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Albert's twisted field construction using division algebras with a multiplicative norm

S. Pumplün
School of Mathematical Sciences, University of Nottingham
University Park, Nottingham NG7 2RD, UK
susanne.pumpluen@nottingham.ac.uk

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We generalize Albert's twisted field construction, applying it to unital division algebras with a multiplicative norm. We give conditions for the resulting algebras to be division algebras. Four- and eight-dimensional real unital and non-unital division algebras with large derivation algebras are constructed out of Hamilton's quaternion and Cayley's octonion algebra.

Keywords: Real division algebras; Albert's twisted fields; composition algebra.

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0. Introduction

Albert's classical construction resulting in twisted semifields [1, 2] is a well-known tool to build n-dimensional unital nonassociative division algebras out of an n-dimensional cyclic field extension K/F with Galois group $\mathrm{Gal}(K/F) = \langle\sigma\rangle$: For any choice of $c \in K^\times$ such that $N_{K/F}(c) \neq 1$ and $1 \leq i, j < n$, K equipped with the new multiplication

$$x \circ y = xy - c\sigma^i(x)\sigma^j(y),$$

is a division algebra over F. For finite base fields F, Kaplanski's trick is then used to associate to any such presemifield $(K, \circ)$ an isotopic semifield. Any semifield isotopic to $(K, \circ)$ is called a *twisted field*. If n is prime and q large enough, any division algebra of dimension n over $\mathbb{F}_q$ is either a field or a twisted field [7]. On the other hand, this construction indeed produces fewer than o non-isotopic semifields of order o [5].

In particular, the commutative twisted fields play a prominent role in the theory of semifields: For $c = -1$, the division algebra obtained from a finite field extension K/F of odd degree, F of odd characteristic, which is given by the multiplication $x \circ y = xy + \alpha(x)\alpha^{-1}(y)$ for a nontrivial $\alpha \in \mathrm{Gal}(K/F)$, $\alpha^{-1} \neq \alpha$, is a presemifield. Although this $(K, \circ)$ itself is not commutative, its isotope $x \star y = x \circ \alpha(y) = x\alpha(y) + \alpha(x)y$ is, thus every associated unital isotope obtained by using Kaplanski's trick is a commutative twisted semifield. All commutative twisted semifields are obtained this way. This construction works for every base field of odd characteristic which admits a cyclic field extension of degree n and yields a unital central commutative division algebra for all odd $n > 2$.

Generalizations of twisted fields also occur in the classification of three-dimensional nonassociative algebras over a field F [4].

The purpose of this paper is to show that Albert's construction can be applied to finite-dimensional algebras over a field F that possess a multiplicative norm $N : A \to F$ of degree n, i.e. $N(xy) = N(x)N(y)$ for all $x, y \in A$, and their isotopes. As possible application, we obtain examples of real division algebras with a large derivation algebra.

We focus on unital algebras A with a multiplicative norm. We choose $c \in A$ and two similarities f, g of N to define a new multiplication on A. Since A need not be commutative or associative, there are several options for a possible generalization of Albert's approach: The possibilities include defining a new multiplication on A as

$$x \circ y = xy - c(f(x)g(y)) \quad \text{or} \quad x \circ y = xy - (cf(x))g(y),$$

but we can also define a multiplicative structure by putting c in the middle, i.e.

$$x \circ y = xy - (f(x)c)g(y) \quad \text{or} \quad x \circ y = xy - f(x)(cg(y)),$$

or on the right-hand side:

$$x \circ y = xy - (f(x)g(y))c \quad \text{or} \quad x \circ y = xy - f(x)(g(y)c).$$

We can moreover choose to swap around the factors in the second part of the equation and define

$$x \circ y = xy - c(f(y)g(x))$$

and so on, exhausting all possible combinations of this type. If A is a division algebra, the right choice of $0 \neq c \in A$ yields a division algebra $(A, \circ)$.

We then apply Kaplanski's trick to $(A, \circ)$ to obtain a unital algebra. Different choices of elements $d \in A$ used for Kaplanski's trick yield isotopic unital division algebras $(A, *_d)$. In the theory of semifields algebras are usually classified only up to isotopy, so the choice of d is not relevant in that setting. Choosing the unit element of A when applying Kaplanski's trick to unital algebras helps to compute the inverses of the left and right multiplication, so that we can write down the multiplication $*$ explicitly.

After the preliminaries in Secs. 1 and 2 explain how to generalize Albert's approach to (isotopes of) algebras with a multiplicative norm and gives criteria when the unital *twisted algebras* $(A, *)$ are division algebras. We construct examples of unital division algebras $(A, *)$ of dimension $n^2 > 4$ over F which contain a commutative subalgebra of odd degree n provided the field F has odd characteristic and permits a cyclic division algebra of odd degree n.

In Sec. 3, we obtain some first results on the automorphism groups and derivation algebras of some of the $(A, *)$. Although our main interest is to understand the twisted algebras, we will then discuss the automorphisms and derivations of some of the algebras $(A, \circ)$ in Sec. 4. Examples of real division algebras $(A, \circ)$ and $(A, *)$, most of them with large derivation algebras, are given in Corollary 21, Examples 25 and 26.

The problem with the construction probably becomes most apparent in the explicitly computed examples of Sec. 3: Although it is rather straightforward to check whether a given twisted algebra is a division algebra or not, it is not easy to check whether it may or may not be isomorphic to already known algebras due to its usually rather complicated multiplicative structure. We leave this more in depth investigation to another paper.

1. Preliminaries

Let F be a field.

1.1. *Nonassociative algebras*

By an "F-algebra" we mean a finite-dimensional nonassociative algebra over F. A nonassociative algebra $A \neq 0$ is called a *division algebra* if for any $a \in A$, $a \neq 0$, the left multiplication with a, $L_a(x) = ax$, and the right multiplication with a, $R_a(x) = xa$, are bijective. A is a division algebra if and only if A has no zero divisors.

For an F-algebra A, commutativity is measured by the *commutator* $[x, y] = xy - yx$ and associativity is measured by the *associator* $[x, y, z] = (xy)z - x(yz)$. The *left nucleus* of A is defined as $\mathrm{Nuc}_l(A) = \{x \in A \,|\, [x, A, A] = 0\}$, the *middle nucleus* of A is defined as $\mathrm{Nuc}_m(A) = \{x \in A \,|\, [A, x, A] = 0\}$ and the *right nucleus* of A is defined as $\mathrm{Nuc}_r(A) = \{x \in A \,|\, [A, A, x] = 0\}$. Their intersection $\mathrm{Nuc}(A) = \{x \in A \,|\, [x, A, A] = [A, x, A] = [A, A, x] = 0\}$ is the *nucleus* of A. The nucleus is an associative subalgebra of A and $x(yz) = (xy)z$ whenever one of the elements x, y, z is in $\mathrm{Nuc}(A)$. An anti-automorphism $\sigma : A \to A$ of period 2 is called an *involution* on A. Since A is a finite-dimensional algebra over F, the Lie algebra of the automorphism group $\mathrm{Aut}(A)$, viewed as algebraic group, is a subalgebra of the derivation algebra $\mathrm{Der}(A)$ and for $F = \mathbb{R}$, we have $\dim \mathrm{Aut}(A) = \dim \mathrm{Der}(A)$.

