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Quantization of Lorentzian free BV theories: factorization algebra vs algebraic quantum field theory

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Abstract

We construct and compare two alternative quantizations, as a time-orderable prefactorization algebra and as an algebraic quantum field theory valued in cochain complexes, of a natural collection of free BV theories on the category of m-dimensional globally hyperbolic Lorentzian manifolds. Our comparison is realized as an explicit isomorphism of time-orderable prefactorization algebras. The key ingredients of our approach are the retarded and advanced Green's homotopies associated with free BV theories, which generalize retarded and advanced Green's operators to cochain complexes of linear differential operators.

Keywords Factorization algebras · Algebraic quantum field theories · Homological methods in gauge theory · Globally hyperbolic Lorentzian manifolds · Green hyperbolic operators

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1 Introduction and summary

Several mathematical axiomatizations of quantum field theory (QFT) on Lorentzian manifolds have been proposed in the literature, such as algebraic quantum field theories (AQFTs) [10, 11, 15] and time-orderable prefactorization algebras [7], i.e. a Lorentzian variant of prefactorization algebras [12, 13]. These two approaches are a priori quite different. For instance, while the former emphasizes the algebraic structure carried by the quantum observables on each spacetime, the latter focuses on their time-ordered products. The differences become even more striking when one tries to construct simple QFT models, such as the free Klein–Gordon field of mass $m \geq 0$: while the corresponding time-orderable prefactorization algebra is constructed out of the (-1)-shifted Poisson structure (antibracket) $\tau_{(-1)}(\varphi \otimes \varphi^{\ddagger}) := \int_M \varphi\varphi^{\ddagger} \, \mathrm{vol}_M$ only, the corresponding AQFT relies crucially also on the retarded and advanced Green's operators $G_{\pm}$ for the Klein–Gordon operator $\Box + m^2$ through the unshifted Poisson structure $\tau_{(0)}(\varphi_1 \otimes \varphi_2) := \int_M \varphi_1 (G_+ - G_-)\varphi_2 \, \mathrm{vol}_M$.

Because of these differences, it is interesting to compare time-orderable prefactorization algebras and AQFTs. This task was undertaken first in a model-based approach by [18] and then in a model-independent fashion by [7]. In [18], it is shown that the time-orderable prefactorization algebra and the AQFT of the free Klein–Gordon field encode equivalent information as a consequence of the *time-slice axiom*, i.e. the property that any spacetime embedding whose image contains a Cauchy surface of the codomain induces an isomorphism at the level of quantum observables. (The results of [18] can be adapted with minor modifications to encompass any field theoretic model that is ruled by a Green hyperbolic operator.) In [7], a model-independent comparison is developed in the form of an equivalence (actually isomorphism) between the categories of time-orderable prefactorization algebras and of AQFTs, both satisfying the time-slice axiom (and an additional technical requirement, called additivity, that is fulfilled by many examples).

Unfortunately, the results of [7, 18] do not cover the examples of linear gauge theories. On the one hand, the equation of motion of a linear gauge theory (with gauge transformations acting non-trivially) must be degenerate. In particular, the corresponding linear differential operator is not a Green hyperbolic operator, see [2] and also Definition 2.8. As a consequence, the results of [18] cannot be applied directly. On the other hand, linear gauge theories are most naturally encoded by cochain complexes in the spirit of the BV formalism, see [8, 12, 13, 16, 17, 19]. In this context, a weaker version of the time-slice axiom holds, where isomorphisms are replaced by quasi-isomorphisms, see [5] and also Examples 4.4 and 4.7. Motivated by this fact, linear gauge theories on Lorentzian manifolds are realized by means of time-orderable prefactorization algebras or AQFTs that take values in the ∞-category $\mathbf{Ch}_{\mathbb{C}}$ of cochain complexes with equivalences given by quasi-isomorphisms, see Definitions 2.11 and 2.9 and also Remark 2.10.

While we are currently not able to upgrade the model-independent comparison of [7] to the case where the target is the ∞-category $\mathbf{Ch}_{\mathbb{C}}$, with the present paper we extend the results of [18] to linear gauge (and also higher gauge) theories. The key ingredient to achieve this goal is a generalization of Green hyperbolic operators, namely the recently developed *Green hyperbolic complexes* [6]. In contrast to Green hyperbolic operators, Green hyperbolic complexes cover many important examples of linear gauge theories, see [6] and also Examples 3.6, 3.7 and 3.8. Their key feature is that they admit *retarded and advanced Green's homotopies* $\Lambda_{\pm}$, generalizing the familiar retarded and advanced Green's operators $G_{\pm}$ for Green hyperbolic operators.

The input of our construction is a *free BV theory* $(F, Q, (-,-), W)$ on an m-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, consisting of a *complex of linear differential operators* (F, Q) with a compatible (-1)*-shifted fiber metric* $(-,-)$ and a *(formally self-adjoint) Green's witness*, see Definitions 3.1, 3.3 and 3.5. Let us provide some interpretation of these data and some information about the structures that can be defined out of it. In the spirit of the BV formalism, one may think of the graded vector bundle F as encoding both gauge and ghost fields, and the respective antifields. In the same spirit, the differential Q, which is degree-wise a linear differential operator, encodes both the action of gauge transformations and the equation of motion. The compatible (-1)-shifted fiber metric $(-,-)$ is a suitable generalization of the more familiar concept of a fiber metric on a vector bundle. $(-,-)$ is closely related to the antibracket from the BV formalism in the sense that, upon integration, it defines the (-1)*-shifted Poisson structure* $\tau_{(-1)}$ on the 1-shift $\mathfrak{F}_{\mathrm{c}}(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$ of the cochain complex of compactly supported smooth sections of (F, Q), see (3.13). Finally, the role of the Green's witness W is to give rise to the Green hyperbolic operator $P := Q\,W + W\,Q$, which allows one to find particularly simple retarded and advanced Green's homotopies $\Lambda_{\pm} := W\,G_{\pm}$, where $G_{\pm}$ denote the retarded and advanced Green's operator for P. In this sense, W "witnesses" the fact that (F, Q) is a Green hyperbolic complex. Taking the difference of Λ_+ and Λ_- defines the *retarded-minus-advanced cochain map* $\Lambda := \Lambda_+ - \Lambda_-$ and taking their average defines the *Dirac homotopy* $\Lambda_D := \frac{1}{2}(\Lambda_+ + \Lambda_-)$, which generalize the familiar retarded-minus-advanced $G := G_+ - G_-$ and Dirac $G_D := \frac{1}{2}(G_+ + G_-)$ propagators. In combination with the (-1)-shifted fiber metric $(-,-)$, Λ and Λ_D define, upon integration, the *unshifted Poisson structure* $\tau_{(0)}$ and respectively the *Dirac pairing* τ_D on the cochain complex $\mathfrak{F}_{\mathrm{c}}(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$ of compactly supported smooth sections, see (3.15) and (3.17). The (-1)-shifted Poisson structure $\tau_{(-1)}$ plays a crucial role in the first step of our construction (quantization as a time-orderable prefactorization algebra), the unshifted Poisson structure $\tau_{(0)}$ in the second step (quantization as an AQFT) and the Dirac pairing τ_D in the last step (comparison).

In the first step, which is carried out in Sect. 4.1, we construct a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ out of a collection $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ of free BV theories that is natural with respect to the morphisms $f : M \to N$ in $\mathbf{Loc}_m$ (see Appendix A for the technical details). The first part of this construction relies only on the complexes of linear differential operators (F_M, Q_M) and on the compatible (-1)-shifted fiber metrics $(-,-)_M$, for all $M \in \mathbf{Loc}_m$. These data are used to define the (-1)-shifted Poisson structures $\tau_{(-1)}$, whose BV quantization provides the time-orderable prefactorization algebra $\mathcal{F}$ of interest to us. Explicitly, from $\tau_{(-1)}$ we define the *BV Laplacian* Δ_{BV} on the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$, see (4.3), and then we deform the original differential $\mathcal{Q}$ to the quantized differential $\mathcal{Q}_\hbar := \mathcal{Q} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$. Even though $\mathcal{Q}_\hbar$ is not compatible with the commutative multiplication μ of the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$, it is compatible with the *time-ordered products* constructed out of μ, see Proposition 4.2. Hence, by defining for all $M \in \mathbf{Loc}_m$ the cochain complexes $\mathcal{F}(M) := (\mathrm{Sym}(\mathfrak{F}_c(M)[1]), \mathcal{Q}_\hbar) \in \mathbf{Ch}_{\mathbb{C}}$ that consist of the graded vector space underlying $\mathrm{Sym}(\mathfrak{F}_c(M)[1])$ with the quantized differential $\mathcal{Q}_\hbar$, we obtain the time-orderable prefactorization algebra $\mathcal{F}$ with time-ordered products constructed out of the symmetric algebra multiplication μ. At this point, it is unclear whether $\mathcal{F}$ fulfills the time-slice axiom. The Green's witnesses W_M become crucial for this purpose, see Theorem 3.13 and Proposition 4.3.

In the second step, which is carried out in Sect. 4.2, we construct an AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ out of the same data. Explicitly, instead of using $\tau_{(-1)}$ to deform the differential, here one uses the unshifted Poisson structure $\tau_{(0)}$ to deform the commutative multiplication μ of the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$ to the (in general non-commutative) *Moyal–Weyl star product* $\mu_\hbar$, see (4.13). The deformed multiplication $\mu_\hbar$ is compatible with the original differential $\mathcal{Q}$ and with the pushforward of compactly supported sections along $\mathbf{Loc}_m$-morphisms. Hence, we obtain the AQFT $\mathcal{A}$ by defining for all $M \in \mathbf{Loc}_m$ the differential graded algebras $\mathcal{A}(M) := (\mathrm{Sym}(\mathfrak{F}_c(M)[1]), \mu_\hbar, \mathbb{1}) \in \mathbf{dgAlg}_{\mathbb{C}}$ that consist of (the cochain complex underlying) $\mathrm{Sym}(\mathfrak{F}_c(M)[1])$ with the Moyal–Weyl star product $\mu_\hbar$ and the unit $\mathbb{1} \in \mathrm{Sym}(\mathfrak{F}_c(M)[1])$, and extending the pushforward of compactly supported sections.

In the last step, which is carried out in Sect. 4.3, we compare the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ and the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ obtained in the previous steps. Explicitly, we construct a comparison isomorphism $T : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ in $\mathbf{tPFA}_m$ between $\mathcal{F} \in \mathbf{tPFA}_m$ and the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ associated with $\mathcal{A} \in \mathbf{AQFT}_m$, whose time-ordered products are constructed out of the Moyal–Weyl star product $\mu_\hbar$. ($\mathcal{F}_{\mathcal{A}}$ is just the evaluation on $\mathcal{A}$ of the functor $\mathbf{AQFT}_m \to \mathbf{tPFA}_m$ from [7].) The comparison isomorphism $T := \exp(\mathrm{i}\,\hbar\,\Delta_D)$ is defined as the exponential of the *Dirac Laplacian* Δ_D, which is obtained from the Dirac pairing τ_D, see Theorem 4.9. In particular, we show that T intertwines the quantized differential $\mathcal{Q}_\hbar$ with the original symmetric algebra differential $\mathcal{Q}$ and the time-ordered products constructed out of the original symmetric algebra multiplication μ with those constructed out of the quantized multiplication $\mu_\hbar$.

The outline of the rest of the paper is the following: Section 2 contains the background material needed later on. In particular, Sect. 2.1 reviews some basic aspects of the theory of cochain complexes $\mathbf{Ch}_{\mathbb{K}}$ over a field $\mathbb{K}$ of characteristic zero; Sect. 2.2 describes the extension of (anti-)symmetric pairings τ of degree $p \in \mathbb{Z}$ on a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ to suitable bi-derivations $\{\!\{-,-\}\!\}_\tau$ and, in the symmetric case, to suitable Laplacians Δ_τ on the symmetric algebra $\mathrm{Sym}\, V \in \mathbf{dgCAlg}_{\mathbb{K}}$; Sect. 2.3 recalls some relevant concepts from Lorentzian geometry and Green hyperbolic operators; Sect. 2.4 reviews the concepts of time-orderable prefactorization algebras and AQFTs valued in cochain complexes $\mathbf{Ch}_{\mathbb{C}}$, including the Einstein causality and time-slice axioms (the latter in the form of a quasi-isomorphism). Section 3 focuses on the concept of a Green's witness and on the structures that can be constructed out of it. More in detail, Sect. 3.1 recalls the concepts of a complex of linear differential operators (F, Q), of a compatible (-1)-shifted fiber metric $(-,-)$ and of a (formally self-adjoint) Green's witness W, which together form a free BV theory $(F, Q, (-,-), W)$ on $M \in \mathbf{Loc}_m$, and out of these data it constructs the (-1)-shifted Poisson structure $\tau_{(-1)}$, the unshifted Poisson structure $\tau_{(0)}$ and the Dirac pairing τ_D; Sect. 3.2 investigates the properties of the structures $\tau_{(-1)}$, $\tau_{(0)}$ and τ_D associated with a natural collection $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ of free BV theories, proving in particular classical analogs of the Einstein causality and time-slice axioms, see Theorem 3.13. The core of the paper is Sect. 4, which is devoted to the construction and comparison of two alternative quantizations of a natural collection $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ of free BV theories. The starting point of both quantization schemes is the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$, where $\mathfrak{F}_\mathrm{c}(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$ denotes the 1-shift of the cochain complex of compactly supported smooth sections of the complex of linear differential operators (F_M, Q_M). Section 4.1 quantizes $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ as a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ by deforming the original differential $\mathcal{Q}$ of $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])$ to the quantized differential $\mathcal{Q}_\hbar := \mathcal{Q} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$ by means of the BV Laplacian Δ_{BV} defined from the (-1)-shifted Poisson structure $\tau_{(-1)}$; Sect. 4.2 quantizes $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ as an AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ by deforming the original commutative multiplication μ of the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$ to the (in general non-commutative) Moyal–Weyl star product $\mu_\hbar$ by means of the bi-derivation $\{\!\{-,-\}\!\}_{(0)}$ extending the unshifted Poisson structure $\tau_{(0)}$; Sect. 4.3 concludes the paper with constructing in Theorem 4.9 an isomorphism $T := \exp(\mathrm{i}\,\hbar\,\Delta_D) : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ in $\mathbf{tPFA}_m$, where Δ_D denotes the Dirac Laplacian defined from the Dirac pairing τ_D. T intertwines the quantized differential $\mathcal{Q}_\hbar$ and original (i.e. constructed out of μ) time-ordered products of $\mathcal{F} \in \mathbf{tPFA}_m$ with the original differential $\mathcal{Q}$ and quantized (i.e. constructed out of $\mu_\hbar$) time-ordered products of the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ associated with $\mathcal{A} \in \mathbf{AQFT}_m$ according to [7]. Appendix A discusses some technical details about naturality of vector bundles, fiber metrics and differential operators, which we require to introduce the concept of natural free BV theories.

2 Preliminaries

2.1 Cochain complexes

We review some basic aspects of the theory of cochain complexes to fix our notation and conventions. More details on the well-known topics recalled here are covered by the classical literature, see e.g. [20, 22]. Let us fix a field $\mathbb{K}$ of characteristic zero. In the main part of this paper, $\mathbb{K}$ will be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Definition 2.1 A *cochain complex* $V = (V, Q_V)$ consists of a $\mathbb{Z}$-graded $\mathbb{K}$-vector space $V = (V^n)_{n\in\mathbb{Z}}$ and a differential Q_V, that is a collection $Q_V = (Q_V^n)_{n\in\mathbb{Z}}$ of degree increasing $\mathbb{K}$-linear maps $Q_V^n : V^n \to V^{n+1}$ such that $Q_V^{n+1} Q_V^n = 0$, for all $n \in \mathbb{Z}$. A *cochain map* $f : V \to W$ is a family $f = (f^n)_{n\in\mathbb{Z}}$ of $\mathbb{K}$-linear maps $f^n : V^n \to W^n$ that is compatible with the differentials, i.e. $Q_W^n\, f^n = f^{n+1}\, Q_V^n$, for all $n \in \mathbb{Z}$. We denote by $\mathbf{Ch}_\mathbb{K}$ the category whose objects are cochain complexes and whose morphisms are cochain maps.

