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AN AGGLOMERATION-BASED MASSIVELY PARALLEL NON-OVERLAPPING ADDITIVE SCHWARZ PRECONDITIONER FOR HIGH-ORDER DISCONTINUOUS GALERKIN METHODS ON POLYTOPIC GRIDS

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Abstract. In this article we design and analyze a class of two-level non-overlapping additive Schwarz preconditioners for the solution of the linear system of equations stemming from discontinuous Galerkin discretizations of second-order elliptic partial differential equations on polytopic meshes. The preconditioner is based on a coarse space and a non-overlapping partition of the computational domain where local solvers are applied in parallel. In particular, the coarse space can potentially be chosen to be non-embedded with respect to the finer space; indeed it can be obtained from the fine grid by employing agglomeration and edge coarsening techniques. We investigate the dependence of the condition number of the preconditioned system with respect to the diffusion coefficient and the discretization parameters, i.e., the mesh size and the polynomial degree of the fine and coarse spaces. Numerical examples are presented which confirm the theoretical bounds.

1. Introduction

The process of defining a computational grid characterized by standard triangular/tetrahedral or quadrilateral/hexahedral-shaped elements is one of the potential bottlenecks when traditional finite element methods are employed for the numerical approximation of problems characterized by strong complexity of the physical domain, such as, for example, in geophysical applications, fluid-structure interaction, or crack propagation problems. In order to overcome this issue, during the last decade a wide strand of literature has focused on the design of numerical methods that support the use of computational meshes composed of general polygonal and polyhedral elements. In the conforming setting we mention, for example, the Composite Finite Element Method [49, 7], the Mimetic Finite Difference Method [50, 28, 27, 24, 6], the Polygonal Finite Element Method [60], the Extended Finite Element Method [61, 45], the Virtual Element Method [22, 23, 2], and the Hybrid High-Order Method [37, 35, 36]. A major issue in the design of conforming methods on such general polytopic meshes is the definition of a suitable space of continuous piecewise polynomial functions; in this context, this is far from being

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a trivial task, particularly for high-order approximations. An alternative strand of literature has focused on the non-conforming setting, where the ease of defining spaces of piecewise polynomial functions is naturally associated with the flexibility provided by polytopic meshes. Here, we mention, for example, Hybridizable Discontinuous Galerkin Methods [33, 34], non-conforming Virtual Element Methods [19, 12, 32], Gradient Schemes [39], and Discontinuous Galerkin (DG) Methods [3, 21, 20, 7, 8, 31, 30, 10, 4, 13, 14, 1, 5]. In particular, DG methods represent a class of powerful non-conforming numerical schemes in which the use of numerical grids characterized by general polytopic elements couples very well with the possibility to build the underlying discrete space in the physical frame, thereby avoiding the need to map polynomial spaces from a reference/canonical element.

However, as was shown in [9], the condition number of the matrix in a system of linear equations stemming from DG methods may be prohibitively large; indeed, by writing h to denote the mesh-size and p the polynomial degree, the condition number of the DG approximation to Poisson's equation grows like $\mathcal{O}(p^4/h^2)$ as h tends to zero and p tends to infinity. For this reason, in recent years, the development of fast solvers and preconditioners for systems of linear equations stemming from (high-order) DG discretizations has been an active area of research. A variety of two-level and multigrid/multilevel techniques have been proposed, both in the geometric and algebraic settings, for the solution of DG discretizations; see, for example, [47, 38, 26, 25, 16]. In particular, the availability of efficient geometric multilevel solvers is strongly related to the possibility of employing general-shaped polytopic grids; indeed, if polytopic grids can be employed, then the sequence of grids which are required within a multilevel iteration can be defined by agglomeration; see [10, 15] for details. Besides multigrid, a recent strand in the literature has focused on Schwarz domain decomposition methods; see, for example, [62], for a general abstract overview of these methods. In the DG setting where standard triangular/tetrahedral or quadrilateral/hexahedral grids are employed, one of the first contributions in terms of domain decomposition solvers was presented for the solution of elliptic problems in [44], where bounds of order $\mathcal{O}(H/\delta)$ and $\mathcal{O}(H/h)$ were obtained for the condition number of the preconditioned system in the framework of overlapping and non-overlapping Schwarz methods, respectively; here, $H,\ h$, and δ represent the size of the coarse grid, the fine grid, and the amount of overlap, respectively. Dryja and Sarkis proposed in [42] an additive Schwarz preconditioner for the solution of second-order elliptic problems with discontinuous coefficients. There, the authors showed that the condition number of the matrix of the preconditioned system is independent of the jumps of the coefficients across the substructure boundaries and outside a thin layer along the substructure boundaries. A further development of this algorithm, which is very well suited for parallel computation, can be found in [41]. Concerning the setting of high-order DG methods, we mention the work in [9], where additive and multiplicative Schwarz preconditioners were introduced for efficiently solving systems of linear equations arising from the discretization of second-order symmetric elliptic boundary-value problems using hp-version DG methods; there, hp-spectral bounds of order $\mathcal{O}(\sigma p^2 H/h)$ were derived for a class of domain decomposition preconditioners for DG discretizations, where σ is the coefficient of the interior penalty stabilization parameter, p is the polynomial approximation degree, and $H,\ h$ is the size of the coarse and fine mesh, respectively. Recently, in [11], this condition number estimate was improved to yield the optimal rate of $\mathcal{O}(\sigma p^2 H/qh)$, where q denotes the polynomial approximation degree employed within the coarse grid solver; cf. [53] for related work. We also mention the recent work presented in [51], where the influence of the penalty terms, as well as the choice of coarse mesh spaces, on the condition number of the matrix of the system of linear equations preconditioned with additive Schwarz methods were investigated.

The goal of this article is to design and analyze a class of two-level non-overlapping additive Schwarz preconditioners for *hp*-version DG discretizations of second-order elliptic problems on general polytopic grids. Given the DG discrete problem defined on a fine mesh of granularity h, the preconditioner is designed by introducing two additional partitions employed to define the local solver operators and the coarse space correction. On the one hand, the partition employed to build the local solvers is related to a suitable splitting of the DG space and hence it is assumed to be nested with respect to the fine polytopic mesh. On the other hand no conditions are imposed on the coarse partition, which can be non-nested with respect to the fine grid. In particular, we consider the *massively parallel setting*, whereby the partition employed for the local solvers are finer than the grid employed within the coarse solver. Here, we investigate the dependence of the condition number with respect to both the diffusion coefficient and the discretization parameters of the fine and coarse spaces, as well as the granularity of the partition for the local solver. We stress that our analysis is carried out in a very general setting, and, in particular, for nested meshes, it allows the computational domain where the model problem is posed to be non-convex; cf. Section 2 below.

The rest of the paper is organized as follows. In Section 2 we introduce the DG method on polytopic grids for the numerical approximation of second-order elliptic problems. In Section 3 we formulate the additive Schwarz preconditioner analyzed in this article. In Section 4 we then outline some key analytical results that are required for the analysis that follows. Section 5 is devoted to deriving some preliminary results required to obtain the desired bound on the condition number of the matrix of the preconditioned system stated in Section 6. Numerical experiments are presented in Section 7 to confirm the theoretical bounds derived in this article.

2. DG method on polytopic grids

In this article we consider the following second-order elliptic problem. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal/polyhedral domain with boundary $\partial\Omega$ and let $f \in L^2(\Omega)$ be a given function. We consider the following weak formulation: find $u \in V = H_0^1(\Omega)$ such that

$$\mathcal{A}(u, v) := \int_\Omega \rho \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_\Omega f v \, \mathrm{d}\mathbf{x} \quad \forall\, v \in V. \tag{1}$$

Here, $\rho \in L^\infty(\Omega)$ denotes the diffusion coefficient, which we assume to be such that $0 < \rho_0 \leq \rho$; since (1) can always be scaled by $1/\rho_0$, we assume that $\rho_0 = 1$.

Let $\mathcal{T}_h$ be a tessellation of Ω consisting of disjoint open polytopic elements κ of diameter h_κ such that $\overline{\Omega} = \cup \overline{\kappa}$. Here, we denote by $\mathcal{F}_h$ the set of *faces* F, which are defined as the $(d-1)$-dimensional planar facets of the elements κ present in the mesh $\mathcal{T}_h$. For $d = 3$, we assume that each planar face of an element $\kappa \in \mathcal{T}_h$ can be subdivided into a set of co-planar $(d-1)$-dimensional simplices and we refer to this set as the set of *faces*, cf. [30]. Moreover, we write $\mathcal{F}_h^B := \{F \in \mathcal{F}_h \colon F \subset \partial\Omega\}$ to denote the set of boundary faces and $\mathcal{F}_h^I := \mathcal{F}_h \setminus \mathcal{F}_h^B$ the set of interior faces. We set $h := \max_{\kappa\in\mathcal{T}_h} h_\kappa$ and, to ease the presentation, we assume that $h \eqsim h_\kappa$ for all $\kappa \in \mathcal{T}_h$.

Remark 2.1. We adopt the hypothesis that the diffusion coefficient ρ is piecewise constant on each polytopic element $\kappa \in \mathcal{T}_h$ and write $\rho_\kappa = \rho|_\kappa$ to denote its restriction to κ; we refer to [46] for the more general case when ρ violates this assumption.

We assume that $\mathcal{T}_h$ satisfies the following assumptions; cf. [31, 29, 30] for details.

Assumption 2.2. *For any $\kappa \in \mathcal{T}_h$ there exists a set of non-overlapping d-dimensional simplices $\kappa_\flat^F \subset \kappa$, for $F \subset \partial\kappa$, such that for any face $F \subset \partial\kappa$, we have that $\overline{F} = \partial\kappa \cap \partial\kappa_\flat^F$, $\bigcup_{F\subset\partial\kappa} \overline{\kappa_\flat^F} \subset \overline{\kappa}$, and the diameter h_κ of κ can be bounded by $h_\kappa \le C_s d|\kappa_\flat^F|/|F|$ for all $F \subset \partial\kappa$, where $|F|$ and $|\kappa_\flat^F|$ denote the Hausdorff measure of F and $\kappa_\flat^F$, respectively, and C_s is a positive constant independent of the mesh parameters.*

Assumption 2.3. *We assume that there exists a covering $\mathcal{T}_h^\# := \{\mathcal{S}_\kappa\}_\kappa$ of $\mathcal{T}_h$ consisting of shape-regular d-dimensional simplices $\mathcal{S}_\kappa$, such that, for any $\kappa \in \mathcal{T}_h$, there is an $\mathcal{S}_\kappa \in \mathcal{T}_h^\#$ satisfying $\kappa \subset \mathcal{S}_\kappa$ and $\mathrm{diam}(\mathcal{S}_\kappa) \lesssim h_\kappa$. We also assume that*

$$\max_{\kappa\in\mathcal{T}_h} card\big\{\kappa' \in \mathcal{T}_h : \kappa' \cap \mathcal{S}_\kappa \neq \emptyset, \mathcal{S}_\kappa \in \mathcal{T}_h^\# \text{ such that } \kappa \subset \mathcal{S}_\kappa\big\} \le \mathcal{O}_\Omega,$$

where $\mathcal{O}_\Omega$ is a positive constant independent of the mesh parameters.

We write $\langle\cdot\rangle$ to denote the harmonic average operator defined as follows: let η be a sufficiently smooth function; then, for any $F \subset \partial\kappa,\ F \in \mathcal{F}_h$, we define

$$\langle\eta\rangle|_F := \begin{cases} \dfrac{2\ \eta_{\kappa^+}\ \eta_{\kappa^-}}{\eta_{\kappa^+}\ +\ \eta_{\kappa^-}}, & F \in \mathcal{F}_h^I,\ F \subset \partial\kappa^+ \cap \partial\kappa^-, \\ \eta_\kappa, & F \in \mathcal{F}_h^B,\ F \subset \partial\kappa \cap \partial\Omega, \end{cases}$$

where η_κ, $\kappa \in \mathcal{T}_h$, denotes the trace of η on $\partial\kappa$. Moreover, for sufficiently smooth vector- and scalar-valued functions $\boldsymbol{\tau}$ and v, respectively, we define the following jump and weighted average operators across $F \in \mathcal{F}_h$: for $F \subset \partial\kappa^+ \cap \partial\kappa^-$, $F \in \mathcal{F}_h^I$, we write

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{\kappa^+} \cdot \mathbf{n}_{\kappa^+} + \boldsymbol{\tau}_{\kappa^-} \cdot \mathbf{n}_{\kappa^-}, \qquad \{\!\{\boldsymbol{\tau}\}\!\}_\omega := \omega\boldsymbol{\tau}_{\kappa^+} + (1-\omega)\boldsymbol{\tau}_{\kappa^-},$$
$$[\![v]\!] := v_{\kappa^+}\mathbf{n}_{\kappa^+} + v_{\kappa^-}\mathbf{n}_{\kappa^-}, \qquad \{\!\{v\}\!\}_\omega := \omega v_{\kappa^+} + (1-\omega)v_{\kappa^-},$$

where $\mathbf{n}_{\kappa^\pm}$ denotes the unit outward normal vector to $\kappa^\pm$, respectively. For $F \subset \partial\kappa$, $F \in \mathcal{F}_h^B$, we set $\{\!\{\boldsymbol{\tau}\}\!\}_\omega := \boldsymbol{\tau}_\kappa$, $[\![v]\!] := v_\kappa\mathbf{n}$, cf. [18]. Here, $\omega \in [0,1]$ represents the weight employed for the definition of $\{\!\{\cdot\}\!\}_\omega$. Given $\mathcal{T}_h^s \subseteq \mathcal{T}_h$ and $\mathcal{F}_h^s \subseteq \mathcal{F}_h$, we write $\int_{\mathcal{T}_h^s} v \ \mathrm{d}x := \sum_{\kappa\in\mathcal{T}_h^s} \int_\kappa v \ \mathrm{d}x$, $|v|^2_{H^1(\mathcal{T}_h^s)} := \sum_{\kappa\in\mathcal{T}_h^s} \|\nabla v\|^2_{L^2(\kappa)}$, $\int_{\mathcal{F}_h^s} v \ \mathrm{d}s := \sum_{F\in\mathcal{F}_h^s} \int_F v \ \mathrm{d}s$, and $\|v\|^2_{L^2(\mathcal{F}_h^s)} := \sum_{F\in\mathcal{F}_h^s} \|v\|^2_{L^2(F)}$. Finally, writing $\mathcal{P}_p(\kappa)$ to denote the space of polynomials of total degree $p \ge 1$ on κ, $\kappa \in \mathcal{T}_h$, the DG space is given by

$$V_h := \{v_h \in L^2(\Omega) : v_h|_\kappa \in \mathcal{P}_p(\kappa)\ \forall\, \kappa \in \mathcal{T}_h\}.$$

With this notation, we introduce the Symmetric Interior Penalty DG (SIPDG) discretization of (1), cf. [63, 17, 43]: find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v_h) = \int_\Omega f v_h \mathrm{d}x \quad \forall\, v_h \in V_h, \tag{2}$$

where $\mathcal{A}_h : V_h \times V_h \to \mathbb{R}$ is the bilinear form of the DG method defined as

$$\mathcal{A}_h(u_h, v_h) := \int_\Omega \rho\Big[\nabla_h u_h \cdot \nabla_h v_h \ + \nabla_h u_h \cdot \mathcal{R}_\rho([\![v_h]\!]) + \nabla_h v_h \cdot \mathcal{R}_\rho([\![u_h]\!])\Big]\mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h} \sigma_{h,\rho}[\![u_h]\!] \cdot [\![v_h]\!] \ \mathrm{d}s, \tag{3}$$

and ∇_h denotes the piecewise gradient operator on $\mathcal{T}_h$. Here, $\mathcal{R}_\rho : [L^1(\mathcal{F}_h)]^d \to [V_h]^d$ denotes the lifting operator defined by

$$\int_\Omega \mathcal{R}_\rho(\mathbf{q}) \cdot \boldsymbol{\eta} \ \mathrm{d}\mathbf{x} := -\int_{\mathcal{F}_h} \mathbf{q} \cdot \{\!\{\boldsymbol{\eta}\}\!\}_\omega \ \mathrm{d}s \quad \forall \boldsymbol{\eta} \in [V_h]^d, \tag{4}$$

where, we take $\omega|_F := \frac{\rho_{\kappa^-}}{\rho_{\kappa^+} + \rho_{\kappa^-}}$ on each internal face $F \in \mathcal{F}_h^I$ shared by $\kappa^\pm$. In (3), according to [40, 30], $\sigma_{h,\rho} \in L^\infty(\mathcal{F}_h)$ denotes the interior penalty stabilization function, which is defined by

$$\sigma_{h,\rho}|_F := C_\sigma \langle \rho_\kappa \rangle \frac{p^2}{\langle h_\kappa \rangle} \quad \forall F \in \mathcal{F}_h, \tag{5}$$

with $C_\sigma > 0$ independent of ρ, p, $|F|$, $|\kappa|$ and h_κ. It is well known that the condition number of the operator $\mathcal{A}_h$ is potentially prohibitively large and depends on the size of the partition $\mathcal{T}_h$ and the polynomial degree p employed for the discretization; cf. [9] for standard triangular/tetrahedral/hexahedral grids and [16] for polytopic grids. Our goal is to introduce a massively parallel non-overlapping additive Schwarz preconditioner, which can be employed as a preconditioner to accelerate the convergence of iterative solvers such as the Conjugate Gradient method.