1.2. *Isotopes and Kaplanski's trick*

Following the notation introduced in [8, Sec. 1], denote the set of possibly non-unital algebra structures on an F-vector space V by $\mathrm{Alg}(V)$. Given $A \in \mathrm{Alg}(V)$, we write xAy for the product of $x, y \in V$ in the algebra in this subsection.

For $f, g, h \in \mathrm{Gl}(V)$ define the algebra $A^{(f,g,h)}$, called an *isotope* of A, as V together with the new multiplication

$$xA^{(f,g,h)}y = h(f(x)Ag(y)) \quad x, y \in V.$$

$A^{(f,g,h)}$ is a division algebra if A is a division algebra. Two algebras $A, A' \in \mathrm{Alg}(V)$ are called *isotopic* if $xAy = h(f(x)A'g(y))$ for all $x, y \in V$. If $f = g = h^{-1}$ then $A \cong A'$.

For $h = id$, we call $A^{(f,g)} = A^{(f,g,h)}$ a *principal Albert isotope* of A. Division algebras are principal Albert isotopes of unital division algebras [8, 1.5].

Let $L_a : A \to A, x \mapsto ax$ denote the left multiplication with a nonzero $a \in A$, and $R_a : A \to A, x \mapsto xa$ the right multiplication. Fix nonzero $a, b \in A$. The unital division algebras isotopic to a division algebra A then are, up to isomorphism, the algebras A' given by the new multiplication

$$xA'y = (R_a^{-1}x)A(L_b^{-1}y)$$

for all $x, y \in A$. A' is a division algebra with identity element aAb and is an isotope of A [7, Proposition 9]. If we choose $a = b$, this construction is called *Kaplanski's trick*. Kaplanski's trick applied to an algebra A yields a unital algebra which is an Albert isotope of A. Two unital algebras obtained from A by choosing two elements $a, a' \in A$ and using Kaplanski's trick with a and a', respectively, are isotopic to each other.

1.3. *Composition algebras*

A quadratic form $N_A \colon A \to F$ on an algebra A is *multiplicative* if $N_A(uv) = N_A(u)N_A(v)$ for all $u, v \in A$. An algebra A is called a *composition algebra* over F if it admits a multiplicative quadratic form $N_A \colon A \to F$. The form N_A is unique [6, p. 454]. It is called the *norm* of A and we will often just write $N = N_A$. A unital composition algebra is called a *Hurwitz algebra*. Hurwitz algebras are quadratic alternative and $N(1_A) = 1$; the norm of a Hurwitz algebra C is the unique nondegenerate quadratic form on A that is multiplicative. Hurwitz algebras exist only in dimensions 1, 2, 4 or 8. Those of dimension 2 are exactly the quadratic étale F-algebras, those of dimension 4 exactly the well-known quaternion algebras. The ones of dimension 8 are called *octonion algebras*. The scalar involution $\overline{x} = T_C(x)1_A - x$ of a Hurwitz algebra C is called the *standard involution* of C, where $T_A \colon A \to F$, $T_A(x) = N_A(1_A, x)$, is the *trace* of A.

Every composition algebra is a principal Albert isotope of a Hurwitz algebra: There are isometries φ_1, φ_2 of the norm N_C for a suitable Hurwitz algebra C over F such that its multiplication can be written as $x \star y = \varphi_1(x)C\varphi_2(y)$ [6].

2. How to Obtain Division Algebras out of a Given Division Algebra with Multiplicative Norm Using Albert's Approach

Unless stated otherwise, let A be a finite-dimensional nonassociative division algebra over F, not necessarily unital, which possesses a nondegenerate multiplicative norm $N : A \to F$ of degree n. Let $O(N)$ denote the group of isometries and $S(N)$ the group of similarities of N. We choose a nonzero $c \in A$ and some $h_i, h, f, g \in S(N)$ with similarity factors $\gamma_i, \gamma, \alpha, \beta \in F^\times$, $1 \leq i \leq 3$, respectively.

2.1. *The general construction*

Take the isotope $(A, \cdot) = A^{(h_1,h_2,h_3)}$ of A, then $N(x \cdot y) = \gamma_1\gamma_2\gamma_3 N(x)N(y)$ for all $x, y \in A$. If N is anisotropic then it is straightforward to see that $(A, \cdot)$ is a division algebra.

Generalizing Albert's approach, we have many options to define new multiplications $\circ$ on $(A, \cdot)$, for instance via

$$x \circ y = x \cdot y - c \cdot h(f(x) \cdot g(y)),$$

$$x \circ y = x \cdot y - h((c \cdot f(x)) \cdot g(y)),$$

$$x \circ y = x \cdot y - h(f(x) \cdot c) \cdot g(y),$$

$$x \circ y = x \cdot y - h(f(x) \cdot (c \cdot g(y))),$$

$$x \circ y = x \cdot y - h((f(x) \cdot g(y)) \cdot c),$$

$$x \circ y = x \cdot y - h(f(x) \cdot (g(y) \cdot c)), \quad \text{etc.}$$

for all $x, y \in A$. For noncommutative algebras, we can also swap around the factors $f(x)$ and $g(y)$ in the second part of the equation. Albert's proof applied to this more general setup yields the following result.

Theorem 1. *Let A be an algebra over F with an anisotropic multiplicative norm N of degree n and $(A, \cdot) = A^{(h_1,h_2,h_3)}$. If*

$$N(c) \neq \frac{1}{\alpha\beta\gamma\gamma_1\gamma_2\gamma_3},$$

then $(A, \circ)$ is a division algebra over F.

Proof. Since N is anisotropic, the isotope $(A, \cdot)$ is a division algebra. Let $x, y \in A$ be nonzero. Suppose that $x \circ y = 0$ and rearrange the resulting equation so that one side consists of the term xy. Apply the norm on both sides. Cancelling the nonzero term $N(x)N(y)$, we obtain an equation that contradicts our assumption. This argument applies to all multiplications $\circ$ of the types indicated above. For instance, if $x \circ y = x \cdot y - c \cdot h(f(x) \cdot g(y))$ then this yields $\gamma_1\gamma_2\gamma_3 N(x)N(y) = \gamma_1\gamma_2\gamma_3 N(c)\gamma\gamma_1\gamma_2\gamma_3 N(f(x))N(g(y))$, thus $N(x)N(y) = \gamma\gamma_1\gamma_2\gamma_3\alpha\beta N(c)N(x)N(y)$. $\square$

Considering isotopes of algebras with a multiplicative norm in the construction does not necessarily yield a larger class of algebras, if we only study them up to isotopy, as the following example shows.