The tensor product $V \otimes W \in \mathbf{Ch}_\mathbb{K}$ of two cochain complexes $V, W \in \mathbf{Ch}_\mathbb{K}$ consists of

$$(V \otimes W)^n := \bigoplus_{p\in\mathbb{Z}} (V^p \otimes W^{n-p}), \tag{2.1a}$$

for all $n \in \mathbb{Z}$, and of the differential $Q_\otimes$ given by the graded Leibniz rule

$$Q_\otimes(v \otimes w) := Q_V v \otimes w + (-1)^{|v|}\, v \otimes Q_W w, \tag{2.1b}$$

for all homogeneous $v \in V$ and $w \in W$, where $|-|$ denotes the degree. The monoidal unit of the tensor product is given by $\mathbb{K} \in \mathbf{Ch}_\mathbb{K}$, regarded as a cochain complex concentrated in degree zero with trivial differential. The symmetric braiding is given by the cochain maps $\gamma : V \otimes W \to W \otimes V$ in $\mathbf{Ch}_\mathbb{K}$ that are defined by the Koszul sign rule

$$\gamma(v \otimes w) := (-1)^{|v|\,|w|}\, w \otimes v, \tag{2.2}$$

for all homogeneous $v \in V$ and $w \in W$. The internal hom $[V, W] \in \mathbf{Ch}_\mathbb{K}$ is the cochain complex that consists of

$$[V, W]^n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_\mathbb{K}(V^p, W^{n+p}), \tag{2.3a}$$

for all $n \in \mathbb{Z}$, where $\mathrm{Hom}_\mathbb{K}$ denotes the vector space of linear maps, and of the differential ∂ defined by

$$\partial f := Q_W \circ f - (-1)^{|f|}\, f \circ Q_V, \tag{2.3b}$$

for all homogeneous $f \in [V, W]$.

To each cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$, one can assign its cohomology $H^\bullet(V) = (H^n(V))_{n\in\mathbb{Z}}$, that is the graded vector space defined degree-wise by $H^n(V) := \mathrm{Ker}(Q_V^n)/\mathrm{Im}(Q_V^{n-1})$, for all $n \in \mathbb{Z}$. The compatibility of cochain maps with differentials entails that cohomology extends to a functor $H^\bullet$ from $\mathbf{Ch}_{\mathbb{K}}$ to the category of graded vector spaces. A cochain map $f : V \to W$ in $\mathbf{Ch}_{\mathbb{K}}$ is called a *quasi-isomorphism* if it induces an isomorphism $H^\bullet(f) : H^\bullet(V) \to H^\bullet(W)$ in cohomology. In many circumstances, quasi-isomorphic cochain complexes should be regarded as "being the same", which can be made precise by using techniques from model category theory. It is proven in [20] that $\mathbf{Ch}_{\mathbb{K}}$ carries the structure of a closed symmetric monoidal model category, whose weak equivalences are the quasi-isomorphisms and whose fibrations are the degree-wise surjective cochain maps.

Remark 2.2 Let us briefly recall how one may interpret the cohomology of the internal hom $[V, W] \in \mathbf{Ch}_{\mathbb{K}}$ between cochain complexes $V, W \in \mathbf{Ch}_{\mathbb{K}}$ in terms of higher cochain homotopies. Given two n-cocycles $f, g \in \mathrm{Ker}(\partial^n)$ in $[V, W]$, one defines a *cochain homotopy* from f to g as an $(n-1)$-cochain $\lambda \in [V, W]^{n-1}$ such that $\partial\lambda = g - f$. Since $\partial\lambda \in \mathrm{Im}(\partial^{n-1})$ is an n-coboundary in $[V, W]$, the cohomology classes $[f] = [g] \in H^n([V, W])$ coincide if and only if a cochain homotopy from f to g exists. In particular, for $n = 0$ one recovers the ordinary concept of cochain homotopies between two cochain maps $f, g : V \to W$ in $\mathbf{Ch}_{\mathbb{K}}$. △

Let us also fix our convention for shifts of cochain complexes. For a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ and an integer $q \in \mathbb{Z}$, we define the q-shift $V[q] \in \mathbf{Ch}_{\mathbb{K}}$ of V as the cochain complex consisting of $V[q]^n := V^{q+n}$, for all $n \in \mathbb{Z}$, and of the differential $Q_{V[q]} := (-1)^q\, Q_V$. Note that $V[p][q] = V[p+q]$, for all $p, q \in \mathbb{Z}$, and that $V[0] = V$. Furthermore, recalling the definition of the tensor product (2.1), one obtains natural cochain isomorphisms $V[q] \cong \mathbb{K}[q] \otimes V$ for all $q \in \mathbb{Z}$.

2.2 Extension of pairings to symmetric algebras

In this paper, we will encounter various types of pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ of degree $p \in \mathbb{Z}$ on a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$. These pairings are either symmetric or anti-symmetric, i.e.

$$\tau \circ \gamma = s\,\tau \tag{2.4}$$

with $s = +1$ in the symmetric case and $s = -1$ in the anti-symmetric case, where γ denotes the symmetric braiding of $\mathbf{Ch}_{\mathbb{K}}$. In particular, we shall consider shifted and also unshifted (i.e. 0-shifted) (linear) Poisson structures as defined below.

Definition 2.3 A *p-shifted (linear) Poisson structure* on a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ consists of a symmetric (respectively, anti-symmetric) pairing $\tau \in [V \otimes V, \mathbb{K}]^p$ of odd (respectively, even) degree $p \in \mathbb{Z}$ that is closed $\partial\tau = 0$ with respect to the internal hom differential (2.3).

The aim of this subsection is to describe an extension of such pairings to suitable bi-derivations and, in the symmetric case, to suitable Laplacians on the symmetric algebra $\mathrm{Sym}\, V \in \mathbf{dgCAlg}_{\mathbb{K}}$. The latter is the commutative differential graded algebra defined by $\mathrm{Sym}\, V = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n\, V$, with unit element $\mathbb{1} := 1 \in \mathrm{Sym}^0\, V = \mathbb{K}$ and multiplication

$$\mu(v_1 \cdots v_n \otimes v_1' \cdots v_{n'}') := v_1 \cdots v_n\, v_1' \cdots v_{n'}', \tag{2.5}$$

for all $n, n' \geq 0$ and all $v_1, \dots, v_n, v_1', \dots, v_{n'}' \in V$. (By convention, the length $n = 0$ corresponds to the unit $\mathbb{1}$.)

Definition 2.4 Given an (anti-)symmetric pairing $\tau \in [V \otimes V, \mathbb{K}]^p$ of degree p, we define

$$\{\!\{-,-\}\!\}_\tau \in \big[\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V, \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V\big]^p \tag{2.6}$$

as the unique graded linear map of degree p that fulfills the following conditions:

(i) $\{\!\{-,-\}\!\}_\tau$ is (anti-)symmetric, i.e.

$$\gamma \circ \{\!\{-,-\}\!\}_\tau \circ \gamma = s\, \{\!\{-,-\}\!\}_\tau \tag{2.7}$$

with $s = +1$ in the symmetric case and $s = -1$ in the anti-symmetric case;

(ii) for all $v_1, v_2 \in V$, $\{\!\{v_1, v_2\}\!\}_\tau = \tau(v_1 \otimes v_2)\, \mathbb{1} \otimes \mathbb{1} \in \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$;

(iii) for all homogeneous $a \in \mathrm{Sym}\, V$, $\{\!\{a, -\}\!\}_\tau : \mathrm{Sym}\, V \to \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$ is a graded derivation of degree $|a| + p$ with respect to the $(\mathrm{Sym}\, V)$-module structure on $\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$ given by multiplication on the second tensor factor, i.e.

$$\{\!\{a, bc\}\!\}_\tau = \{\!\{a, b\}\!\}_\tau\, (\mathbb{1} \otimes c) + (-1)^{(|a|+p)|b|}\, (\mathbb{1} \otimes b)\, \{\!\{a, c\}\!\}_\tau, \tag{2.8}$$

for all homogeneous $b, c \in \mathrm{Sym}\, V$.

An immediate consequence of the previous definition is that

$$\partial \{\!\{-,-\}\!\}_\tau = \{\!\{-,-\}\!\}_{\partial\tau} \quad . \tag{2.9}$$

Furthermore, given two cochain complexes $V, W \in \mathbf{Ch}_{\mathbb{K}}$ endowed with (anti-)symmetric pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ and $\omega \in [W \otimes W, \mathbb{K}]^p$ of degree p and a cochain map $f : V \to W$ in $\mathbf{Ch}_{\mathbb{K}}$ preserving them, i.e. $\tau = \omega \circ (f \otimes f)$, one has

$$(\mathrm{Sym}\, f \otimes \mathrm{Sym}\, f) \circ \{\!\{-,-\}\!\}_\tau = \{\!\{-,-\}\!\}_\omega \circ (\mathrm{Sym}\, f \otimes \mathrm{Sym}\, f) \quad . \tag{2.10}$$

Remark 2.5 A p-shifted (linear) Poisson structure τ on V can be extended to a p-shifted Poisson bracket $\{-,-\}_\tau$ on $\mathrm{Sym}\, V$. Indeed, from Definitions 2.3 and 2.4 it follows that

$$\{-,-\}_\tau := \mu \circ \{\!\{-,-\}\!\}_\tau \in \big[\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V, \mathrm{Sym}\, V\big]^p \tag{2.11}$$

defines a p-shifted Poisson bracket, i.e. a graded linear map of degree p that is closed $\partial\{-,-\}_\tau = 0$, symmetric (respectively, anti-symmetric) for p odd (respectively, even) and fulfills the graded Leibniz rule and the Jacobi identity. △

Definition 2.6 Given a symmetric pairing $\tau \in [V \otimes V, \mathbb{K}]^p$ of degree p, we define the *Laplacian*

$$\Delta_\tau \in \big[\mathrm{Sym}\, V, \mathrm{Sym}\, V\big]^p \tag{2.12}$$

as the unique graded linear map of degree p that fulfills the following conditions:

(i) $\Delta_\tau(\mathbb{1}) = 0$;
(ii) for all $v \in V$, $\Delta_\tau(v) = 0$;
(iii) for all $v_1, v_2 \in V$, $\Delta_\tau(v_1\, v_2) = \tau(v_1 \otimes v_2)\, \mathbb{1}$;
(iv) for all homogeneous $a, b \in \mathrm{Sym}\, V$,

$$\Delta_\tau(a\, b) = \Delta_\tau(a)\, b + (-1)^{p|a|}\, a\, \Delta_\tau(b) + \mu(\{\!\{a, b\}\!\}_\tau) \quad . \tag{2.13}$$

The defining properties of Δ_τ imply the explicit formula

$$\Delta_\tau(v_1 \cdots v_n) = \sum_{i<j} (-1)^{p \sum_{k=1}^{i-1} |v_k| + |v_j| \sum_{k=i+1}^{j-1} |v_k|}\, \tau(v_i \otimes v_j)\, v_1 \cdots \check{v}_i \cdots \check{v}_j \cdots v_n, \tag{2.14}$$

for all $n \geq 1$ and all homogeneous $v_1, \dots, v_n \in V$, where $\check{\cdot}$ means to omit the corresponding factor. Furthermore, for p even, iterating (2.13) and observing that both $\Delta_\tau \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta_\tau$ graded commute with $\{\!\{-, -\}\!\}_\tau$, one finds that, for all $n \geq 1$,

$$\Delta_\tau^n \circ \mu = \mu \circ \big(\Delta_{\tau\otimes} + \{\!\{-, -\}\!\}_\tau\big)^n = \sum_{k=0}^{n} \binom{n}{k}\, \mu \circ \{\!\{-, -\}\!\}_\tau^{n-k} \circ \Delta_{\tau\otimes}^k, \tag{2.15}$$

where $\Delta_{\tau\otimes} := \Delta_\tau \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_\tau$. (For $n = 1$ this recovers (2.13). For p odd, the left-hand side vanishes identically for $n \geq 2$, see (2.17) below.) Taking also (2.9) into account, one shows that

$$\partial \Delta_\tau = \Delta_{\partial\tau} \quad . \tag{2.16}$$

Given two symmetric pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ and $\tau' \in [V \otimes V, \mathbb{K}]^{p'}$ of degrees p and p', respectively, the explicit formula (2.14) for the Laplacian entails

$$\Delta_\tau \circ \Delta_{\tau'} = (-1)^{pp'}\, \Delta_{\tau'} \circ \Delta_\tau \quad . \tag{2.17}$$

Furthermore, given two cochain complexes $V, W \in \mathbf{Ch}_\mathbb{K}$ endowed with symmetric pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ and $\omega \in [W \otimes W, \mathbb{K}]^p$ of degree p and a cochain map $f : V \to W$ in $\mathbf{Ch}_\mathbb{K}$ preserving them, i.e. $\tau = \omega \circ (f \otimes f)$, it follows from (2.14) that

$$\mathrm{Sym}\, f \circ \Delta_\tau = \Delta_\omega \circ \mathrm{Sym}\, f \quad . \tag{2.18}$$

2.3 Lorentzian geometry and Green's operators

In this subsection, we recall some relevant concepts from Lorentzian geometry and Green hyperbolic differential operators. We refer to [2, 4, 21] for an in-depth introduction to these topics.

A *Lorentzian manifold* (M, g) is a smooth manifold M endowed with a metric g of signature $(-, +, \dots, +)$. Given a nonzero tangent vector $0 \neq v \in T_x M$ at a point $x \in M$, we say that v is *spacelike* if $g(v, v) > 0$, *lightlike* if $g(v, v) = 0$ and *timelike* if $g(v, v) < 0$. v is also called *causal* if $g(v, v) \leq 0$, that is v is either timelike or lightlike. Let $I \subseteq \mathbb{R}$ be an open interval. A curve $c : I \to M$ is called *spacelike* (*lightlike*, *timelike* or *causal*) if its tangent vectors $\dot{c}(t)$ are spacelike (lightlike, timelike or causal, respectively), for all $t \in I$. A Lorentzian manifold M is called *time-orientable* if there exists an everywhere timelike vector field $\mathfrak{t} \in \Gamma(TM)$. Such $\mathfrak{t}$ determines a *time-orientation* on M. We will denote *oriented and time-oriented Lorentzian manifolds* by $M = (M, g, \mathfrak{o}, \mathfrak{t})$, where $\mathfrak{o}$ is the chosen orientation. A timelike or causal curve $c : I \to M$ is said to be *future directed* if $g(\mathfrak{t}, \dot{c}) < 0$ and *past directed* if $g(\mathfrak{t}, \dot{c}) > 0$. The *chronological future/past* $I_M^{\pm}(S) \subseteq M$ of a subset $S \subseteq M$ consists of all points that can be reached by a future/past directed timelike curve stemming from S. Similarly, the *causal future/past* $J_M^{\pm}(S) \subseteq M$ consists of S itself and of all points that can be reached by a future/past directed causal curve stemming from S. By definition, $I_M^{\pm}(S) \subseteq J_M^{\pm}(S)$; moreover, recall from e.g. [21, Chapter 14] that

$$I_M^{\pm}(J_M^{\pm}(S)) = I_M^{\pm}(S) = J_M^{\pm}(I_M^{\pm}(S)) \subseteq M \tag{2.19}$$

is always an open subset. A subset $S \subseteq M$ is called *causally convex* if $J_M^{+}(S) \cap J_M^{-}(S) \subseteq S$, i.e. when all causal curves with endpoints in S lie in S. An example of a causally convex subset is the *causally convex hull*

$$J_M^{+\cap-}(S) := J_M^{+}(S) \cap J_M^{-}(S) \subseteq M \tag{2.20}$$

of a subset $S \subseteq M$, i.e. the smallest causally convex subset of M that contains S.

Definition 2.7 An oriented and time-oriented Lorentzian manifold M is called *globally hyperbolic* if it admits a *Cauchy surface* $\Sigma \subset M$, i.e. a subset that is met exactly once by any inextendible future directed timelike curve in M. $\mathbf{Loc}_m$ denotes the category whose objects are all m-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifolds M and whose morphisms are all orientation and time-orientation-preserving isometric embeddings $f : M \to M'$ with open and causally convex image $f(M) \subseteq M'$.

For $M \in \mathbf{Loc}_m$ and $O \subseteq M$ open, one has that the causal future/past

$$J_M^{\pm}(O) = I_M^{\pm}(O) \tag{2.21}$$

coincides with the chronological one. (Indeed, any $p \in J_M^{\pm}(O)$ lies along a future/past directed causal curve emanating from some $q \in O$. Since O is open, q can be reached via a future/past directed timelike curve emanating from some $r \in O$. But then $p \in J^{\pm}_M(q) \subseteq J^{\pm}_M(I^{\pm}_M(r)) = I^{\pm}_M(r) \subseteq I^{\pm}_M(O)$.) In particular, when $O \subseteq M$ is open, the causal future/past $J^{\pm}_M(O) \subseteq M$ and the causally convex hull $J^{+\cap -}_M(O) \subseteq M$ are open subsets.

