \mathbb{R}^d$ is called a *Cayley polytope* of lattice polytopes $P_1, \ldots, P_k \subseteq \mathbb{R}^m$ if $k \geq 2$ and P is unimodularly equivalent to $\mathrm{conv}(P_1 \times \mathbf{e}_1, \ldots, P_k \times \mathbf{e}_k) \subseteq \mathbb{R}^m \times \mathbb{R}^k$ where $\mathbf{e}_1, \ldots, \mathbf{e}_k$ denotes the standard basis of $\mathbb{Z}^k$. In particular, note that the lattice points of a Cayley polytope lie on two parallel affine hyperplanes of lattice distance one.

Corollary 3.7. *Let* P *be a* d-*dimensional lattice polytope with* $h_{i+1}^* = 0$. *If* $d > \frac{i^2 + 19i - 4}{2} =: d'$, *then* $\tilde{P}$ *is a Cayley polytope of lattice polytopes in dimension at most* d'. *In this case, every lattice point in* P *lies on one of two parallel hyperplanes.*

Proof. By, Corollary 3.2, $\deg(\tilde{P}) \leq i$. Now, we apply [HNP09, Theorem 1.2] to $\tilde{P}$. $\qquad\square$

Remark 3.8. Let us briefly discuss the relation of the results of this section to the study of point configurations of small *combinatorial degree*, i.e., the maximal degree of the h-vector of lattice triangulations of P. We refer to [NP15] for terminology and background. Let us observe that the h-vector of a lattice triangulation $\mathcal{T}$ of P has no internal zeros. This can be deduced from the fact that the h-vector is an M-sequence (see [BH93]); an alternative, direct proof can also be given using Lemma 5.4. Now, it follows from the Betke–McMullen formula (see [BM85]) that $h_{i+1}^*(P) = 0$ implies $h_{i+1}(\mathcal{T}) = 0$. Hence, the combinatorial degree of P is bounded by i if $h_i^* = 0$. This shows that Corollary 3.7 sharpens in this case the conclusion in [NP15, Theorem A] which only guaranteed a so-called "weak Cayley" condition.

4. Ehrhart Theory and Half-open Triangulations

4.1. **Half-open triangulations.** In this subsection let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope. The polynomial h_P^* can be computed by means of the *cone* C *over* P, i.e., $C = \mathrm{cone}(\{1\} \times P) \subseteq \mathbb{R}^{d+1}$, equipped with a triangulation which we now outline. For details and references on Ehrhart Theory, we refer to [BR07]. Our approach is in the spirit of [KV08] (see also [HNP12]).

In this paper, by a *triangulation* $\mathcal{T}$ of C, we mean a regular triangulation of C such that primitive ray generators of every face of the triangulation are contained in the affine hyperplane $\{1\} \times \mathbb{R}^d$. The set of faces of dimension k we denote by $\mathcal{T}^{(k)}$.

A point $\xi \in \mathbb{R}^{d+1}$ is called *generic* with respect to a triangulation $\mathcal{T}$ of C, if it is not contained in any of the linear subspaces generated by the faces in $\mathcal{T}^{(d)}$.

We define

$$\Upsilon_C := \{\mathcal{T} \text{ triangulation of } C\},$$
$$\Xi_C := \{\xi \in C \text{ generic with respect to any } \mathcal{T} \in \Upsilon_C\}.$$

The set of primitive generators in $\mathbb{Z}^{d+1}$ of the extremal rays of a polyhedral cone $\sigma \subseteq \mathbb{R}^{d+1}$, we denote by $\sigma^{(1)}$.

Definition 4.1. A *half-open triangulation* of C consists of a choice $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$. For every maximal cell $\sigma \in \mathcal{T}^{(d+1)}$ the corresponding *half-open cell* $\sigma[\xi]$ is given as follows: Write $\xi = \sum_{\mathbf{v} \in \sigma^{(1)}} \lambda_{\mathbf{v}} \mathbf{v}$ for $\lambda_{\mathbf{v}} \in \mathbb{R} \setminus \{0\}$ and set $I_\xi(\sigma) := \{\mathbf{v} \in \sigma^{(1)} \colon \lambda_{\mathbf{v}} < 0\}$. Then

$$\sigma[\xi] = \left\{ \sum_{\mathbf{v} \in \sigma^{(1)}} \mu_{\mathbf{v}} \mathbf{v} \colon \mu_{\mathbf{v}} \in \mathbb{R}_{\geq 0}, \ \mu_{\mathbf{v}} > 0 \text{ for all } \mathbf{v} \in I_\xi(\sigma) \right\}.$$

The proofs of the following results in Section 4.1 are standard and can be done as in [HNP12].

Proposition 4.2. *Let $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$ be a half-open triangulation of C. The half-open cells $\sigma[\xi]$ for $\sigma \in \mathcal{T}^{(d+1)}$ yield a partition of C, i. e., we have a disjoint union*

$$C = \bigcup_{\sigma \in \mathcal{T}^{(d+1)}} \sigma[\xi].$$

Definition 4.3. Let $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$ be a half-open triangulation of C. The *half-open fundamental parallelepiped* $\Pi_\sigma[\xi]$ of a half-open cell $\sigma[\xi]$ for $\sigma \in \mathcal{T}^{(d+1)}$ is given by

$$\Pi_\sigma[\xi] = \left\{ \sum_{\mathbf{v} \in \sigma^{(1)}} \lambda_{\mathbf{v}} \mathbf{v} \colon \lambda_{\mathbf{v}} \in [0, 1[\text{ for } \mathbf{v} \notin I_\xi(\sigma), \lambda_{\mathbf{v}} \in]0, 1] \text{ for } \mathbf{v} \in I_\xi(\sigma) \right\}.$$

Remark 4.4. Let $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$ be a half-open triangulation of C and take $\sigma \in \mathcal{T}^{(d+1)}$. If $\xi \in \sigma^\circ$, then note that $\Pi_\sigma[\xi]$ is the usual half-open parallelepiped, i. e.,

$$\Pi[\xi] = \left\{ \sum_{\mathbf{v} \in \sigma^{(1)}} \lambda_{\mathbf{v}} \mathbf{v} \colon \lambda_{\mathbf{v}} \in [0, 1[\right\}.$$

Proposition 4.5. *Let $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$ be a half-open triangulation and fix a half-open cell $\sigma[\xi]$ for $\sigma \in \mathcal{T}^{(d+1)}$. The translates of $\Pi[\xi]$ by vectors in $\mathcal{M} := \sum_{\mathbf{w} \in \sigma^{(1)}} \mathbb{Z}_{\geq 0} \mathbf{w}$ yield a partition of the half-open cell $\sigma[\xi]$, i. e., we have a disjoint union*

$$\sigma[\xi] = \bigcup_{\mathbf{v} \in \mathcal{M}} \mathbf{v} + \Pi[\xi].$$

Definition 4.6. We define a map

$$h^* \colon \left(C \cap \mathbb{Z}^{d+1} \right) \times \Upsilon_C \times \Xi_C \to \mathbb{Z}_{\geq 0},$$

as follows: For given $\mathbf{v} \in \left(C \cap \mathbb{Z}^{d+1} \right)$, $\mathcal{T} \in \Upsilon_C$ and $\xi \in \Xi_C$, there is exactly one $\sigma \in \mathcal{T}^{(d+1)}$ such that $\mathbf{v}$ is contained in the half-open cell $\sigma[\xi]$. There is a unique $\overline{\{\mathbf{v}\}}_{\mathcal{T},\xi} \in \Pi_\sigma[\xi] \cap \mathbb{Z}^{d+1}$ such that $\mathbf{v} - \overline{\{\mathbf{v}\}}_{\mathcal{T},\xi} \in \sum_{\mathbf{w} \in \sigma^{(1)}} \mathbb{Z}_{\geq 0} \mathbf{w}$. Then $h^*_{\mathcal{T},\xi}(\mathbf{v})$ is, by definition, equal to the first coordinate of $\overline{\{\mathbf{v}\}}_{\mathcal{T},\xi}$.
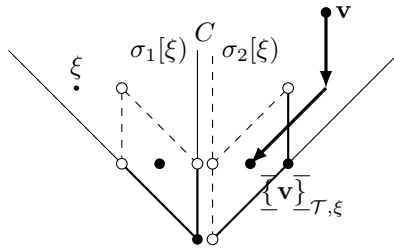