Example 2. Let A be a unital algebra over F with an anisotropic multiplicative norm N of degree n and $(A,\cdot) = A^{(h_1,h_2,h_3)}$. For instance, consider $x \circ y = x \cdot y - c \cdot (f(x) \cdot g(y))$ then

$$x \circ y = x \cdot y - c \cdot (f(x) \cdot g(y)) = h_3(h_1(x)h_2(y)) - c \cdot (h_3(h_1(f(x))h_2(g(y))))$$
$$= h_3(h_1(x)h_2(y)) - h_3(h_1(c)h_2((h_3(h_1(f(x))h_2(g(y))))).$$

Now note that

$$h_3^{-1}(h_1^{-1}(x) \circ h_2^{-1}(y)) = xy - h_1(c)h_2(h_3(h_1(f(h_1^{-1}(x))))h_2(g(h_2^{-1}(y)))))$$
$$= xy - dh_2((h_3(h_1(f(h_1^{-1}(x)))h_2(g(h_2^{-1}(y))))),$$

therefore here $(A,\circ)$ is isotopic to an algebra $(A,\diamond)$ with multiplication $x \diamond y = xy - dH(F(x)G(y))$, where $d \in A$ and $H, F, G \in S(N)$ are suitably chosen.

2.2. *The construction starting with a unital algebra*

We will only look at the possible multiplications which occur when we apply the construction to a unital algebra A with a multiplicative norm. Let F be a field of characteristic 0 or $> d$. Then the unital division algebras with multiplicative norms are exactly the Hurwitz division algebras and the central simple associative division algebras over separable field extensions K of F [9]. All have anisotropic nondegenerate norms.

From now on let the following.

A be a unital division algebra over F with multiplicative norm N of degree n.

The multiplication of A will be denoted by juxtaposition. Let $f, g \in S(N)$ with similarity factors $\alpha, \beta \in F^\times$, respectively. Theorem 1 yields.

Corollary 3. *If $N(c) \neq \frac{1}{\alpha\beta}$, then $(A,\circ)$ is a division algebra over F.*

Theorem 4 (cf. [7, p. 85]). *Let K be a cyclic field extension of F of degree n with norm N_K and $\mathrm{Gal}(K/F) = \langle\sigma\rangle$. For $c \in K^\times$, define*

$$x \circ y = xy - c\sigma^s(x)\sigma^t(y), \quad 0 \leq s, \ t \leq n-1.$$

If s or t is prime to n and $(K,\circ)$ is a division algebra then $N_K(c) \neq 1$.

From Theorems 1 and 4, we obtain the following theorem.

Theorem 5. *Let K/F be a cyclic field extension of degree n with $\mathrm{Gal}(K/F) = \langle\sigma\rangle$, which is a subalgebra of A. Let $c \in K^\times$. Suppose that $f|_K, g|_K \in S(N_{K/F})$ and $f|_K(x) = a\sigma^s(x)$, $g|_K(x) = b\sigma^t(x)$, where $a, b \in K^\times$, $0 \leq s, t \leq n-1$, and s or t is prime to n. Then*

$$(A,\circ) \quad \textit{is a division algebra if and only if } N_{K/F}(abc) \neq 1.$$

Proof. Obviously, here $\alpha = N(a)$ and $\beta = N(b)$. By Corollary 3, $(A, \circ)$ is a division algebra if $N(c) \neq 1/N(a)N(b)$, i.e. $N_{K/F}(abc) \neq 1$.

Conversely, suppose that $(A, \circ)$ is a division algebra. Since $c \in K^\times$, $(K, \circ)$ is a subalgebra of $(A, \circ)$ with multiplication $x \circ y = xy - abc\sigma^s(x)\sigma^t(y)$. By assumption, s or t is prime to n, and $(K, \circ)$ is a division algebra, so by Theorem 4 it follows that $N_{K/F}(abc) \neq 1$. □

Corollary 6. *Let A be a quaternion or octonion division algebra over F. Let $c \in A$ such that $K = F(c)$ is a separable quadratic field extension with nontrivial automorphism σ. Suppose that $f|_K(x) = a\sigma^s(x)$, $g|_K(x) = b\sigma^t(x)$ for all $x \in K$, where $a, b \in K^\times$, and $s = 1$ or $t = 1$. Then $(A, \circ)$ is a division algebra if and only if $N_{K/F}(abc) \neq 1$.*

Corollary 7. *Let K/F be a Galois field extension of degree $n \geq 2$ with $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ and $A = (K/F, \sigma, d)$ be a cyclic division algebra over F of degree n. Let $c \in K^\times$ and $f|_K, g|_K \in S(N_{K/F})$. Let $f|_K(x) = a\sigma^s(x)$, $g|_K(x) = b\sigma^t(x)$, where $a, b \in K^\times$, and s, t be integers, $0 \leq s, t \leq n - 1$, with s or t prime to n. Then $(A, \circ)$ is a division algebra if and only if $N_{K/F}(abc) \neq 1$.*

To obtain a unital algebra $(A, *)$ from $(A, \circ)$ we apply Kaplanski's trick. Different choices of $d \in A$ used for Kaplanski's trick yield isotopic unital division algebras $(A, *_d)$. In the theory of semifields this is not relevant as division algebras over finite fields are usually classified up to isotopy. Let L_x and R_y be the left and right multiplication on $(A, \circ)$ and let $e = 1_A$ be the unit element of A. Define a multiplication using Kaplanski's trick via

$$x * y = R_e^{-1}(x) \circ L_e^{-1}(y)$$

for all $x, y \in A$. We call the unital algebra $(A, *)$ a *twisted algebra*. By construction, $(A, *)$ is isotopic to $(A, \circ)$, and is a division algebra if and only if so is $(A, \circ)$. Note that choosing the unit element of A when applying Kaplanski's trick to unital algebras helps to compute the inverses of the left and right multiplication for when we want to write down the multiplication $*$ explicitly.

We note that Menichetti gives a geometric condition for when a division algebra of dimension n is isomorphic to a twisted algebra, involving its left and right zero divisor hypersurfaces [7, Corollary 32].

Lemma 8. *Let B be a subalgebra of A with norm N_B and $0 \neq c \in B$. Assume $f|_B, g|_B \in S(N_B)$. Then $(B, *)$ is a unital subalgebra of $(A, *)$.*

Proof. Let $x, y \in B$. Since A is unital, clearly $e = 1_A \in B$ and $(B, \circ)$ is a subalgebra of $(A, \circ)$ by construction. We know that $x \circ e$, $e \circ y \in B$, and thus the restricted maps $R_e : B \to B$, $L_e : B \to B$ are isomorphisms onto B. We conclude that $x * y = R_e^{-1}(x) \circ L_e^{-1}(y) \in B$ as well. □

Corollary 9. *Let A be a quaternion or octonion division algebra over F. Choose $c \in A$ such that $K = F(c)$ is a separable quadratic field extension with automorphism σ. Suppose that $f|_K(x) = a\sigma^s(x)$, $g|_K(x) = b\sigma^t(x)$ for all $x \in K$, where $a, b \in K^\times$ and $0 \le s, t \le 1$. Then $(A, *)$ is a division algebra if and only if $N_{K/F}(abc) \neq 1$.*

In particular, $(A, *)$ contains the two-dimensional subalgebra $(K, *)$.