Consider an oriented and time-oriented globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ of dimension $m \geq 2$. Let $E \to M$ be a real or complex vector bundle of finite rank. Denote the vector space of smooth sections of E by $\Gamma(E)$ and the vector subspace of compactly supported sections by $\Gamma_{\mathrm{c}}(E) \subseteq \Gamma(E)$.

Definition 2.8 A *Green hyperbolic operator* is a linear differential operator $P : \Gamma(E) \to \Gamma(E)$ that admits *retarded and advanced Green's operators* $G_{\pm}$, which are linear maps $G_{\pm} : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ such that, for all $\varphi \in \Gamma_{\mathrm{c}}(E)$, the following conditions hold:

(i) $PG_{\pm}\varphi = \varphi$;
(ii) $G_{\pm}P\varphi = \varphi$;
(iii) $\mathrm{supp}(G_{\pm}\varphi) \subseteq J^{\pm}_M(\mathrm{supp}(\varphi))$.

The difference $G := G_+ - G_- : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ between the retarded and advanced Green's operators is called the *retarded-minus-advanced propagator* and their average $G_D := \frac{1}{2}(G_+ + G_-) : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ is called the *Dirac propagator*.

In [2], it is shown that the retarded and advanced Green's operators associated with a Green hyperbolic operator are unique.

Given a real vector bundle $E \to M$ endowed with a *fiber metric* $\langle -, - \rangle$, i.e. a fiber-wise non-degenerate, symmetric, bilinear form, and denoting the volume form on M by vol_M, one defines the integration pairing

$$\langle\langle \varphi, \varphi' \rangle\rangle := \int_M \langle \varphi, \varphi' \rangle \, \mathrm{vol}_M, \tag{2.22}$$

for all sections $\varphi, \varphi' \in \Gamma(E)$ with compact overlapping support, i.e. such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\varphi') \subseteq M$ is compact. Given two vector bundles $(E_1, \langle -, - \rangle_1)$, $(E_2, \langle -, - \rangle_2)$ endowed with fiber metrics and a linear differential operator $Q : \Gamma(E_1) \to \Gamma(E_2)$, one defines its *formal adjoint* $Q^* : \Gamma(E_2) \to \Gamma(E_1)$ as the unique linear differential operator such that

$$\langle\langle Q^* \varphi_2, \varphi_1 \rangle\rangle_1 := \langle\langle \varphi_2, Q\varphi_1 \rangle\rangle_2, \tag{2.23}$$

for all sections $\varphi_1 \in \Gamma(E_1)$, $\varphi_2 \in \Gamma(E_2)$ with compact overlapping support. A linear differential operator $P : \Gamma(E) \to \Gamma(E)$ on $(E, \langle -, - \rangle)$ is *formally self-adjoint* if $P^* = P$. When $P : \Gamma(E) \to \Gamma(E)$ is a formally self-adjoint Green hyperbolic operator, the associated retarded and advanced Green's operators $G_{\pm}$ are "formal adjoints" of each other, i.e.

$$\langle\langle G_{\pm}\varphi, \varphi' \rangle\rangle = \langle\langle \varphi, G_{\mp}\varphi' \rangle\rangle, \tag{2.24}$$

for all compactly supported sections $\varphi, \varphi' \in \Gamma_c(E)$. This entails that the retarded-minus-advanced propagator G is “formally skew-adjoint”, i.e.

$$\langle\!\langle G\varphi, \varphi' \rangle\!\rangle = -\langle\!\langle \varphi, G\varphi' \rangle\!\rangle, \tag{2.25}$$

for all compactly supported sections $\varphi, \varphi' \in \Gamma_c(E)$.

2.4 Algebraic QFTs and time-orderable prefactorization algebras

Algebraic quantum field theories (AQFTs) [10, 11, 15] and factorization algebras [7, 12, 13] provide two axiomatic frameworks to describe the algebraic structures on the observables of a quantum field theory in various geometric settings. In this subsection, we review some basic concepts from these two frameworks in the Lorentzian setting.

We say that two $\mathbf{Loc}_m$-morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ to a common target are *causally disjoint* if there exists no causal curve in N connecting their images, i.e. $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$, where $J_M(S) := J_M^+(S) \cup J_M^-(S)$ denotes the union of the causal future and past of a subset $S \subseteq M$. Furthermore, a morphism $f : M \to N$ in $\mathbf{Loc}_m$ is *Cauchy* if its image $f(M) \subseteq N$ contains a Cauchy surface of N.

Definition 2.9 A $\mathbf{Ch}_{\mathbb{C}}$-valued *algebraic quantum field theory (AQFT)* $\mathcal{A}$ on $\mathbf{Loc}_m$ is a functor $\mathcal{A} : \mathbf{Loc}_m \to \mathbf{dgAlg}_{\mathbb{C}}$ taking values in the category $\mathbf{dgAlg}_{\mathbb{C}}$ of differential graded algebras that satisfies the following axioms:

(i) *Einstein causality:* For all causally disjoint morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd}
\mathcal{A}(M_1) \otimes \mathcal{A}(M_2) \arrow[r, "\mathcal{A}(f_1)\otimes\mathcal{A}(f_2)"] \arrow[d, "\mathcal{A}(f_1)\otimes\mathcal{A}(f_2)"'] & \mathcal{A}(N) \otimes \mathcal{A}(N) \arrow[d, "\mu_N^{\mathrm{op}}"] \\
\mathcal{A}(N) \otimes \mathcal{A}(N) \arrow[r, "\mu_N"'] & \mathcal{A}(N)
\end{tikzcd} \tag{2.26}$$

in $\mathbf{Ch}_{\mathbb{C}}$ commutes, where μ_N and $\mu_N^{\mathrm{op}} := \mu_N \circ \gamma$ are the multiplication and the opposite multiplication of $\mathcal{A}(N) \in \mathbf{dgAlg}_{\mathbb{C}}$;

(ii) *Time-slice:* For all Cauchy morphisms $f : M \to N$ in $\mathbf{Loc}_m$, the morphism $\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ in $\mathbf{dgAlg}_{\mathbb{C}}$ is a quasi-isomorphism.

A morphism $\kappa : \mathcal{A} \to \mathcal{B}$ between AQFTs is a natural transformation. This defines the category $\mathbf{AQFT}_m$ of $\mathbf{Ch}_{\mathbb{C}}$-valued AQFTs as the full subcategory $\mathbf{AQFT}_m \subseteq \mathbf{dgAlg}_{\mathbb{C}}^{\mathbf{Loc}_m}$ of the functor category consisting of all functors that satisfy the Einstein causality and time-slice axioms.

Remark 2.10 There exists a more elegant and powerful operadic description [10] of the category $\mathbf{AQFT}_m$. This more abstract perspective is particularly useful to endow $\mathbf{AQFT}_m$ with a model category structure [9], which provides a solid foundation for the study of $\mathbf{Ch}_{\mathbb{C}}$-valued AQFTs. To prove the results of our present paper, we do not have to make explicit use of these techniques. △

Our goal is to construct and compare AQFTs and prefactorization algebras in the Lorentzian setting. For this purpose, we recall below a Lorentzian version of the prefactorization algebras from [12], called *time-orderable* prefactorization algebras [7]. This requires some preliminaries. A tuple of $\mathbf{Loc}_m$-morphisms $(f_1 : M_1 \to N, \dots, f_n : M_n \to N)$, also denoted $\underline{f} : \underline{M} \to N$, to a common target is called *time-ordered* if $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$, for all $i < j$. Given a tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ of length n, a *time-ordering permutation* $\rho \in \Sigma_n$ is a permutation such that the ρ-permuted tuple $\underline{f}\rho := (f_{\rho(1)}, \dots, f_{\rho(n)}) : \underline{M}\rho \to N$ of $\mathbf{Loc}_m$-morphisms is time-ordered. When a time-ordering permutation exists, one says that $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ is *time-orderable*. (Note that the time-ordering permutation for a tuple may not be unique. For instance, two morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ in $\mathbf{Loc}_m$ are causally disjoint precisely when both (f_1, f_2) and (f_2, f_1) are time-ordered pairs.) A time-orderable 1-tuple $(f) : \underline{M} \to N$ in $\mathbf{Loc}_m$ is denoted simply as a morphism $f : M \to N$ in $\mathbf{Loc}_m$ and, for each $N \in \mathbf{Loc}_m$, we define a unique time-orderable empty tuple $\emptyset \to N$. Time-orderable tuples are composable and carry permutation group actions, see [7]. These facts are crucial for the next definition.

Definition 2.11 A $\mathbf{Ch}_{\mathbb{C}}$-valued *time-orderable prefactorization algebra* $\mathcal{F}$ on $\mathbf{Loc}_m$ consists of the data listed below:

(a) For each $M \in \mathbf{Loc}_m$, a cochain complex $\mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}}$.
(b) For each time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, a morphism $\mathcal{F}(\underline{f}) : \mathcal{F}(\underline{M}) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$, called *time-ordered product*, where $\mathcal{F}(\underline{M}) := \bigotimes_{i=1}^n \mathcal{F}(M_i) \in \mathbf{Ch}_{\mathbb{C}}$ denotes the tensor product. By convention, the time-ordered product assigned to an empty tuple $\emptyset \to N$ is a morphism $\mathbb{C} \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ from the monoidal unit.

These data are subject to the following axioms:

(i) For all time-orderable tuples $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and $\underline{g}_i = (g_{i1}, \dots, g_{ik_i}) : \underline{L}_i \to M_i$, $i = 1, \dots, n$, in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd} \bigotimes_{i=1}^{n} \mathcal{F}(\underline{L}_i) \arrow[r, "\bigotimes_i \mathcal{F}(\underline{g}_i)"] \arrow[dr, "\mathcal{F}(\underline{f}(\underline{g}_1, \dots, \underline{g}_n))"'] & \mathcal{F}(\underline{M}) \arrow[d, "\mathcal{F}(\underline{f})"] \\ & \mathcal{F}(N) \end{tikzcd} \tag{2.27}$$

in $\mathbf{Ch}_{\mathbb{C}}$ commutes, where $\underline{f}(\underline{g}_1, \dots, \underline{g}_n) := (f_1 g_{11}, \dots, f_n g_{nk_n}) : (\underline{L}_1, \dots, \underline{L}_n) \to N$ is the time-orderable tuple given by composition in $\mathbf{Loc}_m$.
(ii) For all $M \in \mathbf{Loc}_m$, $\mathcal{F}(\mathrm{id}_M) = \mathrm{id}_{\mathcal{F}(M)} : \mathcal{F}(M) \to \mathcal{F}(M)$ in $\mathbf{Ch}_{\mathbb{C}}$ is the identity.
(iii) For all time-orderable tuples $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ and permutations $\sigma \in \Sigma_n$, the diagram

$$\begin{tikzcd} \mathcal{F}(\underline{M}) \arrow[r, "\mathcal{F}(\underline{f})"] \arrow[d, "\gamma_\sigma"'] & \mathcal{F}(N) \\ \mathcal{F}(\underline{M}\sigma) \arrow[ur, "\mathcal{F}(\underline{f}\sigma)"'] & \end{tikzcd} \tag{2.28}$$

in $\mathbf{Ch}_{\mathbb{C}}$ commutes, where γ_σ is defined by the symmetric braiding γ of $\mathbf{Ch}_{\mathbb{C}}$.

We say that a time-orderable prefactorization algebra $\mathcal{F}$ satisfies the *time-slice axiom* if, for all Cauchy morphism $f : M \to N$ in $\mathbf{Loc}_m$, $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ is a quasi-isomorphism.

A morphism $\zeta = (\zeta_M)_{M \in \mathbf{Loc}_m} : \mathcal{F} \to \mathcal{G}$ of time-orderable prefactorization algebras is a collection of cochain maps $\zeta_M : \mathcal{F}(M) \to \mathcal{G}(M)$ in $\mathbf{Ch}_{\mathbb{C}}$, indexed by objects $M \in \mathbf{Loc}_m$, that is compatible with the time-ordered products in the sense that, for all time-orderable tuples $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd}
\mathcal{F}(\underline{M}) \arrow[r, "\mathcal{F}(\underline{f})"] \arrow[d, "\zeta_{\underline{M}}"'] & \mathcal{F}(N) \arrow[d, "\zeta_N"] \\
\mathcal{G}(\underline{M}) \arrow[r, "\mathcal{G}(\underline{f})"'] & \mathcal{G}(N)
\end{tikzcd} \tag{2.29}$$

in $\mathbf{Ch}_{\mathbb{C}}$ commutes, where $\zeta_{\underline{M}} := \bigotimes_i \zeta_{M_i}$. We denote the category of time-orderable prefactorization algebras on $\mathbf{Loc}_m$ satisfying the time-slice axiom by $\mathbf{tPFA}_m$.

3 Green's witnesses

In this section, we briefly recall the concept of a *Green's witness* for a complex of linear differential operators, see [6] for more details. This consists of a collection of degree decreasing linear differential operators that enable the explicit construction of *retarded and advanced Green's homotopies*. The latter are differential graded analogs of the usual retarded and advanced Green's operators, see e.g. [2, 4] and also Sect. 2.3, and they will play a key role in our construction of AQFTs and their comparison to time-orderable prefactorization algebras in Sect. 4. Given a Green's witness, we shall endow the underlying complex of linear differential operators with the following three structures: 1.) a (-1)-shifted Poisson structure $\tau_{(-1)}$, 2.) an unshifted Poisson structure $\tau_{(0)}$ and 3.) a symmetric pairing τ_D, that we call Dirac pairing, trivializing the (-1)-shifted Poisson structure, i.e. $\tau_{(-1)} = \partial \tau_D$. We shall show that $\tau_{(-1)}$, $\tau_{(0)}$ and τ_D are natural when all input data are natural (with respect to the category $\mathbf{Loc}_m$ of m-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifolds). In particular, we shall construct a functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau_{(0)}) : \mathbf{Loc}_m \to \mathbf{PoCh}_{\mathbb{R}}$ that assigns to each $M \in \mathbf{Loc}_m$ a Poisson cochain complex $(\mathfrak{F}_{\mathrm{c}}(M)[1], \tau_{(0)}^M)$ whose cochains may be interpreted field-theoretically as linear observables. (Here $\mathbf{PoCh}_{\mathbb{R}}$ denotes the category whose objects are Poisson cochain complexes (V, τ), consisting of a cochain complex $V \in \mathbf{Ch}_{\mathbb{R}}$ endowed with an unshifted linear Poisson structure τ, see Definition 2.3, and whose morphisms $f : (V, \tau) \to (W, \omega)$ are cochain maps $f : V \to W$ in $\mathbf{Ch}_{\mathbb{R}}$ preserving the Poisson structures, i.e. $\omega \circ (f \otimes f) = \tau$.) In Theorem 3.13, we shall prove that the functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau_{(0)})$ satisfies the classical analogs of the Einstein causality and time-slice axioms.

3.1 $\mathcal{T}_{(-1)}$, $\mathcal{T}_{(0)}$ and $\mathcal{T}_D$ over a fixed globally hyperbolic Lorentzian manifold

Given a ($\mathbb{Z}$-)graded ($\mathbb{R}$-)vector bundle $F \to M$ (degree-wise of finite rank) over an oriented and time-oriented globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, we denote by

$$\mathfrak{F}(M)^n := \Gamma(F^n) \tag{3.1}$$

the vector space of degree n smooth sections, i.e. the smooth sections of the degree n vector bundle $F^n \to M$, and by

$$\mathfrak{F}_c(M)^n := \Gamma_c(F^n) \tag{3.2}$$

the vector space of degree n smooth sections with compact support.

Definition 3.1 A *complex of linear differential operators* (F, Q) over $M \in \mathbf{Loc}_m$ consists of a graded vector bundle $F \to M$ and of a collection $Q = (Q^n : \mathfrak{F}(M)^n \to \mathfrak{F}(M)^{n+1})_{n\in\mathbb{Z}}$ of degree increasing linear differential operators such that $Q^{n+1} Q^n = 0$, for all $n \in \mathbb{Z}$. We denote by $\mathfrak{F}(M) \in \mathbf{Ch}_{\mathbb{R}}$ the cochain complex of sections associated with the complex of linear differential operators (F, Q).