FIGURE 1. Illustration of Definition 4.6 for $P = [-2, 2]$ (here, $h^*_{\mathcal{T},\xi}(\mathbf{v}) = 1$).

For fixed $\mathcal{T} \in \Upsilon_C$ and $\xi \in \Xi_C$, the h^*-*polynomial* of P is given by

$$(9) \qquad h_P^*(t) = \sum_{k=0}^{s} h_k^* t^k = \sum_{\sigma \in \mathcal{T}^{(d+1)}} \sum_{\mathbf{v} \in \Pi_\sigma[\xi] \cap \mathbb{Z}^{d+1}} t^{h_{\mathcal{T},\xi}^*(\mathbf{v})}.$$

From this equality it is evident that the coefficients h_k^* are non-negative integers. In particular, we observe that

$$(10) \qquad \{k = 0, \ldots, s \colon h_k^* \neq 0\} = \big\{ h_{\mathcal{T},\xi}^*(\mathbf{v}) \colon \mathbf{v} \in C \cap \mathbb{Z}^{d+1} \big\}.$$

4.2. **A general Ehrhart-theoretic result and the proof of Theorem 1.3.** Theorem 1.3 is an immediate consequence of the following main result of this paper.

Theorem 4.7. *Let $P \subseteq \mathbb{R}^d$ be a d-dimensional lattice polytope and let $C \subseteq \mathbb{R}^{d+1}$ be the cone over it. Let Γ_P be the sublattice of $\mathbb{Z}^{d+1}$ spanned by the lattice points in $\{1\} \times P$. Then for every $\mathbf{v} \in \mathbb{Z}^{d+1}$ and all tuples $(\mathcal{T}_0, \xi_0) \in \Upsilon_C \times \Xi_C$, there exist nonnegative integers $a_{\mathbf{v}} \leq b_{\mathbf{v}}$ (independent of the choice $(\mathcal{T}_0, \xi_0)$) such that*

$$[a_{\mathbf{v}}, b_{\mathbf{v}}] \cap \mathbb{Z} = h_{\mathcal{T}_0, \xi_0}^*(C \cap (\mathbf{v} + \Gamma_P)).$$

The proof of Theorem 4.7 will be developed in Section 5. Let us show here how to use Theorem 4.7 to prove Theorem 1.3.

Proof of Theorem 1.3. As in the statement of Theorem 4.7, let Γ_P be the sublattice spanned by the lattice points in $\{1\} \times P$. Since P is spanning, we obtain $\Gamma_P = \mathbb{Z}^{d+1}$. The statement follows from Equation (10) by applying Theorem 4.7 with $\mathbf{v} := \mathbf{0}$. $\square$

Remark 4.8. One can also interpret Theorem 4.7 as follows. We use the notation from that theorem. Assume that $h_b^* = h_B^* = 0$ for two integers $b < B$ such that $h_k^* \neq 0$ for all $k = b+1, \ldots, B-1$. Fix $(\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$ and take a vector $\mathbf{v} \in C \cap \mathbb{Z}^{d+1}$ with $b < h_{\mathcal{T},\xi}^*(\mathbf{v}) < B$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_a, \mathbf{v}_1', \ldots, \mathbf{v}_A' \in (\{1\} \times P) \cap \mathbb{Z}^{d+1}$ such that $\mathbf{v}' := \mathbf{v} + \sum_{i=1}^{a} \mathbf{v}_i - \sum_{j=1}^{A} \mathbf{v}_j' \in C$. Then $b < h_{\mathcal{T},\xi}^*(\mathbf{v}') < B$, i.e., the lattice points in C that can be reached from $\mathbf{v}$ by adding or subtracting lattice points from $\{1\} \times P$ contribute only to the h^*-coefficients with index in the interval $]b, B[$.

5. PROOF OF THEOREM 4.7

5.1. **Overview.** We give an overview of the proof of Theorem 4.7. We use the notation from that theorem with $\Gamma := \Gamma_P$. We start with the following observation.

Lemma 5.1. *For all pairs $(\mathcal{T}_0, \xi_0), (\mathcal{T}, \xi) \in \Upsilon_C \times \Xi_C$, it holds that*

$$h_{\mathcal{T}_0, \xi_0}^*(C \cap (\mathbf{v} + \Gamma)) = h_{\mathcal{T}, \xi}^*(C \cap (\mathbf{v} + \Gamma)).$$

Proof. We note that

$$\sum_{k=0}^{\infty} |(\{k\} \times (kP)) \cap (\mathbf{v} + \Gamma)| t^k = \frac{\sum_{\sigma \in \mathcal{T}^{(d+1)}} \sum_{\mathbf{w} \in \Pi_\sigma[\xi] \cap (\mathbf{v}+\Gamma)} t^{\mathrm{ht}(\mathbf{w})}}{(1-t)^{d+1}} =: \frac{h_{P, \mathbf{v}+\Gamma}^*(t)}{(1-t)^{d+1}}$$

where $\mathrm{ht} \colon \mathbb{R}^{d+1} \to \mathbb{R}; \mathbf{w} = (w_0, \ldots, w_d) \mapsto w_0$ is the projection onto the first coordinate. In particular, for $\mathbf{v} = \mathbf{0}$ and $\Gamma = \mathbb{Z}^{d+1}$ this yields the usual equality $\sum_{k=0}^{\infty} |kP \cap \mathbb{Z}^d| t^k = h_P^*(t)/(1-t)^{d+1}$. We obtain

$$h_{P, \mathbf{v}+\Gamma}^*(t) = \sum_{k=0}^{s} h_{k, \mathbf{v}+\Gamma}^* t^k = \sum_{\sigma \in \mathcal{T}^{(d+1)}} \sum_{\mathbf{w} \in \Pi_\sigma[\xi] \cap (\mathbf{v}+\Gamma)} t^{h_{\mathcal{T},\xi}^*(\mathbf{w})}.$$

Analogous to Equation (10), it follows that

$$h_{\mathcal{T}, \xi}^*(C \cap (\mathbf{v} + \Gamma)) = \big\{ k \in \mathbb{N} \colon h_{k, \mathbf{v}+\Gamma}^* \neq 0 \big\},$$

so, in particular, this set is independent of the choice of $(\mathcal{T}_0, \xi_0) \in \Upsilon_C \times \Xi_C$. $\square$

The following two propositions will be used in our proof of Theorem 4.7. We will prove them below in Section 5.2 and Section 5.3, respectively.