For $n > 2$, unital division algebras of dimension n^2 containing isotopes of commutative twisted algebras of dimension n as subalgebras can be constructed out of cyclic division algebras as follows:

Proposition 10. *Let K/F be a Galois field extension of degree $n \ge 2$ with $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ and $A = (K/F, \sigma, d)$ be a cyclic division algebra over F of degree n. Let $c \in K^\times$ and $f|_K, g|_K \in S(N_{K/F})$. Let $f|_K(x) = a\sigma^s(x)$, $g|_K(x) = b\sigma^t(x)$, where $a, b \in K$, and s, t are integers, $0 \le s, t \le n-1$, with s or t prime to n. Then $(A, *)$ is a unital division algebra if and only if $N_{K/F}(abc) \neq 1$.*

If additionally $s \neq t$, $s \neq 0$, $t \neq 0$, then $(A, *)$ has the twisted algebra $(K, *)$ as a subalgebra.

In particular, suppose n is odd, $s + t = n$, $s \neq 0$, $t \neq 0$ and $abc = -1$. Then the subalgebra $(K, *)$ of $(A, *)$ is isotopic to an n-dimensional commutative twisted algebra.

Proof. For all $c \in A$ such that $N_{K/F}(abc) \neq 1$, the twisted algebra $(A, *)$ is a division algebra by Theorem 1. Since $c \in K^\times$ and $f|_K, g|_K \in S(N_K)$, $(A, \circ)$ contains the n-dimensional subalgebra $(K, \circ)$, and $(A, *)$ contains the n-dimensional unital subalgebra $(K, *)$. Moreover, $(A, \circ)$ and thus $(A, *)$ are division algebras if and only if $N(abc) \neq 1$ (Theorem 5). If additionally $s \neq t$, $s \neq 0$, $t \neq 0$, then the subalgebra $(K, *)$ is a twisted division algebra.

If n is odd, $s + t = n$, $s \neq 0$, $t \neq 0$ and $abc = -1$, then $(K, \circ)$, $x \circ y = xy + \sigma^s(x)\sigma^{-s}(y)$, is a subalgebra of $(A, \circ)$. The isotope $(K, \star)$ given by $x \star y = x \circ \sigma^s(y) = x\sigma^s(y) + \sigma^s(x)y$ of $(K, \circ)$ is a commutative algebra, and Kaplanski's trick applied to $(K, \star)$ yields a unital n-dimensional commutative division algebra. □

More precisely, we now define $\circ_i$ on A by

(1) $x \circ_{(1)} y = xy - c(f(x)g(y))$,
(2) $x \circ_{(2)} y = xy - (cf(x))g(y)$,
(3) $x \circ_{(3)} y = xy - (f(x)c)g(y)$,
(4) $x \circ_{(4)} y = xy - f(x)(cg(y))$,
(5) $x \circ_{(5)} y = xy - (f(x)g(y))c$,
(6) $x \circ_{(6)} y = xy - f(x)(g(y)c)$

for all $x, y \in A$. For noncommutative algebras A, we can swap around the factors in the second part of the equation and define

(7) $x \circ_{(7)} y = xy - c(f(y)g(x))$,
(8) $x \circ_{(8)} y = xy - (cf(y))g(x)$,
(9) $x \circ_{(9)} y = xy - (f(y)c)g(x)$,
(10) $x \circ_{(10)} y = xy - f(y)(cg(x))$,
(11) $x \circ_{(11)} y = xy - (f(y)g(x))c$,
(12) $x \circ_{(12)} y = xy - f(y)(g(x)c)$

for all $x, y \in A$. Note that if B is a subalgebra of A, $f|_B, g|_B \in S(N_B)$ and $c \in B^\times$ then $(B, \circ)$ is a subalgebra of $(A, \circ)$.

If the twisted algebra $(A, *)$ is obtained from $(A, \circ_{(i)})$, we denote it by $(A, *_{(i)})$. We still write $(A, *) = (A, *_{(i)})$, $i = 1, \ldots, 12$, for the unital algebra obtained by applying Kaplanski's trick employing the unit element of A, if it is clear from the context which multiplication $*_{(i)}$ we use.

Note that all the possible multiplications $\circ_{(i)}$ in Proposition 10 are given by $i = 1, 3, 5, 7, 9, 11$ since A is associative here. All of them become the same multiplication on the subalgebra $(K, \circ)$.

Remark 11. The algebras $(A, \circ_{(i)})$ we construct can be isotopic or even isomorphic to A: Suppose that $f = g \in \mathrm{Aut}(A)$. Put $G(x) = x - f(x)c$. Then

$$x \circ_{(1)} y = xy - c(f(x)f(y)) = (id - cf)(xy) \quad \text{and}$$

$$x \circ_{(5)} y = xy - f(xy)c = G(xy),$$

and for all $c \in A$ such that $N(c) \neq 1$, $(A, \circ_{(1)})$ and $(A, \circ_{(5)})$ are division algebras by Theorem 1. However, $(A, \circ_{(1)})$ and $(A, \circ_{(5)})$ are isotopic to A, since the maps $F = id - cf$ and $G(x) = x - f(x)c$ are bijective if $N(c) \neq 1$. Hence if A is associative, we obtain $(A, *_{(1)}) \cong (A, *_{(5)}) \cong A$. The same argument applies when A is a quaternion algebra and $f = g = \sigma$ is the canonical involution of A. Again, $(A, *_{(1)}) \cong (A, *_{(5)}) \cong A$.

The question when two division algebras we construct are not isotopic will be dealt with in another paper.

3. Some Observations on the Automorphisms and Derivations of the Unital Division Algebras $(A, *)$

Let A be a unital division algebra with (anisotropic) multiplicative norm N, $0 \neq c \in A$ and $f, g \in S(N)$. We look at a few special cases where we can say something on the automorphism group and the derivation algebra of $(A, *)$.

3.1. *Central simple algebras*

Let A be a central simple division algebra over F of degree n. The possible choices for the multiplication $\circ$ are $\circ_{(i)}$, $i = 1, 3, 5, 7, 9, 11$.