A *compatible* (-1)-*shifted fiber metric* $(-,-)$ on (F, Q) is a fiber-wise non-degenerate, graded anti-symmetric, graded vector bundle map $(-,-) : F \otimes F \to M \times \mathbb{R}[-1]$ such that the identity

$$\int_M (Q\varphi_1, \varphi_2)\, \mathrm{vol}_M + (-1)^{|\varphi_1|} \int_M (\varphi_1, Q\varphi_2)\, \mathrm{vol}_M = 0 \tag{3.3}$$

holds for all homogeneous sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support.

Remark 3.2 The compatibility condition (3.3) implies that the integration pairing

$$(\!(-,-)\!) : \mathfrak{F}_c(M) \otimes \mathfrak{F}(M) \longrightarrow \mathbb{R}[-1], \tag{3.4a}$$

defined by

$$(\!(\psi, \varphi)\!) := \int_M (\psi, \varphi)\, \mathrm{vol}_M, \tag{3.4b}$$

for all $\psi \in \mathfrak{F}_c(M)$ and $\varphi \in \mathfrak{F}(M)$, is a cochain map. △

Definition 3.3 A *(formally self-adjoint) Green's witness* $W = (W^n)_{n\in\mathbb{Z}}$ for a complex of linear differential operators (F, Q) endowed with a compatible (-1)-shifted fiber metric $(-,-)$ consists of a collection of degree decreasing linear differential operators $W^n : \mathfrak{F}(M)^n \to \mathfrak{F}(M)^{n-1}$ such that the following conditions hold:

(i) For all $n \in \mathbb{Z}$, $P^n := Q^{n-1} W^n + W^{n+1} Q^n : \mathfrak{F}(M)^n \to \mathfrak{F}(M)^n$ are Green hyperbolic operators.

(ii) $Q\,W\,W = W\,W\,Q$.
(iii) $\int_M (W\varphi_1, \varphi_2)\,\mathrm{vol}_M = (-1)^{|\varphi_1|} \int_M (\varphi_1, W\varphi_2)\,\mathrm{vol}_M$, for all homogeneous sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support.

Remark 3.4 Some direct consequences of Definition 3.3 are listed below:

(1) For all $n \in \mathbb{Z}$, there exist unique retarded and advanced Green's operators $G^n_\pm : \mathfrak{F}_\mathrm{c}(M)^n \to \mathfrak{F}(M)^n$ associated with the Green hyperbolic operators P^n;
(2) It follows that $P\,W = W\,P$ and $P\,Q = Q\,P$, hence also $G_\pm\,W = W\,G_\pm$ and $G_\pm\,Q = Q\,G_\pm$;
(3) P is formally self-adjoint, i.e. $\int_M (P\varphi_1, \varphi_2)\,\mathrm{vol}_M = \int_M (\varphi_1, P\varphi_2)\,\mathrm{vol}_M$, for all sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support. It follows that $\int_M (\psi_1, G_\pm\psi_2)\,\mathrm{vol}_M = \int_M (G_\mp\psi_1, \psi_2)\,\mathrm{vol}_M$, for all sections $\psi_1, \psi_2 \in \mathfrak{F}_\mathrm{c}(M)$ with compact support, and hence also $\int_M (\psi_1, G\psi_2)\,\mathrm{vol}_M = -\int_M (G\psi_1, \psi_2)\,\mathrm{vol}_M$ and $\int_M (\psi_1, G_D\psi_2)\,\mathrm{vol}_M = \int_M (G_D\psi_1, \psi_2)\,\mathrm{vol}_M$, where $G := G_+ - G_-$ and $G_D := \frac{1}{2}(G_+ + G_-)$ denote, respectively, the retarded-minus-advanced and Dirac propagators.

These observations will be used frequently in our constructions in this paper. △

In analogy with the Riemannian setting [13], we introduce the following terminology.

Definition 3.5 A *free BV theory* $(F, Q, (-,-), W)$ on $M \in \mathbf{Loc}_m$ consists of a complex of linear differential operators (F, Q) with a compatible (-1)-shifted fiber metric $(-,-)$ and a Green's witness W.

Several examples of free BV theories, closely related to the examples from [1, 5, 6], are presented below.

Example 3.6 Our first example of a free BV theory over $M \in \mathbf{Loc}_m$ is obtained from an ordinary field theory, which is defined by a formally self-adjoint Green hyperbolic operator P acting on sections of a vector bundle E over M endowed with a fiber metric $\langle -,- \rangle$. To these data, one assigns the free BV theory $(F_P, Q_P, (-,-)_P, W_P)$ consisting of the complex of linear differential operators

$$(F_P, Q_P) := \big(E \xrightarrow{\;P\;} E \big) \tag{3.5a}$$

concentrated in degrees 0 and 1, of the compatible (-1)-shifted fiber metric $(-,-)_P$ uniquely determined by

$$(\varphi^\ddagger, \varphi)_P := \langle \varphi^\ddagger, \varphi \rangle, \tag{3.5b}$$

for all $\varphi \in F^0_P = E$ and $\varphi^\ddagger \in F^1_P = E$ over the same base point, and of the Green's witness

$$W_P := \big(E \xleftarrow{\;\mathrm{id}\;} E \big). \tag{3.5c}$$

Here and in the following examples, we decided to use a convenient graphical visualization for a Green's witness as a sequence of linear differential operators, which is pointing from right to left because W decreases the degree. It is important to emphasize that this sequence is in general *not* a chain complex because a Green's witness is not necessarily square-zero.

Example 3.7 The free BV theory $(F_{\mathrm{CS}}, Q_{\mathrm{CS}}, (-,-)_{\mathrm{CS}}, W_{\mathrm{CS}})$ associated with linear Chern–Simons theory on $M \in \mathbf{Loc}_3$ consists of the complex of linear differential operators

$$(F_{\mathrm{CS}}, Q_{\mathrm{CS}}) := \big(\Lambda^0 M \xrightarrow{\ \mathrm{d}\ } \Lambda^1 M \xrightarrow{\ \mathrm{d}\ } \Lambda^2 M \xrightarrow{\ \mathrm{d}\ } \Lambda^3 M \big) \tag{3.6a}$$

concentrated between degrees -1 and 2 (this is the 1-shift of the de Rham complex up to a global sign), of the (-1)-shifted fiber metric $(-,-)_{\mathrm{CS}}$ uniquely determined by

$$(A^{\ddagger}, A)_{\mathrm{CS}} := *^{-1}(A^{\ddagger} \wedge A), \qquad (c^{\ddagger}, c)_{\mathrm{CS}} := - *^{-1} (c^{\ddagger} \wedge c), \tag{3.6b}$$

for all $c \in F_{\mathrm{CS}}^{-1} = \Lambda^0 M$, $A \in F_{\mathrm{CS}}^{0} = \Lambda^1 M$, $A^{\ddagger} \in F_{\mathrm{CS}}^{1} = \Lambda^2 M$ and $c^{\ddagger} \in F_{\mathrm{CS}}^{2} = \Lambda^3 M$ over the same base point, where $\wedge$ denotes the wedge product on differential forms and $*$ denotes the Hodge operator on M, and of the Green's witness

$$W_{\mathrm{CS}} := \big(\Lambda^0 M \xleftarrow{\ \delta\ } \Lambda^1 M \xleftarrow{\ \delta\ } \Lambda^2 M \xleftarrow{\ \delta\ } \Lambda^3 M \big), \tag{3.6c}$$

where $\delta := (-1)^k *^{-1} \mathrm{d} *$ denotes the de Rham codifferential on M on k-forms, for $k = 1, 2, 3$. (It is useful to keep in mind that $*^{-1} = -*$ in odd dimension and Lorentzian signature.)

Example 3.8 The free BV theory $(F_{\mathrm{MW}}, Q_{\mathrm{MW}}, (-,-)_{\mathrm{MW}}, W_{\mathrm{MW}})$ associated with Maxwell p-forms on $M \in \mathbf{Loc}_m$, for $p = 0, \dots, m-1$, consists of the complex of linear differential operators

$$(F_{\mathrm{MW}}, Q_{\mathrm{MW}}) := \big(\Lambda^0 M \xrightarrow{\ \mathrm{d}\ } \cdots \xrightarrow{\ \mathrm{d}\ } \Lambda^p M \xrightarrow{\ \delta\mathrm{d}\ } \Lambda^p M \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } \Lambda^0 M \big) \tag{3.7a}$$

concentrated between degrees $-p$ and $p+1$, of the (-1)-shifted fiber metric $(-,-)_{\mathrm{MW}}$ uniquely determined by

$$(a^{\ddagger}, a)_{\mathrm{MW}} := s_{k+1} *^{-1} (a^{\ddagger} \wedge *a), \tag{3.7b}$$

for all $k = 0, \dots, p$, $a \in F_{\mathrm{MW}}^{-k} = \Lambda^{p-k} M$ and $a^{\ddagger} \in F_{\mathrm{MW}}^{k+1} = \Lambda^{p-k} M$ over the same base point, where $s_1 := 1$ and $s_k := (-1)^k s_{k-1}$, for $k = 2, \dots, p+1$, and of the Green's witness

$$W_{\mathrm{MW}} := \big(\Lambda^0 M \xleftarrow{\ \delta\ } \cdots \xleftarrow{\ \delta\ } \Lambda^p M \xleftarrow{\ \mathrm{id}\ } \Lambda^p M \xleftarrow{\ \mathrm{d}\ } \cdots \xleftarrow{\ \mathrm{d}\ } \Lambda^0 M \big). \tag{3.7c}$$

Note that for $p = 1$ Maxwell p-forms recover linear Yang–Mills theory.

Let $(F, Q, (-,-), W)$ be free BV theory. We define the *retarded/advanced Green's homotopy*

$$\Lambda_\pm := W\, G_\pm = G_\pm\, W \in [\mathfrak{F}_c(M), \mathfrak{F}(M)]^{-1}, \tag{3.8}$$

where $G_\pm$ denotes the retarded/advanced Green's operator associated with P, see Definition 3.3 and Remark 3.4. (In (3.8), we used Remark 3.4 (2) and that W preserves supports.) Note that the retarded/advanced Green's homotopy $\Lambda_\pm \in [\mathfrak{F}_c(M), \mathfrak{F}(M)]^{-1}$ is a cochain homotopy that trivializes the cochain map $j : \mathfrak{F}_c(M) \to \mathfrak{F}(M)$ in $\mathbf{Ch}_\mathbb{R}$ forgetting compact supports. More explicitly, one computes

$$\partial \Lambda_\pm = Q\, W\, G_\pm + W\, G_\pm\, Q = P\, G_\pm = j, \tag{3.9}$$

where in the first step we used the definition of the internal hom differential ∂, the second step follows from Remark 3.4 (2) and in the last step we used that $G_\pm$ is the retarded/advanced Green's operator associated with P.

Remark 3.9 $\Lambda_\pm$ as defined in (3.8) is a specific choice of a retarded/advanced Green's homotopy in the more general sense of [6, Definition 3.5]. Such level of generality plays a crucial role to ensure uniqueness of retarded/advanced Green's homotopies, see [6, Proposition 3.9]. This general and more abstract concept of a retarded/advanced Green's homotopy is not needed for the present paper because a Green's witness W for the complex of linear differential operators (F, Q) is given, which allows us to consider the explicit choices $\Lambda_\pm$ from (3.8). This considerably simplifies our analysis, in particular in view of naturality with respect to $M \in \mathbf{Loc}_m$, see Sect. 3.2 below. △

We shall now endow the complex $\mathfrak{F}_c(M)[1] \in \mathbf{Ch}_\mathbb{R}$ of linear observables with both a (-1)-shifted Poisson structure $\tau_{(-1)}$ and an unshifted one $\tau_{(0)}$. Furthermore, we shall construct a symmetric pairing τ_D, called Dirac pairing, that trivializes $\tau_{(-1)}$, i.e. $\partial \tau_D = \tau_{(-1)}$. The key ingredients for our construction are the integration pairing $(\!(-,-)\!)$ from (3.4) and the retarded and advanced Green's homotopies $\Lambda_\pm$ from (3.8). By taking their difference, we define the *retarded-minus-advanced cochain map*

$$\Lambda := \Lambda_+ - \Lambda_- : \mathfrak{F}_c(M)[1] \longrightarrow \mathfrak{F}(M) \tag{3.10}$$

in $\mathbf{Ch}_\mathbb{R}$, where $\Lambda_\pm$ are regarded here as 0-cochains in $[\mathfrak{F}_c(M)[1], \mathfrak{F}(M)] \in \mathbf{Ch}_\mathbb{R}$ (under the isomorphism $[\mathfrak{F}_c(M)[1], \mathfrak{F}(M)] \cong [\mathfrak{F}_c(M), \mathfrak{F}(M)][-1]$ in $\mathbf{Ch}_\mathbb{R}$ given by $(-1)^n$ in degree n). Note that Λ is a cochain map because $\partial \Lambda_\pm = j$. Similarly, we define the *Dirac homotopy*

$$\Lambda_D := \frac{1}{2}\big(\Lambda_+ + \Lambda_-\big) \in [\mathfrak{F}_c(M)[1], \mathfrak{F}(M)]^0 \tag{3.11}$$

as a graded linear map of degree 0. We have seen in (3.9) that the cochain map $j : \mathfrak{F}_c(M) \to \mathfrak{F}(M)$ is trivialized by $\Lambda_\pm$. It follows that a similar result is achieved by the Dirac homotopy Λ_D, namely

$$\partial\Lambda_D = j \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^1. \tag{3.12}$$

First, we define the (-1)*-shifted Poisson structure*

$$\begin{tikzcd}
\mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2} \arrow[d, "\cong"'] \arrow[rr, dashed, "\tau_{(-1)}"] & & \mathbb{R}[1] \\
\mathfrak{F}_{\mathrm{c}}(M)[1]\otimes\mathbb{R}[1]\otimes\mathfrak{F}_{\mathrm{c}}(M) \arrow[r, "\gamma\otimes j"'] & \mathbb{R}[1]\otimes\mathfrak{F}_{\mathrm{c}}(M)[1]\otimes\mathfrak{F}(M) \arrow[r, "\mathrm{id}\otimes(\!(-,-)\!)[1]"'] & \mathbb{R}[1]\otimes\mathbb{R} \arrow[u, "\cong"']
\end{tikzcd} \tag{3.13}$$

in $\mathbf{Ch}_{\mathbb{R}}$, where γ denotes the symmetric braiding. To confirm that (3.13) defines a (-1)-shifted Poisson structure we have to check symmetry $\tau_{(-1)}\circ\gamma = \tau_{(-1)}$. Indeed, for all homogeneous sections $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)[1]$ with compact support, one has

$$\begin{aligned}
\tau_{(-1)}\gamma(\psi_1\otimes\psi_2) &= (-1)^{(|\psi_1|+1)|\psi_2|}\int_M (\psi_2, \psi_1)\,\mathrm{vol}_M \\
&= (-1)^{|\psi_1|}\int_M (\psi_1, \psi_2)\,\mathrm{vol}_M \\
&= \tau_{(-1)}(\psi_1\otimes\psi_2),
\end{aligned} \tag{3.14}$$

where in the first and last steps we used the definition of $\tau_{(-1)}$ from (3.13) and in the second step we used that the fiber metric $(-,-)$ is graded anti-symmetric, see Definition 3.1.

Second, we define the *unshifted Poisson structure*

$$\begin{tikzcd}
\mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2} \arrow[d, "\cong\otimes\Lambda"'] \arrow[r, dashed, "\tau_{(0)}"] & \mathbb{R} \\
\mathbb{R}[1]\otimes\mathfrak{F}_{\mathrm{c}}(M)\otimes\mathfrak{F}(M) \arrow[r, "\mathrm{id}\otimes(\!(-,-)\!)"'] & \mathbb{R}[1]\otimes\mathbb{R}[-1] \arrow[u, "\cong"']
\end{tikzcd} \tag{3.15}$$

in $\mathbf{Ch}_{\mathbb{R}}$. To confirm that (3.15) defines an unshifted Poisson structure, we have to check anti-symmetry $\tau_{(0)}\circ\gamma = -\tau_{(0)}$. Indeed, for all homogeneous sections $\psi_1, \psi_2 \in$

$\mathfrak{F}_c(M)[1]$ with compact support, one has

$$\begin{aligned}
\tau_{(0)}\gamma(\psi_1 \otimes \psi_2) &= (-1)^{|\psi_1||\psi_2|} \int_M (\psi_2, GW\psi_1)\,\mathrm{vol}_M \\
&= -(-1)^{|\psi_1|} \int_M (GW\psi_1, \psi_2)\,\mathrm{vol}_M \\
&= -\int_M (\psi_1, WG\psi_2) \\
&= -\tau_{(0)}(\psi_1 \otimes \psi_2),
\end{aligned} \tag{3.16}$$

where in the first and last steps we used the definition of $\tau_{(0)}$ from (3.15), in the second step we used that the fiber metric $(-,-)$ is graded anti-symmetric, see Definition 3.1, and in the third step we used Definition 3.3 (iii) and Remark 3.4 (3).