Proposition 5.2 (Changing the generic vector)**.** *Let $\mathcal{T} \in \Upsilon_C$ and $\mathbf{x} \in C \cap \mathbb{Z}^{d+1}$. Then there exists $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$ such that*

$$\left\{ h^*_{\mathcal{T},\xi}(\mathbf{x}) \colon \xi \in \Xi_C \right\} = [a, b] \cap \mathbb{Z}.$$

Proposition 5.3 (Changing the triangulation)**.** *Let $\mathbf{x} \in C \cap (\mathbf{v} + \Gamma)$, $\xi \in \Xi_C$ and $\mathcal{T}, \mathcal{T}' \in \Upsilon_C$ be two triangulations. Then there exist $(\mathcal{S}_1, \xi_1, \mathbf{y}_1), \ldots, (\mathcal{S}_R, \xi_R, \mathbf{y}_R) \in \Upsilon_C \times \Xi_C \times (C \cap (\mathbf{v} + \Gamma))$ such that*

$$\left\{ h^*_{\mathcal{S}_i, \xi_i}(\mathbf{y}_i) \colon i = 1, \ldots, R \right\} \cup \left\{ h^*_{\mathcal{T}, \xi}(\mathbf{x}), h^*_{\mathcal{T}', \xi}(\mathbf{x}) \right\} = [a, b] \cap \mathbb{Z},$$

for two integers $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$.

Proof of Theorem 4.7. Let $a_{\mathbf{v}} := \min \left\{ h^*_{\mathcal{T}, \xi}(C \cap (\mathbf{v} + \Gamma)) \right\}$ and $b_{\mathbf{v}} := \max \left\{ h^*_{\mathcal{T}, \xi}(C \cap (\mathbf{v} + \Gamma)) \right\}$.

Note that we may replace $\mathbf{v}$ by any element in $\mathbf{v} + \Gamma$ without changing the statement. In particular, we may assume that $\mathbf{v} \in C$ and that $h^*_{\mathcal{T}, \xi}(\mathbf{v}) = a_{\mathbf{v}}$. Moreover, there exists an element $\mathbf{w} \in C \cap (\mathbf{v} + \Gamma)$ with $h^*_{\mathcal{T}, \xi}(\mathbf{w}) = b_{\mathbf{v}}$. We can write $\mathbf{w} = \mathbf{v} + \sum_{i=1}^{r} \mathbf{v}_i - \sum_{i=1}^{s} \mathbf{v}'_i$ with $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}'_1, \ldots, \mathbf{v}'_s, \in (\{1\} \times P) \cap \mathbb{Z}^{d+1}$.

Let $\mathbf{x}_k := \mathbf{v} + \sum_{i=1}^{k} \mathbf{v}_i$ for $0 \leq k \leq r$. We are going to show that for each $k = 1, \ldots, r$, there are $(\mathcal{S}_1, \xi_1, \mathbf{w}_1), \ldots, (\mathcal{S}_q, \xi_q, \mathbf{w}_q) \in \Upsilon_C \times \Xi_C \times (C \cap (\mathbf{v} + \Gamma))$ such that the corresponding $h^*_{\mathcal{S}_j, \xi_j}(\mathbf{w}_j)$ fill the gap between $h^*_{\mathcal{T}, \xi}(\mathbf{x}_{k-1})$ and $h^*_{\mathcal{T}, \xi}(\mathbf{x}_k)$. Thus, we can fill the gap between $\mathbf{v}$ and $\mathbf{v} + \sum_{i=1}^{r} \mathbf{v}_i$. By symmetry, we then can also fill the gap between $\mathbf{w}$ and $\mathbf{w} + \sum_{i=1}^{s} \mathbf{v}'_i = \mathbf{v} + \sum_{i=1}^{r} \mathbf{v}_i$, so the claim follows.

By Proposition 5.2 and Proposition 5.3, it is in fact sufficient to show that there exists a triangulation $\mathcal{T}' \in \Upsilon_C$ and a generic vector $\xi' \in \Xi_C$ such that the gap between $h^*_{\mathcal{T}', \xi'}(\mathbf{x}_{k-1})$ and $h^*_{\mathcal{T}', \xi'}(\mathbf{x}_k)$ can be filled.

For this, let $\mathcal{T}' \in \Upsilon_C$ be a pulling triangulation (see, for instance, [DLRS10, Section 4.3.2]) which uses $\mathbf{v}_k$ as its *last* vertex. Then $\mathbf{v}_k$ is an extremal ray generator of every full-dimensional cone in $\mathcal{T}'$, cf. [DLRS10, Lemma 4.3.6 (2)]. Choose $\sigma \in (\mathcal{T}')^{(d+1)}$ such that $\mathbf{x}_{k-1} \in \sigma$ and choose $\xi' \in \sigma^\circ \cap \Xi_C$. Then $\mathbf{x}_{k-1} \in \sigma[\xi']$ and it holds that

$$h^*_{\mathcal{T}', \xi'}(\mathbf{x}_k) = h^*_{\mathcal{T}', \xi'}(\mathbf{x}_{k-1} + \mathbf{v}_k) = h^*_{\mathcal{T}', \xi'}(\mathbf{x}_{k-1}).$$

Thus, the claim follows. The precise way in which we apply Proposition 5.2 and Proposition 5.3 is also indicated in Figure 2, where an arrow "$\leftrightarrow$" means that the gap between the two endpoints can be filled. $\qquad\square$

FIGURE 2. How to fill the gap between $h^*_{\mathcal{T}, \xi}(\mathbf{x}_{k-1})$ and $h^*_{\mathcal{T}, \xi}(\mathbf{x}_k)$ in the proof of Theorem 4.7.

5.2. **Changing the generic vector.** In this subsection, we are going to prove Proposition 5.2. The next lemma is used in that proof.

Lemma 5.4. *Let $\mathcal{T} \in \Upsilon_C$ and $\sigma \in \mathcal{T}$. Then the set*

$$\Lambda_{\mathcal{T}, \sigma} := \left\{ I_\xi(\sigma') \colon \sigma' \in \mathcal{T}^{(d+1)}, \xi \in \Xi_C, \sigma^\circ \subseteq \sigma'[\xi] \right\}$$

is an abstract simplicial complex, i.e., closed under taking subsets.

Proof. We show that if $S \in \Lambda_{\mathcal{T}, \sigma}$ and $\mathbf{v} \in S$, then $(S \setminus \{\mathbf{v}\}) \in \Lambda_{\mathcal{T}, \sigma}$ as well. Hence every subset of S, which can be achieved by repeatedly removing vectors from S, is contained in $\Lambda_{\mathcal{T}, \sigma}$.

Take $S \in \Lambda_{\mathcal{T}, \sigma}$, choose $\sigma' \in \mathcal{T}^{(d+1)}$, $\xi \in \Xi_C$ such that $\sigma^\circ \subseteq \sigma'[\xi]$ and $S = I_\xi(\sigma')$, and let $\mathbf{v} \in S$. For $t \geq 0$ let $\xi_t := \xi + t\mathbf{v}$. Clearly $\xi_t \in C$ for all $t \geq 0$. Also, as $\xi_0 = \xi$ is generic it follows that ξ_t is

generic for all but finitely many values of t. For a sufficiently large choice of t the coefficient of $\mathbf{v}$ in the linear combination $\xi_t = \sum_{\mathbf{w}\in(\sigma')^{(1)}} \mu_{\mathbf{w},t}\mathbf{w}$ is positive and hence $I_{\xi_t}(\sigma') = S \setminus \{\mathbf{v}\}$ (see Figure 3).