3. Non-overlapping additive Schwarz preconditioner

The definition of the additive Schwarz preconditioner requires the introduction of two additional partitions (besides $\mathcal{T}_h$): a partition $\mathcal{T}_\mathbb{H}$ composed of disjoint polytopic subdomains where local solvers are applied in parallel and a non-overlapping partition $\mathcal{T}_H$ employed for the coarse space correction. To this end, we introduce the following notation:

- $\mathcal{T}_\mathbb{H} := \{\Omega_1, \ldots, \Omega_{N_\mathbb{H}}\}$ of size $\mathbb{H} := \max_{1\le i\le N_\mathbb{H}}\{\mathrm{diam}(\Omega_i)\}$ consisting of disjoint open subdomains Ω_i, such that $\overline{\Omega} = \cup_{i=1}^{N_\mathbb{H}} \overline{\Omega}_i$, and each subdomain Ω_i is the union of some elements $\kappa \in \mathcal{T}_h$; we assume that $\mathbb{H} \approx \mathrm{diam}(\Omega_i)$ for all $i = 1, \ldots, N_\mathbb{H}$. We also assume that a colouring property holds, i.e., there exists a positive integer $N_\mathbb{S}$ such that

$$\max_{i=1,\ldots,N_\mathbb{H}} \mathrm{card}\{\Omega_j : \ \partial\Omega_i \cap \partial\Omega_j \neq \emptyset\} \le N_\mathbb{S}, \tag{6}$$

  i.e., $N_\mathbb{S}$ represents the maximum number of neighbours that any subdomain $\Omega_i \in \mathcal{T}_\mathbb{H}$ may possess.
- $\mathcal{T}_H := \{\mathcal{D}_1, \ldots, \mathcal{D}_{N_H}\}$ of size $H := \max_{1\le j\le N_H}\{\mathrm{diam}(\mathcal{D}_j)\}$ such that $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for any $i \neq j$, $\overline{\Omega} = \cup_{j=1}^{N_H} \overline{\mathcal{D}}_j$, and $H \approx \mathrm{diam}(\mathcal{D}_j)$ for all $j = 1, \ldots, N_H$.

We remark that the grids $\mathcal{T}_H$ and $\mathcal{T}_h$ are possibly non-nested, cf. Figure 1.

Remark 3.1. Given that $\mathcal{T}_\mathbb{H}$ is defined by agglomeration of fine-grid-elements $\kappa \in \mathcal{T}_h$, we write $\mathcal{T}_h \subseteq \mathcal{T}_\mathbb{H}$, since, for all $\kappa \in \mathcal{T}_h$, there exists a $\mathcal{K} \in \mathcal{T}_\mathbb{H}$ such that $\kappa \subseteq \mathcal{K}$. However, we point out that no further assumptions are needed on the relationship between $\mathcal{T}_\mathbb{H}$ and $\mathcal{T}_H$ for the definition of our method. Classical

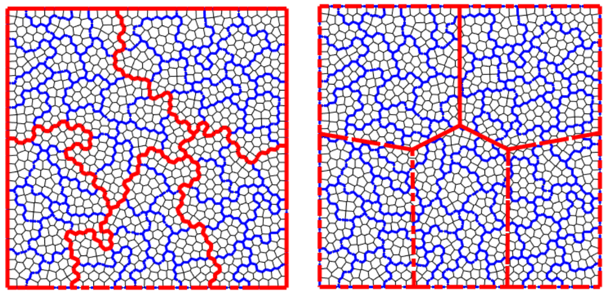

FIGURE 1. Example of polygonal $\mathcal{T}_h$ (black), $\mathcal{T}_{\mathbb{H}}$ (blue) and $\mathcal{T}_H$ (red), when the coarse and fine grids are nested, i.e., $\mathcal{T}_h \subseteq \mathcal{T}_H$, (left) and non-nested, i.e., $\mathcal{T}_h \nsubseteq \mathcal{T}_H$ (right).

additive Schwarz methods have typically been defined based on the assumption that $\mathcal{T}_h \subseteq \mathcal{T}_H \subseteq \mathcal{T}_{\mathbb{H}}$. In this article we take a different approach: firstly, we *assume* that the granularity of $\mathcal{T}_{\mathbb{H}}$ is finer than that of $\mathcal{T}_H$; indeed, we are particularly interested in the massively parallel case whereby $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$, cf. [41]. Secondly, we also permit the use of non-nested coarse and fine partitions, i.e., when $\mathcal{T}_h \nsubseteq \mathcal{T}_H$.

The main ingredients of the additive Schwarz method are defined as follows.

Local Solvers. Consider the subdomain partition $\mathcal{T}_{\mathbb{H}}$ with cardinality $N_{\mathbb{H}}$. Then, for each subdomain $\Omega_i \in \mathcal{T}_{\mathbb{H}}$ we define a local space V_i as the restriction of the DG finite element space V_h to Ω_i, i.e., for $i = 1, \dots, N_{\mathbb{H}}$,

$$V_i := \{v_i \in L^2(\Omega_i) : \ v_i|_\kappa \in \mathcal{P}_p(\kappa) \ \forall \kappa \in \mathcal{T}_h, \ \kappa \subseteq \Omega_i\} \equiv V_h|_{\Omega_i}.$$

The associated local bilinear form on $V_i \times V_i$ is defined by

$$\mathcal{A}_i : V_i \times V_i \to \mathbb{R}, \quad \mathcal{A}_i(u_i, v_i) := \mathcal{A}_h(R_i^\top u_i, R_i^\top v_i) \quad \forall u_i, v_i \in V_i,$$

where $R_i^\top : V_i \to V_h$ denotes the classical extension-by-zero operator from the local space V_i to the global space V_h. The restriction operator $R_i : V_h \to V_i$, $i = 1, \dots, N_{\mathbb{H}}$, is defined as the transpose of $R_i^\top$ with respect to the $L^2(\Omega_i)$ inner product.

Coarse Solver. For $1 \leq q \leq p$, the coarse solver is defined on the partition $\mathcal{T}_H$. To this end, let V_0 be the DG finite element space defined on $\mathcal{T}_H$ given by

$$V_0 \equiv V_H := \{v_H \in L^2(\Omega) : v_H|_{\mathcal{D}_j} \in \mathcal{P}_q(\mathcal{D}_j), \ j = 1, \dots, N_H\}.$$

Further, let $R_0^\top$ be the L^2-prolongation operator from V_0 to V_h, defined as:

$$R_0^\top : v_0 \in V_0 \longmapsto R_0^\top v_0 \in V_h : \quad \int_\Omega R_0^\top v_0 w_h \mathrm{d}\mathbf{x} := \int_\Omega v_0 w_h \mathrm{d}\mathbf{x} \quad \forall w_h \in V_h. \tag{7}$$

In this way $R_0^\top$ is well defined also when $\mathcal{T}_H$ and $\mathcal{T}_h$ are non-nested. Then, the bilinear form associated to V_0 is defined by

$$\mathcal{A}_0 : V_0 \times V_0 \to \mathbb{R}, \quad \mathcal{A}_0(u_0, v_0) := \mathcal{A}_h(R_0^\top u_0, R_0^\top v_0) \quad \forall u_0, v_0 \in V_0. \tag{8}$$

Introducing the projection operators $P_i := R_i^\top \widetilde{P}_i : V_h \to V_h,\ i = 0, 1, \ldots, N_{\mathbb{H}}$, where

$$\widetilde{P}_i : V_h \to V_i,\ \mathcal{A}_i(\widetilde{P}_i v_h, w_i) := \mathcal{A}_h(v_h, R_i^\top w_i) \quad \forall\, w_i \in V_i,\ i = 0,1, \ldots, N_{\mathbb{H}},$$

the additive Schwarz operator is defined by $P_{ad} := \sum_{i=0}^{N_{\mathbb{H}}} P_i$. For an upper bound on the condition number of P_{ad}, we refer to Section 6 below.

4. Analytical background

Before we embark on developing the preliminary results needed to analyze the condition number of the additive Schwarz operator introduced in the previous section, we first state two key theorems, which are essential in the forthcoming analysis, and which may be of relevance in other more general settings. To this end, we write $D \subset \mathbb{R}^d$, $d \geq 2$, to denote a bounded, open, simply connected domain, with boundary ∂D; in the proceeding analysis, D will be selected to be an element $\mathcal{D}_j$, $j = 1, \ldots, N_H$, from the coarse mesh $\mathcal{T}_H$. Following [59], cf. also [56, 57], we introduce the definition of a *special Lipschitz domain*, and the notion of a *domain with a minimally smooth boundary.*

Definition 4.1. *(*[59, Section 3.2],[56, Definition 1]*) Let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a function that satisfies the Lipschitz condition*

$$|\phi(\mathbf{x}) - \phi(\mathbf{y})| \leq M|\mathbf{x} - \mathbf{y}| \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}. \tag{9}$$

The smallest M for which (9) *holds is denoted by C_ϕ. Based on this function, we define the special Lipschitz domain it determines to be the set of points lying above the hypersurface $y = \phi(\mathbf{x})$ in $\mathbb{R}^d$, i.e.,*

$$\chi = \left\{\mathbf{x} \in \mathbb{R}^d : x_d > \phi(x_1, x_2, \ldots, x_{d-1})\right\}.$$

The Lipschitz constant of the domain χ is defined by $C_\chi := C_\phi$.

Equipped with this definition, we now introduce the concept of a *minimally smooth boundary.*

Definition 4.2. *(*[59, Section 3.3],[56, Definition 2]*) The boundary ∂D of D is said to be minimally smooth if there exists an $\epsilon > 0$, a positive integer N, an $M > 0$, and a sequence $U_1, U_2, \ldots$ of open sets such that:*

1. *If $\mathbf{x} \in \partial D$, then $B(\mathbf{x}, \epsilon) \subset U_i$ for some i, where $B(\mathbf{x}, \epsilon)$ denotes the ball with centre $\mathbf{x}$ and radius ϵ;*
2. *No point $\mathbf{x} \in \mathbb{R}^d$ is contained in more than N of the U_i's;*
3. *For each i there exists a special Lipschitz domain χ_i with $C_{\chi_i} \leq M$ such that*
$$U_i \cap D = U_i \cap \chi_i.$$

Based on the previous definition, we now introduce the following classical extension operator.

Theorem 4.3. *(*[59, Theorem 5], [56, Theorem 3]*) Let D be a domain with minimally smooth boundary. Then, there exists a linear extension operator $\mathfrak{E} : H^s(D) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}$, such that $\mathfrak{E}v|_D = v$ and*

$$\|\mathfrak{E}v\|_{H^s(\mathbb{R}^d)} \leq C_{\mathfrak{E}} \|v\|_{H^s(D)},$$

where $C_{\mathfrak{E}}$ is a positive constant depending only on s and the constants ϵ, N, and M defined in Definition 4.2, which characterize the boundary ∂D.

Remark 4.4. We highlight that, crucially, the constant $C_{\mathfrak{E}}$ appearing in Theorem 4.3 is independent of the measure of the underlying domain D.

Secondly, we now study the regularity of the following Neumann boundary-value problem: find z such that

$$-\Delta z = f \quad \text{in } D, \qquad \nabla z \cdot \mathbf{n} = 0 \quad \text{on } \partial D, \tag{10}$$

with $f \in L^2_0(D) := \{v \in L^2(D) \ : \ \int_D v \ \mathrm{d}\mathbf{x} = 0\}$.

Theorem 4.5. *Let D be a bounded, open, convex (and therefore Lipschitz) domain. Then, there exists a unique solution $z \in H^2(D) \cap L^2_0(D)$ to the homogeneous Neumann problem (10). Moreover, the following stability bound holds:*

$$\|z\|_{H^2(D)} \leq 2\sqrt{6}\left(\|f\|_{L^2(D)} + \frac{1}{\pi}\mathrm{diam}(D)\|\nabla z\|_{L^2(D)}\right). \tag{11}$$

Proof. We defer the proof to Appendix A. □

Remark 4.6. An analogous bound to (11), with constant equal to unity, has been derived in [48, Theorem 4.3.1.4] for convex polygonal domains in $\mathbb{R}^2$; there the term $\frac{1}{\pi}\mathrm{diam}(D)\|\nabla z\|_{L^2(\Omega)}$ on the right-hand side of (11) is replaced by $\|z\|_{H^1(D)}$.

5. Preliminary results

We first present some preliminary results, which will be employed within the analysis contained in Section 6. Throughout this article, we use the notation $x \lesssim y$ to signify that there exists a positive constant C, independent of the diffusion coefficient ρ, the polynomial approximation degrees p and q, and the mesh parameters of $\mathcal{T}_h$, $\mathcal{T}_{\mathbb{H}}$ and $\mathcal{T}_H$, such that $x \leq Cy$. Similarly we write $x \gtrsim y$ in lieu of $x \geq Cy$, while $x \approx y$ is used if both $x \lesssim y$ and $x \gtrsim y$ hold. For the sake of simplicity of the presentation, we make the following assumption.

Assumption 5.1. *We assume that the fine and coarse grids $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, are nested, i.e., $\mathcal{T}_h \subseteq \mathcal{T}_H$.*

Remark 5.2. The extension of the theoretical analysis to the general case $\mathcal{T}_h \nsubseteq \mathcal{T}_H$ is deferred to Appendix B.

Here, we introduce the following energy norm:

$$\|w\|^2_{h,\rho} := \int_{\mathcal{T}_h} \rho|\nabla_h w|^2 \ \mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h} \sigma_{h,\rho}|[\![w]\!]|^2 \ \mathrm{d}s. \tag{12}$$

The well-posedness of problem (2) with respect to the norm (12) is then established in the following lemma, cf. [30].

Lemma 5.3. *Suppose that $\mathcal{T}_h$ satisfies Assumption 2.2; then,*

$$\begin{aligned} \mathcal{A}_h(u_h, v_h) &\lesssim \|u_h\|_{h,\rho}\|v_h\|_{h,\rho} \quad && \forall\, u_h, v_h \in V_h, \\ \mathcal{A}_h(u_h, u_h) &\gtrsim \|u_h\|^2_{h,\rho} && \forall\, u_h \in V_h. \end{aligned}$$

The second bound holds provided that C_σ appearing in (5) is sufficiently large.