Finally, we define the *Dirac pairing*

$$\begin{tikzcd}
\mathfrak{F}_c(M)[1]^{\otimes 2} \arrow[rr, dashed, "\tau_D"] \arrow[d, "\cong \otimes \Lambda_D"'] & & \mathbb{R} \\
\mathbb{R}[1] \otimes \mathfrak{F}_c(M) \otimes \mathfrak{F}(M) \arrow[rr, "\mathrm{id}\otimes(\!(-,-)\!)"'] & & \mathbb{R}[1] \otimes \mathbb{R}[-1] \arrow[u, "\cong"']
\end{tikzcd} \tag{3.17}$$

as a graded linear map of degree 0, i.e. $\tau_D \in [\mathfrak{F}_c(M)[1]^{\otimes 2}, \mathbb{R}]^0$. The same calculation as in (3.16) (with the Dirac propagator G_D replacing the retarded-minus-advanced one G) proves symmetry $\tau_D \circ \gamma = \tau_D$. Note that τ_D trivializes the (-1)-shifted Poisson structure $\tau_{(-1)}$, i.e.

$$\partial \tau_D = \tau_{(-1)}. \tag{3.18}$$

Indeed, for all homogeneous sections $\psi_1, \psi_2 \in \mathfrak{F}_c(M)[1]$ with compact support, one has

$$\begin{aligned}
\partial\tau_D(\psi_1 \otimes \psi_2) &= \int_M (Q\psi_1, \Lambda_D\psi_2)\,\mathrm{vol}_M - (-1)^{|\psi_1|} \int_M (\psi_1, \Lambda_D Q_{[1]}\psi_2)\,\mathrm{vol}_M \\
&= (-1)^{|\psi_1|} \int_M (\psi_1, (Q\,\Lambda_D - \Lambda_D\, Q_{[1]})\psi_2)\,\mathrm{vol}_M \\
&= (-1)^{|\psi_1|} \int_M (\psi_1, \psi_2)\,\mathrm{vol}_M \\
&= \tau_{(-1)}(\psi_1 \otimes \psi_2),
\end{aligned} \tag{3.19}$$

where in the first step we used the definition of τ_D from (3.17), in the second step we used (3.3), in the third step we used (3.12) and in the last step we used the definition of $\tau_{(-1)}$ from (3.13).

3.2 Properties of $\tau_{(-1)}$, $\tau_{(0)}$ and τ_D

Let us now consider a collection $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ of free BV theories, indexed by $M \in \mathbf{Loc}_m$. We assume that $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ is natural with respect to the morphisms $f : M \to N$ in $\mathbf{Loc}_m$ in the sense of the next definition.

Definition 3.10 A *natural collection of free BV theories* $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ consists of natural vector bundles F^n, natural linear differential operators $Q^n : \Gamma(\mathsf{F}^n) \to \Gamma(\mathsf{F}^{n+1})$ and $W^n : \Gamma(\mathsf{F}^n) \to \Gamma(\mathsf{F}^{n-1})$ and natural fiber metrics $(-,-)^n : \mathsf{F}^n \otimes \mathsf{F}^{1-n} \to \mathbb{R}$, for all $n \in \mathbb{Z}$, such that, for all $M \in \mathbf{Loc}_m$, $(F_M, Q_M, (-,-)_M, W_M)$ is a free BV theory in the sense of Definition 3.5.

The concepts of natural vector bundles, natural fiber metrics and natural differential operators, which are relevant for the definition above, are recalled in Appendix A.

Example 3.11 Let us upgrade Examples 3.6, 3.7 and 3.8 to natural collections of free BV theories as formalized in Definition 3.10.

Concerning the natural upgrade of Example 3.6, it suffices to take as input a natural vector bundle E endowed with a natural fiber metric $\langle -,-\rangle$ and a natural linear differential operator P defined on E, whose components P_M are formally self-adjoint Green hyperbolic operators, for all $M \in \mathbf{Loc}_m$. Taking the natural Green's witness given, for all $M \in \mathbf{Loc}_m$, by the identity as in Example 3.6, one obtains a natural collection of free BV theories. For instance, the natural collection of free BV theories associated with the real Klein–Gordon field of mass $m \geq 0$ is obtained by taking $\mathsf{E} = \underline{\mathbb{R}}$ to be the natural trivial line bundle, whose components are the trivial line bundles $M \times \mathbb{R} \to M$, endowed with its canonical natural fiber metric $\langle -,-\rangle$, given component-wise by the multiplication on $\mathbb{R}$, and the Klein–Gordon operator $P = \Box + m^2$, whose naturality follows from the fact that morphisms of $\mathbf{Loc}_m$ are isometries.

The upgrade of Examples 3.7 and 3.8 to natural collections of free BV theories is obtained as follows. First, consider the natural vector bundles of differential k-forms Λ^k, whose components are the vector bundles $\Lambda^k M \to M$. (Naturality follows from the fact that morphisms of $\mathbf{Loc}_m$ are open embeddings.) Second, note that the wedge product $\wedge$ of differential forms, the Hodge operator $*$ and the de Rham differential d are natural with respect to the morphisms in $\mathbf{Loc}_m$. (This relies also on the fact that morphisms of $\mathbf{Loc}_m$ are orientation-preserving isometries.) This defines a natural structure in the sense of Definition 3.10 on the collection of free BV theories from Examples 3.7 and 3.8, which describe linear Chern–Simons theory and Maxwell p-forms.

We summarize below the key facts that will play a crucial role in the rest of the paper. These are part of Definition 3.10, or follow from it and the constructions outlined in Appendix A, especially (A.1), (A.5), (A.7) and (A.8). For all $f : M \to N$ in $\mathbf{Loc}_m$, one has the following:

(1) A pushforward cochain map $f_* : \mathfrak{F}_\mathrm{c}(M) \to \mathfrak{F}_\mathrm{c}(N)$ in $\mathbf{Ch}_\mathbb{R}$ for compactly supported sections and a pullback cochain map $f^* : \mathfrak{F}(N) \to \mathfrak{F}(M)$ in $\mathbf{Ch}_\mathbb{R}$ for sections. (In particular, $Q_N\, f_* = f_*\, Q_M$ and $Q_M\, f^* = f^*\, Q_N$.)

(2) Naturality of the integration pairing (3.4), i.e. the diagram

$$
\begin{tikzcd}
\mathfrak{F}_{\mathrm{c}}(M) \otimes \mathfrak{F}(N) \arrow[r, "\mathrm{id}\otimes f^*"] \arrow[d, "f_*\otimes\mathrm{id}"'] & \mathfrak{F}_{\mathrm{c}}(M) \otimes \mathfrak{F}(M) \arrow[d, "(\!(-,-)\!)_M"] \\
\mathfrak{F}_{\mathrm{c}}(N) \otimes \mathfrak{F}(N) \arrow[r, "(\!(-,-)\!)_N"'] & \mathbb{R}[-1]
\end{tikzcd}
\tag{3.20}
$$

in $\mathbf{Ch}_{\mathbb{R}}$ commutes.

(3) Naturality of Green's witnesses, i.e. $W_N\, f_* = f_*\, W_M$ and $W_M\, f^* = f^*\, W_N$.

(1) and (3) entail that also the Green hyperbolic operators $P_M := Q_M\, W_M + W_M\, Q_M$ are natural, hence $P_N\, f_* = f_*\, P_M$ and $P_M\, f^* = f^*\, P_N$, for all $f : M \to N$ in $\mathbf{Loc}_m$. As a consequence of the naturality of $P = (P_M)_{M\in\mathbf{Loc}_m}$ and of the theory of Green hyperbolic operators, for all $f : M \to N$ in $\mathbf{Loc}_m$, one has the usual naturality property $f^*\, G^N_\pm\, f_* = G^M_\pm$ for the retarded/advanced Green's operator $G^M_\pm$ associated with P_M, see [3], as well as the analogs $f^*\, G^N\, f_* = G^M$ and $f^*\, G^N_D\, f_* = G^M_D$ for the retarded-minus-advanced propagator $G^M := G^M_+ - G^M_-$ and for the Dirac propagator $G^M_D := \frac{1}{2}(G^M_+ + G^M_-)$. Therefore, the retarded/advanced Green's homotopies $\Lambda^M_\pm := W_M\, G^M_\pm$, the retarded-minus advanced cochain maps $\Lambda^M := \Lambda^M_+ - \Lambda^M_-$ and the Dirac homotopies $\Lambda^M_D := \frac{1}{2}(\Lambda^M_+ + \Lambda^M_-)$ inherit the same naturality, that is, for all $\mathbf{Loc}_m$-morphisms $f : M \to N$, one has

$$
f^*\, \Lambda^N_\pm\, f_* = \Lambda^M_\pm\,, \qquad f^*\, \Lambda^N\, f_* = \Lambda^M\,, \qquad f^*\, \Lambda^N_D\, f_* = \Lambda^M_D\,. \tag{3.21}
$$

Finally, for all $M \in \mathbf{Loc}_m$, let us consider the (-1)-shifted Poisson structures $\tau^M_{(-1)}$ from (3.13), the unshifted Poisson structures $\tau^M_{(0)}$ from (3.15) and the Dirac pairings τ^M_D from (3.17), with additional superscripts emphasizing the underlying object in $\mathbf{Loc}_m$. As a consequence of (3.21) and of the naturality of the integration pairing, see (3.20), one obtains the following result.

Lemma 3.12 *For all $f : M \to N$ in $\mathbf{Loc}_m$, the following holds*

$$
\tau^N_{(-1)} \circ (f_* \otimes f_*) = \tau^M_{(-1)}\,, \quad \tau^N_{(0)} \circ (f_* \otimes f_*) = \tau^M_{(0)}\,, \quad \tau^N_D \circ (f_* \otimes f_*) = \tau^M_D\,. \tag{3.22}
$$

Proof The first equality follows immediately from (3.13) and (3.20). To prove also the second equality, recall (3.15) and, for all $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)$, compute

$$
\tau^N_{(0)}(f_*\psi_1 \otimes f_*\psi_2) = (\!(\psi_1, f^*\Lambda^N f_*\psi_2)\!)_M = (\!(\psi_1, \Lambda^M\psi_2)\!)_M = \tau^M_{(0)}(\psi_1 \otimes \psi_2)\,, \tag{3.23}
$$

where we used (3.20) in the first and (3.21) in the second step. Recalling (3.17), the proof of the third equality is the same. □

This means that $\tau^M_{(-1)}$, $\tau^M_{(0)}$ and τ^M_D are the components at $M \in \mathbf{Loc}_m$ of the natural transformations $\tau_{(-1)}$, $\tau_{(0)}$ and τ_D, respectively. In particular, the assignment to each object $M \in \mathbf{Loc}_m$ of the Poisson cochain complex $(\mathfrak{F}_c(M)[1], \tau^M_{(0)}) \in \mathbf{PoCh}_{\mathbb{R}}$ and to each morphism $f : M \to N$ in $\mathbf{Loc}_m$ of the pushforward $f_* : (\mathfrak{F}_c(M)[1], \tau^M_{(0)}) \to (\mathfrak{F}_c(N)[1], \tau^N_{(0)})$ in $\mathbf{PoCh}_{\mathbb{R}}$ defines a functor $(\mathfrak{F}_c[1], \tau_{(0)}) : \mathbf{Loc}_m \to \mathbf{PoCh}_{\mathbb{R}}$. (Note that (3.22) expresses the necessary compatibility of f_* with the unshifted Poisson structures $\tau^M_{(0)}$ and $\tau^N_{(0)}$.)

The next result shows that classical analogs of the Einstein causality and time-slice axioms hold. To simplify our notation, from now on we shall suppress the superscripts and subscripts emphasizing the underlying object of $\mathbf{Loc}_m$, whenever this information can be inferred from the context.

Theorem 3.13 *Let $(F_M, Q_M, (-,-)_M, W_M)_{M \in \mathbf{Loc}_m}$ be a natural collection of free BV theories.*

(a) For all causally disjoint morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ in $\mathbf{Loc}_m$,

$$\tau_{(0)} \circ (f_{1\,*} \otimes f_{2\,*}) = 0 \tag{3.24}$$

vanishes.

(b) For all Cauchy morphisms $f : M \to N$ in $\mathbf{Loc}_m$, the pushforward cochain map

$$f_* : \mathfrak{F}_c(M)[1] \longrightarrow \mathfrak{F}_c(N)[1] \tag{3.25}$$

in $\mathbf{Ch}_{\mathbb{R}}$ is a quasi-isomorphism.

Proof Item (a) follows from $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$ (because f_1 and f_2 are causally disjoint), the definition of the unshifted Poisson structure $\tau_{(0)}$ and the support properties of retarded and advanced Green's operators, see (3.15) and Definition 2.8.

To prove also item (b), we shall construct a quasi-inverse $g : \mathfrak{F}_c(N)[1] \to \mathfrak{F}_c(M)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ for f_* and homotopies $\eta \in [\mathfrak{F}_c(N)[1], \mathfrak{F}_c(N)[1]]^{-1}$, witnessing that $f_*\, g \sim \mathrm{id}$, and $\zeta \in [\mathfrak{F}_c(M)[1], \mathfrak{F}_c(M)[1]]^{-1}$, witnessing that $g\, f_* \sim \mathrm{id}$. Recalling that $f : M \to N$ in $\mathbf{Loc}_m$ is by hypothesis a Cauchy morphism, let us consider two spacelike Cauchy surfaces $\Sigma_\pm \subset N$ lying inside the image of f such that $\Sigma_+ \subset I^+_N(\Sigma_-)$ is contained in the chronological future of Σ_-. Choose a partition of unity $\{\chi_+, \chi_-\}$ subordinate to the open cover $\{I^+_N(\Sigma_-), I^-_N(\Sigma_+)\}$ of N.

Quasi-inverse g**:** We construct a candidate quasi-inverse as the (unique) cochain map

$$g : \mathfrak{F}_c(N)[1] \longrightarrow \mathfrak{F}_c(M)[1] \tag{3.26a}$$

in $\mathbf{Ch}_{\mathbb{R}}$ that satisfies the equation

$$j\, f_*\, g = \mp \partial(\chi_\pm\, \Lambda) : \mathfrak{F}_c(N)[1] \longrightarrow \mathfrak{F}(N)[1] \tag{3.26b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, where $j : \mathfrak{F}_{\mathrm{c}}(N)[1] \to \mathfrak{F}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ denotes the inclusion forgetting compact supports, the (-1)-cochain $\chi_{\pm} \in [\mathfrak{F}(N), \mathfrak{F}(N)[1]]^{-1}$ denotes multiplication by the partition function $\chi_{\pm}$ and ∂ denotes the internal hom differential of $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]] \in \mathbf{Ch}_{\mathbb{R}}$. Such cochain map g exists (uniquely) because $\mp\partial(\chi_{\pm}\, \Lambda)$ is manifestly a cochain map and, for all sections $\psi \in \mathfrak{F}_{\mathrm{c}}(N)[1]$, the section $-(\partial(\chi_{+}\, \Lambda))\psi = (\partial(\chi_{-}\, \Lambda))\psi \in \mathfrak{F}(N)[1]$ lies in the image of the degree-wise injective cochain map $j\, f_{*}$. Indeed, the support of the section $-(\partial(\chi_{+}\, \Lambda))\psi = (\partial(\chi_{-}\, \Lambda))\psi$ is contained in the compact subset $J_N(\mathrm{supp}(\psi)) \cap J_N^{+}(\Sigma_{-}) \cap J_N^{-}(\Sigma_{+}) \subseteq f(M)$. (The latter subset is compact by [4, Corollary A.5.4] and contained in $f(M)$ because by construction $J_N^{+}(\Sigma_{-}) \cap J_N^{-}(\Sigma_{+}) \subseteq f(M)$.)