Moreover, note that $\sigma^\circ \subseteq \sigma'[\xi]$ if and only if $I_\xi(\sigma') \subseteq \sigma^{(1)}$. As $I_{\xi_t}(\sigma') \subseteq I_\xi(\sigma')$, it follows that $\sigma^\circ \subseteq \sigma'[\xi_t]$ and thus $S \setminus \{\mathbf{v}\} \in \Lambda_{\mathcal{T},\sigma}$. $\qquad\square$
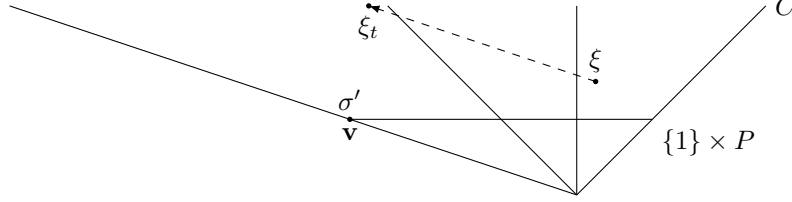

FIGURE 3. The point ξ_t for large $t > 0$.

Proof of Proposition 5.2. There exists a unique cone $\sigma \in \mathcal{T}$ such that $\mathbf{x} \in \sigma^\circ$. This cone does not need to be full-dimensional. We can represent $\mathbf{x}$ as a linear combination $\mathbf{x} = \sum_{\mathbf{v}\in\sigma^{(1)}} \lambda_\mathbf{v}\mathbf{v}$ for positive real numbers $\lambda_\mathbf{v} > 0$. For a given $\xi \in \Xi_C$, there exists a unique full-dimensional cone $\sigma' \in \mathcal{T}^{(d+1)}$, such that $\sigma^\circ \subseteq \sigma'[\xi]$, and hence

$$(11) \qquad h^*_{\mathcal{T},\xi}(\mathbf{x}) = \sum_{\mathbf{v}\in\sigma^{(1)}} \{\lambda_\mathbf{v}\} + \left| I_\xi(\sigma') \cap \left\{\mathbf{v} \in \sigma^{(1)} \colon \lambda_\mathbf{v} \in \mathbb{Z}\right\}\right|,$$

where $\{\lambda_\mathbf{v}\}$ denotes the *fractional part* of $\lambda_\mathbf{v}$, i.e., $\lambda_\mathbf{v} - \lfloor\lambda_\mathbf{v}\rfloor$. By Lemma 5.4, $\Lambda_{\mathcal{T},\sigma,\mathbf{x}} := \left\{ S \cap \left\{\mathbf{v} \in \sigma^{(1)} \colon \lambda_\mathbf{v} \in \mathbb{Z}\right\} \colon S \in \Lambda_{\mathcal{T},\sigma}\right\}$ is an abstract simplicial complex (a subcomplex of $\Lambda_{\mathcal{T},\sigma}$). It follows from Eq. (11) that

$$\left\{h^*_{\mathcal{T},\xi}(\mathbf{x}) \colon \xi \in \Xi_C\right\} \subseteq \sum_{\mathbf{v}\in\sigma^{(1)}} \{\lambda_\mathbf{v}\} + \{|S'| \colon S' \in \Lambda_{\mathcal{T},\sigma,\mathbf{x}}\}.$$

The other inclusion "$\supseteq$" follows by the fact that every $S' \in \Lambda_{\mathcal{T},\sigma,\mathbf{x}}$ has a presentation $S' = I_\xi(\sigma')$ for $\xi \in \Xi_C$ and $\sigma' \in \mathcal{T}^{(d+1)}$ with $\sigma^\circ \subseteq \sigma'[\xi]$ (see Lemma 5.4). Hence

$$\left\{h^*_{\mathcal{T},\xi}(\mathbf{x}) \colon \xi \in \Xi_C\right\} = [a, a+b] \cap \mathbb{Z},$$

with $a = \sum_{\mathbf{v}\in\sigma^{(1)}} \{\lambda_\mathbf{v}\}$ and $b = \dim\Lambda_{\mathcal{T},\sigma,\mathbf{x}} + 1$ where the dimension of an abstract simplicial complex is the largest dimension of any of its faces S which in turn is $\dim S = |S| - 1$. $\qquad\square$

Example 5.5. If we let $\mathcal{T} \in \Upsilon_C$ also vary in Proposition 5.2, then the analogous statement is false in general.

Denote the standard basis of $\mathbb{R}^6$ by $\mathbf{e}_1, \ldots, \mathbf{e}_6$ and consider the lattice polytope

$$P := \mathrm{conv}(5\mathbf{e}_1 - 4(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) - 3(\mathbf{e}_5 + \mathbf{e}_6), \mathbf{e}_2, \ldots, \mathbf{e}_6, \mathbf{0}, 5\mathbf{e}_1 - \mathbf{e}_2 - \ldots - \mathbf{e}_6),$$

whose only lattice points are its vertices (such polytopes are called *empty*). We denote the vertices of $\{1\} \times P \subseteq \mathbb{R}^7$ by $\mathbf{v}_i$ for $i = 1, \ldots, 8$ where the order is taken to be the one as they appear in the definition above. Let $C \subseteq \mathbb{R}^7$ be the cone over P. As P is a circuit (see Remark 5.10), Υ_C consists of two triangulations $\mathcal{T}_+, \mathcal{T}_-$ where

$$\mathcal{T}_+ = \{\mathrm{cone}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_8) \colon i = 1, \ldots, 6\},$$
$$\mathcal{T}_- = \{\mathrm{cone}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_8) \colon i = 7, 8\}.$$

We take the lattice point $\mathbf{x} := 4\mathbf{e}_0 + \mathbf{e}_1$ in C which has representations

$$\mathbf{x} = \tfrac{1}{5}\mathbf{v}_1 + \tfrac{4}{5}(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) + \tfrac{3}{5}(\mathbf{v}_5 + \mathbf{v}_6) + \tfrac{1}{5}\mathbf{v}_7 = \tfrac{1}{5}(\mathbf{v}_2 + \ldots + \mathbf{v}_6 + \mathbf{v}_8) + \tfrac{14}{5}\mathbf{v}_7.$$

For every $\xi \in \Xi_C$, we obtain $h^*_{\mathcal{T}_-,\xi}(\mathbf{x}) = 4$ while $h^*_{\mathcal{T}_+,\xi}(\mathbf{x}) = 2$, and thus $h^*_{\Upsilon_C,\Xi_C}(\mathbf{x}) = \{2,4\}$ is missing the number 3. On the other hand if we fix $\mathcal{T} \in \Upsilon_C$, then $\left|h^*_{\mathcal{T},\Xi_C}(\mathbf{x})\right| = 1$ and hence does not has a gap.

5.3. **Changing the triangulation.** The proof of Proposition 5.3 relies on flips in triangulations and the fact that any two regular triangulations can be connected by a sequence of flips. We recall some notions and results and refer to [DLRS10] (see also [GKZ08, Chapter 7, Section 2]) for details and references.

Definition 5.6. A *(homogeneous) vector set* in $\mathbb{R}^{d+1}$ is a finite subset $\mathcal{A} \subseteq \mathbb{R}^{d+1}$, such that the first component of each $\mathbf{v} \in \mathcal{A}$ is 1. The number $|\mathcal{A}| - \dim(\mathrm{span}(\mathcal{A}))$ is called its *corank*. We will never consider inhomogeneous vector sets, hence we will omit the specifier "homogeneous".