We also recall the following trace inequality on polytopic domains, introduced in [29].

Lemma 5.4. *Suppose that $\mathcal{T}_h$ satisfies Assumption 2.2; then, the following inequality holds:*

$$\|v\|^2_{L^2(\partial\kappa)} \lesssim \frac{p^2}{h_\kappa} \|v\|^2_{L^2(\kappa)} \quad \forall\, v \in \mathcal{P}_p(\kappa)\ \forall\, \kappa \in \mathcal{T}_h.$$

Next we also recall a result regarding the approximation operator presented in [31, Theorem 5.2] and [29, Lemma 5.5], to which we refer for details. However, for the purposes of this article, we consider a slight generalization: given a connected subdomain $D \subseteq \Omega$, we assume that D is formed from the union of a subset of elements $\kappa \in \mathcal{T}_h$; we denote the collection of such elements by $\mathcal{T}_{h,D}$, i.e., $\bar{D} := \cup_{\kappa\in\mathcal{T}_{h,D}}\bar{\kappa}$. With this definition, the approximant relies on the properties of the extension operator $\mathfrak{E} : H^s(D) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_0$, introduced in Theorem 4.3. Hence, following [31, Theorem 5.2] and [29, Lemma 5.5], we deduce the following lemma.

Lemma 5.5. *Suppose that Assumption 2.3 is satisfied, and let $v \in L^2(D)$ be such that, for some $k \geq 0$, $v|_\kappa \in H^k(\kappa)$ and $\mathfrak{E}v|_{\mathcal{S}_\kappa} \in H^k(\mathcal{S}_\kappa)$ for $\kappa \in \mathcal{T}_{h,D}$, with $\mathcal{S}_\kappa \in \mathcal{T}_h^{\#}$ as defined in Assumption 2.3. Then, there exists a projection operator $\Pi^p_{h,\kappa} : L^2(D) \to \mathcal{P}_p(\kappa)$ such that*

$$\|v - \Pi^p_{h,\kappa} v\|_{H^l(\kappa)} \lesssim \frac{h^{s-l}}{p^{k-l}} \|\mathfrak{E}v\|_{H^k(\mathcal{S}_\kappa)}, \qquad 0 \leq l \leq k.$$

$$\|v - \Pi^p_{h,\kappa} v\|_{L^2(\partial\kappa)} \lesssim \frac{h^{s-1/2}}{p^{k-1/2}} \|\mathfrak{E}v\|_{H^k(\mathcal{S}_\kappa)}, \qquad k \geq 1,$$

where $s := \min\{p+1, k\}$ and $p \geq 1$.

Remark 5.6 (Global approximant)**.** Given that Lemma 5.5 holds for all $\kappa \in \mathcal{T}_{h,D}$ we can define the global approximation operator $\Pi^p_h : L^2(D) \to V_h|_D$ such that $\Pi^p_h|_\kappa = \Pi^p_{h,\kappa}$. If $v \in H^k(D)$, $k \geq 0$, then, by noting Assumption 2.3, together with Theorem 4.3, the following bound holds:

$$\|v - \Pi^p_h v\|_{H^l(\mathcal{T}_{h,D})} \lesssim \frac{h^{s-l}}{p^{k-l}} \|v\|_{H^k(D)}, \quad 0 \leq l \leq k.$$

For the purposes of the proceeding analysis, it is assumed that the hidden constant $C_{\mathfrak{E}}$ in the above error estimate is uniformly bounded with respect to ∂D, when D is selected to be any coarse element $\mathcal{D}_j \in \mathcal{T}_H$, $j = 1, \ldots, N_H$.

A key ingredient in our analysis is the conforming approximant defined in [11]. In particular, to ensure that the preconditioner is scalable in the presence of jumps in the diffusion coefficient, here we define the conforming approximant in a slightly different manner, in order to obtain an approximation of discontinuous discrete functions $v_h \in V_h$ on each local domain $\mathcal{D}_j \in \mathcal{T}_H$, $j = 1, \ldots, N_H$. To this end, we adopt the following assumption on the coarse mesh $\mathcal{T}_H$.

Assumption 5.7. *We assume that $\mathcal{D}_j \in \mathcal{T}_H$ is a convex polytope with boundary $\partial\mathcal{D}_j$, for any $j = 1, \ldots, N_H$, and that $|\mathcal{D}_j| \approx H^d$, $j = 1, \ldots, N_H$, where $|\mathcal{D}_j|$ represents the Hausdorff measure of $\mathcal{D}_j$.*

For the sake of the analysis, we define the following local sets generated from $\mathcal{T}_h$, $\mathcal{T}_{\mathbb{H}}$ and $\mathcal{T}_H$: for $\Omega_i \in \mathcal{T}_{\mathbb{H}}$, $i = 1, \ldots, N_{\mathbb{H}}$, we write

$$\mathcal{F}^I_h(\Omega_i) := \{F \in \mathcal{F}_h : F \subset \Omega_i\}. \tag{13}$$

Furthermore, for $\mathcal{D}_j \in \mathcal{T}_H$, $j = 1, \ldots, N_H$, we write

$$\begin{aligned} \mathcal{T}_{h,j} &:= \{\kappa \in \mathcal{T}_h : \kappa \subset \mathcal{D}_j\}, \quad \mathcal{F}^I_{h,j} := \{F \in \mathcal{F}_h : F \subset \mathcal{D}_j\}, \\ \mathcal{F}^B_{h,j} &:= \{F \in \mathcal{F}_h : F \subset \partial\mathcal{D}_j\}, \quad \mathcal{F}_{h,j} := \mathcal{F}^I_{h,j} \cup \mathcal{F}^B_{h,j}. \end{aligned} \tag{14}$$

Remark 5.8. Note that, as consequence of Assumption 5.1, we have that $\mathcal{T}_{h,j} \subseteq \mathcal{T}_h$, $j = 1, \ldots, N_H$, and, consequently, $\mathcal{T}_{h,j}$ also satisfies Assumptions 2.2 and 2.3, for all $j = 1, \ldots, N_H$.

The local conforming approximant is then defined as follows.

Definition 5.9. *Let $\mathcal{D}_j \in \mathcal{T}_H$ satisfy Assumption 5.7, $j = 1, \ldots, N_H$, and let the discrete gradient operator of $v_h \in V_h$ inside $\mathcal{D}_j$ be defined by the equality $\mathcal{G}_{h,j}(v_h) = (\nabla_h v_h + \mathcal{R}_1(\mathcal{J}_j(v_h)))\mathbb{1}_{\{\mathcal{D}_j\}}$, $j = 1, \ldots, N_H$, where $\mathbb{1}_{\{\mathcal{D}_j\}}$ is the characteristic function on $\mathcal{D}_j$, while $\mathcal{J}_j : V_h \to [L^1(\mathcal{F}_h)]^d$ is defined by*

$$\mathcal{J}_j(v_h)|_F := [\![v_h]\!]|_F \quad \text{if } F \in \mathcal{F}^I_{h,j}, \qquad \mathcal{J}_j(v_h)|_F := 0 \quad \text{otherwise,} \tag{15}$$

where $\mathcal{F}^I_{h,j}$ is the set defined in (14). Here, $\mathcal{R}_1 : [L^1(\mathcal{F}_h)]^d \to [V_h]^d$ is the lifting operator with $\rho = 1$ and $\omega = 1/2$ in its definition given in (4). Then, $\widetilde{v}_{h,j}$ is defined as the solution of the following problem: find $\widetilde{v}_{h,j} \in \widetilde{V}_j = H^1(\mathcal{D}_j) \cap L^2_0(\mathcal{D}_j)$ such that

$$\int_{\mathcal{D}_j} \nabla\widetilde{v}_{h,j} \cdot \nabla w \, \mathrm{d}\mathbf{x} = \int_{\mathcal{D}_j} \mathcal{G}_{h,j}(v_h) \cdot \nabla w \, \mathrm{d}\mathbf{x} \quad \forall\, w \in \widetilde{V}_j. \tag{16}$$

Before we proceed, we first recall the following Poincaré inequality, cf. [64, Corollary 3.4].

Lemma 5.10 (Poincaré's inequality)**.** *Given $v \in H^1(D)$, where D is an open bounded convex domain in $\mathbb{R}^d$, $d \geq 1$, then the following bound holds*

$$\|v\|_{L^2(D)} \leq 3\,diam(D)^{1+d/2}|D|^{-1/2}\|\nabla v\|_{L^2(D)} + |D|^{-1/2}\left|\int_D v\,\mathrm{d}\mathbf{x}\right|,$$

where $diam(D)$ and $|D|$ denote the diameter and Hausdorff measure of D, respectively.

Remark 5.11. Since $\widetilde{v}_{h,j} \in \widetilde{V}_j$ and $\mathcal{D}_j$ satisfies Assumption 5.7, $j = 1, \ldots, N_H$, upon application of Lemma 5.10 (and exploiting Assumption 5.7), we deduce that

$$\|\widetilde{v}_{h,j}\|^2_{H^1(\mathcal{D}_j)} \leq (1 + 9H^2)\|\nabla\widetilde{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)} \lesssim \|\nabla\widetilde{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)},$$

where we also made use of the fact that $\mathrm{diam}(\mathcal{D}_j) \eqsim H \lesssim 1$.

By proceeding as in [11] we prove the following approximation result.

Theorem 5.12. *Let Assumption 5.1 be satisfied. Suppose that $\mathcal{T}_h$ satisfies Assumptions 2.2 and 2.3, and $\mathcal{T}_H$ satisfies Assumption 5.7. Given $v_h \in V_h$, for any $\mathcal{D}_j \in \mathcal{T}_H$, $j = 1, \ldots, N_H$, we take $\widetilde{v}_{h,j} \in \widetilde{V}_j$, $j = 1, \ldots, N_H$, to be the conforming approximant given in Definition 5.9 and we define*

$$\overline{v}_{h,j} := \widetilde{v}_{h,j} + \frac{1}{|\mathcal{D}_j|}\int_{\mathcal{D}_j} v_h \mathrm{d}\mathbf{x}.$$

Then, the following approximation and stability bounds hold:

$$\|v_h - \overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)} \lesssim \frac{h}{p}\|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_{h,j}^I)}, \tag{17}$$

$$|\widetilde{v}_{h,j}|^2_{H^1(\Omega)} \lesssim \|\nabla_h v_h\|^2_{L^2(\mathcal{D}_j)} + \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|^2_{L^2(\mathcal{F}_{h,j}^I)}, \tag{18}$$

for $j = 1, \ldots, N_H$.

Proof. Given $\mathcal{D}_j \in \mathcal{T}_H$, $j = 1, \ldots, N_H$, let $z \in H^2(\mathcal{D}_j) \cap L^2_0(\mathcal{D}_j)$ be the solution of problem (10) posed on the domain $\mathcal{D}_j$, i.e., $D = \mathcal{D}_j$, with $f = v_h - \overline{v}_{h,j}$. Then, by employing integration by parts we obtain

$$\begin{aligned}\|v_h - \overline{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)} &= -\int_{\mathcal{D}_j} (v_h - \overline{v}_{h,j})\ \Delta z \ \mathrm{d}\mathbf{x} \\ &= \int_{\mathcal{D}_j} (\nabla_h v_h - \nabla \widetilde{v}_{h,j}) \cdot \nabla z \ \mathrm{d}\mathbf{x} \ - \int_{\mathcal{F}_{h,j}^I} \nabla z \cdot [\![v_h]\!] \ \mathrm{d}s \\ &= -\int_{\mathcal{D}_j} \mathcal{R}_1(\mathcal{J}_j(v_h)) \cdot \nabla z \ \mathrm{d}\mathbf{x} \quad - \int_{\mathcal{F}_{h,j}^I} \nabla z \cdot [\![v_h]\!] \ \mathrm{d}s,\end{aligned}$$

where we have also used the facts that $\nabla \overline{v}_{h,j} = \nabla \widetilde{v}_{h,j}$, $\nabla z \cdot \mathbf{n}|_F = 0$ if $F \subset \partial\mathcal{D}_j$, $[\![\overline{v}_{h,j}]\!]|_F = \mathbf{0}$ for all $F \in \mathcal{F}_{h,j}^I$, since $\overline{v}_{h,j} \in H^1(\mathcal{D}_j)$, and that $\mathcal{G}_{h,j}(v_h) = (\nabla_h v_h + \mathcal{R}_1(\mathcal{J}_j(v_h))\mathbb{1}_{\{\mathcal{D}_j\}}$ together with the definition of $\widetilde{v}_{h,j}$, cf. (16). Using the definition of $\mathcal{R}_1$ and $\mathcal{J}_j$, cf. (4) and (15), respectively, for any $z_h \in V_h|_{\mathcal{D}_j}$, we have

$$\begin{aligned}\|v_h - \overline{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)} &= -\int_{\mathcal{F}_{h,j}^I} \nabla z \cdot [\![v_h]\!] \ \mathrm{d}s \ - \int_{\mathcal{D}_j} \mathcal{R}_1(\mathcal{J}_j(v_h)) \cdot \nabla z \ \mathrm{d}\mathbf{x} \\ &= - \int_{\mathcal{F}_{h,j}^I} \nabla z \cdot [\![v_h]\!] \ \mathrm{d}s \ + \int_{\mathcal{F}_{h,j}^I} [\![v_h]\!] \cdot \{\!\{\nabla_h z_h\}\!\}_{1/2} \ \mathrm{d}s \\ &\quad - \int_{\mathcal{D}_j} \mathcal{R}_1(\mathcal{J}_j(v_h)) \cdot (\nabla z - \nabla_h z_h) \ \mathrm{d}\mathbf{x} \\ &= \int_{\mathcal{F}_{h,j}^I} [\![v_h]\!] \cdot \{\!\{\nabla_h z_h - \nabla z\}\!\}_{1/2} \ \mathrm{d}s \\ &\quad - \int_{\mathcal{D}_j} \mathcal{R}_1(\mathcal{J}_j(v_h)) \cdot (\nabla z - \nabla_h z_h) \ \mathrm{d}\mathbf{x};\end{aligned}$$

here we have used that, since $z \in H^2(\mathcal{D}_j)$, $\{\!\{\nabla z\}\!\}_{1/2}|_F = \nabla z|_F$, $F \in \mathcal{F}_{h,j}^I$. Hence, we get

$$\begin{aligned}\|v_h - \overline{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)} &\lesssim \|\mathcal{R}_1(\mathcal{J}_j(v_h))\|_{L^2(\mathcal{D}_j)}\|\nabla z - \nabla_h z_h\|_{L^2(\mathcal{D}_j)} \\ &\quad + \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_{h,j}^I)}\|\sigma_{h,1}^{-1/2}\{\!\{\nabla z - \nabla_h z_h\}\!\}_{1/2}\ \|_{L^2(\mathcal{F}_{h,j}^I)}.\end{aligned} \tag{19}$$

The first term on the right-hand side of (19) can be written as follows:

$$\begin{aligned}\|\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\mathcal{D}_j)} &\leq \int_{\Omega} \mathcal{R}_1(\mathcal{J}_j(v_h)) \cdot \mathcal{R}_1(\mathcal{J}_j(v_h)) \ \mathrm{d}\mathbf{x} \\ &= -\int_{\mathcal{F}_h} (\mathcal{J}_j(v_h)) \cdot \{\!\{\mathcal{R}_1(\mathcal{J}_j(v_h))\}\!\}_{1/2} \ \mathrm{d}s \\ &= -\int_{\mathcal{F}_{h,j}^I} [\![v_h]\!] \cdot \{\!\{\mathcal{R}_1(\mathcal{J}_j(v_h))\}\!\}_{1/2} \ \mathrm{d}s,\end{aligned} \tag{20}$$

where we have also used the definitions of $\mathcal{R}_1$ and $\mathcal{J}_j$, cf. (4) and (15), respectively. Then, from (20) and the Cauchy–Schwarz inequality we obtain the following bound:

$$\|\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\mathcal{D}_j)} \le \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}^I_{h,j})}\ \|\sigma_{h,1}^{-1/2}\{\!\{\mathcal{R}_1(\mathcal{J}_j(v_h))\}\!\}_{1/2}\|_{L^2(\mathcal{F}^I_{h,j})}. \tag{21}$$