Homotopy η: We construct a candidate homotopy as the (unique) (-1)-cochain

$$\eta \in [\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]]^{-1} \tag{3.27a}$$

that satisfies the equation

$$j\, \eta = -\chi_{-}\, \Lambda_{+} - \chi_{+}\, \Lambda_{-} \in [\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)[1]]^{-1}, \tag{3.27b}$$

where $\Lambda_{\pm}$ are regarded here as 0-cochains in $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)] \in \mathbf{Ch}_{\mathbb{R}}$ (under the isomorphism $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)] \cong [\mathfrak{F}_{\mathrm{c}}(N), \mathfrak{F}(N)][-1]$ in $\mathbf{Ch}_{\mathbb{R}}$ given by $(-1)^n$ in degree n). Such (-1)-cochain η exists (uniquely) because, for all sections $\psi \in \mathfrak{F}_{\mathrm{c}}(N)[1]$, the section $\chi_{\mp}\Lambda_{\pm}\psi \in \mathfrak{F}(N)[1]$ lies in the image of the degree-wise injective cochain map j that forgets compact supports. Indeed, the support of the section $\chi_{\mp}\Lambda_{\pm}\psi \in \mathfrak{F}(N)[1]$ is contained in the compact subset $J_N^{\mp}(\Sigma_{\pm}) \cap J_N^{\pm}(\mathrm{supp}(\psi)) \subseteq N$. Let us check that $\partial\eta = \mathrm{id} - f_{*}\, g$. Since j is degree-wise injective, this follows from

$$j\, (\partial\eta) = \partial(-\chi_{-}\, \Lambda_{+} - \chi_{+}\, \Lambda_{-}) = j + (\partial\chi_{+})\, \Lambda = j\, (\mathrm{id} - f_{*}\, g), \tag{3.28}$$

where in the first step we used that j is a cochain map and the equation defining η, in the second step we used the Leibniz rule of ∂ with respect to the composition, $\chi_{+} + \chi_{-} = 1$ (hence $\partial\chi_{+} = -\partial\chi_{-}$), $\partial\Lambda_{\pm} = j$ and $\Lambda = \Lambda_{+} - \Lambda_{-}$, and in the last step, we used $\partial\Lambda = 0$ and (3.26).

Homotopy ζ: We construct a candidate homotopy as the (unique) (-1)-cochain

$$\zeta \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(M)[1]]^{-1} \tag{3.29a}$$

that satisfies the equation

$$f_{*}\, \zeta = \eta\, f_{*} \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]]^{-1}. \tag{3.29b}$$

Such (-1)-cochain ζ exists (uniquely) because, for all homogeneous sections $\psi \in \mathfrak{F}_{\mathrm{c}}(M)[1]$, the section $\chi_{\mp}\Lambda_{\pm} f_{*}\psi \in \mathfrak{F}(N)[1]$ lies in the image of the degree-wise injective cochain map f_{*}. Indeed, the support of the section $\chi_{\mp}\Lambda_{\pm} f_{*}\psi \in \mathfrak{F}(N)[1]$ is contained in the compact subset $J_N^{\mp}(\Sigma_\pm) \cap J_N^{\pm}(f(\mathrm{supp}(\psi))) \subseteq f(M)$. Let us check that $\partial\zeta = \mathrm{id} - g\, f_*$. Since f_* is degree-wise injective, this follows from

$$f_*\,(\partial\zeta) = (\partial\eta)\, f_* = f_*\,(\mathrm{id} - g\, f_*) \tag{3.30}$$

where in the first step we used that f_* is a cochain map and the definition of ζ and in the last step we used $\partial\eta = \mathrm{id} - f_*\, g$. □

To conclude this section, we record a simple result relating $\tau_{(0)}$ and τ_D via time-ordering.

Proposition 3.14 *Let $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ be a natural collection of free BV theories. Then, for all time-ordered pairs $(f_1, f_2) : (M_1, M_2) \to N$ in $\mathbf{Loc}_m$,*

$$\tau_D \circ (f_{1\,*} \otimes f_{2\,*}) = \frac{1}{2}\,\tau_{(0)} \circ (f_{1\,*} \otimes f_{2\,*}). \tag{3.31}$$

Proof For all $\psi_1 \in \mathfrak{F}_\mathrm{c}(M_1)[1]$ and $\psi_2 \in \mathfrak{F}_\mathrm{c}(M_2)[1]$, recalling the support properties of retarded and advanced Green's operators from Definition 2.8, one computes

$$\begin{aligned}
\frac{1}{2}\,\tau_{(0)}(f_{1\,*}\psi_1 \otimes f_{2\,*}\psi_2) &= \frac{1}{2}\int_N (f_{1\,*}\psi_1, \Lambda\, f_{2\,*}\psi_2)_N\ \mathrm{vol}_N\\
&= \frac{1}{2}\int_N (f_{1\,*}\psi_1, \Lambda_+ f_{2\,*}\psi_2)_N\ \mathrm{vol}_N\\
&= \int_N (f_{1\,*}\psi_1, \Lambda_D\, f_{2\,*}\psi_2)_N\ \mathrm{vol}_N\\
&= \tau_D(f_{1\,*}\psi_1 \otimes f_{2\,*}\psi_2).
\end{aligned} \tag{3.32}$$

The first step uses the definition of the unshifted Poisson structure $\tau_{(0)}$, see (3.15). Both the second and third steps use that $f_1(M_1) \cap J_N^-(f_2(M_2)) = \emptyset$ is empty (because (f_1, f_2) is time-ordered), in combination either with $\Lambda = \Lambda_+ - \Lambda_-$ or with $\Lambda_D = \frac{1}{2}(\Lambda_+ + \Lambda_-)$. The last step uses the definition of the Dirac pairing τ_D, see (3.17). □

4 Quantizations and comparison

In this section, we shall present two a priori different approaches to the quantization of a natural collection $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ of free BV theories. First, in Sect. 4.1 we shall construct a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ by deforming the ordinary differential of the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])$ generated by linear observables with the BV Laplacian, as prescribed by the BV formalism [12, 13]. Second, in Sect. 4.2 we shall construct an AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ by deforming the commutative multiplication of $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgAlg}_\mathbb{C}$ to the non-commutative Moyal–Weyl star product. These two constructions involve different input data. More specifically, the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ relies only on the natural (-1)-shifted fiber metric $(-,-)$ through the natural (-1)-shifted Poisson structure $\tau_{(-1)}$ (except for the time-slice axiom), while the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ relies also on the natural Green's witness W through the natural unshifted Poisson structure $\tau_{(0)}$. Last, we shall show in Sect. 4.3 that, when both $(-,-)$ and W are given, the natural Dirac pairing τ_D leads to an isomorphism $T : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ in $\mathbf{tPFA}_m$ to the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ canonically associated with $\mathcal{A} \in \mathbf{AQFT}_m$, see [7]. Let us mention that the deformation parameter $\hbar > 0$ will not be formal in our constructions below. Indeed, all expansions in powers of $\hbar$ that appear later on actually stop at finite order, see for instance the comment after (4.13).

As a preparatory step, let us present a geometric construction that will be used frequently in the rest of the paper.

Lemma 4.1 *Let $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ be a time-ordered tuple in $\mathbf{Loc}_m$ of length $n \geq 2$. Then there exist $M \in \mathbf{Loc}_m$, $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f'} = (f'_1, \dots, f'_{n-1}) : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ of length $n-1$ such that $(f, f_n) : (M, M_n) \to N$ is a time-ordered pair in $\mathbf{Loc}_m$ and $f \circ f'_i = f_i$, for all $i = 1, \dots, n-1$. In short, each time-ordered n-tuple $\underline{f} : \underline{M} \to N$, for $n \geq 2$, admits a factorization*

$$\begin{tikzcd} \underline{M} \arrow[rr, "\underline{f}"] \arrow[dr, "(\underline{f'}, \mathrm{id}_{M_n})"'] & & N \\ & (M, M_n) \arrow[ur, "(f, f_n)"'] & \end{tikzcd} \tag{4.1}$$

with $\underline{f'}$ a time-ordered $(n-1)$-tuple and (f, f_n) a time-ordered pair.

Proof Recalling Sect. 2.3, we define the subset

$$M := J_N^{+\cap-}\left(\bigcup_{i=1}^{n-1} f_i(M_i)\right) \subseteq N \tag{4.2}$$

as the causally convex hull of the union of the images of f_i, for $i = 1, \dots, n-1$. Since the images are open, $M \subseteq N$ is open and causally convex. Endowing it with the restriction of the orientation, time-orientation and metric of N defines an object $M \in \mathbf{Loc}_m$ and promotes the subset inclusion $M \subseteq N$ to a morphism $f : M \to N$ in $\mathbf{Loc}_m$. Since, for each $i = 1, \dots, n-1$, $f_i(M_i) \subseteq M$ by construction, $f_i : M_i \to N$ in $\mathbf{Loc}_m$ factors as $f_i = f \circ f'_i$, where $f'_i : M_i \to M$ in $\mathbf{Loc}_m$ is the codomain restriction of f_i. To conclude, let us also check that (f, f_n) is a time-ordered pair, i.e. $J_N^+(f(M)) \cap f_n(M_n) = \emptyset$. By contraposition, suppose that the intersection is not empty. Then there exists a future directed causal curve in N emanating from $f(M)$ and reaching $f_n(M_n)$. Since any point in the causally convex hull M is by definition in the causal future of $f_i(M_i)$, for some $i = 1, \dots, n-1$, it follows that there exists a future directed causal curve in N emanating from $f_i(M_i)$, for some $i = 1, \dots, n-1$, and reaching $f_n(M_n)$, leading to a contradiction with the hypothesis that $\underline{f}$ is time-ordered. $\square$

4.1 BV quantization

Let $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ be a natural collection of free BV theories. Consider the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$ generated by the complexification of $\mathfrak{F}_c(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$, whose differential $\mathcal{Q}$ is defined by the differential $Q_{[1]} = -Q$ of $\mathfrak{F}_c(M)[1]$ and the graded Leibniz rule. BV quantization consists of deforming $\mathcal{Q}$ by means of the *BV Laplacian*

$$\Delta_{\mathrm{BV}} := \Delta_{\tau_{(-1)}} \in \big[\mathrm{Sym}(\mathfrak{F}_c(M)[1]), \mathrm{Sym}(\mathfrak{F}_c(M)[1])\big]^1, \tag{4.3}$$

which is the Laplacian associated with the (-1)-shifted Poisson structure $\tau_{(-1)}$, see Definition 2.6 and (3.13). Explicitly, one defines the degree increasing graded linear map

$$\mathcal{Q}_\hbar := \mathcal{Q} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}} \in \big[\mathrm{Sym}(\mathfrak{F}_c(M)[1]), \mathrm{Sym}(\mathfrak{F}_c(M)[1])\big]^1, \tag{4.4}$$

where $\hbar > 0$ is Planck's constant and $\mathrm{i} \in \mathbb{C}$ is the imaginary unit. Note that $\mathcal{Q}_\hbar$ defines a new differential since it squares to zero

$$\mathcal{Q}_\hbar^2 = \mathcal{Q}^2 + \mathrm{i}\,\hbar\,\partial\Delta_{\mathrm{BV}} - \hbar^2\,\Delta_{\mathrm{BV}}^2 = 0, \tag{4.5}$$

where we used $\mathcal{Q}^2 = 0$, $\partial\Delta_{\mathrm{BV}} = \Delta_{\partial\tau_{(-1)}} = 0$ and $\Delta_{\mathrm{BV}}^2 = -\Delta_{\mathrm{BV}}^2 = 0$, see (2.16) and (2.17). We define the *cochain complex of quantum observables*

$$\mathcal{F}(M) := \big(\mathrm{Sym}(\mathfrak{F}_c(M)[1]), \mathcal{Q}_\hbar\big) \in \mathbf{Ch}_{\mathbb{C}} \tag{4.6}$$

by replacing the original differential $\mathcal{Q}$ with the quantized one $\mathcal{Q}_\hbar$. The assignment $\mathbf{Loc}_m \ni M \mapsto \mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}}$ of the cochain complex of quantum observables can be promoted to a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$. For this purpose, we need to define time-ordered products that are compatible with the quantized differential $\mathcal{Q}_\hbar$. This is the goal of the next proposition.

Proposition 4.2 *Let $\underline{f} : \underline{M} \to N$ be a time-orderable tuple in $\mathbf{Loc}_m$ of length n. Then, the time-ordered product*

$$\begin{tikzcd}[column sep=small] \mathcal{F}(\underline{M}) \arrow[rr, dashed, "\mathcal{F}(\underline{f})"] \arrow[dr, "\bigotimes_i f_{i\,*}"'] & & \mathcal{F}(N) \\ & \mathcal{F}(N)^{\otimes n} \arrow[ur, "\mu^{(n)}"'] & \end{tikzcd}, \tag{4.7}$$

is a cochain map, i.e. $\mathcal{Q}_\hbar\,\mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f})\,\mathcal{Q}_{\hbar\,\otimes}$. Here $f_{i\,*}$ denotes the symmetric algebra extension of the pushforward cochain map $f_{i\,*} : \mathfrak{F}_c(M_i)[1] \to \mathfrak{F}_c(N)[1]$ for compactly supported sections, see Sect. 3.2, and $\mu^{(n)}$ denotes the n-ary multiplication on the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(N)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$.

Proof Since $\mathcal{Q}$ is natural and compatible $\mathcal{Q}\,\mu = \mu\,\mathcal{Q}_\otimes$ with the symmetric algebra multiplication μ, one has $\mathcal{Q}\,\mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f})\,\mathcal{Q}_\otimes$. Hence, it suffices to prove the analog $\Delta_{\mathrm{BV}}\,\mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f})\,\Delta_{\mathrm{BV}\otimes}$ for the BV Laplacian Δ_{BV}. Furthermore, since the symmetric algebra multiplication μ is commutative, it suffices to prove the claim for $\underline{f}$ time-ordered. We argue by induction on the length n. For $n = 0$, the time-ordered product $\mathbb{C} \to \mathcal{F}(N)$ defined above assigns the unit $\mu^{(0)} = \mathbb{1}$ of the symmetric algebra; hence, the claim follows from $\Delta_{\mathrm{BV}}(\mathbb{1}) = 0$, see Definition 2.6. For $n = 1$, $\mathcal{F}(f) = f_*$, hence the claim follows because the BV Laplacian Δ_{BV} inherits the naturality of the (-1)-shifted Poisson structure $\tau_{(-1)}$, see (2.18) and (3.22). For $n = 2$, one computes

$$\begin{aligned}\Delta_{\mathrm{BV}} \circ \mathcal{F}(f_1, f_2) &= \Delta_{\mathrm{BV}} \circ \mu \circ (f_{1*} \otimes f_{2*})\\ &= \mu \circ \big(\Delta_{\mathrm{BV}\otimes} + \{\!\{-,-\}\!\}_{(-1)}\big) \circ (f_{1*} \otimes f_{2*})\\ &= \mu \circ (f_{1*} \otimes f_{2*}) \circ \Delta_{\mathrm{BV}\otimes}\\ &= \mathcal{F}(f_1, f_2) \circ \Delta_{\mathrm{BV}\otimes},\end{aligned} \tag{4.8}$$

where in the first and last steps we used the definition of the time-ordered product $\mathcal{F}(f_1, f_2)$, in the second step we used the degree increasing graded endomorphism $\{\!\{-,-\}\!\}_{(-1)} := \{\!\{-,-\}\!\}_{\tau_{(-1)}}$ to spell out the modified Leibniz rule of Δ_{BV}, see Definitions 2.4 and 2.6, and in the third step we used naturality of the BV Laplacian Δ_{BV} and that $\{\!\{-,-\}\!\}_{(-1)}$ vanishes on the image of $f_{1*} \otimes f_{2*}$, because $f_1(M_1) \cap f_2(M_2) = \emptyset$ and $\tau_{(-1)}$ vanishes on sections with disjoint supports, see (3.13). For $n \geq 3$, taking $M \in \mathbf{Loc}_m$, $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f}' : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ as provided by Lemma 4.1, one computes

$$\begin{aligned}\mathcal{F}(\underline{f}) = \mu^{(n)} \circ \bigotimes_{i=1}^{n} f_{i\,*} &= \mu \circ (f_* \otimes f_{n\,*}) \circ \left(\left(\mu^{(n-1)} \circ \bigotimes_{i=1}^{n-1} f'_{i\,*}\right) \otimes \mathrm{id}\right)\\ &= \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \otimes \mathrm{id}),\end{aligned} \tag{4.9}$$

where in the first and last steps we used the definition of the time-ordered product $\mathcal{F}(\underline{f})$ and in the second step we used $\mu^{(n)} = \mu \circ (\mu^{(n-1)} \otimes \mathrm{id})$, $f \circ f'_i = f_i$, for all $i = 1, \dots, n-1$, and the naturality of the symmetric algebra multiplication μ. Hence, the claim for length $n \geq 3$ follows from lengths 2 and $n - 1$. □

With these preparations, we define the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ by the data listed below:

(a) For each $M \in \mathbf{Loc}_m$, the cochain complex $\mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}}$ from (4.6);
(b) For each time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, the time-ordered product $\mathcal{F}(\underline{f}) : \mathcal{F}(\underline{M}) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ from Proposition 4.2.