In this paper under a *polyhedral subdivision* $\mathcal{S}$ of $\mathcal{A}$ we will understand a subset $\mathcal{S}$ of the power set of $\mathcal{A}$ such that

(1) $\{\mathrm{cone}(B) \colon B \in \mathcal{S}\}$ forms a polyhedral subdivision of the cone generated by $\mathcal{A}$, i.e., $C_\mathcal{A} := \mathrm{cone}(\mathcal{A})$, and
(2) for every $B, B' \in \mathcal{S}$

$$B \cap (\mathrm{cone}(B) \cap \mathrm{cone}(B')) = B' \cap (\mathrm{cone}(B) \cap \mathrm{cone}(B')).$$

A *cell* $B \in \mathcal{S}$ is called *simplicial* if it consists of linearly independent vectors. A *triangulation* $\mathcal{T}$ of $\mathcal{A}$ is a polyhedral subdivision such that all its cells are simplicial.

Remark 5.7. The vector sets which we will deal with in this paper come from lattice points on height 1 contained in the cone over lattice polytopes. In particular, subtleties in connection with "double points" won't appear.

Given a simplicial cell B of a polyhedral subdivision $\mathcal{S}$ of a vector set $\mathcal{A}$, the set B necessarily consists of the primitive generators of the extremal rays of $\mathrm{cone}(B)$. In particular, B and $\mathrm{cone}(B)$ uniquely determine each other. Hence there is a natural correspondence between triangulations of cone $C_\mathcal{A}$ as defined in Section 4.1 and triangulations of the vector set $\mathcal{A}$. However, for an arbitrary cell B in a polyhedral subdivision $\mathcal{S}$ of $\mathcal{A}$, it is necessary to remember B, as $\mathrm{cone}(B)$ does not determine B in general. One might want to think of B as the "markings" of $\mathrm{cone}(B)$.

A *refinement* $\mathcal{S}'$ of a polyhedral subdivision $\mathcal{S}$ is a polyhedral subdivision where for each $B' \in \mathcal{S}'$ there exists $B \in \mathcal{S}$ such that $B' \subseteq B$.

An *almost-triangulation* of a vector set $\mathcal{A}$ is a pair $(\mathcal{B}, \mathcal{S})$ of a subset $\mathcal{B} \subseteq \mathcal{A}$ and a polyhedral subdivision $\mathcal{S}$ of simultaneously both $\mathcal{A}$ and $\mathcal{B}$ such that it is not a triangulation but all its proper refinements (with respect to $\mathcal{B}$) are one.

Proposition 5.8 (see [DLRS10, Corollary 2.4.6])**.** *Every almost-triangulation has exactly* two *proper refinements, which are both triangulations.*

Two triangulations $\mathcal{T}_1$, $\mathcal{T}_2$ of the same vector set $\mathcal{A}$ are *connected by a flip* if there is an almost-triangulation $(\mathcal{B}, \mathcal{S})$ of $\mathcal{A}$ such that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the only two triangulations refining $(\mathcal{B}, \mathcal{S})$.

Example 5.9. Consider the vector set $\mathcal{A} := \{\mathbf{v}_1, \ldots, \mathbf{v}_{10}\} \subseteq \mathbb{R}^3$ whose projection to $\mathbb{R}^2$ by forgetting the first coordinate is given in Figure 4. To simplify notation we will abbreviate the subset $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_{10}\}$ by "$i_1 \ldots i_k$". The two triangulations $\mathcal{T}_1 := \{123, 134,$
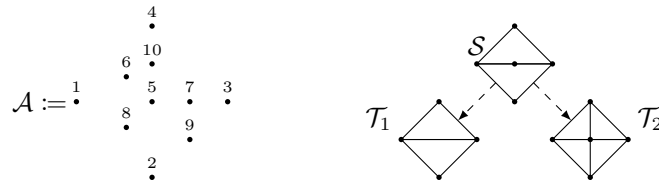

FIGURE 4. The flip of Example 5.9.

$12, 23, 13, 14, 34, 1, 2, 3, 4, \emptyset\}$ and $\mathcal{T}_2 := \{125, 235, 345, 145, 12, 15, 25, 35, 23, 45, 34, 14, 1, 2, 3, 4, 5, \emptyset\}$ are connected by a flip supported on the almost-triangulation $(\mathcal{B}, \mathcal{S})$ where $\mathcal{B} := \{1, \ldots, 5\}$ and $\mathcal{S} = \{1235, 1345, 153, 12, 23, 14, 34, 1, 2, 3, 4, \emptyset\}$.

Vector sets of corank 1 will play an important role in the proof, so let us recall some facts. We refer to [DLRS10, Section 2.4] for details.

Remark 5.10. A corank 1 vector set $\mathcal{A} \subseteq \mathbb{R}^{d+1}$ possesses a unique linear dependence relation, say $\mathbf{0} = \sum_{\mathbf{v} \in \mathcal{A}} \lambda_{\mathbf{v}} \mathbf{v}$, which partitions $\mathcal{A}$ into three subsets $\mathcal{A}_-$, $\mathcal{A}_0$ and $\mathcal{A}_+$:

$$\mathcal{A}_+ := \{\mathbf{v} \in \mathcal{A} \colon \lambda_{\mathbf{v}} > 0\}, \quad \mathcal{A}_0 := \{\mathbf{v} \in \mathcal{A} \colon \lambda_{\mathbf{v}} = 0\}, \quad \mathcal{A}_- := \{\mathbf{v} \in \mathcal{A} \colon \lambda_{\mathbf{v}} < 0\}.$$

The following are the only two triangulations of $\mathcal{A}$, both are regular.

$$\mathcal{T}_+ := \{\text{cone}(B) \colon \mathcal{A}_+ \nsubseteq B \subseteq \mathcal{A}\} \qquad \text{and} \qquad \mathcal{T}_- := \{\text{cone}(B) \colon \mathcal{A}_- \nsubseteq B \subseteq \mathcal{A}\}.$$

By [DLRS10, Theorem 4.4.1], the two triangulations $\mathcal{T}_+$ and $\mathcal{T}_-$ form the prototype of a flip. A lattice polytope $P \subseteq \mathbb{R}^d$ such that its homogenized vertices form a vector set $\mathcal{A}$ of corank 1 with $\mathcal{A}_0 = \emptyset$ is called a *circuit*.

Proposition 5.3 will follow from the following further reduction to the case of corank 1.

Lemma 5.11. *Let $\mathcal{A} \subseteq \mathbb{Z}^{d+1}$ be a vector set of corank 1, $\mathbf{v} \in \mathbb{Z}^{d+1}$, $\xi \in \Xi_{C_{\mathcal{A}}}$, $\Gamma_{\mathcal{A}}$ be the sublattice of $\mathbb{Z}^{d+1}$ generated by $\mathcal{A}$ and $\mathbf{x} \in C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}})$. Then there exist $(\mathcal{S}_1, \xi_1, \mathbf{y}_1), \dots, (\mathcal{S}_R, \xi_R, \mathbf{y}_R) \in \{\mathcal{T}_+, \mathcal{T}_-\} \times \Xi_{C_{\mathcal{A}}} \times (C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}}))$ such that*

$$\left\{ h^*_{\mathcal{S}_i, \xi_i}(\mathbf{y}_i) \colon i = 1, \dots, R \right\} \cup \left\{ h^*_{\mathcal{T}_+, \xi}(\mathbf{x}), h^*_{\mathcal{T}_-, \xi}(\mathbf{x}) \right\} = [a, b] \cap \mathbb{Z},$$

for two nonnegative integers $a \leq b$.