The second term on the right-hand side of (21) can be bounded by invoking Lemma 5.4 as follows:

$$\begin{aligned}\|\sigma_{h,1}^{-1/2}\{\!\{\mathcal{R}_1(\mathcal{J}_j(v_h))\}\!\}_{1/2}\|^2_{L^2(\mathcal{F}^I_{h,j})} &\le \sum_{\kappa\in\mathcal{T}_{h,j}} \|\sigma_{h,1}^{-1/2}\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\partial\kappa)}\\ &= C_\sigma^{-1}\sum_{\kappa\in\mathcal{T}_{h,j}}\frac{\langle h_\kappa\rangle}{p^2}\|\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\partial\kappa)}\\ &\lesssim \sum_{\kappa\in\mathcal{T}_{h,j}}\|\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\kappa)},\end{aligned} \tag{22}$$

where we have used that $\langle h_\kappa\rangle \le 2h_\kappa$ for any $\kappa\in\mathcal{T}_h$. By inserting (22) into (21) we obtain

$$\|\mathcal{R}_1(\mathcal{J}_j(v_h))\|_{L^2(\mathcal{D}_j)} \lesssim \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}^I_{h,j})}. \tag{23}$$

Hence, by selecting $z_h = \Pi_h^p z$ in (19) and employing (23), Lemma 5.5, cf., also, Remark 5.6, and Theorem 4.5, we obtain

$$\begin{aligned}\|v_h-\overline{v}_{h,j}\|^2_{L^2(\mathcal{D}_j)} &\lesssim {}^{h}\!/\!_{p}\,\|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}^I_{h,j})}\ \|z\|_{H^2(\mathcal{D}_j)}\\ &\lesssim {}^{h}\!/\!_{p}\,\|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}^I_{h,j})}\ (\|v_h-\overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)} + \|\nabla z\|_{L^2(\mathcal{D}_j)}).\end{aligned} \tag{24}$$

Here, we note that z also solves

$$\int_{\mathcal{D}_j}\nabla z\cdot\nabla w\ \mathrm{d}\mathbf{x} = \int_{\mathcal{D}_j}(v_h-\overline{v}_{h,j})w\ \mathrm{d}\mathbf{x}\quad \forall\, w\in\widetilde{V}_j,$$

from which, by choosing $w = z$, upon application of the Cauchy–Schwarz inequality and Poincaré's inequality, cf. Remark 5.11, we get that $\|\nabla z\|_{L^2(\mathcal{D}_j)} \lesssim \|v_h-\overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)}$. By inserting this bound into (24) we obtain (17). In order to show (18) we first select $w = \widetilde{v}_{h,j}\in\widetilde{V}_j$ in (16); then, using the Cauchy–Schwarz inequality we obtain:

$$|\widetilde{v}_{h,j}|_{H^1(\mathcal{D}_j)} \lesssim \|\mathcal{G}_{h,j}(v_h)\|_{L^2(\mathcal{D}_j)}.$$

Then, from the definition of $\mathcal{G}_{h,j}$ given in Definition 5.9 we have:

$$\|\mathcal{G}_{h,j}(v_h)\|^2_{L^2(\mathcal{D}_j)} \lesssim \|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\mathcal{R}_1(\mathcal{J}_j(v_h))\|^2_{L^2(\mathcal{D}_j)}. \tag{25}$$

The bound (18) is then obtained by inserting (23) into (25). □

We are now ready to investigate the relationship between the spaces V_h, V_H, and $V_{\mathbb{H}}$ introduced above. The following result concerns the approximation of a function $v_h\in V_h$ with a coarse function $v_H\in V_H$; this represents an extension of the analogous result presented in [11, Lemma 5.1].

Lemma 5.13. *Let Assumption 5.1 be satisfied. Suppose that $\mathcal{T}_h$ satisfies Assumptions 2.2 and 2.3, and $\mathcal{T}_H$ satisfies Assumptions 2.3 and 5.7. Then, for any $v_h\in V_h$*

there exists a coarse function $v_H \in V_H$ *such that*

$$\|v_h - R_0^\top v_H\|_{L^2(\mathcal{D}_j)} \lesssim {}^{H}\!/\!{}_{q}\ (\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})})^{1/2}, \tag{26}$$

$$|v_h - R_0^\top v_H|_{H^1(\mathcal{T}_{h,j})} \lesssim (\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})})^{1/2}, \tag{27}$$

for $j = 1, \ldots, N_H$*, where* $\mathcal{T}_{h,j}$ *and* $\mathcal{F}^I_{h,j}$ *are as defined in* (14).

Proof. We first observe that, as consequence of Assumption 5.1, together with the definition of $R_0^\top$ given in (7), we have that $R_0^\top v_H = v_H$ for any $v_H \in V_H$. Let $v_h \in V_h$ and define v_H by $v_H|_{\mathcal{D}_j} := ({}^{1}\!/\!{}_{|\mathcal{D}_j|} \int_{\mathcal{D}_j} v_h \mathrm{d}\mathbf{x}) + (\Pi_H^q(\widetilde{v}_{h,j}))|_{\mathcal{D}_j}$, $j = 1, \ldots, N_H$, where $\widetilde{v}_{h,j}$ is as defined in Definition 5.9 and Π_H^q denotes the global variant of the *hp*-approximant introduced in Lemma 5.5, cf. also Remark 5.6, defined on the coarse space V_H. Then, the application of the triangle inequality gives

$$\begin{aligned} \|v_h - R_0^\top v_H\|_{L^2(\mathcal{D}_j)} &= \|v_h - v_H\|_{L^2(\mathcal{D}_j)} \\ &\lesssim \|v_h - \overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)} + \|\overline{v}_{h,j} - v_H\|_{L^2(\mathcal{D}_j)} \\ &\lesssim \|v_h - \overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)} + \|\widetilde{v}_{h,j} - \Pi_H^q(\widetilde{v}_{h,j})\|_{L^2(\mathcal{D}_j)}, \end{aligned}$$

with $\overline{v}_{h,j}$ as defined in Theorem 5.12. Employing Lemma 5.5 together with Assumption 2.3, cf. Remark 5.6, gives

$$\|v_h - R_0^\top v_H\|_{L^2(\mathcal{D}_j)} \lesssim \|v_h - \overline{v}_{h,j}\|_{L^2(\mathcal{D}_j)} + {}^{H}\!/\!{}_{q}\, \|\widetilde{v}_{h,j}\|_{H^1(\mathcal{D}_j)}.$$

By applying Poincaré's inequality to $\widetilde{v}_{h,j} \in \widetilde{V}_j$, see also Remark 5.11, and noting the bounds given in Theorem 5.12, we immediately deduce inequality (26) by observing that $h \leq H$ and $q \leq p$. In order to obtain (27) we proceed as follows:

$$\begin{aligned} |v_h - R_0^\top v_H|_{H^1(\mathcal{T}_{h,j})} &\lesssim |v_h|_{H^1(\mathcal{T}_{h,j})} + |R_0^\top v_H|_{H^1(\mathcal{T}_{h,j})} \\ &= |v_h|_{H^1(\mathcal{T}_{h,j})} + |v_H|_{H^1(\mathcal{T}_{h,j})}. \end{aligned} \tag{28}$$

Thanks to the triangle inequality and observing that $v_H|_{\mathcal{D}_j} \in \mathcal{P}_q(\mathcal{D}_j) \subset H^1(\mathcal{D}_j)$ we have that

$$\begin{aligned} |v_H|_{H^1(\mathcal{T}_{h,j})} &\lesssim |\Pi_H^q(\widetilde{v}_{h,j}) - \widetilde{v}_{h,j}|_{H^1(\mathcal{D}_j)} + |\widetilde{v}_{h,j}|_{H^1(\mathcal{D}_j)} \\ &\lesssim |\Pi_H^q(\widetilde{v}_{h,j}) - \widetilde{v}_{h,j}|_{H^1(\mathcal{T}_H)} + |\widetilde{v}_{h,j}|_{H^1(\Omega)} \lesssim |\widetilde{v}_{h,j}|_{H^1(\Omega)}, \end{aligned} \tag{29}$$

where we have used the bound stated in Remark 5.6 and Poincaré's inequality, cf. also Remark 5.11. Inserting (29) into (28) and noting Theorem 5.12 gives (27). □

Before proceeding with the analysis of P_{ad} we also need the following result regarding the properties of the subdomain decomposition introduced in Section 3.

Lemma 5.14. *Suppose that* $\mathcal{T}_h$ *satisfies Assumption 2.2. Given* $v_h \in V_h$*, there exists a unique decomposition, i.e.,* $v_h = \sum_{i=1}^{N_{\mathbb{H}}} R_i^\top v_i$*, with* $v_i \in V_i$ $i = 1, \ldots, N_{\mathbb{H}}$*, such that*

$$\mathcal{A}_h(v_h, v_h) = \sum_{i=1}^{N_{\mathbb{H}}} \mathcal{A}_i(v_i, v_i) + \sum_{i,j=1, i\neq j}^{N_{\mathbb{H}}} \mathcal{A}_h(R_i^\top v_i, R_j^\top v_j),$$

and

$$\Big| \sum_{i,j=1, i\neq j}^{N_{\mathbb{H}}} \mathcal{A}_h(R_i^\top v_i, R_j^\top v_j) \Big| \lesssim \|\sqrt{\rho}\ \nabla_h v_h\|^2_{L^2(\mathcal{T}_h)} + \sum_{i=1}^{N_{\mathbb{H}}} \|\sigma_{h,\rho}^{1/2} v_h\|^2_{L^2(\partial\Omega_i)}.$$

Proof. Given $v_h \in V_h$ set $v_i := R_i v_h,\ i = 1, \ldots, N_{\mathbb{H}}$; then, $\mathcal{A}_h(R_i^\top v_i, R_j^\top v_j) = 0$ if $\partial\Omega_i \cap \partial\Omega_j = \emptyset$. For $i, j = 1, \ldots, N_{\mathbb{H}}$, we have

$$\Big|\sum_{i,j=1, i\neq j}^{N_{\mathbb{H}}} \mathcal{A}_i(R_i^\top v_i, R_j^\top v_j)\Big| \lesssim \sum_{i,j=1, i\neq j}^{N_{\mathbb{H}}} |\mathcal{A}_i(R_i^\top v_i, R_j^\top v_j)| \quad \forall\, v_i \in V_i, v_j \in V_j.$$

Now let $i \neq j$ be such that $\partial\Omega_i \cap \partial\Omega_j \neq \emptyset$ and write $\check{v}_i = R_i^\top v_i$ and $\check{v}_j = R_j^\top v_j$; then,

$$\mathcal{A}_h(\check{v}_i, \check{v}_j) = \int_\Omega \Big[\rho\nabla\check{v}_i \cdot \mathcal{R}_\rho([\![\check{v}_j]\!]) + \rho\nabla\check{v}_j \cdot \mathcal{R}_\rho([\![\check{v}_i]\!])\Big]\mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h} \sigma_{h,\rho}[\![\check{v}_i]\!] \cdot [\![\check{v}_j]\!]\mathrm{d}s. \tag{30}$$

By recalling the definition of $\mathcal{R}_\rho$ given in (4), the first term on the right-hand side of (30) can be written as

$$\int_\Omega \rho\nabla_h\check{v}_i \cdot \mathcal{R}_\rho([\![\check{v}_j]\!])\mathrm{d}\mathbf{x} = -\int_{\mathcal{F}_h} \{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega \cdot [\![\check{v}_j]\!]\mathrm{d}s,$$

since ρ is piecewise constant and $\check{v}_j \in R_i^\top V_i \subseteq V_h$. By observing that $[\![\check{v}_j]\!]|_F = \mathbf{0}$ for all $F \in \mathcal{F}_h$ such that $F \cap \overline{\Omega}_j = \emptyset$, and that $\{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega|_F = \mathbf{0}$ when $F \cap \overline{\Omega}_i = \emptyset$, we have

$$\begin{aligned}\int_\Omega \rho\nabla_h\check{v}_i \cdot \mathcal{R}_\rho([\![\check{v}_j]\!])\mathrm{d}\mathbf{x} &= -\int_{\partial\Omega_i\cap\partial\Omega_j} [\![\check{v}_j]\!] \cdot \{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega\mathrm{d}s \\ &\leq \|\sigma_{h,\rho}^{1/2}[\![\check{v}_j]\!]\|^2_{L^2(\partial\Omega_i\cap\partial\Omega_j)} + \|\sigma_{h,\rho}^{-1/2}\{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega\|^2_{L^2(\partial\Omega_i\cap\partial\Omega_j)} \\ &\leq \|\sigma_{h,\rho}^{1/2}[\![\check{v}_j]\!]\|^2_{L^2(\partial\Omega_j)} + \|\sigma_{h,\rho}^{-1/2}\{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega\|^2_{L^2(\partial\Omega_i)}. \end{aligned}\tag{31}$$

Here, the second term on the right-hand side of (31) can be bounded by applying Lemma 5.4 and noting that $\langle\rho_\kappa\rangle \leq 2\rho_\kappa$ and $\langle h_\kappa\rangle \leq 2h_\kappa$, as follows:

$$\begin{aligned}\|\sigma_{h,\rho}^{-1/2}\{\!\{\rho\nabla_h\check{v}_i\}\!\}_\omega\|^2_{L^2(\partial\Omega_i)} &\leq \sum_{\kappa\subset\Omega_i} \|\sigma_{h,\rho}^{-1/2}\langle\rho_\kappa\rangle\nabla_h\check{v}_i\|^2_{L^2(\partial\kappa)} \\ &= C_\sigma^{-1}\sum_{\kappa\subset\Omega_i} \frac{\langle h_\kappa\rangle}{\langle\rho_\kappa\rangle p^2}\langle\rho_\kappa\rangle^2\|\nabla_h\check{v}_i\|^2_{L^2(\partial\kappa)} \\ &\lesssim \sum_{\kappa\subset\Omega_i} \|\sqrt{\rho_\kappa}\nabla_h\check{v}_i)\|^2_{L^2(\kappa)} = \|\sqrt{\rho}\nabla_h\check{v}_i\|^2_{L^2(\Omega_i)}. \end{aligned}\tag{32}$$

Inserting (32) into (31) gives

$$\int_\Omega \rho\nabla\check{v}_i \cdot \mathcal{R}_\rho([\![\check{v}_j]\!])\mathrm{d}\mathbf{x} \lesssim \|\sqrt{\rho}\ \nabla_h\check{v}_i\|^2_{L^2(\Omega_i)} + \|\sigma_{h,\rho}^{1/2}[\![\check{v}_j]\!]\|^2_{L^2(\partial\Omega_j)}. \tag{33}$$

Similarly, we have that

$$\int_\Omega \rho\nabla\check{v}_j \cdot \mathcal{R}_\rho([\![\check{v}_i]\!])\mathrm{d}\mathbf{x} \lesssim \|\sqrt{\rho}\ \nabla_h\check{v}_j\|^2_{L^2(\Omega_j)} + \|\sigma_{h,\rho}^{1/2}[\![\check{v}_i]\!]\|^2_{L^2(\partial\Omega_i)} \tag{34}$$

and

$$\int_{\mathcal{F}_h} \sigma_{h,\rho}[\![\check{v}_i]\!] \cdot [\![\check{v}_j]\!]\mathrm{d}s \lesssim \|\sigma_{h,\rho}^{1/2}[\![\check{v}_i]\!]\|^2_{L^2(\partial\Omega_i)} + \|\sigma_{h,\rho}^{1/2}[\![\check{v}_j]\!]\|^2_{L^2(\partial\Omega_j)}. \tag{35}$$

Substituting (33), (34) and (35) into (30) we obtain

$$\begin{aligned}\mathcal{A}_h(\check{v}_i, \check{v}_j) &\lesssim \|\sqrt{\rho}\ \nabla_h\check{v}_i\|^2_{L^2(\Omega_i)} + \|\sqrt{\rho}\ \nabla_h\check{v}_j\|^2_{L^2(\Omega_j)} \\ &\quad + \|\sigma_{h,\rho}^{1/2}[\![\check{v}_i]\!]\|^2_{L^2(\partial\Omega_i)} + \|\sigma_{h,\rho}^{1/2}[\![\check{v}_j]\!]\|^2_{L^2(\partial\Omega_j)}.\end{aligned}$$

The result follows by summing over $i, j = 1, \dots, N_{\mathbb{H}},\ i \neq j$, and exploiting (6). $\square$

For the forthcoming analysis we also require an extension of the trace-inverse inequality introduced by Feng and Karakashian in [44]; cf. also Smears [58, Lemma 5]. To this end, we introduce the following assumption on $\mathcal{T}_{\mathbb{H}}$.