Note that these data satisfy the axioms of Definition 2.11 because $\mathfrak{F}_{\mathrm{c}}[1] : \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{R}}$ is a functor, see Sect. 3.2, and the symmetric algebra multiplication μ is associative, unital and commutative. The resulting time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ satisfies the time-slice axiom, as explained by the next proposition.

Proposition 4.3 *If $f : M \to N$ in $\mathbf{Loc}_m$ is a Cauchy morphism, then $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ is a quasi-isomorphism.*

Proof For any $L \in \mathbf{Loc}_m$, consider the filtration of $\mathcal{F}(L) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathfrak{F}_c(L)[1]) \in \mathbf{Ch}_{\mathbb{C}}$ associated with symmetric powers. Explicitly, we denote the subcomplex of $\mathcal{F}(L)$ consisting of symmetric powers up to $p \geq 0$ by

$$F_p(\mathcal{F}(L)) := \left(\bigoplus_{n=0}^{p} \mathrm{Sym}^n(\mathfrak{F}_c(L)[1]),\, \mathcal{Q}_\hbar \right) \subseteq \mathcal{F}(L). \tag{4.10}$$

(Note that this filtration is compatible with the quantized differential $\mathcal{Q}_\hbar = \mathcal{Q} + \mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}$ because the original differential $\mathcal{Q}$ preserves the symmetric power and the BV Laplacian Δ_{BV} lowers the symmetric power by 2.) The resulting filtration is bounded from below, i.e. $F_p(\mathcal{F}(L)) = 0$ vanishes, for all $p < 0$. The quotient maps $\mathcal{F}(L) \to \mathcal{F}(L)/F_p(\mathcal{F}(L))$ in $\mathbf{Ch}_{\mathbb{C}}$, for all $p \in \mathbb{Z}$, form a universal cone, i.e. $\mathcal{F}(L) \cong \lim_{p\in\mathbb{Z}} \mathcal{F}(L)/F_p(\mathcal{F}(L))$. This shows that the filtration is complete, see [14]. Furthermore, for $p \geq 0$, the p-th component of the associated graded cochain complex

$$E_p^\circ(L) := F_p(\mathcal{F}(L))/F_{p-1}(\mathcal{F}(L)) \cong \mathrm{Sym}^p(\mathfrak{F}_c(L)[1]) \in \mathbf{Ch}_{\mathbb{C}} \tag{4.11}$$

is isomorphic to the p-th symmetric power of $\mathfrak{F}_c(L)[1] \in \mathbf{Ch}_{\mathbb{C}}$ (endowed with the original differential $\mathcal{Q}$) because the BV Laplacian Δ_{BV} lowers the symmetric power by 2, see (2.14). Functoriality with respect to $L \in \mathbf{Loc}_m$ of the filtration (4.10) and naturality of the isomorphism (4.11) entail that, for all $f : M \to N$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd} E_p^\circ(M) \arrow[r, "E_p^\circ(f_*)"] \arrow[d, "\simeq"'] & E_p^\circ(N) \arrow[d, "\simeq"] \\ \mathrm{Sym}^p(\mathfrak{F}_c(M)[1]) \arrow[r, "f_*"'] & \mathrm{Sym}^p(\mathfrak{F}_c(N)[1]) \end{tikzcd} \tag{4.12}$$

in $\mathbf{Ch}_{\mathbb{C}}$ commutes. Since the bottom cochain map is a quasi-isomorphism by Theorem 3.13, the claim follows from [14, Theorem 7.4]. □

Example 4.4 Taking the natural free BV theories from Example 3.11 as inputs, the constructions and results from this subsection produce time-orderable prefactorization algebras satisfying the time-slice axiom that quantize ordinary field theories, linear Chern–Simons theory and Maxwell p-forms (including linear Yang–Mills theory for $p = 1$).

4.2 Moyal–Weyl star product

Let $(F_M, Q_M, (-,-)_M, W_M)_{M\in\mathbf{Loc}_m}$ be a natural collection of free BV theories and consider again the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_c(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$ generated by the complexification of $\mathfrak{F}_c(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$. Canonical quantization can be realized by deforming the commutative multiplication μ of the symmetric algebra

$\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgCAlg}_\mathbb{C}$ to the *Moyal–Weyl star product*

$$\begin{tikzcd}[column sep=small]
\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])^{\otimes 2} \arrow[rr, dashed, "\mu_\hbar"] \arrow[dr, "\exp\left(\frac{\mathrm{i}\hbar}{2}\{\!\{-,-\}\!\}_{(0)}\right)"'] & & \mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \\
& \mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])^{\otimes 2} \arrow[ur, "\mu"'] &
\end{tikzcd} \quad , \tag{4.13}$$

where $\{\!\{-,-\}\!\}_{(0)} := \{\!\{-,-\}\!\}_{\tau_{(0)}}$ is the degree preserving graded endomorphism associated with the unshifted Poisson structure $\tau_{(0)}$, see Definition 2.4 and (3.15). Note that, for all polynomials $a, b \in \mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])$, the exponential series defining $\mu_\hbar(a \otimes b)$ truncates to a finite sum. In particular, there is no need to regard $\hbar$ as a formal parameter.

Remark 4.5 The Moyal–Weyl star product $\mu_\hbar$ is a non-commutative deformation of the commutative multiplication μ of the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgCAlg}_\mathbb{C}$ in the sense that the multiplications

$$\mu_\hbar = \mu + \mathcal{O}(\hbar) \tag{4.14}$$

coincide up to terms of order at least $\hbar$ and, moreover, the $\mu_\hbar$-commutator

$$[-,-]_\hbar = \mathrm{i}\,\hbar\,\{-,-\}_{(0)} + \mathcal{O}(\hbar^2) \tag{4.15}$$

is proportional to the Poisson bracket $\{-,-\}_{(0)} := \mu \circ \{\!\{-,-\}\!\}_{(0)}$, see Remark 2.5, up to terms of order at least $\hbar^2$. △

The Moyal–Weyl star product $\mu_\hbar$ is manifestly a degree preserving graded linear map. Furthermore, it is associative and unital with respect to $\mathbb{1} \in \mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])$ as a consequence of the properties of the degree preserving graded endomorphism $\{\!\{-,-\}\!\}_{(0)} = \{\!\{-,-\}\!\}_{\tau_{(0)}}$ and of the exponential. Let us also check that the Moyal–Weyl star product $\mu_\hbar$ is compatible with the differential $\mathcal{Q}$ of $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1])$, i.e.

$$\begin{aligned}
\partial\mu_\hbar &= \mu \circ \partial \exp\left(\frac{\mathrm{i}\,\hbar}{2}\{\!\{-,-\}\!\}_{(0)}\right) \\
&= \mu \circ \left(\sum_{n\geq 1}\frac{1}{n!}\left(\frac{\mathrm{i}\,\hbar}{2}\right)^n \sum_{k=0}^{n-1}\{\!\{-,-\}\!\}_{(0)}^k \circ \big(\partial\{\!\{-,-\}\!\}_{(0)}\big) \circ \{\!\{-,-\}\!\}_{(0)}^{n-1-k}\right) \\
&= 0,
\end{aligned} \tag{4.16}$$

where in the first step we used the compatibility $\partial\mu = 0$ of the symmetric algebra multiplication μ with the differential $\mathcal{Q}$, in the second step we expanded the exponential series and applied the Leibniz rule for ∂ and in the last step we used that $\partial\{\!\{-,-\}\!\}_{(0)} = \{\!\{-,-\}\!\}_{\partial\tau_{(0)}} = 0$ vanishes, see (2.9) and recall that $\tau_{(0)}$ is a cochain map. Therefore, we define the quantized differential graded algebra

$$\mathcal{A}(M) := \big(\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]), \mu_\hbar, \mathbb{1}\big) \in \mathbf{dgAlg}_\mathbb{C}. \tag{4.17}$$

To promote the assignment $\mathbf{Loc}_m \ni M \mapsto \mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$ of the quantized differential graded algebra to a functor, we check the naturality of the Moyal–Weyl star product $\mu_\hbar$ with respect to morphisms $f : M \to N$ in $\mathbf{Loc}_m$, i.e.

$$f_* \circ \mu_\hbar = \mu \circ (f_* \otimes f_*) \circ \exp\left(\frac{\mathrm{i}\,\hbar}{2}\{\!\{-,-\}\!\}_{(0)}\right) = \mu_\hbar \circ (f_* \otimes f_*), \tag{4.18}$$

where in the first step we used naturality of the symmetric algebra multiplication μ and in the second step we used the naturality of the unshifted Poisson structure $\tau_{(0)}$, see (3.22), in combination with (2.10) at all orders in $\hbar$. We are now ready to define the functor

$$\mathcal{A} : \mathbf{Loc}_m \longrightarrow \mathbf{dgAlg}_{\mathbb{C}} \tag{4.19}$$

that assigns to any object $M \in \mathbf{Loc}_m$ the differential graded algebra $\mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$ and to any morphism $f : M \to N$ in $\mathbf{Loc}_m$ the morphism $\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ in $\mathbf{dgAlg}_{\mathbb{C}}$, whose underlying cochain map is the symmetric algebra extension of the pushforward cochain maps $f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$. The next proposition shows that $\mathcal{A}$ is an AQFT.

Proposition 4.6 *The functor $\mathcal{A} : \mathbf{Loc}_m \to \mathbf{dgAlg}_{\mathbb{C}}$ from (4.19) satisfies the Einstein causality and time-slice axioms of Definition 2.9, hence $\mathcal{A} \in \mathbf{AQFT}_m$ is an AQFT.*

Proof First, let us check the Einstein causality axiom. For causally disjoint morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ in $\mathbf{Loc}_m$, Definition 2.4 applied to the unshifted Poisson structure $\tau_{(0)}$ and Theorem 3.13 entail that

$$\{\!\{-,-\}\!\}_{(0)} \circ (f_{1\,*} \otimes f_{2\,*}) = 0 \tag{4.20}$$

vanishes. Therefore, on the image of $f_{1\,*} \otimes f_{2\,*}$ the Moyal–Weyl star product $\mu_\hbar$, see (4.13), coincides

$$\mu_\hbar \circ (f_{1\,*} \otimes f_{2\,*}) = \mu \circ \big(f_{1\,*} \otimes f_{2\,*}\big) \tag{4.21}$$

with the symmetric algebra multiplication μ. Since the latter is commutative, the Einstein causality axiom follows.

Second, let us check the time-slice axiom. Given a Cauchy morphism $f : M \to N$ in $\mathbf{Loc}_m$, it suffices to show that the cochain map underlying $\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ in $\mathbf{dgAlg}_{\mathbb{C}}$ is a quasi-isomorphism. This is the case because the cochain map underlying $\mathcal{A}(f)$ is by definition the symmetric algebra extension of the pushforward cochain map $f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$, which is a quasi-isomorphism by Theorem 3.13. □

Example 4.7 Taking the natural free BV theories from Example 3.11 as inputs, the constructions and results of this subsection produce AQFTs that quantize ordinary field theories, linear Chern–Simons theory and Maxwell p-forms (including linear Yang–Mills theory for $p = 1$). In the case of the Klein–Gordon field and of Maxwell p-forms, earlier constructions of the same AQFTs can be found in [1, 5].

4.3 Comparison

This subsection compares the two different quantization schemes from Sects. 4.1 and 4.2. More specifically, we establish an isomorphism between the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ constructed using the BV formalism and the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ canonically associated with the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$. For this purpose, recall from [7] that $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ consists of the data listed below:

(a) For each $M \in \mathbf{Loc}_m$, the cochain complex $\mathcal{F}_{\mathcal{A}}(M) := \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \in \mathbf{Ch}_{\mathbb{C}}$ underlying $\mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$;

(b) For each time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, the time-ordered product

$$\begin{tikzcd}
\mathcal{F}_{\mathcal{A}}(\underline{M}) \arrow[rr, dashed, "\mathcal{F}_{\mathcal{A}}(\underline{f})"] \arrow[dr, "\bigotimes_i f_{i\,*}"'] & & \mathcal{F}_{\mathcal{A}}(N) \\
& \mathcal{F}_{\mathcal{A}}(N)^{\otimes n} \arrow[ur, "\mu_{\hbar}^{(\rho)}"'] &
\end{tikzcd} \tag{4.22}$$

in $\mathbf{Ch}_{\mathbb{C}}$, where n denotes the length of the tuple $\underline{f}$, ρ is a time-ordering permutation for $\underline{f}$ and $\mu_{\hbar}^{(\rho)} := \mu_{\hbar}^{(n)} \circ \gamma_\rho$ denotes the n-ary Moyal–Weyl star product in the order prescribed by ρ. (The Einstein causality axiom of $\mathcal{A}$ ensures that $\mathcal{F}_{\mathcal{A}}(\underline{f})$ does not depend on the choice of the time-ordering permutation ρ.)

The above data fulfill the axioms of Definition 2.11, see [7] for more details.

In preparation for our comparison result stated in Theorem 4.9, the next lemma explains how the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ captures the usual time-ordered products built out of the *Dirac multiplication*

$$\begin{tikzcd}
\mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[rr, dashed, "\mu_D"] \arrow[dr, "\exp(\,\mathrm{i}\,\hbar\,\{\!\{-,-\}\!\}_D)"'] & & \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])\,, \\
& \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[ur, "\mu"'] &
\end{tikzcd} \tag{4.23}$$

where $\{\!\{-,-\}\!\}_D := \{\!\{-,-\}\!\}_{\tau_D}$ denotes the degree preserving graded endomorphism associated with the natural Dirac pairing τ_D (3.17). Note that the Dirac multiplication μ_D is associative, unital with respect to $\mathbb{1} \in \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ and commutative because the Dirac pairing τ_D is symmetric; however, it is not compatible with the differential $\mathcal{Q}$ of $\mathcal{F}_{\mathcal{A}}(M) \in \mathbf{Ch}_{\mathbb{C}}$ because $\partial\tau_D = \tau_{(-1)}$ does not vanish, see (2.9) and (3.18). Furthermore, the naturality of τ_D and that of the symmetric algebra multiplication μ entail that the Dirac multiplication μ_D is natural too.