We will prove Lemma 5.11 in Section 5.4. The following technical lemma will be needed to make a generic point "more" generic.

Lemma 5.12. *Let $C \subseteq \mathbb{R}^{d+1}$ be a full-dimensional cone, $\sigma \subseteq C$ a simplicial full-dimensional subcone and $H_1, \dots, H_R \subseteq \mathbb{R}^{d+1}$ a family of (linear) hyperplanes. For every $\xi \in C$, there exists $\xi' \in C^\circ \setminus \left(\bigcup_{j=1}^R H_j \right)$ with $I_\xi(\sigma) = I_{\xi'}(\sigma)$ (see Definition 4.1).*

Proof. As σ is full-dimensional and simplicial, there exists a unique representation $\xi = \sum_{\mathbf{v} \in \sigma^{(1)}} \lambda_{\mathbf{v}} \mathbf{v}$. Further, there exists $\mathbf{x} \in \sigma^\circ \setminus \left(\bigcup_{j=1}^R H_j \right)$, which has a representation $\mathbf{x} = \sum_{\mathbf{v} \in \sigma^{(1)}} \mu_{\mathbf{v}} \mathbf{v}$ with $\mu_{\mathbf{v}} > 0$ for all $\mathbf{v} \in \sigma^{(1)}$.

For $0 \leq t \leq 1$ let $\xi(t) := (1-t)\xi + t\mathbf{x}$. Clearly $\xi(t) \in C^\circ$ for all $0 < t \leq 1$. As $\xi(1) \notin \bigcup_{j=1}^R H_j$, the points $\xi(t)$ avoid the hyperplanes H_i for all but finitely many values of t. If we choose t close to 0, then $\mu_{\mathbf{v}} > 0$ for every $\mathbf{v} \in \sigma^{(1)}$ implies that the nonzero coefficients of $\xi(t)$ in the basis $\sigma^{(1)}$ have the same signs as the $\lambda_{\mathbf{v}}$, and the zero coefficients become positive. Hence for such a choice of t, $\xi' := \xi(t)$ satisfies the claim. $\qquad \square$

Proof of Proposition 5.3. As both $\mathcal{T}$ and $\mathcal{T}'$ are regular triangulations of the same vector set $\mathcal{A}_0 := (\{1\} \times P) \cap \mathbb{Z}^{d+1}$, they are connected by a sequence of flips (see, for instance, [DLRS10, Theorem 5.3.7]), i.e., there is a finite sequence of triangulations of the vector set $\mathcal{A}_0$ such that every two consecutive triangulations differ in a flip. It is sufficient to prove our claim for every pair of consecutive triangulations in this sequence, and hence we assume from now on that $\mathcal{T}$ and $\mathcal{T}'$ differ only by a flip.

Let $(\mathcal{B}, \mathcal{S})$ be the almost-triangulation such that $\mathcal{T}$ and $\mathcal{T}'$ are the two proper refinements of it. According to [DLRS10, Lemma 2.4.5], each cell $\mathcal{A} \in \mathcal{S}^{(d+1)}$ has corank at most 1. Take $\mathcal{A} \in \mathcal{S}^{(d+1)}$ such that $\mathbf{x} \in \sigma := \text{cone}(\mathcal{A})$ and fix $\xi' \in \Xi_C \cap \sigma^\circ$.

If $\mathcal{A}$ has corank 0, then $\mathcal{A} \in \mathcal{T} \cap \mathcal{T}'$, and thus $h^*_{\mathcal{T}, \xi'}(\mathbf{x}) = h^*_{\mathcal{T}', \xi'}(\mathbf{x})$. The statement follows by Proposition 5.2.

If $\mathcal{A}$ has corank 1, then (up to swapping $\mathcal{T}_+$ and $\mathcal{T}_-$) we may assume $\mathcal{T}_+ \subseteq \mathcal{T}$ and $\mathcal{T}_- \subseteq \mathcal{T}'$ where $\mathcal{T}_\pm$ denote the two triangulations of $\mathcal{A}$ (see Remark 5.10). We obtain

$$h^*_{\mathcal{T}, \xi'}(\mathbf{x}) = h^*_{\mathcal{T}_+, \xi'}(\mathbf{x}) \qquad \text{and} \qquad h^*_{\mathcal{T}', \xi'}(\mathbf{x}) = h^*_{\mathcal{T}_-, \xi'}(\mathbf{x}).$$

By Lemma 5.11, there exist $(\mathcal{S}_1, \xi_1, \mathbf{y}_1), \dots, (\mathcal{S}_R, \xi_R, \mathbf{y}_R) \in \{\mathcal{T}_+, \mathcal{T}_-\} \times \Xi_{C_{\mathcal{A}}} \times (C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}}))$ such that the $h^*_{\mathcal{S}_i, \xi_i}(\mathbf{y}_i)$ fill the gap between $h^*_{\mathcal{T}_+, \xi'}(\mathbf{x})$ and $h^*_{\mathcal{T}_-, \xi'}(\mathbf{x})$. By Lemma 5.12, we may assume $\xi_i \in \Xi_C$ without changing the value of $h^*_{\mathcal{S}_i, \xi_i}(\mathbf{y}_i)$. Moreover $C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}}) \subseteq C \cap (\mathbf{v} + \Gamma)$. If $\mathcal{S}_i = \mathcal{T}_+$ for some $i = 1, \dots, R$, then $h^*_{\mathcal{S}_i, \xi_i}(\mathbf{y}_i) = h^*_{\mathcal{T}, \xi_i}(\mathbf{y}_i)$ (and analogously for $\mathcal{S}_i = \mathcal{T}_-$). In particular, we can fill the gap between $h^*_{\mathcal{T}, \xi'}(\mathbf{x})$ and $h^*_{\mathcal{T}', \xi'}(\mathbf{x})$. The statement follows by Proposition 5.2. $\qquad \square$

5.4. **The corank** 1 **case.** In this section we will prove Lemma 5.11. In its proof we will consider certain functions which we want to discuss separately here. We denote by $\{x\}$ the *fractional part* of a real number x, i.e., $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. For two finite families of positive integers $(\lambda_i)_{i \in I}$ and $(\mu_j)_{j \in J}$ with $\gcd(\lambda_i, \mu_j : i \in I, j \in J) = 1$ and $\sum_{i \in I} \lambda_i = \sum_{j \in J} \mu_j$ and a further family $(x_k)_{k \in I \cup J}$ of rational numbers, we define

$$f \colon \mathbb{R} \to \mathbb{Q}; t \mapsto \sum_{i \in I} \{x_i - \lambda_i t\} + \sum_{j \in J} \{x_j + \mu_j t\},$$

which is a periodic bounded step function with period 1. Such functions have already appeared in number theory and algebraic geometry (see, for instance, [Vas99, Bor08, BB09, Bob09]).