Remark 12. The multiplications

$$x \circ_{(1)} y = xy - cf(x)y = (id - cf)(x)y,$$

$$x \circ_{(1)} y = xy - cxg(y) = x(id - cg)(y) \quad \text{for } c \in F^\times,$$

$$x \circ_{(1)} y = xy - cf(x)f(y) = (id - cf)(xy) \quad \text{if } f \in \mathrm{Aut}(A),$$

$$x \circ_{(5)} y = xy - xg(y)c = x(y - g(y)c),$$

$$x \circ_{(5)} y = xy - f(x)f(y)c = T(xy) \quad \text{if } f \in \mathrm{Aut}(A), \text{ with } T(z) = z - f(z)c,$$

$$x \circ_{(7)} y = xy - cf(y)f(x) = (id - cf)(xy) \quad \text{if } f(xy) = f(y)f(x),$$

$$x \circ_{(11)} y = xy - f(y)f(x)c$$

$$= xy - f(xy)c = S(xy) \quad \text{if } f(xy) = f(y)f(x) \text{ with } S(z) = z - f(z)c,$$

all yield isotopes of A, provided the maps $id - cf$, T, S etc. are bijective, which they are when we choose $c \in A$ suitable for $(A, \circ)$ to be division algebras. Therefore in these cases, the twisted unital algebras $(A, *)$ are isotopic to the unital associative algebra A, thus have the same nucleus, and therefore are isomorphic to A.

Suppose $f, g \in \mathrm{Aut}(A)$, $f \neq id$, $g \neq id$.
Let τ be an involution on A and $c \in F^\times$, $c \neq \pm 1$, and

$$(7.1) \quad x \circ_{(7.1)} y = xy - c\tau(y)x,$$

$$(7.2) \quad x \circ_{(7.2)} y = xy - cy\tau(x),$$

$$(7.3) \quad x \circ_{(7.3)} y = xy - c\tau(x)\tau(y).$$

Then we can explicitly compute

$$(7.1) \quad x *_{(7.1)} y = \frac{1}{(1-c)(1-c\tau(c))}(xy - c\tau(y)x + cx\tau(y) - c^2 yx),$$

$$(7.2) \quad x *_{(7.2)} y = \frac{1}{(1-c)(1-c\tau(c))}(xy - cy\tau(x) + c\tau(x)y - c^2 yx),$$

$$(7.3) \quad x *_{(7.3)} y = (1 - c\tau(c))^{-2}((1 - c\tau(c)^2)xy - c(1-c)\tau(x)\tau(y) + c(1 - \tau(c))(x\tau(y) + \tau(x)y)).$$

An easy calculation shows.

Proposition 13. *For the multiplications* $*_{(7.i)}$, $i \in \{1, 2, 3\}$, *we obtain*

(i) *If* $H \in \mathrm{Aut}(A)$ *such that* $H \circ \tau = \tau \circ H$, *then* $H \in \mathrm{Aut}(A, *_{(7.i)})$.

(ii) *If* $D \in \mathrm{Der}(A)$ *such that* $D(\tau(x)) = \tau(D(x))$ *for all* $x \in A$, *then* $D \in \mathrm{Der}(A, *_{(7.i)})$.

Corollary 14. *Suppose* A *contains a quaternion subalgebra* C *with canonical involution* σ *and assume that* $\tau|_C = \sigma$. *For the multiplications* $*_{(7.i)}$, $i \in \{1, 2, 3\}$, *we obtain*

(i) $\{H \in \mathrm{Aut}(A) \,|\, H(x) = dxd^{-1}, d \in C\} \subset \mathrm{Aut}(A, *_{(7.i)})$ *and* $\mathrm{Aut}(A, *_{(7.i)})$ *contains a subgroup isomorphic to* $SU(2)$.
(ii) $\{D \in \mathrm{Der}(A) \,|\, D(x) = ax - xa, a \in C\} \subset \mathrm{Der}(A, *_{(7.i)})$ *and* $\mathrm{Der}(A, *_{(7.i)})$ *contains a subalgebra isomorphic to* $su(2)$.

Proof. (i) We know that $H(x) = dxd^{-1}$ for some $d \in A$, therefore $H(\tau(x)) = d\tau(x)d^{-1}$, $\tau(H(x)) = \tau(dxd^{-1}) = \tau(d^{-1})\tau(x)\tau(d)$, so $H(\tau(x)) = \tau(H(x))$ if and only if $d\tau(x)d^{-1} = \tau(d^{-1})\tau(x)\tau(d)$ if and only if $\tau(x) = d^{-1}\tau(d^{-1})\tau(x)\tau(d)d$ if and only if $x\tau(d)d = \tau(d)dx$ for all $x \in A$ if and only if $\tau(d)d \in \mathrm{Comm}(A) = F$. Now $\tau(d)d = \sigma(d)d = N_C(d) \in F^\times$ for all $d \in C^\times$. The canonical embedding

$$\{H \in \mathrm{Aut}(C) \,|\, H(x) = dxd^{-1}, d \in C\}$$
$$\hookrightarrow \{H \in \mathrm{Aut}(A, *_{(7.i)}) \,|\, H(x) = dxd^{-1}\}, \quad H \mapsto H$$

implies that $SU(2) \cong \mathrm{Aut}(C)$ is isomorphic to a subgroup of $\mathrm{Aut}(A, *_{(7.i)})$.
(ii) We have $D(x) = ax - xa$ for some nonzero $a \in A$, therefore $D(\tau(x)) = \tau(D(x))$ if and only if $(a + \tau(a))\tau(x) = \tau(x)(a + \tau(a))$ for all $x \in A$, which is equivalent to $a + \tau(a) \in \mathrm{Comm}(A) = F$. Now, $a + \tau(a) = a + \sigma(a)$ for all $d \in D^\times$. The canonical embedding

$$\{D \in \mathrm{Der}(C) \,|\, D(x) = ax - xa, a \in C\}$$
$$\hookrightarrow \{D \in \mathrm{Der}(A, *_{(7.i)}) \,|\, D(x) = ax - xa\}, \quad D \mapsto D$$

implies that $su(2)$ is isomorphic to a subalgebra of $\mathrm{Der}(A, *_{(7.i)})$. □

For $c \in A$, define

$$\mathrm{Aut}_c(A) = \{f \in \mathrm{Aut}(A) \,|\, f(c) = c\} \quad \text{and} \quad \mathrm{Der}_c(A) = \{D \in \mathrm{Der}(A) \,|\, D(c) = 0\}.$$

If $c \in F^\times$ then $\mathrm{Aut}_c(A) = \mathrm{Aut}(A)$ and $\mathrm{Der}_c(A) = \mathrm{Der}(A)$.

For our next setup suppose now that $f, g \in \mathrm{Aut}(A)$ with $f^m = id$, $g^m = id$ for some $m \geq 2$ and $c \in A$ such that $1 \neq cf(c)f^2(c) \cdots f^{m-1}(c)$, $1 \neq cg(c)g^2(c) \cdots g^{m-1}(c)$ are chosen to define $\circ$. Note that if F does not contain a primitive mth root of unity, then the conditions on c are trivially satisfied, if $c \in F$. The proof of the next lemma is a simple computation, once we have computed the multiplication $*$ explicitly (which is straightforward but tedious).