Assumption 5.15. *We assume that for each $\Omega_i \in \mathcal{T}_{\mathbb{H}},\ i = 1, \dots, N_{\mathbb{H}}$, there exists an $\mathbf{x}_{0,i} \in \Omega_i$ such that $(\mathbf{x} - \mathbf{x}_{0,i}) \cdot \mathbf{n}_i \gtrsim \mathbb{H}$ for all $\mathbf{x} \in \partial\Omega_i$, where $\mathbf{n}_i$ is the unit outward normal vector to $\partial\Omega_i$.*

Equipped with Assumption 5.15, the trace-inverse inequality can be stated as follows.

Lemma 5.16 (Trace inverse inequality)**.** *Let $\mathcal{T}_h$ and $\mathcal{T}_{\mathbb{H}}$ be a pair of nested polytopic grids. We assume that $\mathcal{T}_{\mathbb{H}}$ is obtained by agglomeration of elements of $\mathcal{T}_h$ and that $\mathcal{T}_h$ satisfies Assumption 2.2, while $\mathcal{T}_{\mathbb{H}}$ is assumed to satisfy Assumption 5.15. Then, for any $v_h \in V_h$, the following bound holds:*

$$\begin{aligned}\|v_h\|^2_{L^2(\partial\Omega_i)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega_i)}\|v_h\|_{L^2(\Omega_i)} + \frac{1}{\mathbb{H}}\|v_h\|^2_{L^2(\Omega_i)} \\ + \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h^I(\Omega_i))}\|v_h\|_{L^2(\Omega_i)},\end{aligned}$$

where $\mathcal{F}_h^I(\Omega_i)$ is defined in (13)*.*

Proof. Since $\mathcal{T}_{\mathbb{H}}$ satisfies Assumption 5.15, for any fixed $\Omega_i \in \mathcal{T}_{\mathbb{H}}$, following [58, Lemma 5], we have

$$\|v_h\|^2_{L^2(\partial\Omega_i)} = \int_{\partial\Omega_i} |v_h|^2 \mathrm{d}\mathbf{x} \lesssim \frac{1}{\mathbb{H}} \int_{\partial\Omega_i} |v_h|^2 (\mathbf{x} - \mathbf{x}_{0,i}) \cdot \mathbf{n}_i \mathrm{d}s. \tag{36}$$

Employing integration by parts, we deduce that

$$\begin{aligned}&\sum_{\kappa \subset \Omega_i} -\int_\kappa (\mathbf{x} - \mathbf{x}_{0,i}) \cdot \nabla(|v_h|^2)\mathrm{d}\mathbf{x} \\ &\quad= \sum_{\kappa \subset \Omega_i} \Big[\int_\kappa \nabla \cdot (\mathbf{x} - \mathbf{x}_{0,i})\ |v_h|^2 \mathrm{d}\mathbf{x}\ -\ \int_{\partial\kappa} (\mathbf{x} - \mathbf{x}_{0,i}) \cdot \mathbf{n}_i\ |v_h|^2 \mathrm{d}s\Big] \\ &\quad= \sum_{\kappa \subset \Omega_i} \Big[\int_\kappa \nabla \cdot (\mathbf{x} - \mathbf{x}_{0,i})\ |v_h|^2 \mathrm{d}\mathbf{x}\Big]\ -\ \int_{\partial\Omega_i} (\mathbf{x} - \mathbf{x}_{0,i}) \cdot \mathbf{n}_i\ |v_h|^2 \mathrm{d}s \\ &\quad - \sum_{F \in \mathcal{F}_h^I(\Omega_i)} \int_F [\![|v_h|^2]\!] \cdot \{\!\{(\mathbf{x} - \mathbf{x}_{0,i})\}\!\} \mathrm{d}s,\end{aligned}$$

where the last equality follows from $\mathbf{x}-\mathbf{x}_{0,i} = \{\!\!\{(\mathbf{x}-\mathbf{x}_{0,i})\}\!\!\}$. Then, by observing that $[\![|v_h|^2]\!] = 2[\![v_h]\!]\cdot\{\!\!\{v_h\}\!\!\}$, we have

$$\begin{aligned}\int_{\partial\Omega_i}(\mathbf{x}-\mathbf{x}_{0,i})\cdot\mathbf{n}_i\ |v_h|^2\mathrm{d}s &= \sum_{\kappa\subset\Omega_i}\int_\kappa\Big[\nabla\cdot(\mathbf{x}-\mathbf{x}_{0,i})\ |v_h|^2 -\ (\mathbf{x}-\mathbf{x}_{0,i})\cdot\nabla(|v_h|^2)\Big]\mathrm{d}\mathbf{x}\\ &\quad - \sum_{F\in\mathcal{F}_h^I(\Omega_i)}\int_F [\![|v_h|^2]\!]\cdot\{\!\!\{(\mathbf{x}-\mathbf{x}_{0,i})\}\!\!\}\mathrm{d}s\\ &= \sum_{\kappa\subset\Omega_i}\int_\kappa\Big[d\ |v_h|^2 -\ (\mathbf{x}-\mathbf{x}_{0,i})\cdot 2v_h\nabla v_h\Big]\mathrm{d}\mathbf{x}\\ &\quad - \sum_{F\in\mathcal{F}_h^I(\Omega_i)}\int_F 2[\![v_h]\!]\{\!\!\{v_h\}\!\!\}\cdot\{\!\!\{(\mathbf{x}-\mathbf{x}_{0,i})\}\!\!\}\mathrm{d}s.\end{aligned}$$

Employing Hölder's inequality and the Cauchy–Schwarz inequality gives

$$\begin{aligned}\int_{\partial\Omega_i}(\mathbf{x}_{0,i}-\mathbf{x})\cdot\mathbf{n}_i\ |v_h|^2\mathrm{d}s &\lesssim \|v_h\|^2_{L^2(\Omega_i)} + \|\mathbf{x}-\mathbf{x}_{0,i}\|_{[L^\infty(\Omega_i)]^d}\|v\nabla_h v_h\|_{[L^1(\Omega_i)]^d}\\ &\quad +\ \|\mathbf{x}-\mathbf{x}_{0,i}\|_{[L^\infty(\mathcal{F}_h^I(\Omega_i))]^d}\|[\![v_h]\!]\{\!\!\{v_h\}\!\!\}\|_{[L^1(\mathcal{F}_h^I(\Omega_i))]^d}\\ &\lesssim \|v_h\|^2_{L^2(\Omega_i)} + \mathbb{H}\ \|v_h\|_{L^2(\Omega_i)}\ \|\nabla_h v_h\|_{L^2(\Omega_i)}\\ &\quad + \mathbb{H}\ \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h^I(\Omega_i))}\ \|\sigma_{h,1}^{-1/2}\{\!\!\{v_h\}\!\!\}\|_{L^2(\mathcal{F}_h^I(\Omega_i))}\\ &\lesssim \|v_h\|^2_{L^2(\Omega_i)} + \mathbb{H}\ \|v_h\|_{L^2(\Omega_i)}\ \|\nabla_h v_h\|_{L^2(\Omega_i)}\\ &\quad + \mathbb{H}\ \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h^I(\Omega_i))}\ \|v_h\|_{L^2(\Omega_i)},\end{aligned}$$

where we have employed Lemma 5.4 in the last inequality. By inserting the previous bound into (36) we obtain the statement of the lemma. □

6. Condition number estimates

In this section we derive an upper bound on the condition number of P_{ad} by following the analysis presented in [62]; see also [54]. To this end, we show that the following three properties are satisfied.

Property 6.1 (Local stability)**.** *There exists an $\alpha\in(0,2)$ such that*

$$\mathcal{A}_h(R_i^\top v_i, R_i^\top v_i) \le \alpha\mathcal{A}_i(v_i,v_i) \quad \forall\, v_i\in V_i,\ i=0,1,\dots,N_H.$$

Property 6.2 (Strengthened Cauchy–Schwarz inequality)**.** *There exist constants $\epsilon_{ij}\in[0,1]$, for $1\le i,j\le N_{\mathbb{H}}$, such that*

$$|\mathcal{A}_h(R_i^\top v_i, R_j^\top v_j)| \le \epsilon_{ij}\mathcal{A}_h(R_i^\top v_i, R_i^\top v_i)^{1/2}\mathcal{A}_h(R_j^\top v_j, R_j^\top v_j)^{1/2}$$

for all $v_i\in V_i$, $v_j\in V_j$. Define $\Theta(\mathcal{E})$ to be the spectral radius of $(\mathcal{E})_{ij}=\{\epsilon_{ij}\}_{i,j=1,\dots,N_{\mathbb{H}}}$.

Property 6.3 (Stable decomposition)**.** *Each $v_h\in V_h$ admits a decomposition of the form $v_h=\sum_{i=0}^{N_{\mathbb{H}}}R_i^\top v_i$, $v_i\in V_i$, $i=1,\dots,N_{\mathbb{H}}$, and $v_0\in V_0$, such that*

$$\sum_{i=0}^{N_{\mathbb{H}}}\mathcal{A}_i(u_i,u_i)\le C_\sharp^2\mathcal{A}_h(u_h,u_h).$$

If Properties 6.1–6.3 hold, following [62, Theorem 2.7] the upper bound on the condition number of P_{ad} is stated in the following theorem.

Theorem 6.4. *Supposing that Properties 6.1–6.3 hold, the condition number $K(P_{ad})$ of the additive Schwarz operator P_{ad} is bounded as follows:*

$$K(P_{ad}) \lesssim C_\sharp^2 \alpha(\Theta(\mathcal{E})+1),$$

where α, $\Theta(\mathcal{E})$, and $C_\sharp$ are as defined in Properties 6.1, 6.2, and 6.3, respectively.

In our setting Property 6.1 immediately follows with $\alpha = 1$ from the definition of $\mathcal{A}_i(\cdot,\cdot)$ given in Section 3, cf. [9]. Property 6.2 immediately follows since each subdomain $\Omega_i \in \mathcal{T}_{\mathbb{H}}$, $i = 1,\dots,N_{\mathbb{H}}$, can possess only a finite number of neighbours, cf. (6). In particular, by observing that if $\partial\Omega_i \cap \partial\Omega_j = \emptyset$, then $\mathcal{A}_h(R_i^\top v_i, R_j^\top v_j) = 0$ for all $v_i \in V_i$, $v_j \in V_j$, we deduce that $\epsilon_{ij} = 0$ if $\partial\Omega_i \cap \partial\Omega_j = \emptyset$, $\epsilon_{ij} = 1$, otherwise. Then $\Theta(\mathcal{E})$ is uniformly bounded by $N_{\mathbb{S}} + 1$, where $N_{\mathbb{S}}$ is the maximum number of neighbours that each subdomain may possess, cf. (6). Hence, it remains to prove that Property 6.3, which ensures that a stable (in the sense of the energy norm) decomposition can be found for the local spaces and the coarse one, holds. This is undertaken in the following theorem.

Theorem 6.5. *Given that Assumption 5.1 holds, let $v_h \in V_h$, and assume that the grid $\mathcal{T}_h$ satisfies Assumptions 2.2 and 2.3. We also assume that $\mathcal{T}_{\mathbb{H}}$ is obtained by agglomeration of elements of $\mathcal{T}_h$, $\mathcal{T}_H$ is obtained by agglomeration of elements of $\mathcal{T}_{\mathbb{H}}$, and that both $\mathcal{T}_{\mathbb{H}}$ and $\mathcal{T}_H$ satisfy Assumptions 2.2 and 2.3. Moreover, we require that $\mathcal{T}_{\mathbb{H}}$ satisfies Assumption 5.15 and $\mathcal{T}_H$ satisfies Assumption 5.7. Then, Property 6.3 holds with*

$$C_\sharp^2 \approx \Big[\max_{j=1,\dots,N_H}\Big(\frac{\overline{\rho}_j}{\underline{\rho}_j}\Big)\Big]\Big(\frac{p^2}{q}\frac{H}{h} + \frac{p^2}{q^2}\frac{H^2}{h\mathbb{H}}\Big),$$

where $\underline{\rho}_j = \min_{\mathbf{x}\in\mathcal{D}_j}(\rho(\mathbf{x}))$ and $\overline{\rho}_j = \max_{\mathbf{x}\in\mathcal{D}_j}(\rho(\mathbf{x}))$.