The function f is piecewise constant and the interesting t-values are the ones where $f(t)$ is different from its left-handed or right-handed limit. We call those t *potential jump discontinuities* and observe that this is the case if and only if $x_i - \lambda_i t \in \mathbb{Z}$ for some $i \in I$ or $x_j + \mu_j t \in \mathbb{Z}$ for some $j \in J$. We define for a potential jump discontinuity t

$$l(t) := |\{j \in J : x_j + \mu_j t \in \mathbb{Z}\}| \qquad \text{and} \qquad r(t) := |\{i \in I : x_i - \lambda_i t \in \mathbb{Z}\}|.$$

In the following $\lim_{t \to t_0-} f(t)$ (resp. $\lim_{t \to t_0+} f(t)$) will denote the left-handed (resp. right-handed) limit of a function $f \colon \mathbb{R} \to \mathbb{R}$.

Lemma 5.13. *For a potential jump discontinuity $t_0 \in \mathbb{R}$ the relationship between $f(t_0)$ and its left- resp. right-handed limit is given as follows:*

$$\lim_{t \to t_0-} f(t) = f(t_0) + l(t_0), \qquad \text{and} \qquad \lim_{t \to t_0+} f(t) = f(t_0) + r(t_0).$$
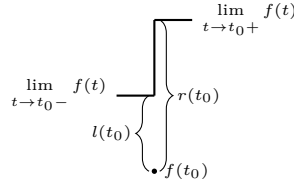

FIGURE 5. Relationship between $f(t_0)$ and its left- resp. right-handed limit at a jump discontinuity t_0.

Proof. As $\sum_{i \in I} \lambda_i = \sum_{j \in J} \mu_j$, we can rewrite f as follows

$$f \colon \mathbb{R} \to \mathbb{Q}; t \mapsto \sum_{i \in I \cup J} x_i - \sum_{i \in I} \lfloor x_i + \lambda_i t \rfloor - \sum_{j \in J} \lfloor x_j - \mu_j t \rfloor.$$

The statement follows by the following properties of the floor-function. Let $x, t_0 \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{Z}_{>0}$ with $x + \lambda t_0, x - \mu t_0 \in \mathbb{Z}$. Then

$$\lfloor x + \lambda t_0 \rfloor = \lim_{t \to t_0-} \lfloor x + \lambda t \rfloor + 1, \qquad \lfloor x + \lambda t_0 \rfloor = \lim_{t \to t_0+} \lfloor x + \lambda t \rfloor,$$
$$\lfloor x - \mu t_0 \rfloor = \lim_{t \to t_0-} \lfloor x - \mu t \rfloor, \qquad \lfloor x - \mu t_0 \rfloor = \lim_{t \to t_0+} \lfloor x - \mu t \rfloor + 1.$$

$\square$

Proof of Lemma 5.11. The coefficients in the linear dependence relation $\sum_{\mathbf{v} \in \mathcal{A}_+} \lambda_{\mathbf{v}} \mathbf{v} = \sum_{\mathbf{v} \in \mathcal{A}_-} \mu_{\mathbf{v}} \mathbf{v}$ can be chosen to be positive integers with $\gcd(\mu_{\mathbf{v}_-}, \lambda_{\mathbf{v}_+} : \mathbf{v}_- \in \mathcal{A}_-, \mathbf{v}_+ \in \mathcal{A}_+) = 1$. Let $\sigma' \in \mathcal{T}_-$ be the unique cone such that $\mathbf{x} \in \sigma'[\xi]$ and denote by $\mathbf{v}' \in \mathcal{A}_-$ the unique element of $\mathcal{A}_- \setminus (\sigma')^{(1)}$. We can represent $\mathbf{x}$ as a linear combination $\mathbf{x} = \sum_{\mathbf{v} \in \mathcal{A} \setminus \{\mathbf{v}'\}} x_{\mathbf{v}} \mathbf{v}$ for nonnegative rational numbers $x_{\mathbf{v}}$. Moreover we set $x_{\mathbf{v}'} := 0$.

Take $\mathbf{v}'' \in \mathcal{A}_+$ such that $\frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}} = \min\left\{\frac{x_{\mathbf{v}}}{\lambda_{\mathbf{v}}} \colon \mathbf{v} \in \mathcal{A}_+\right\}$, so that $x_{\mathbf{v}} - \frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\lambda_{\mathbf{v}} \geq 0$ for all $\mathbf{v} \in \mathcal{A}_+$. Let $\sigma'' \in \mathcal{T}_+$ be the unique cone such that $\mathbf{v}''$ does not generate a ray of σ''. We use the dependence relation to change the representation of $\mathbf{x}$ to

$$\mathbf{x} = \sum_{\mathbf{v} \in \mathcal{A}_+} (x_{\mathbf{v}} - t\lambda_{\mathbf{v}})\mathbf{v} + \sum_{\mathbf{v} \in \mathcal{A}_0} x_{\mathbf{v}}\mathbf{v} + \sum_{\mathbf{v} \in \mathcal{A}_-} (x_{\mathbf{v}} + t\mu_{\mathbf{v}})\mathbf{v}.$$

We let $t \in \left[0, \frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\right]$, so that the coefficients in all representations of $\mathbf{x}$ are nonnegative. Further we consider the following periodic bounded step function with period 1:

$$f\colon \mathbb{R} \to \mathbb{Z}; t \mapsto \sum_{\mathbf{v} \in \mathcal{A}_+} \underline{\{x_{\mathbf{v}} - t\lambda_{\mathbf{v}}\}} + \sum_{\mathbf{v} \in \mathcal{A}_0} \underline{\{x_{\mathbf{v}}\}} + \sum_{\mathbf{v} \in \mathcal{A}_-} \underline{\{x_{\mathbf{v}} + t\mu_{\mathbf{v}}\}}.$$

Observe that f takes integer values, as $\sum_{\mathbf{v} \in \mathcal{A}} x_{\mathbf{v}} \in \mathbb{Z}$. Then $f(0) = h^*_{\mathcal{T}_-,\xi'}(\mathbf{x})$ and $f\left(\frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\right) = h^*_{\mathcal{T}_+,\xi''}(\mathbf{x})$ for $\xi' \in \Xi_{C_{\mathcal{A}}} \cap (\sigma')^{\circ}$ and $\xi'' \in \Xi_{C_{\mathcal{A}}} \cap (\sigma'')^{\circ}$. The gap between $h^*_{\mathcal{T}_-,\xi}(\mathbf{x})$ and $h^*_{\mathcal{T}_-,\xi'}(\mathbf{x})$ (resp. the gap between $h^*_{\mathcal{T}_+,\xi}(\mathbf{x})$ and $h^*_{\mathcal{T}_+,\xi''}(\mathbf{x})$) can be filled by using Proposition 5.2, so it remains to show that the gap between $h^*_{\mathcal{T}_-,\xi'}(\mathbf{x})$ and $h^*_{\mathcal{T}_+,\xi''}(\mathbf{x})$ can be also filled.