Lemma 4.8 *Let $\underline{f} : \underline{M} \to N$ be a time-orderable tuple in $\mathbf{Loc}_m$ of length n. Then, the time-ordered product $\mathcal{F}_\mathcal{A}(\underline{f})$ can be computed using the Dirac multiplication μ_D, i.e.*

$$\begin{tikzcd} \mathcal{F}_\mathcal{A}(\underline{M}) \arrow[rr, "\mathcal{F}_\mathcal{A}(\underline{f})"] \arrow[dr, "\bigotimes_i f_{i\,*}"'] & & \mathcal{F}_\mathcal{A}(N)\,. \\ & \mathcal{F}_\mathcal{A}(N)^{\otimes n} \arrow[ur, "\mu_D^{(n)}"'] & \end{tikzcd} \tag{4.24}$$

Proof Since the Dirac multiplication μ_D is commutative and $\mathcal{F}_\mathcal{A} \in \mathbf{tPFA}_m$ is a time-orderable factorization algebra (hence its time-ordered products are equivariant with respect to permutations, see Definition 2.11), it suffices to check the claim for $\underline{f}$ time-ordered. We argue by induction on the length n. For $n = 0$ and $n = 1$, the claim holds because $\mu_D^{(0)} = \mathbb{1} = \mu_\hbar^{(0)}$ and $\mu_D^{(1)} = \mathrm{id} = \mu_\hbar^{(1)}$. For $n = 2$, Proposition 3.14 entails that, for all $k \geq 1$,

$$\{\!\{-,-\}\!\}_D^k \circ (f_{1\,*} \otimes f_{2\,*}) = \left(\frac{1}{2}\{\!\{-,-\}\!\}_{(0)}\right)^k \circ (f_{1\,*} \otimes f_{2\,*}). \tag{4.25}$$

Then, one computes

$$\begin{aligned} \mu_D \circ (f_{1\,*} \otimes f_{2\,*}) &= \mu \circ \exp\left(\mathrm{i}\,\hbar\,\{\!\{-,-\}\!\}_D\right) \circ (f_{1\,*} \otimes f_{2\,*}) \\ &= \mu \circ \exp\left(\frac{\mathrm{i}\,\hbar}{2}\,\{\!\{-,-\}\!\}_{(0)}\right) \circ (f_{1\,*} \otimes f_{2\,*}) \\ &= \mathcal{F}_\mathcal{A}(f_1, f_2), \end{aligned} \tag{4.26}$$

where in the first step we used the definition of the Dirac multiplication μ_D, see (4.23), in the second step we used (4.25) and in the last step we used the definition of the time-ordered product $\mathcal{F}_\mathcal{A}(f_1, f_2)$, see (4.22). For $n \geq 3$, taking $M \in \mathbf{Loc}_m$, $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f'} : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ as provided by Lemma 4.1, one computes

$$\mu_D^{(n)} \circ \bigotimes_{i=1}^{n} f_{i\,*} = \mu_D \circ (f_* \otimes f_{n\,*}) \circ \left(\left(\mu_D^{(n-1)} \circ \bigotimes_{i=1}^{n-1} f'_{i\,*}\right) \otimes \mathrm{id}\right), \tag{4.27}$$

where we used $\mu_D^{(n)} = \mu_D \circ (\mu_D^{(n-1)} \otimes \mathrm{id})$, $f \circ f'_i = f_i$, for all $i = 1, \dots, n-1$, and the naturality of the Dirac multiplication μ_D. Hence, the claim for length $n \geq 3$ follows from lengths 2 and $n-1$. □

The alternative description from Lemma 4.8 of the time-ordered products of $\mathcal{F}_\mathcal{A} \in \mathbf{tPFA}_m$ plays a key role in the proof of our main result.

Theorem 4.9 *The time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ (constructed via the BV formalism in Sect. 4.1) and the time-orderable prefactorization algebra*

$\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ associated with the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ (constructed via the Moyal–Weyl star product in Sect. 4.2) are isomorphic. Explicitly, the time-ordering map

$$T := \exp(\mathrm{i}\,\hbar\,\Delta_D) : \mathcal{F} \xrightarrow{\cong} \mathcal{F}_{\mathcal{A}} \tag{4.28}$$

in $\mathbf{tPFA}_m$ is an isomorphism. Here $\Delta_D := \Delta_{\tau_D}$, called Dirac Laplacian, is the Laplacian associated with the Dirac pairing τ_D, see Definition 2.6 and (3.17).

Proof Suppose that T as defined above is a morphism of time-orderable prefactorization algebras. Then, it is also an isomorphism with inverse $T^{-1} := \exp(-\mathrm{i}\,\hbar\,\Delta_D) : \mathcal{F}_{\mathcal{A}} \to \mathcal{F}$ in $\mathbf{tPFA}_m$. Therefore, it suffices to check that T is a morphism of time-orderable prefactorization algebras. We split this check in two parts. First, we show the compatibility with differentials, i.e. that, for all $M \in \mathbf{Loc}_m$, the M-component $T_M : \mathcal{F}(M) \to \mathcal{F}_{\mathcal{A}}(M)$ is a cochain map. Second, we show the compatibility with time-ordered products, i.e. that, for all time-orderable tuples $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, $T_N \circ \mathcal{F}(\underline{f}) = \mathcal{F}_{\mathcal{A}}(\underline{f}) \circ T_{\underline{M}}$.

Compatibility with differentials: Recall the BV and Dirac Laplacians $\Delta_{\mathrm{BV}} := \Delta_{\tau_{(-1)}} \in [\mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])]^1$ and $\Delta_D := \Delta_{\tau_D} \in [\mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])]^0$, see Definition 2.6, (3.13) and (3.17). From (2.16), (2.17) and (3.18), it follows that

$$\partial \Delta_D = \Delta_{\partial \tau_D} = \Delta_{\mathrm{BV}}, \qquad \Delta_{\mathrm{BV}} \circ \Delta_D = \Delta_D \circ \Delta_{\mathrm{BV}}. \tag{4.29}$$

Therefore, regarding T_M as a 0-cochain in $[\mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])] \in \mathbf{Ch}_{\mathbb{C}}$, one computes

$$\mathcal{Q} \circ T_M - T_M \circ \mathcal{Q} = \partial T_M = \sum_{n \geq 1} \frac{1}{n!}\, \partial\big((\mathrm{i}\,\hbar\,\Delta_D)^n\big) = T_M \circ (\mathrm{i}\,\hbar\,\Delta_{\mathrm{BV}}). \tag{4.30}$$

In the first step we used the definition of ∂. In the second step we expanded the exponential that defines T_M (recall that the series evaluated on any $a \in \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ truncates to a finite sum) and used that ∂ is linear and vanishes on id. In the last step, we used the Leibniz rule for ∂ with respect to composition, (4.29) and the definition of T_M. Equation (4.30) means that $\mathcal{Q} \circ T_M = T_M \circ \mathcal{Q}_\hbar$, which shows that $T_M : \mathcal{F}(M) \to \mathcal{F}_{\mathcal{A}}(M)$ is a cochain map.

Compatibility with time-ordered products: Since $\mathcal{F}, \mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ are time-orderable prefactorization algebras, their time-ordered products are equivariant with respect to permutations, see Definition 2.11. Hence, it suffices to show the compatibility of T with time-ordered products for $\underline{f} : \underline{M} \to N$ time-ordered. We argue by induction on the length n of $\underline{f}$. For $n = 0$ this is trivial. For $n = 1$, the claim follows from naturality of the Dirac Laplacian $\Delta_D \circ f_* = f_* \circ \Delta_D$, see (2.18) and (3.22). For

$n = 2$, one computes

$$T_N \circ \mathcal{F}(f_1, f_2) = \mu_D \circ (T_N \otimes T_N) \circ \big(\mathcal{F}(f_1) \otimes \mathcal{F}(f_2)\big) = \mathcal{F}_{\mathcal{A}}(f_1, f_2) \circ (T_{M_1} \otimes T_{M_2}), \tag{4.31}$$

where in the first step we used $T_N \circ \mu = \mu_D \circ (T_N \otimes T_N)$, which follows from (2.15), and in the last step we used the claim for $n = 1$ and Lemma 4.8. For $n \geq 3$, taking $M \in \mathbf{Loc}_m$, $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f}' : (M_1, \ldots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ as provided by Lemma 4.1, one computes

$$\begin{aligned} T_N \circ \mathcal{F}(\underline{f}) &= T_N \circ \mathcal{F}(f, f_n) \circ \big(\mathcal{F}(\underline{f}') \otimes \mathrm{id}\big) = \mathcal{F}_{\mathcal{A}}(f, f_n) \circ \big(\mathcal{F}_{\mathcal{A}}(\underline{f}') \otimes \mathrm{id}\big) \circ T_{\underline{M}} \\ &= \mathcal{F}_{\mathcal{A}}(\underline{f}) \circ T_{\underline{M}}, \end{aligned} \tag{4.32}$$

where in the first and last steps we used the composition and identity axioms of time-orderable prefactorization algebras and $f \circ f_i' = f_i$, for all $i = 1, \ldots, n-1$, and in the second step we used the claim for lengths 2 and $n - 1$. □

Example 4.10 The abstract comparison result established in Theorem 4.9, when specialized to the time-orderable prefactorization algebras from Example 4.4 and the corresponding AQFTs from Example 4.7, provides concrete comparison results between the time-orderable prefactorization algebras and the AQFTs quantizing ordinary field theories, linear Chern–Simons theory and Maxwell p-forms (including linear Yang–Mills theory for $p = 1$). Our result generalizes the earlier comparison result in [18], which is formulated only for ordinary field theories, to the case of gauge and also higher gauge theories.

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Data Availability All data generated or analyzed during this study are contained in this document.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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A Natural geometric structures

This appendix provides an explicit description of the constituents of the natural collections of free BV theories from Definition 3.10, namely natural vector bundles, natural fiber metrics and natural linear differential operators. This appendix also outlines the relevant constructions that lead to the key facts (1–3) from Sect. 3.2.

Let us consider the category $\mathbf{VBun}_{\mathbb{R}}$, whose objects are pairs (M, E) consisting of $M \in \mathbf{Loc}_m$ and a finite rank real vector bundle $E \to M$ and whose morphisms are pairs $(f, \bar{f}) : (M_1, E_1) \to (M_2, E_2)$ consisting of a morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ and a vector bundle map $\bar{f} : E_1 \to E_2$ over f that acts as an isomorphism on the fibers, namely such that, for all $p \in M$, the linear map $\bar{f}_p : E_{1\,p} \to E_{2\,f(p)}$ is an isomorphism. Let us consider the evident functor $\pi : \mathbf{VBun}_{\mathbb{R}} \to \mathbf{Loc}_m$, $(M, E) \mapsto M$.

A *natural vector bundle* E is a section of π, i.e. a functor $\mathsf{E} : \mathbf{Loc}_m \to \mathbf{VBun}_{\mathbb{R}}$ such that $\pi \circ \mathsf{E} = \mathrm{id}$. This means that E sends $M \in \mathbf{Loc}_m$ to an object of the form $\mathsf{E}(M) = (M, E_M) \in \mathbf{VBun}_{\mathbb{R}}$ and $f : M_1 \to M_2$ to a morphism of the form $\mathsf{E}(f) = (f, E_f) : \mathsf{E}(M_1) \to \mathsf{E}(M_2)$ in $\mathbf{VBun}_{\mathbb{R}}$. Given a natural vector bundle E and a morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$, one constructs the pullback and pushforward linear maps

$$f^* : \Gamma(E_{M_2}) \longrightarrow \Gamma(E_{M_1}), \qquad f_* : \Gamma_{\mathrm{c}}(E_{M_1}) \longrightarrow \Gamma_{\mathrm{c}}(E_{M_2}) \tag{A.1}$$

for sections and sections with compact support, respectively. Explicitly, given $\varphi \in \Gamma(E_{M_2})$, $f^*\varphi \in \Gamma(E_{M_1})$ is defined by $f^*\varphi := E_f^{-1} \circ \varphi \circ f$, where we note that E_f^{-1} can be inverted as only fibers over $f(M_1)$ are involved. Furthermore, given $\psi \in \Gamma_{\mathrm{c}}(E_{M_1})$, $f_*\psi \in \Gamma_{\mathrm{c}}(E_{M_2})$ is defined as the extension by zero along the open embedding $f(M_1) \subseteq M_2$ of the compactly supported section $E_f \circ \psi \circ f^{-1} : f(M_1) \to E_{M_2}$. Note that f^* and f_* upgrade the assignments $M \mapsto \Gamma(E_M)$ and $M \mapsto \Gamma_{\mathrm{c}}(E_M)$ to functors

$$\Gamma(\mathsf{E}) : \mathbf{Loc}_m^{\mathrm{op}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}, \qquad \Gamma_{\mathrm{c}}(\mathsf{E}) : \mathbf{Loc}_m \longrightarrow \mathbf{Vec}_{\mathbb{R}} \tag{A.2}$$

with values in the category of real vector spaces and linear maps. By constructions of f^* and f_*, it follows that

$$f^*\, f_*\, \psi = \psi, \qquad f_*\, f^*\, \varphi = \varphi, \tag{A.3}$$

for all $\psi \in \Gamma_{\mathrm{c}}(E_{M_1})$ and all $\varphi \in \Gamma_{\mathrm{c}}(E_{M_2})$ with $\mathrm{supp}(\varphi) \subseteq f(M_1)$.

A *natural fiber metric* $\langle -, - \rangle$ on a natural vector bundle E is a natural transformation $\langle -, - \rangle : \mathsf{E} \otimes \mathsf{E} \to \underline{\mathbb{R}}$ to the natural trivial line bundle $\underline{\mathbb{R}} : \mathbf{Loc}_m \to \mathbf{VBun}_{\mathbb{R}}$, $M \mapsto M \times \mathbb{R}$, whose components $\langle -, - \rangle_M : E_M \otimes E_M \to M \times \mathbb{R}$ are fiber metrics, for all $M \in \mathbf{Loc}_m$. Given $M \in \mathbf{Loc}_m$, one defines the integration pairing

$$\langle\!\langle -, - \rangle\!\rangle_M : \Gamma_{\mathrm{c}}(E_M) \otimes \Gamma(E_M) \longrightarrow \mathbb{R} \tag{A.4a}$$

by

$$\langle\!\langle \psi, \varphi \rangle\!\rangle_M := \int_M \langle \psi, \varphi \rangle_M \, \mathrm{vol}_M , \tag{A.4b}$$

for all $\psi \in \Gamma_c(E_M)$ and $\varphi \in \Gamma(E_M)$, see also (2.22). Because $\langle -, - \rangle$ is a natural fiber metric, it follows that the integration pairing $\langle\!\langle -, - \rangle\!\rangle$ is natural in the following sense: for all $f : M_1 \to M_2$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd} \Gamma_c(E_{M_1}) \otimes \Gamma(E_{M_2}) \arrow[r, "\mathrm{id}\otimes f^*"] \arrow[d, "f_*\otimes\mathrm{id}"'] & \Gamma_c(E_{M_1}) \otimes \Gamma(E_{M_1}) \arrow[d, "\langle\!\langle -,- \rangle\!\rangle_{M_1}"] \\ \Gamma_c(E_{M_2}) \otimes \Gamma(E_{M_2}) \arrow[r, "\langle\!\langle -,- \rangle\!\rangle_{M_2}"'] & \mathbb{R} \end{tikzcd} \tag{A.5}$$

in $\mathbf{Vec}_{\mathbb{R}}$ commutes. Indeed, for all $\psi \in \Gamma_c(E_{M_1})$ and $\varphi \in \Gamma(E_{M_2})$, one has

$$\langle\!\langle \psi, f^*\varphi \rangle\!\rangle_{M_1} = \langle\!\langle f^* f_* \psi, f^*\varphi \rangle\!\rangle_{M_1} = \int_{M_1} \big(\langle f_*\psi, \varphi \rangle_{M_2} \circ f \big) \, \mathrm{vol}_{M_1} = \langle\!\langle f_*\psi, \varphi \rangle\!\rangle_{M_2} , \tag{A.6}$$

where we used (A.3) in the first step, naturality of $\langle -, - \rangle$ and the definitions (A.1) of f^* and f_* in the second step and naturality of the integral with respect to $\mathbf{Loc}_m$-morphisms in the last step.

A *natural linear differential operator* P between natural vector bundles E and E' is a natural transformation $P : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}')$ whose components $P_M : \Gamma(E_M) \to \Gamma(E'_M)$ are linear differential operators, for all $M \in \mathbf{Loc}_m$. Explicitly, this means that, for all $f : M_1 \to M_2$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd} \Gamma(E_{M_2}) \arrow[r, "P_{M_2}"] \arrow[d, "f^*"'] & \Gamma(E'_{M_2}) \arrow[d, "f^*"] \\ \Gamma(E_{M_1}) \arrow[r, "P_{M_1}"'] & \Gamma(E'_{M_1}) \end{tikzcd} \tag{A.7}$$

in $\mathbf{Vec}_{\mathbb{R}}$ commutes. The latter diagram entails that P defines also a natural transformation $P : \Gamma_c(\mathsf{E}) \to \Gamma_c(\mathsf{E}')$. Indeed, for all $f : M_1 \to M_2$ in $\mathbf{Loc}_m$, the diagram

$$\begin{tikzcd} \Gamma_c(E_{M_1}) \arrow[r, "P_{M_1}"] \arrow[d, "f_*"'] & \Gamma_c(E'_{M_1}) \arrow[d, "f_*"] \\ \Gamma_c(E_{M_2}) \arrow[r, "P_{M_2}"'] & \Gamma_c(E'_{M_2}) \end{tikzcd} \tag{A.8}$$

in $\mathbf{Vec}_{\mathbb{R}}$ commutes, as it follows from the straightforward computation

$$f_* \, P_{M_1} = f_* \, P_{M_1} \, f^* \, f_* = f_* \, f^* \, P_{M_2} \, f_* = P_{M_2} \, f_*, \tag{A.9}$$

where we used (A.3) in the first and last steps and (A.7) in the second step.

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