Proof. Given $v_h \in V_h$, we select $v_0 = v_H$, where $v_H \in V_H$ is defined as in the proof of Lemma 5.13. Then, by employing Lemma 5.14, $v_h - R_0^\top v_0$ can be uniquely decomposed as $v_h - R_0^\top v_0 = \sum_{i=1}^{N_{\mathbb{H}}} R_i^\top v_i$, where $v_i = R_i(v_h - R_0^\top v_0)$, $i = 1,\dots,N_{\mathbb{H}}$, and

$$\mathcal{A}_h(v_h - R_0^\top v_0, v_h - R_0^\top v_0) = \sum_{i=1}^{N_{\mathbb{H}}} \mathcal{A}_i(v_i,v_i) + \sum_{i,j=1,i\neq j}^{N_{\mathbb{H}}} \mathcal{A}_h(R_i^\top v_i, R_j^\top v_j). \tag{37}$$

Adding $\mathcal{A}_0(v_0,v_0)$ to both sides of (37) we obtain the following inequality:

$$\begin{aligned}\Big|\sum_{i=0}^{N_{\mathbb{H}}} \mathcal{A}_i(v_i,v_i)\Big| &\le \Big|\mathcal{A}_h(v_h - R_0^\top v_0, v_h - R_0^\top v_0)\Big| + \Big|\mathcal{A}_0(v_0,v_0)\Big| \\ &\quad + \Big|\sum_{i,j=1,i\neq j}^{N_{\mathbb{H}}} \mathcal{A}_h(R_i^\top v_i, R_j^\top v_j)\Big| \\ &\equiv I + II + III.\end{aligned} \tag{38}$$

From the definition of $\mathcal{A}_0(\cdot,\cdot)$, cf. (8), we have that

$$\begin{aligned} II &\le |\mathcal{A}_h(R_0^\top v_0 - v_h, R_0^\top v_0)| + |\mathcal{A}_h(v_h, R_0^\top v_0)| \\ &\le |\mathcal{A}_h(R_0^\top v_0 - v_h, R_0^\top v_0 - v_h)| + 2|\mathcal{A}_h(R_0^\top v_0 - v_h, v_h)| + |\mathcal{A}_h(v_h,v_h)|.\end{aligned} \tag{39}$$

Recalling the continuity of $\mathcal{A}_h$, cf. Lemma 5.3, and applying the triangle inequality and Young's inequality gives

$$|\mathcal{A}_h(R_0^\top v_0 - v_h, v_h)| \lesssim \|v_h - R_0^\top v_0\|_{h,\rho}\|v_h\|_{h,\rho} \lesssim \|v_h - R_0^\top v_0\|_{h,\rho}^2 + \|v_h\|_{h,\rho}^2.$$

Then, by inserting the above bound into (39) and using the continuity and coercivity of $\mathcal{A}_h$, cf. Lemma 5.3, we deduce that

$$I + II \lesssim \|v_h - R_0^\top v_0\|_{h,\rho}^2 + \mathcal{A}_h(v_h, v_h). \tag{40}$$

In particular, we observe that from the definition of $\|\cdot\|_{h,\rho}$ we have

$$\|v_h - R_0^\top v_0\|_{h,\rho}^2 = \|\sqrt{\rho}\nabla_h(v_h - R_0^\top v_0)\|_{L^2(\mathcal{T}_h)}^2 + \|\sigma_{h,\rho}^{1/2}[\![v_h - R_0^\top v_0]\!]\|_{L^2(\mathcal{F}_h)}^2. \tag{41}$$

Writing $\mathcal{F}_H$ to denote the set of faces of $\mathcal{T}_H$, and observing that $\mathcal{F}_H \subseteq \mathcal{F}_h$ thanks to Assumption 5.1, the second term on the right-hand side of (41) can be bounded as follows:

$$\begin{aligned}
\|\sigma_{h,\rho}^{1/2}[\![v_h - R_0^\top v_0]\!]\|_{L^2(\mathcal{F}_h)}^2 &= \|\sigma_{h,\rho}^{1/2}[\![v_h - R_0^\top v_0]\!]\|_{L^2(\mathcal{F}_h\setminus\mathcal{F}_H)}^2 + \|\sigma_{h,\rho}^{1/2}[\![v_h - R_0^\top v_0]\!]\|_{L^2(\mathcal{F}_H)}^2 \\
&= \|\sigma_{h,\rho}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h\setminus\mathcal{F}_H)}^2 + \|\sigma_{h,\rho}^{1/2}[\![v_h - R_0^\top v_0]\!]\|_{L^2(\mathcal{F}_H)}^2 \\
&\leq \|v_h\|_{h,\rho}^2 \ + \ \sum_{j=1}^{N_H} \|\sigma_{h,\rho}^{1/2}(v_h - R_0^\top v_0)\|_{L^2(\partial\mathcal{D}_j)}^2 \\
&\leq \|v_h\|_{h,\rho}^2 \ + \ \sum_{i=1}^{N_{\mathbb{H}}} \|\sigma_{h,\rho}^{1/2}(v_h - R_0^\top v_0)\|_{L^2(\partial\Omega_i)}^2, \qquad (42)
\end{aligned}$$

where we have used that $[\![R_0^\top v_0]\!] = \mathbf{0}$ on each face $F \in \mathcal{F}_h \setminus \mathcal{F}_H$ and, in the last step, the fact that $\mathcal{T}_{\mathbb{H}} \subseteq \mathcal{T}_H$; cf. Remark 3.1. Hence, inserting (42) into (41) and employing Lemma 5.3, inequality (40) becomes

$$I + II \lesssim \|\sqrt{\rho}\ \nabla_h(v_h - R_0^\top v_0)\|_{L^2(\mathcal{T}_h)}^2 + \sum_{i=1}^{N_{\mathbb{H}}} \|\sigma_{h,\rho}^{1/2}(v_h - R_0^\top v_0)\|_{L^2(\partial\Omega_i)}^2 + \Big|\mathcal{A}_h(v_h, v_h)\Big|.$$

From Lemma 5.14 we get

$$III \lesssim \|\sqrt{\rho}\ \nabla_h(v_h - R_0^\top v_0)\|_{L^2(\mathcal{T}_h)}^2 \ + \ \sum_{i=1}^{N_{\mathbb{H}}} \|\sigma_{h,\rho}^{1/2}(v_h - R_0^\top v_0)\|_{L^2(\partial\Omega_i)}^2. \tag{43}$$

Thereby, (38) can be bounded as follows

$$\begin{aligned}
\Big|\sum_{i=0}^{N_{\mathbb{H}}} \mathcal{A}_i(v_i, v_i)\Big| &\lesssim \Big|\mathcal{A}_h(v_h, v_h)\Big| + \|\sqrt{\rho}\ \nabla_h(v_h - R_0^\top v_0)\|_{L^2(\mathcal{T}_h)}^2 \\
&\quad + \sum_{i=1}^{N_{\mathbb{H}}} \|\sigma_{h,\rho}^{1/2}(v_h - R_0^\top v_0)\|_{L^2(\partial\Omega_i)}^2 \\
&\equiv IV + V + VI.
\end{aligned} \tag{44}$$

Thanks to Lemma 5.13 we have that

$$
\begin{aligned}
V &= \sum_{j=1}^{N_H} \|\sqrt{\rho}\nabla_h(v_h - R_0^\top v_0)\|^2_{L^2(\mathcal{D}_j)} \lesssim \sum_{j=1}^{N_H} \overline{\rho}_j \|\nabla_h(v_h - R_0^\top v_0)\|^2_{L^2(\mathcal{D}_j)} \\
&\lesssim \sum_{j=1}^{N_H} \overline{\rho}_j \Big[\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})}\Big] \\
&\lesssim \sum_{j=1}^{N_H} {}^{\overline{\rho}_j}\!/\!_{\underline{\rho}_j} \Big[\|\sqrt{\rho}\ \nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,\rho}^{1/2}[\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})}\Big] \\
&\lesssim \max_{j=1,\dots,N_H} \Big({}^{\overline{\rho}_j}\!/\!_{\underline{\rho}_j}\Big) \|v_h\|^2_{h,\rho} \lesssim \max_{j=1,\dots,N_H} \Big({}^{\overline{\rho}_j}\!/\!_{\underline{\rho}_j}\Big) \mathcal{A}_h(v_h, v_h),
\end{aligned} \tag{45}
$$

where we have used the coercivity bound from Lemma 5.3 in the last inequality. The bound on term VI can be deduced by using the inverse trace inequality of Lemma 5.16. To this end, we first observe that

$$
VI \lesssim \sum_{i=1}^{N_{\mathbb{H}}} \max_{\kappa\subset\Omega_i}(\rho_\kappa)\ \frac{p^2}{h}\ \|v_h - R_0^\top v_0\|^2_{L^2(\partial\Omega_i)}, \tag{46}
$$

where we have also employed the definition of $\sigma_{h,\rho}$ and the fact that $\langle\rho_\kappa\rangle|_F \le 2\rho_{\kappa^\pm}$ for any $F \subset \partial\Omega_i$, $F \subset \partial\kappa^\pm$, for some $\kappa^\pm \in \mathcal{T}_h$, which implies that $\langle\rho_\kappa\rangle|_F \le 2\max_{\{\kappa\subset\Omega_i\}}\rho_\kappa$ for all $F \in \mathcal{F}_h$ such that $F \subset \partial\Omega_i$. Then, by applying Lemma 5.16 to each $\Omega_i \in \mathcal{T}_{\mathbb{H}}$, $i = 1, \dots, N_{\mathbb{H}}$, from (46) we obtain the following bound:

$$
\begin{aligned}
VI \lesssim \sum_{i=1}^{N_{\mathbb{H}}} \max_{\kappa\subset\Omega_i} \rho_\kappa\ \frac{p^2}{h} \Big[&\|\nabla_h(v_h - R_0^\top v_0)\|_{L^2(\Omega_i)} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \\
&+ {}^1\!/\!_{\mathbb{H}}\, \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} \\
&+ \Big(\sum_{F\in\mathcal{F}^I_h(\Omega_i)} \|\sigma_{h,1}^{1/2}[\![v_h - R_0^\top v_0]\!]\|^2_{L^2(F)}\Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)}\Big].
\end{aligned}
$$

Since $\mathcal{T}_{\mathbb{H}} \subseteq \mathcal{T}_H$, we denote by $\mathcal{I}_j := \{k : 1 \le k \le N_{\mathbb{H}},\ \Omega_k \in \mathcal{T}_{\mathbb{H}} \text{ and } \Omega_k \subset \mathcal{D}_j\}$ the set of indices that correspond to the subdomains inside $\mathcal{D}_j \in \mathcal{T}_H$, for all $j = 1, \dots, N_H$.

Hence, $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$ for any $j \neq k,\ 1 \leq j,k \leq N_H$, and $\cup_{j=1}^{N_H} \mathcal{I}_j = \{1,\dots,N_{\mathbb{H}}\}$. Then,

$$\begin{aligned} VI \lesssim \sum_{j=1}^{N_H} \sum_{i\in\mathcal{I}_j} \max_{\kappa\subset\Omega_i} \rho_\kappa \, \frac{p^2}{h} \Big[& \|\nabla_h (v_h - R_0^\top v_0)\|_{L^2(\Omega_i)} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \\ & + {}^1\!/\!_{\mathbb{H}} \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} \\ & + \Big(\sum_{F\in\mathcal{F}_h^I(\Omega_i)} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(F)} \Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \Big] \\ \lesssim \sum_{j=1}^{N_H} \frac{p^2 \overline{\rho}_j}{h} \Big[& \sum_{i\in\mathcal{I}_j} \|\nabla_h (v_h - R_0^\top v_0)\|_{L^2(\Omega_i)} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \\ & + {}^1\!/\!_{\mathbb{H}} \sum_{i\in\mathcal{I}_j} \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} \\ & + \sum_{i\in\mathcal{I}_j} \Big(\sum_{F\in\mathcal{F}_h^I(\Omega_i)} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(F)} \Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \Big]. \end{aligned} \tag{47}$$

We now proceed by bounding each term present in the bracket in (47); to this end, using the Cauchy–Schwarz inequality for sums, we get

$$\begin{aligned} \sum_{i\in\mathcal{I}_j} & \|\nabla_h (v_h - R_0^\top v_0)\|_{L^2(\Omega_i)} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \\ & \leq \Big(\sum_{i\in\mathcal{I}_j} \|\nabla_h (v_h - R_0^\top v_0)\|^2_{L^2(\Omega_i)} \Big)^{1/2} \Big(\sum_{i\in\mathcal{I}_j} \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} \Big)^{1/2} \\ & = |v_h - R_0^\top v_0|_{H^1(\mathcal{T}_{h,j})} \|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)}. \end{aligned}$$

Similarly, by noting that $\mathcal{F}_h^I(\Omega_i)$ is the set of faces $F \in \mathcal{F}_h$ strictly contained in Ω_i, and therefore $\cup_{i\in\mathcal{I}_j} \mathcal{F}_h^I(\Omega_i) \subset \mathcal{F}_{h,j}^I$, we deduce that

$$\begin{aligned} \sum_{i\in\mathcal{I}_j} & \Big(\sum_{F\in\mathcal{F}_h^I(\Omega_i)} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(F)} \Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\Omega_i)} \\ & \leq \Big(\sum_{i\in\mathcal{I}_j} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(\mathcal{F}_h^I(\Omega_i))} \Big)^{1/2} \Big(\sum_{i\in\mathcal{I}_j} \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} \Big)^{1/2} \\ & \leq \Big(\sum_{F\in\mathcal{F}_{h,j}^I} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(F)} \Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)}. \end{aligned}$$

Noting that $\sum_{i\in\mathcal{I}_j} \|v_h - R_0^\top v_0\|^2_{L^2(\Omega_i)} = \|v_h - R_0^\top v_0\|^2_{L^2(\mathcal{D}_j)}$ gives

$$\begin{aligned} VI \lesssim \sum_{j=1}^{N_H} \frac{p^2 \overline{\rho}_j}{h} \Big[& |v_h - R_0^\top v_0|_{H^1(\mathcal{T}_{h,j})} \|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)} + {}^1\!/\!_{\mathbb{H}} \|v_h - R_0^\top v_0\|^2_{L^2(\mathcal{D}_j)} \\ & + \Big(\sum_{F\in\mathcal{F}_{h,j}^I} \|\sigma_{h,1}^{1/2} [\![v_h - R_0^\top v_0]\!] \|^2_{L^2(F)} \Big)^{1/2} \|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)} \Big]. \end{aligned} \tag{48}$$

The last term on the right-hand side of (48) can rewritten as

$$\Big(\sum_{F\in\mathcal{F}^I_{h,j}} \|\sigma_{h,1}^{1/2}[\![v_h - R_0^\top v_0]\!]\|^2_{L^2(F)}\Big)^{1/2}\|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)}$$
$$= \Big(\sum_{F\in\mathcal{F}^I_{h,j}} \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|^2_{L^2(F)}\Big)^{1/2}\|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)}$$
$$= \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}^I_{h,j})}\|v_h - R_0^\top v_0\|_{L^2(\mathcal{D}_j)};$$

here we observe that $[\![R_0^\top v_0]\!]|_F = \mathbf{0}$ on each $F \in \mathcal{F}^I_{h,j}$, since $\mathcal{T}_h$ and $\mathcal{T}_H$ are nested. Then, by employing the above estimate together with Lemma 5.13, we deduce that

$$\begin{aligned} VI &\lesssim \sum_{j=1}^{N_H}\Big[\frac{p^2\overline{\rho}_j}{h}\Big(\frac{H}{q} + \frac{1}{q^2}\frac{H^2}{\mathbb{H}}\Big)\Big(\|\nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,1}^{1/2}[\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})}\Big)\Big] \\ &\lesssim \sum_{j=1}^{N_H}\frac{\overline{\rho}_j}{\underline{\rho}_j}\Big(\frac{p^2}{q}\frac{H}{h} + \frac{p^2}{q^2}\frac{H^2}{h\mathbb{H}}\Big)\Big(\|\sqrt{\rho}\ \nabla_h v_h\|^2_{L^2(\mathcal{T}_{h,j})} + \|\sigma_{h,\rho}^{1/2}[\![v_h]\!]\|^2_{L^2(\mathcal{F}^I_{h,j})}\Big) \qquad (49) \\ &\lesssim \max_{j=1,\dots,N_H}\Big(\frac{\overline{\rho}_j}{\underline{\rho}_j}\Big)\Big(\frac{p^2}{q}\frac{H}{h} + \frac{p^2}{q^2}\frac{H^2}{h\mathbb{H}}\Big)\mathcal{A}_h(v_h, v_h), \end{aligned}$$

where we have also made use of the coercivity bound of Lemma 5.3 in the last inequality. Inserting the estimates (45) and (49) into (44) we obtain the desired result. □

Remark 6.6. According to the statement of Theorem 6.5, given that Properties 6.1 and 6.2 hold, using Theorem 6.4 we deduce that

$$K(P_{ad}) \lesssim \max_{1\le j\le N_H}\Big(\frac{\overline{\rho}_j}{\underline{\rho}_j}\Big)\Big(\frac{p^2}{q}\frac{H}{h} + \frac{p^2}{q^2}\frac{H^2}{h\mathbb{H}}\Big)(N_{\mathbb{S}} + 2). \qquad (50)$$

In particular, in the lowest order case, i.e., when $p = q = 1$, we have $K_h(P_{ad}) \lesssim {}^{H^2}/_{h\mathbb{H}}$, which is in agreement with the corresponding bound derived in [41]. On the other hand if the size of the coarse subdomain and fine meshes are fixed, we deduce that $K_p(P_{ad}) \lesssim {}^{p^2}/_{q}$. Moreover, we also observe that if the diffusion coefficient ρ is constant on each subdomain $\mathcal{D}_j,\ j = 1,\dots,N_H$, then the condition number is independent of the jump in ρ.