Proposition 15. (i) *If* $H \in \mathrm{Aut}_c(A)$ *such that* $H \circ f = f \circ H$, $H \circ g = g \circ H$, *then* $H \in \mathrm{Aut}(A, *)$.

(ii) *If* $D \in \mathrm{Der}_c(A)$ *such that* $D(f(x)) = f(D(x))$ *and* $D(g(x)) = g(D(x))$ *for all* $x \in A$, *then* $D \in \mathrm{Der}(A, *)$.

If the multiplication $\circ$ is defined instead by using an involution τ of A and an automorphism f, such that $f^m = id$ for some $m \geq 2$ and $1 \neq cf(c)f^2(c) \cdots f^{m-1}(c)$, or using two different involutions of A, the multiplication $(A, *)$ can be explicitly computed analogously and results corresponding to Propositions 15 and 13 hold.

3.2. *Hurwitz algebras*

Let A be a quaternion or octonion division algebra over F. Recall that $h \in \mathrm{Aut}(A)$ is a *reflection* of A, if $h^2 = id$. Let $f, g \in \mathrm{Aut}(A)$ be two reflections of A.

Proposition 16. *Let* $c \in A$ *such that* $N(c) \neq 1$.

(i) *Let* $f \neq id$, $g \neq id$, $c \in F^\times$ *and* $x \circ_{(1)} y = xy - cf(x)g(y)$, *then* $(A, *_{(1)})$ *is a division algebra.*
(ii) *Let* A *be a quaternion division algebra and* $f \neq id$, $g \neq id$, $cf(c) \neq 1$, $cg(c) \neq 1$, *and* $x \circ_{(1)} y = xy - cf(x)g(y)$, $x \circ_{(3)} y = xy - f(x)cg(y)$, $x \circ_{(5)} y = xy - f(x)g(y)c$, $x \circ_{(7)} y = xy - cg(y)f(x)$, $x \circ_{(9)} y = xy - g(y)cf(x)$, *or* $x \circ_{(11)} y = xy - g(y)f(x)c$. *then* $(A, *_i)$ *is a division algebra for* $i \in \{1, 3, 5, 7, 9, 11\}$.
(iii) *Let* A *be a quaternion division algebra,* $f = id$, $c \in A \setminus F$ *such that* $cg(c) \neq 1$, *and* $x \circ_{(1)} y = xy - cxg(y)$. *Then* $(A, *_{(1)})$ *is a division algebra.*

This is clear from our main result. Moreover, in all the above cases, we can compute the multiplication $*$ explicitly, sometimes under additional assumptions on c. Put $u = (1 - cf(c))^{-1}$, $v = (1 - f(c)c)^{-1}$, $w = (1 - cg(c))^{-1}$, $s = (1 - g(c)c)^{-1} \in A$.

In (i), if $c^2 \neq 1$, then

$$x *_{(1)} y = (1 - c^2)^{-2}(1 - c^3)xy + (1 + c)^{-2}c(1 - c)^{-1}(xg(y) + f(x)y + f(x)g(y)).$$

In (ii), if $0 \neq c \in A$, $cf(c) \neq 1 \neq cg(c)$, then

$$x *_{(1)} y = u(x + cf(x))w(y + cg(y)) - c(u(f(x) + f(c)x))s(g(y) + g(c)y)),$$

$$x *_{(3)} y = v(x + f(x)c)w(y + cg(y)) - u(f(x) + xf(c))cs(g(y) + g(c)y)),$$

$$x *_{(5)} y = v(x + f(x)c)s(x + g(x)c) - u(f(x) + xf(c)))w(g(x) + xg(c)))c,$$

$$x *_{(7)} y = u(x + cf(x))w(x + cg(x)) - c(s(g(x) + g(c)x))v(f(x) + f(c)x),$$

$$x *_{(9)} y = u(x + cf(x))s(x + g(x)c) - w(g(x) + xg(c))cv(f(x) + f(c)x),$$

$$x *_{(11)} y = u(x + cf(x))s(x + g(x)c) - (w(g(x) + xf(c))v(f(x) + f(c)x)c.$$

In (iii), if $c \in A \setminus F$ $c \neq 1$, $cg(c) \neq 1$, $f = id$, then

$$x *_{(1)} y = (1 - c)^{-1}xw(y + cg(y)) - c(1 - c)^{-1}xs(g(y) + g(c)y).$$

Note that if $f = g$ in Proposition 16(i), (ii), then the unital algebra $(A, *)$ is clearly an isotope of A. In (i), the choice of $f = id$ or $g = id$ also yields unital algebras $(A, *)$ isotopic to A (and thus isomorphic to A if they are four-dimensional).

We do not explicitly compute the remaining possible cases in (ii), where we choose id and a reflection in multiplications $\circ_{(i)}$ for $i \in \{1,3,5,7,9,11\}$, since they are analogous to the case given in (ii). Some of these $(A,*)$ again yield algebras isomorphic to quaternion algebras.

Suppose now that A is a quaternion algebra and $0 \neq c \in A$ such that $N(c) \neq 1$ and $cf(c) \neq 1$, $cg(c) \neq 1$. We look at some explicitly computed multiplications:

(7.1) $x \circ_{(7.1)} y = xy - cf(y)x$, so

$$x *_{(7.1)} y = (1-c)^{-1}xu(y + cf(y)) - cv(f(y) + f(c)y)(1-c)^{-1}y.$$

(7.2) $x \circ_{(7.2)} y = xy - cyg(x)$, so

$$x *_{(7.2)} y = w(x + cg(x))(1-c)^{-1}y - c(1-c)^{-1}ys(g(x) + g(c)x).$$

(7.3) $x \circ_{(7.3)} y = xy - cf(y)g(x)$, so

$$x *_{(7.3)} y = w(x + cg(x))u(y + cf(y)) - cv(f(y) + f(c)y)s(g(x) + g(c)x).$$

The proofs are straightforward but tedious calculations. The algebras $(A, *_{(7.1)})$, $(A, *_{(7.2)})$. $(A, *_{(7.3)})$ are four-dimensional unital division algebras.

Lemma 17. (a) *Suppose* $c \in F^\times$, $c^2 \neq 1$, *then any* $H \in \mathrm{Aut}(A)$ *such that* $f \circ H = H \circ f$, $g \circ H = H \circ g$ *lies in* $\mathrm{Aut}(A, *_{(1)})$.