Let D be the set of potential jump discontinuities of f which lie in the interval $\left[0, \frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\right]$. Let $t_1 < t_2$ be two successive potential jump discontinuities. Then, if $f(t_1) < f(t_2)$ it also holds that $f(t_1) + r(t_1) \geq f(t_2)$, see Figure 6. Similarly, if $f(t_1) > f(t_2)$, then $f(t_1) \leq f(t_2) + l(t_2)$. To finish our proof it is therefore sufficient to prove the following two claims:

For each $t \in D \setminus \left\{\frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\right\}$ with $r(t) > 0$, we claim that

$$[f(t), f(t) + r(t) - 1] \cap \mathbb{Z} \subseteq \left\{h^*_{\mathcal{T}_+,\widetilde{\xi}}(\mathbf{y}) \colon \widetilde{\xi} \in \Xi_{C_{\mathcal{A}}}, \mathbf{y} \in C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}})\right\},$$

and similarly for $t \in D \setminus \{0\}$ with $l(t) > 0$, we claim that

$$[f(t), f(t) + l(t) - 1] \cap \mathbb{Z} \subseteq \left\{h^*_{\mathcal{T}_-,\widetilde{\xi}}(\mathbf{y}) \colon \widetilde{\xi} \in \Xi_{C_{\mathcal{A}}}, \mathbf{y} \in C_{\mathcal{A}} \cap (\mathbf{v} + \Gamma_{\mathcal{A}})\right\}.$$

We only show the first claim, as the proof of the second one is analogous. For this, fix $t \in D \setminus \left\{\frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}}\right\}$ with $r(t) > 0$. Choose $\mathbf{v}_0 \in \mathcal{A}_+$ with $x_{\mathbf{v}_0} - t\lambda_{\mathbf{v}_0} \in \mathbb{Z}$ and let $\sigma_0 := \mathrm{cone}(\mathcal{A} \setminus \{\mathbf{v}_0\}) \in \mathcal{T}_+$ be the unique cone such that $\mathbf{v}_0$ does not generate a ray of σ_0. Let $\mathbf{y} := \mathbf{x} - (x_{\mathbf{v}_0} - t\lambda_{\mathbf{v}_0})\mathbf{v}_0 + \sum_{\mathbf{v} \in \mathcal{A}_+ \setminus \{\mathbf{v}_0\}} \mathbf{v}$. For $\xi_0 \in \sigma_0^{\circ} \cap \Xi_{C_{\mathcal{A}}}$ it follows from Eq. (11) that $h^*_{\mathcal{T}_+,\xi_0}(\mathbf{y}) = f(t)$. On the other hand, it holds that $\mathbf{v}_0 = 1/\lambda_{\mathbf{v}_0}\left(\sum_{\mathbf{v} \in \mathcal{A}_-} \mu_{\mathbf{v}}\mathbf{v} - \sum_{\mathbf{v} \in \mathcal{A}_+ \setminus \{\mathbf{v}_0\}} \lambda_{\mathbf{v}}\mathbf{v}\right)$ and thus $I_{\mathbf{v}_0}(\sigma_0) = \mathcal{A}_+ \setminus \{\mathbf{v}_0\}$. By Lemma 5.12, we can find an element $\xi_1 \in \Xi_{C_{\mathcal{A}}}$ with $I_{\xi_1}(\sigma_0) = I_{\mathbf{v}_0}(\sigma_0)$. Using Eq. (11) again, it follows that $h^*_{\mathcal{T}_+,\xi_1}(\mathbf{y}) = f(t) + r(t) - 1$. Finally, the gap between $f(t)$ and $f(t) + r(t) - 1$ can be filled by using Proposition 5.2.
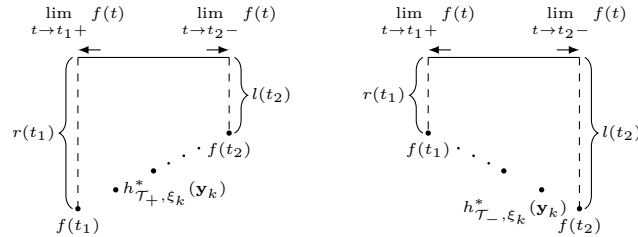

FIGURE 6. The possible cases for two successive potential jump discontinuities.

□

Example 5.14 (Continuation of Example 5.5)**.** The lattice polytope in Example 5.5 is an empty circuit with unique dependence relation

$$\mathbf{v}_1 + 3(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) + 2(\mathbf{v}_5 + \mathbf{v}_6) = 13\mathbf{v}_7 + \mathbf{v}_8.$$

Let $\mathcal{A} \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_8\}$ be the associated vector set of homogenized lattice points in $\{1\} \times P$. We have $\mathcal{A}_+ \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$ and $\mathcal{A}_- \coloneqq \{\mathbf{v}_7, \mathbf{v}_8\}$. In the proof of Lemma 5.11, the coefficients of this dependence relation were denoted by $(\lambda_{\mathbf{v}})_{\mathbf{v} \in \mathcal{A}_+}$ and $(\mu_{\mathbf{v}})_{\mathbf{v} \in \mathcal{A}_-}$ respectively. There we also introduced $\mathbf{v}'' \in \mathcal{A}_+$ such that $x_{\mathbf{v}} - \frac{x_{\mathbf{v}''}}{\lambda_{\mathbf{v}''}} \lambda_{\mathbf{v}} \geq 0$ for all $\mathbf{v} \in \mathcal{A}_+$. In this example we have $\mathbf{v}'' = \mathbf{v}_1$. The periodic bounded step function (with period 1) associated to $\mathbf{x}$ is given as follows (see Figure 7)

$$f \colon \mathbb{R} \to \mathbb{Z}; t \mapsto \left\{ \tfrac{1}{5} - t \right\} + 3\left\{ \tfrac{4}{5} - 3t \right\} + 2\left\{ \tfrac{3}{5} - 2t \right\} + \left\{ \tfrac{1}{5} + 13t \right\} + \left\{ t \right\}.$$

Let us consider, e. g., the possible jump discontinuity $t_0 = \frac{3}{5} = 0.6$. From Figure 7 we can read off $f(t_0) = 2$ while $\lim_{t \to t_0-} f(t) = 3$ and $\lim_{t \to t_0+} f(t) = 5$ which implies that $l(t_0) = 1$ and $r(t_0) = 3$.
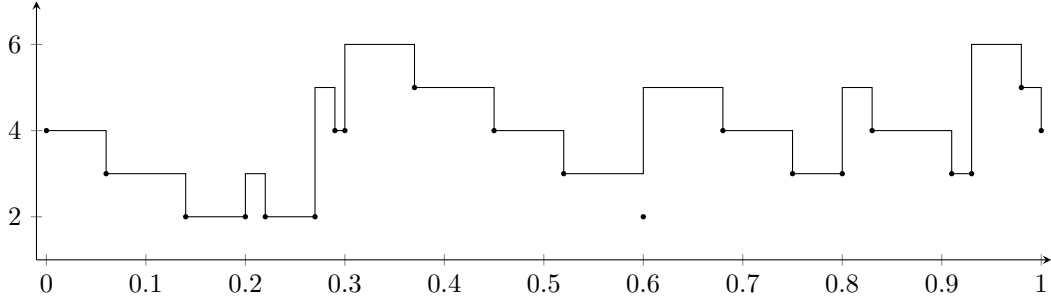

FIGURE 7. The periodic bounded step function of Example 5.14. The dots indicate the value of f at the potential jump discontinuities.

Finally, we have $f(0) = h^*_{\mathcal{T}_-, \Xi_C}(\mathbf{x}) = 4$ while $f\left(\tfrac{1}{5}\right) = h^*_{\mathcal{T}_+, \Xi_C}(\mathbf{x}) = 2$, and thus there is a gap at 3. We can fill it by looking at the potential jump discontinuity $\frac{4}{65}$: $f\left(\tfrac{4}{65}\right) = h^*_{\mathcal{T}_-, \xi}(\mathbf{x} - \mathbf{v}_7) = 3$ for $\xi \in \Xi_C \cap \sigma^\circ$ where $\sigma \in \mathcal{T}_-$ is the unique cone such that $\mathbf{v}_7$ does not generate a ray of it.

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