Remark 6.7. We remark that Assumption 5.7 is needed in order to obtain the local estimates of Lemma 5.13, which allow to bound $K(P_{ad})$ with $\max_{1\le j\le N_H}({}^{\overline{\rho}_j}/_{\underline{\rho}_j})$, cf. (50). However, if the diffusion coefficient ρ is constant on Ω, we point out that the analysis can be simplified by employing the global variant of the bounds derived in Lemma 5.13, without making Assumption 5.7. We refer to [11] for further details.

7. Numerical results

In this section, we present a series of numerical experiments to demonstrate the sharpness of the condition number bounds stated in Remarks 6.6 and 6.7. Throughout this section we solve (2) by using the additive Schwarz Preconditioned Conjugate Gradient method; here, we report the number of iterations needed to reduce the Euclidean norm of the relative residual vector below a tolerance of 10^{-8}, based on starting from the trivial initial guess. Furthermore, we estimate

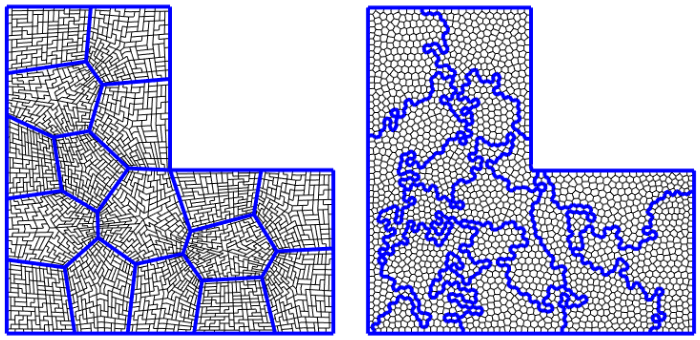

FIGURE 2. Example 1. Nested polygonal grids $\mathcal{T}_h$ (thin) and $\mathcal{T}_H$ (thick) on an L-shaped domain, when the elements of $\mathcal{T}_H$ are convex (left) and non-convex (right).

the condition number $K(P_{ad})$ by using the extreme-eigenvalue-estimate based on the preconditioned Conjugate Gradient iterations. For the sake of simplicity of the presentation, we only consider the massively parallel case, i.e., when $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$. Furthermore, here we select the penalty parameter $C_\sigma = 10$.

7.1. **Example 1.** In this first example we investigate the dependence of the condition number of P_{ad} on the diffusion coefficient ρ. Based on Remark 6.6, we expect the condition number of the preconditioned system to be dependent on the choice of the coarse grid $\mathcal{T}_H$. More precisely, if $\mathcal{T}_H$ is chosen to be aligned with the discontinuities of ρ, then $K(P_{ad})$ should be independent of the jump in the coefficient ρ; otherwise it will depend on the maximum ratio between the maximum and the minimum value of ρ present inside the subdomains $\mathcal{D}_j \in \mathcal{T}_H$. To verify this behavior, we consider two experiments based on fine/coarse grids $\mathcal{T}_h/\mathcal{T}_H$, respectively, where $\mathcal{T}_H$ is a Voronoi polygonal grid on the L-shaped domain Ω depicted in Figure 2 with 16 polygonal elements and $\mathcal{T}_h$ is obtained by successive refinement of elements of $\mathcal{T}_H$. Here, we observe that the elements of $\mathcal{T}_H$ are convex and satisfy Assumption 5.7, cf. Figure 2 (left). Moreover, we choose the forcing function $f = 1$ and the polynomial degrees to be either $p = q = 1$ or $p = q = 2$. In the first experiment we fix $\rho|_{\mathcal{D}_j} = \rho_o = 1$ on the elements $\mathcal{D}_j \in \mathcal{T}_H$ with odd index j and set $\rho|_{\mathcal{D}_j} = \rho_e \in \{10^1, 10^2, \ldots, 10^6\}$, in each test case, on the polygonal subdomains with even index j. The results shown in the first two lines of Table 1 confirm the independence with respect to the jumps of ρ when those jumps are aligned with the subdomains of $\mathcal{T}_H$. In the second experiment we proceed similarly, but here we take different values of ρ on odd and even polygonal elements $\kappa \in \mathcal{T}_h$: in this way $\mathcal{T}_H$ is not aligned with the discontinuities of ρ, and hence the ratio between the maximum and the minimum value of ρ inside the polygonal subdomains $\mathcal{D}_j \in \mathcal{T}_H$ is given by ρ_e. As expected from the theory, the results presented in the last two lines of Table 1 show that the condition number of P_{ad} grows linearly with ρ_e. Finally, we repeat the same set of experiments by first selecting $\mathcal{T}_h$ as a Voronoi polygonal grid consisting of 2000 elements and, subsequently, define the coarse grid $\mathcal{T}_H$ by successive agglomerations of elements of $\mathcal{T}_h$. The agglomeration is undertaken based on employing *Metis*, cf. [52]. As shown in Figure 2 (right), the elements of $\mathcal{T}_H$ are clearly non-convex in this example. Although Assumption 5.7 is not satisfied

		ρ_e →					
		10	10^2	10^3	10^4	10^5	10^6
Aligned	$p = 1$	$2.31 \cdot 10^2$	$2.33 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$	$2.34 \cdot 10^2$
	$p = 2$	$6.12 \cdot 10^2$	$6.14 \cdot 10^2$	$6.14 \cdot 10^2$	$6.12 \cdot 10^2$	$6.11 \cdot 10^2$	$6.10 \cdot 10^2$
Not Aligned	$p = 1$	$2.92 \cdot 10^2$	$1.04 \cdot 10^3$	$7.73 \cdot 10^3$	$7.42 \cdot 10^4$	$7.39 \cdot 10^5$	$7.38 \cdot 10^6$
	$p = 2$	$6.54 \cdot 10^2$	$2.26 \cdot 10^3$	$1.71 \cdot 10^4$	$1.65 \cdot 10^5$	$1.64 \cdot 10^6$	$1.64 \cdot 10^7$

TABLE 1. Example 1. Condition number $K(P_{ad})$ as a function of the ratio between the maximum and minimum value of ρ when the polygonal elements of $\mathcal{T}_H$ are convex and $\mathcal{T}_H$ is aligned (top) and not aligned (bottom) with the discontinuities of ρ.

		ρ_e →					
		10	10^2	10^3	10^4	10^5	10^6
Aligned	$p = 1$	$9.65 \cdot 10^2$	$1.15 \cdot 10^3$	$1.20 \cdot 10^3$	$1.21 \cdot 10^3$	$1.21 \cdot 10^3$	$1.21 \cdot 10^3$
	$p = 2$	$2.47 \cdot 10^3$	$2.86 \cdot 10^3$	$3.18 \cdot 10^3$	$3.25 \cdot 10^3$	$3.26 \cdot 10^3$	$3.26 \cdot 10^3$
Not Aligned	$p = 1$	$8.02 \cdot 10^2$	$2.09 \cdot 10^3$	$1.68 \cdot 10^4$	$1.65 \cdot 10^5$	$1.64 \cdot 10^6$	$1.64 \cdot 10^7$
	$p = 2$	$2.36 \cdot 10^3$	$6.10 \cdot 10^3$	$4.74 \cdot 10^4$	$4.62 \cdot 10^5$	$4.61 \cdot 10^6$	$4.61 \cdot 10^7$

TABLE 2. Example 1. Condition number $K(P_{ad})$ as a function of the ratio between the maximum and minimum value of ρ when the polygonal elements of $\mathcal{T}_H$ are non-convex and $\mathcal{T}_H$ is aligned (top) and not aligned (bottom) with the discontinuities of ρ.

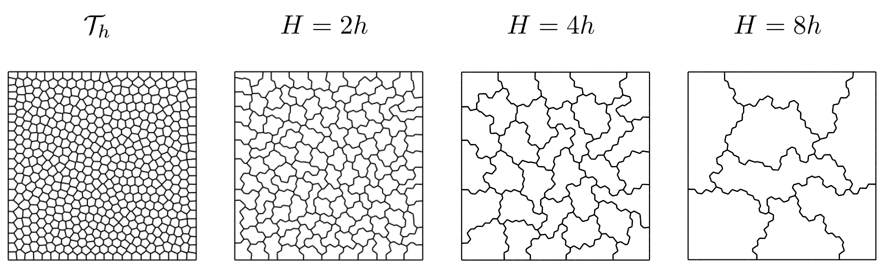

FIGURE 3. Example 2. Example of a sequence of nested polygonal grids.

in this case, the results of Table 2 illustrate analogous behavior to that observed in the previous setting when the coarse elements were convex.

7.2. **Example 2.** In this example, we investigate the performance of the proposed preconditioning algorithm on a set of Voronoi polygonal fine grids $\mathcal{T}_h$, where $\Omega = (0,1)^2$, $\rho = 1$ and $f = 1$. For each grid $\mathcal{T}_h$ we construct a sequence of nested polygonal grids $\mathcal{T}_H$ obtained by successive levels of agglomeration, cf. [10]. For each fine Voronoi grid of size h the agglomeration process has been performed in order to ensure that the size of the coarser partitions is approximately $H = 2h,\ 4h, \dots,$ cf. Figure 3, for example. In Tables 3 and 4 we report the condition number and the iteration counts for the proposed preconditioning algorithm for $p = q = 1$ and $p = q = 3$, respectively. Here, we clearly observe that the condition number and the iteration counts grow quadratically and linearly, respectively, as h tends to

$\downarrow \mathcal{T}_H$	$\mathcal{T}_h \rightarrow$ $h = 8\overline{h}$	$h = 4\overline{h}$	$h = 2\overline{h}$	$h = \overline{h}$
$H = 16\overline{h}$	20.70 (45)	72.31 (86)	269.70 (163)	818.09 (289)
$H = 8\overline{h}$	-	21.89 (46)	73.42 (86)	261.36 (163)
$H = 4\overline{h}$	-	-	20.91 (46)	83.77 (91)
$H = 2\overline{h}$	-	-	-	23.08 (48)

TABLE 3. Example 2. Condition number (and iteration counts): nested polygonal grids, with $p = q = 1$. Here, $\overline{h}$ is the diameter of a grid with $N_h = 4096$ elements.

$\downarrow \mathcal{T}_H$	$\mathcal{T}_h \rightarrow$ $h = 8\overline{h}$	$h = 4\overline{h}$	$h = 2\overline{h}$	$h = \overline{h}$
$H = 16\overline{h}$	88.63 (80)	291.77 (145)	1137.55 (276)	3241.64 (513)
$H = 8\overline{h}$	-	102.96 (82)	278.15 (140)	949.21 (271)
$H = 4\overline{h}$	-	-	90.30 (79)	343.19 (148)
$H = 2\overline{h}$	-	-	-	104.24 (82)

TABLE 4. Example 2. Condition number (and iteration counts): nested polygonal grids with, $p = q = 3$. Here, $\overline{h}$ is the diameter of a grid with $N_h = 4096$ elements.

zero for fixed H. Moreover, if the ratio of h and H is kept fixed, then we observe that the condition number and iteration counts are approximately constant; cf. the diagonals and superdiagonals of Tables 3 and 4. This behavior is in agreement with the theoretical bound stated in Remark 6.6, where $K(P_{ad}) = \mathcal{O}(H^2/h^2)$.

7.3. **Example 3.** We now consider the performance of the additive Schwarz preconditioning algorithm on tetrahedral meshes in three dimensions. To this end, we set $\Omega = (0,1)^3$, $\rho = 1$, and $f = 3\pi^2 \sin(\pi x)\cos(\pi y)\cos(\pi z)$; furthermore, the elements of the coarse mesh are general-shaped polyhedra obtained by successive agglomeration, cf. the previous example. The results for $p = q = 1$ and $p = q = 3$ are reported in Tables 5 and 6, respectively. Here, we have also added a line with the condition number of the operator $A_h : V_h \times V_h \rightarrow V_h$ defined by $(A_h u_h, v_h)_{L^2(\Omega)} = \mathcal{A}_h(u_h, v_h)$ for all $u_h, v_h \in V_h$, and, in parentheses, the iteration counts of the Conjugate Gradient method for solving (2) without preconditioning. Analogous behaviour of the condition number and iteration counts to those presented in the previous example are observed. In particular, we observe that the condition number is roughly constant on the diagonals and superdiagonals of the two tables, while, along each row, i.e., when $\mathcal{T}_H$ is fixed, the expected quadratic

$\downarrow N_H$ \ $N_h \rightarrow$	384	3072	24576	196608	1572864	12582912
48	107 (85)	411 (156)	1497 (294)	6216 (580)	25791 (1089)	94276 (2012)
384	-	136 (95)	499 (169)	1878 (311)	7407 (584)	28762 (1106)
3072	-	-	146 (96)	480 (165)	1904 (306)	7861 (578)
24576	-	-	-	144 (94)	491 (164)	1973 (306)
196608	-	-	-	-	144 (94)	496 (164)
1572864	-	-	-	-	-	145 (94)
$K(A_h)$	734 (166)	2859 (289)	11407 (507)	45618 (933)	182199 (1649)	708509 (3012)

TABLE 5. Example 3. Condition number (and iteration counts): tetrahedral fine meshes and agglomerated polyhedral coarse grids, with $p = q = 1$.

$\downarrow N_H$ \ $N_h \rightarrow$	384	3072	24576	196608
48	607 (174)	2120 (309)	6760 (515)	26674 (924)
384	-	655 (179)	2334 (314)	7507 (536)
3072	-	-	693 (182)	2295 (316)
24576	-	-	-	697 (182)
$K(A_h)$	12247 (679)	40675 (1056)	154934 (1722)	602050 (2991)

TABLE 6. Example 3. Condition number (and iteration counts): tetrahedral fine meshes and agglomerated polyhedral coarse grids, with $p = q = 3$.

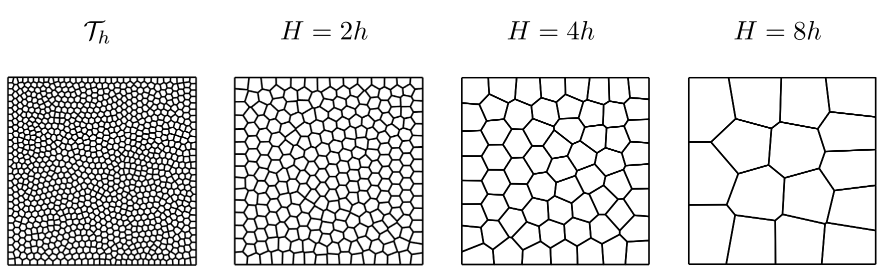

FIGURE 4. Example 4. Sequence of non-nested Voronoi polygonal grids employed.

growth in $K(P_{ad})$ is observed. Similar considerations are also noted for the iteration counts.

7.4. **Example 4.** Given the definition of $R_0^\top$, the proposed preconditioning algorithm naturally admits the use of non-nested coarse spaces, i.e., when $V_H \not\subseteq V_h$. In

$\downarrow N_H$ \ $N_h \rightarrow$	64	256	1024	4096	16384
16	23.29 (38)	92.13 (77)	387.61 (159)	1624.26 (324)	6370.86 (657)
64	-	25.91 (39)	106.42 (84)	411.02 (167)	1774.19 (342)
256	-	-	26.73 (41)	100.89 (82)	425.97 (169)
1024	-	-	-	31.81 (44)	118.61 (86)
4096	-	-	-	-	30.56 (43)

TABLE 7. Example 4. Condition number (and iteration counts): non-nested polygonal grids, with $p = q = 1$.