(b) *Let A be a quaternion division algebra over F. Let* $f, g \in \mathrm{Aut}(A)$, $f \neq id$, $g \neq id$ *be two reflections and* $0 \neq c \in A$, $cf(c) \neq 1$, $cg(c) \neq 1$. *Let* $H \in \mathrm{Aut}_c(A)$. *Then*:

(i) *if* $H \circ f = f \circ H$, *then* $H \in \mathrm{Aut}(A, *_{(7.1)})$,
(ii) *if* $H \circ g = g \circ H$, *then* $H \in \mathrm{Aut}(A, *_{(7.2)})$,
(iii) *if* $H \circ f = f \circ H$ *and* $H \circ g = g \circ H$, *then* $H \in \mathrm{Aut}(A, , *_{(7.3)})$.

Corollary 18. *Let A be a quaternion algebra and* $f \neq id$, $g \neq id$ *be two reflections,* $f(x) = sxs^{-1}$, $g(x) = txt^{-1}$, *and* $0 \neq c \in A$, $cf(c) \neq 1$, $cg(c) \neq 1$. *Then*

$$\{H \in \mathrm{Aut}_c(A) \,|\, H(x) = dxd^{-1}, d \in F(s)\}$$
$$\subset \{H \in \mathrm{Aut}_c(A) \,|\, H \circ f = f \circ H\} \subset \mathrm{Aut}(A, *_{(7.1)}),$$
$$\{H \in \mathrm{Aut}_c(A) \,|\, H(x) = dxd^{-1}, d \in F(t)\}$$
$$\subset \{H \in \mathrm{Aut}_c(A) \,|\, H \circ g = g \circ H\} \subset \mathrm{Aut}(A, *_{(7.2)}),$$
$$\{H \in \mathrm{Aut}_c(A) \,|\, H(x) = dxd^{-1}, d \in F(t) \cap F(s)\}$$
$$\subset \{H \in \mathrm{Aut}_c(A) \,|\, H \circ f = f \circ H,\ H \circ g = g \circ H\} \subset \mathrm{Aut}(A, *_{(7.3)}).$$

Let now τ be an involution on A and $c \in F^\times$ such that $1 \neq c\tau(c)$. Then for $c \in F^\times$, we obtain

$$(7.1) \quad x \circ_{(7.1)} y = xy - c\tau(y)x,$$

$$x *_{(7.1)} y = \frac{1}{(1-c)(1-c\tau(c))}(xy - c\tau(y)x + cx\tau(y) - c^2yx),$$

$$(7.2) \quad x \circ_{(7.2)} y = xy - cy\tau(x),$$

$$x *_{(7.2)} y = \frac{1}{(1-c)(1-c\tau(c))}(xy - cy\tau(x) + c\tau(x)y - c^2yx),$$

$$(7.3) \quad x \circ_{(7.3)} y = xy - c\tau(x)\tau(y),$$

$$x *_{(7.3)} y = (1-c\tau(c))^{-2}((1-c\tau(c)^2)xy - c(1-c)\tau(x)\tau(y) + c(1-\tau(c))(x\tau(y) + \tau(x)y)).$$

Proposition 19. *Let A be a quaternion or octonion division algebra and $c \in F^\times$ such that $c \neq \pm 1$ and $c\tau(c) \neq 1$. Let $(A, *)$ be as in cases* (7.1), (7.2) *or* (7.3).

(i) *If $f \in \mathrm{Aut}(A)$ such that $f \circ \tau = \tau \circ f$ then $f \in \mathrm{Aut}(A, *)$.*
(ii) *If τ is the canonical involution of A then $(A, *)$ is a division algebra, and*

$$SU(2) \cong \mathrm{Aut}(A) \subset \mathrm{Aut}(A, *)$$

if A is a quaternion algebra and

$$G_2 \cong \mathrm{Aut}(A) \subset \mathrm{Aut}(A, *)$$

if A is an octonion algebra.
(iii) *If τ is the canonical involution of A and A a quaternion algebra then*

$$su(2) \cong \mathrm{Der}(A) \subset \mathrm{Der}(A, *).$$

Proof. (i) and (iii) are straightforward calculations, (ii) follows from (i) and [7, p. 85]. □

More generally, similar considerations yield the following proposition.

Proposition 20. *Let A be a quaternion or octonion division algebra over F with canonical involution σ, $c \in A$ such that $N(c) \neq 1$ and $f, g \in \{\mathrm{id}, \sigma\}$. Take any possible definition of $\circ$ using these f and g. If A has dimension* 8, *we additionally assume that $c \in F^\times$. Then*

$$\mathrm{Aut}_c(A) \subset \mathrm{Aut}(A, *).$$

In particular, for $c \in F^\times$,

$$SU(2) \cong \mathrm{Aut}(A) \subset \mathrm{Aut}(A, *)$$

if A is a quaternion algebra and

$$G_2 \cong \mathrm{Aut}(A) \subset \mathrm{Aut}(A,*)$$

if A is an octonion algebra.

Proof. This follows from the fact that the canonical involution commutes with any automorphism of A. □

We point out that in the above setting, $(A,*)$ is a division algebra if and only if $N(c) \neq 1$. Using the classification of real division algebras [3] we obtain the following corollary.

Corollary 21. *Let $F = \mathbb{R}$ and A a quaternion or octonion division algebra over $\mathbb{R}$ with canonical involution σ. Let $c \in \mathbb{R}^\times$ such that $c \neq \pm 1$. Take any possible definition of $\circ$ using $f, g \in \{\mathrm{id}, \sigma\}$. Then*

$$su(2) \cong \mathrm{Der}(A,*)$$

if A is a quaternion division algebra and

$$G_2 \cong \mathrm{Der}(A,*)$$

if A is an octonion division algebra. In particular, this is true for the multiplications (7.1), (7.2), (7.3) *with $\tau = \sigma$ from Proposition* 19.

Concerning subalgebras, since σ restricted to any subalgebra of A is the canonical involution of that subalgebra, we state the following lemma.

Lemma 22. *Let A be a quaternion or octonion division algebra over F. Take any possible definition of $\circ$ using $f, g \in \{\mathrm{id}, \sigma\}$.*

(i) *If $c \in F^\times$ then every subalgebra B of A yields a subalgebra $(B,\circ)$ of $(A,\circ)$ and $(B,*)$ of $(A,*)$.*
(ii) *If $c \in A \backslash F$ then every subalgebra B of A such that $c \in B$ yields a subalgebra $(B,\circ)$ of $(A,\circ)$ and $(B,*)$ of $(A,*)$.*
(iii) *Suppose D is a subalgebra of A. If $c \notin D$ then $(D,\circ)$ is not a subalgebra of $(A,\circ)$.*

Proof. We only show (iii), as the rest is trivial: Assume that $(D,\circ)$ is a subalgebra of $(A,\circ)$, then $x \circ y \in D$ for all $x, y \in D$, so in particular, $1 \circ 1 = 1 - c \in D$ and thus also $c \in D$, contradiction. □

4. Automorphisms and Derivations of the Non-Unital Algebras $(A,\circ)$

Let A be an algebra over F with a nondegenerate multiplicative norm N of degree n, and $0 \neq c \in A$.

Lemma 23. *Let $f, g \in O(N)$.*

(i) *Let $H \in \mathrm{Aut}_c(A)$ such that $H(f(x)) = f(H(x))$ and $H(g(x)) = g(H(x))$ for all $x \in A$. Then $H \in \mathrm{Aut}(A, \circ)$.*
(ii) *If $D \in \mathrm{Der}(A)$ such that $D(c) = 0$, $D(f(x)) = f(D(x))$ and $D(g(x)) = g(D(x))$ for all $x \in A$, then $D \in \mathrm{Der}(A, \circ)$.*

Proof. (i) Consider $x \circ_{(1)} y = xy - c(f(x)g(y))$. For $H \in \mathrm{Aut}(A)$, $H(x \circ_{(1)} y) = H(x)H(y) - H(c)(H(f(x))H(g(y)))$ and $H(x)H(y) = H(x)H(y) - c(f(H(x))$ $g(H(y)))$. So if $H(c) = c$ and $H(f(x)) = f(H(x))$ as well as $H(g(x)) = g(H(x))$ for all $x \in A$, $H \in \mathrm{Aut}(A, \circ)$. A similar argument applies for all $\circ_{(i)}$.

(ii) The proof is a simple computation. □

Proposition 24. *Let A be a central simple associative algebra over F and $f, g \in \mathrm{Aut}(A)$, i.e. $f(x) = sxs^{-1}$ and $g(x) = txt^{-1}$ where $s, t \in A^\times$.*

(i) *If $c \in A \backslash F$ then*

$$\{d_a \in \mathrm{Der}(A) \,|\, a \in F(s) \cap F(t) \cap F(c)\} \subset \{d_a \in \mathrm{Der}_c(A) \,|\, a \in F(s) \cap F(t)\}$$
$$\subset \{d_a \in \mathrm{Der}_c(A) \,|\, f, g \in \mathrm{Aut}_a(A)\} \subset \mathrm{Der}(A, \circ).$$

If $c \in F^\times$ then

$$\{d_a \in \mathrm{Der}(A) \,|\, a \in F(s) \cap F(t)\} \subset \{d_a \in \mathrm{Der}(A) \,|\, f, g \in \mathrm{Aut}_a(A)\} \subset \mathrm{Der}(A, \circ).$$

(ii) *Suppose $c \in A \backslash F$. If $s, t \in F(c)$ or $c \in F(s) \cap F(t)$ then $d_c \in \mathrm{Der}(A, \circ)$.*

Proof. (i) Every derivation of A has the form $d_a(x) = ax - xa$ for all $x \in A$. Let $f(x) = sxs^{-1}$ and $g(x) = txt^{-1}$, $s, t \in A^\times$. Then $D(f(x)) = f(D(x))$ and $D(g(x)) = g(D(x))$ implies that if $f(a) = a$, $g(a) = a$, it follows that $d_a \in \mathrm{Der}(A, \circ)$. Now, $f(a) = a$ is the same as $sa = as$, so that this holds for all $a \in F(s)$, and analogously, $g(a) = a$ is the same as $ta = at$, so that this holds for all $a \in F(t)$. If $c \in F^\times$ then $D(c) = 0$. The assertions follow.

(ii) We know that $0 \neq d_c \in \mathrm{Der}_c(A)$ by (i). Now if $s, t \in F(c)$ then $f(c) = c$ and $g(c) = c$ and so $d_c \in \mathrm{Der}(A, \circ)$. Alternatively, if $c \in F(s) \cap F(t)$ then also $f(c) = c$ and $g(c) = c$.

(iii) is now clear. □

Example 25. Let $F = \mathbb{R}$, $A = \mathbb{H}$ and $f, g \in S(N)$ with similarity factors α and β. For all $0 \neq c \in \mathbb{H}$ such that $N(c) \neq 1/\alpha\beta$, $(\mathbb{H}, \circ)$ is a division algebra. Suppose that $f, g \in \mathrm{Aut}(\mathbb{H})$ and $f(x) = sxs^{-1}$, $g(x) = txt^{-1}$ with $s, t \in \mathbb{H}^\times$. By [3, Corollary 24], then $\dim \mathrm{Der}(\mathbb{H}, \circ) = 1$ or $\mathrm{Der}(\mathbb{H}, \circ) \cong su(2)$.

If $f|_\mathbb{C}, g|_\mathbb{C} \in S(N_\mathbb{C})$ and $c \in \mathbb{C}^\times$, then $(\mathbb{H}, \circ)$ contains the two-dimensional subalgebra $(\mathbb{C}, \circ)$. Choose any $d \in \mathbb{H}$ and apply Kaplanski's trick. This yields a unital division algebra $(\mathbb{H}, *_d)$. If $d \in \mathbb{C}^\times$, $(\mathbb{C}, *_d)$ is a unital subalgebra isomorphic to $\mathbb{C}$.

For $F = \mathbb{R}$, we have $\dim \mathrm{Aut}(A) = \dim \mathrm{Der}(A)$.

Example 26. Let $F = \mathbb{R}$ and $\mathbb{O}$ be Cayley's octonion division algebra with norm N.

(i) Let $c \in \mathbb{R}^\times$ and $f, g \in \{\mathrm{id}, \sigma\}$. Then $(\mathbb{O}, \circ)$ is a division algebra if and only if $N(c) \neq 1$. Moreover, $G_2 \cong \mathrm{Aut}(\mathbb{O}) \subset \mathrm{Aut}(\mathbb{O}, \circ)$ by Proposition 20. Thus

$$G_2 \cong \mathrm{Aut}(\mathbb{O}, \circ) \quad \text{and} \quad G_2 \cong \mathrm{Der}(\mathbb{O}, \circ)$$

by [3].

(ii) Let $f, g \in S(N)$ with similarity factors α, β. For all $0 \neq c \in \mathbb{O}$ such that $N(c) \neq 1/\alpha\beta$, $(\mathbb{O}, \circ)$ is a division algebra.

If $f|_{\mathbb{H}}, g|_{\mathbb{H}} \in S(N)$ and $c \in \mathbb{H}^\times$, $(\mathbb{O}, \circ)$ contains the four-dimensional subalgebra $(\mathbb{H}, \circ)$. Then choose any $0 \neq d \in \mathbb{O}$ to apply Kaplanski's trick to $(\mathbb{O}, \circ)$. This yields a unital division algebra $(\mathbb{O}, *_d)$. If additionally $d \in \mathbb{H}$, $(\mathbb{H}, *_d)$ is a unital four-dimensional subalgebra of $(\mathbb{O}, *_d)$ and contains $\mathbb{C}$ as subalgebra if $f|_{\mathbb{C}}, g|_{\mathbb{C}} \in S(N_{\mathbb{C}})$ and $c \in \mathbb{C}$.

ORCID

S. Pumplün https://orcid.org/0000-0001-6566-4666

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