$\downarrow N_H$ \ $N_h \rightarrow$	64	256	1024	4096	16384
16	148.36 (83)	429.84 (143)	1602.53 (275)	5405.40 (529)	21263.66 (1058)
64	-	142.42 (76)	405.44 (135)	1525.94 (263)	5170.13 (498)
256	-	-	157.47 (80)	452.41 (137)	1469.08 (249)
1024	-	-	-	147.97 (77)	402.98 (124)
4096	-	-	-	-	135.77 (70)

TABLE 8. Example 4. Condition number (and iteration counts): non-nested polygonal grids, with $p = q = 3$.

order to confirm the condition number bound stated in Remark B.7 when $V_H \nsubseteq V_h$, we consider a set of independently generated Voronoi polygonal tessellations of $(0, 1)^2$ of size h and $H > h$, respectively; in this way, $\mathcal{T}_h$ and $\mathcal{T}_H$ are non-nested, cf. Figure 4. The results (for $\rho = 1$ and $f = 1$) shown in Tables 7 and 8 for $p = q = 1$ and $p = q = 3$, respectively, illustrate analogous behavior to the results for the nested case presented in the previous examples; this is in agreement with the condition number bound stated in Remark B.7.

7.5. **Example 5.** In this final example we investigate the dependence of the condition number on the polynomial degree p in both the nested and non-nested cases with $\rho = 1$ and $f = 1$. For the nested case, we consider a total of four tests: two of them are characterized by quadrilateral fine grids with $N_h = 256$ and $N_h = 1024$ elements, while the two other tests are based on employing the polygonal fine grids depicted in Figure 5, where the fine meshes have $N_h = 262$ and $N_h = 516$ polygonal elements. For each test the coarse mesh $\mathcal{T}_H$ is obtained by agglomeration of $\mathcal{T}_h$ in order to guarantee $H \approx h/4$. Analogous fine meshes are also considered in the non-nested setting; however, here the coarse mesh $\mathcal{T}_H$ is selected to be a Voronoi grid generated independently of $\mathcal{T}_h$, cf. Figure 6. In Figure 7 we plot the condition number $K(P_{ad})$ on each set of grids as the polynomial degree p is increased, with $p = q$. Here, we observe that, in the nested setting, i.e., when $V_H \subseteq V_h$, for a fixed mesh size $K(P_{ad}) = \mathcal{O}(p)$ as p increases; however, when $V_H \nsubseteq V_h$, then

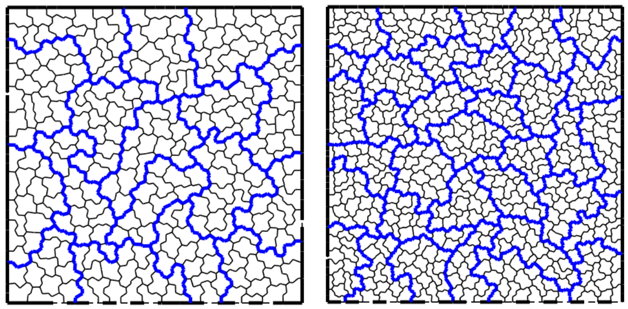

FIGURE 5. Example 5. Pairs of nested grids $\mathcal{T}_h$ (solid) and $\mathcal{T}_H$ (dashed), respectively.

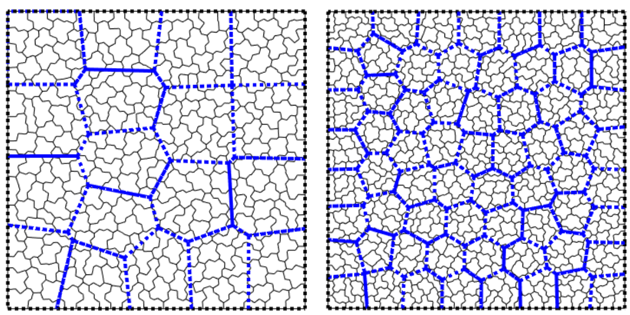

FIGURE 6. Example 5. Pairs of non-nested grids $\mathcal{T}_h$ (solid) and $\mathcal{T}_H$ (dashed), respectively.

$K(P_{ad}) = \mathcal{O}(p^2)$ as p increases. This behaviour is in agreement with the condition number bounds stated in Remarks B.7 and 6.6, respectively.

In order to investigate the dependence of the condition number when $q \neq p$, we first consider the nested polygonal grid depicted Figure 5 where $N_h = 516$ with $q = 1$ and $q = 3$, cf. Figure 8(left). Here, we observe that the condition number behaves as $K(P_{ad}) = \mathcal{O}(p^2)$ as p increases, which is in agreement with (50). Finally, we consider the non-nested setting, based on employing the grid depicted in Figure 6 with $N_h = 516$. For a fixed (coarse) polynomial degree q of V_H, according to Remark B.7, we expect the condition number to behave like $K(P_{ad}) = \mathcal{O}(p^4)$ as p increases. However, Figure 8(right) indicates that $K(P_{ad})$ seems to behave as $\mathcal{O}(p^3)$ as p increases, for this particular case. We note that we have repeated analogous numerical experiments on several pairs of non-nested grids and for all cases, we have observed the same asymptotic behaviour; for brevity these results have been omitted. Such a numerically observed rate might suggest that either the theory of Appendix B is slightly suboptimal when $q \neq p$ or that test cases with less regular agglomerated grids should be considered. This behaviour is currently under investigation and will be the subject of future research.

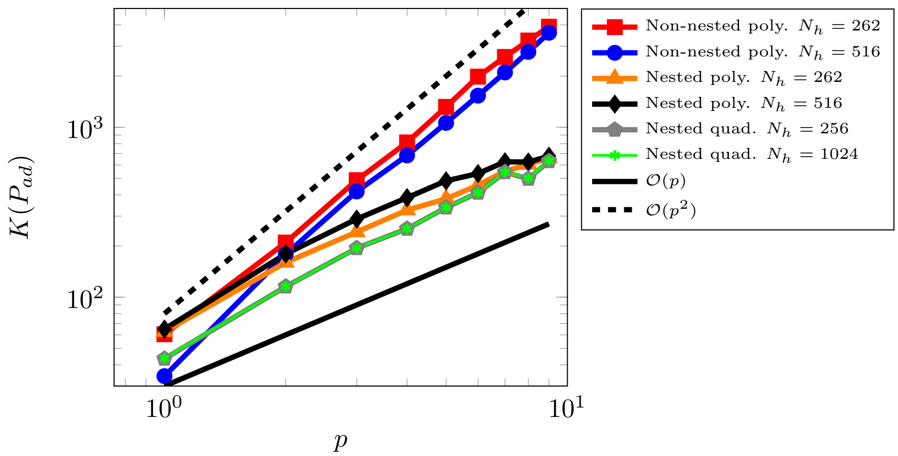

FIGURE 7. Example 5. Condition number $K(P_{ad})$ as function of p, when $p = q$.

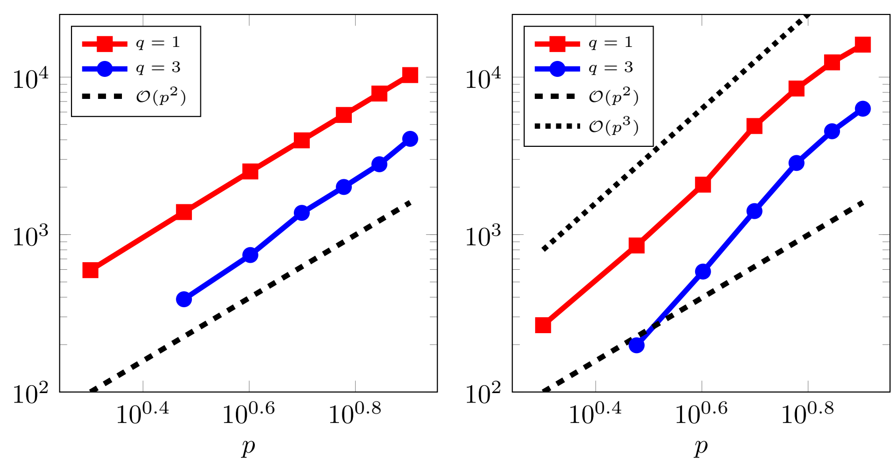

FIGURE 8. Example 5. Condition number $K(P_{ad})$ as function of p when $p \neq q$, for nested grids (left) and non-nested grids (right).

APPENDIX A. PROOF OF THEOREM 4.5

In this section we present the proof of Theorem 4.5. To this end, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded, open, convex domain with boundary $\partial\Omega$. Given $f \in L_0^2(\Omega)$, consider the homogeneous Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{51}$$

The weak formulation of (51) is: find $u \in H^1(\Omega)/\mathbb{R} := H^1(\Omega) \cap L_0^2(\Omega)$, such that

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \ell(v) := \int_\Omega f v \, \mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega)/\mathbb{R},$$

with $H^1(\Omega)/\mathbb{R}$ equipped with the norm $\|v\|_{H^1(\Omega)/\mathbb{R}} := \|\nabla v\|_{L^2(\Omega)}$. From [48, Theorem 3.2.1.3] it follows that $u \in H^2(\Omega)$. Furthermore, if the boundary of the domain Ω is sufficiently smooth, cf. below, we note that the following result holds.

Lemma A.1. *Assuming that Ω is a bounded, open, convex C^2 domain, then the solution u of problem* (51) *satisfies the following bound*

$$\|u\|_{H^2(\Omega)} \leq 2\sqrt{6}\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

Proof. The essential idea is to employ [48, Theorem 3.1.2.3]; however, this cannot be directly applied. Thereby, noting that $u \in H^1(\Omega) \cap L_0^2(\Omega)$ is the unique weak solution satisfying (51), it is clear that u is the unique weak solution of the equivalent problem

$$-\Delta u + \lambda u = f + \lambda u \quad \text{in } \Omega, \qquad \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

for any $\lambda \in \mathbb{R}$. By restricting $\lambda > 0$, upon application of [48, Theorem 3.1.2.3], with f replaced by $f + \lambda u$ and setting $\beta = 0$, we have that

$$\|u\|_{H^2(\Omega)} \leq \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 4\right)^{1/2} \left(\|f + \lambda u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

Therefore, by the triangle inequality, and noting that $1 < 1 + \lambda$, we obtain

$$\|u\|_{H^2(\Omega)} \leq C(\lambda)\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

for any $\lambda > 0$, where $C(\lambda) = \left(\lambda^{-2} + \lambda^{-1} + 4\right)^{1/2}(1 + \lambda)$. By selecting $\lambda = 1$, we deduce that $C(\lambda) = 2\sqrt{6} \approx 4.89898$. Of course, the value of $C(\lambda)$ can be optimized with respect to λ; a simple calculation shows that the minimal value is $C(\lambda) = 4.63918$. □

The proof of (11) when Ω is a general convex (and hence Lipschitz) domain now proceeds with the following steps.

Step 1. [$H^1(\Omega)$ bound] The existence of a unique weak solution to (51) follows by the Lax–Milgram lemma applied to the bilinear form $a(\cdot,\cdot)$, which is bounded and coercive on $H^1(\Omega)/\mathbb{R}$, and noting that the linear functional $\ell(\cdot)$ is bounded on $H^1(\Omega)/\mathbb{R}$. Indeed,

$$\ell(v) = \int_\Omega f v \,\mathrm{d}\mathbf{x} = \int_\Omega f\left[v - v_\Omega\right]\mathrm{d}\mathbf{x} \leq \|f\|_{L^2(\Omega)}\|v - v_\Omega\|_{L^2(\Omega)},$$

where $v_\Omega := {}^1\!/_{|\Omega|}\int_\Omega v \,\mathrm{d}\mathbf{x}$. By Poincaré's inequality,

$$\|v - v_\Omega\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla v\|_{L^2(\Omega)},$$

where $C(\Omega)$ is a positive constant. Recalling that Ω is a bounded, open, convex domain, it can be shown that $C(\Omega) \leq {}^1\!/_\pi \operatorname{diam}(\Omega)$, cf. [55]. Setting $v = u$ in the weak formulation above, we then deduce that

$$\|\nabla u\|_{L^2(\Omega)}^2 = \ell(u) \leq C(\Omega)\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{\pi}\operatorname{diam}(\Omega)\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}.$$

Hence,

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{\pi}\operatorname{diam}(\Omega)\|f\|_{L^2(\Omega)}.$$

Step 2. Suppose that $\Omega \subset \Omega_m$, where Ω_m, $m = 1, 2, \ldots$, is a sequence of bounded, open, convex C^2 domains, such that $\operatorname{dist}(\Omega, \Omega_m) \leq {}^1\!/_m$. The existence of such a sequence $\{\Omega_m\}_{m=1}^\infty$ follows from Eggleston's lemma, cf. [48, Lemma 3.2.1.1]. Hence, in particular $0 \leq \operatorname{diam}(\Omega_m) - \operatorname{diam}(\Omega) \to 0$ as $m \to \infty$.

Step 3. With Ω_m, $m = 1, 2, \dots$, as in Step 2, we extend f by 0 from Ω to Ω_m for each $m = 1, 2, \dots$, and define

$$f_m(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega, \\ 0 & \text{for } \mathbf{x} \in \Omega_m \setminus \Omega. \end{cases}$$

Clearly, $f_m \in L^2_0(\Omega_m)$. We consider the following Neumann problem on Ω_m:

$$-\Delta u_m = f_m \quad \text{in } \Omega_m, \qquad \nabla u_m \cdot \mathbf{n}_m = 0 \quad \text{on } \partial\Omega_m,$$

where $\mathbf{n}_m$ is the unit outward normal vector to $\partial\Omega_m$. We have that the unique weak solution $u_m \in H^1(\Omega_m)/\mathbb{R}$ of the above Neumann problem satisfies the elliptic regularity result given in Lemma A.1; thereby, we deduce that

$$\|u_m\|_{H^2(\Omega_m)} \leq 2\sqrt{6}\left(\|f\|_{L^2(\Omega)} + \|u_m\|_{L^2(\Omega_m)}\right), \qquad m = 1, 2, \dots. \tag{52}$$

Since $\Omega \subset \Omega_m$, upon application of Poincaré's inequality on the right-hand side of (52), followed by recalling the $H^1(\Omega)$ bound derived in Step 1, we get

$$\begin{aligned} \|u_m\|_{H^2(\Omega)} \leq \|u_m\|_{H^2(\Omega_m)} &\leq 2\sqrt{6}\left(\|f\|_{L^2(\Omega)} + \frac{1}{\pi}\mathrm{diam}(\Omega_m)\|\nabla u_m\|_{L^2(\Omega_m)}\right) \\ &\leq 2\sqrt{6}\left(\|f\|_{L^2(\Omega)} + \frac{1}{\pi^2}[\mathrm{diam}(\Omega_m)]^2\|f_m\|_{L^2(\Omega_m)}\right) \\ &= 2\sqrt{6}\left(1 + \frac{1}{\pi^2}[\mathrm{diam}(\Omega_m)]^2\right)\|f\|_{L^2(\Omega)}, \end{aligned} \tag{53}$$

$m = 1, 2, \dots$. Thanks to Step 2, $\lim_{m\to\infty} \mathrm{diam}(\Omega_m) = \mathrm{diam}(\Omega)$. As $(\mathrm{diam}(\Omega_m))_{m=1}^\infty$ is a convergent sequence in $\mathbb{R}$ it is automatically a bounded sequence, and therefore, because of (53), there exists a positive constant C_0, independent of m, such that

$$\|u_m\|_{H^2(\Omega)} \leq \|u_m\|_{H^2(\Omega_m)} \leq C_0 \qquad \text{for all } m = 1, 2, \dots. \tag{54}$$

Thus, $(u_m)_{m=1}^\infty$ is a bounded sequence in $H^2(\Omega)$. Hence, there exists an element $u_\infty \in H^2(\Omega)$ and a weakly convergent subsequence $u_{m_k} \rightharpoonup u_\infty$ in $H^2(\Omega)$. By weak lower semicontinuity of the norm function $\|\cdot\|_{H^2(\Omega)}$, we have that

$$\|u_\infty\|_{H^2(\Omega)} \leq \liminf_{k\to\infty} \|u_{m_k}\|_{H^2(\Omega)}. \tag{55}$$

According to the Rellich–Kondrashov theorem $H^2(\Omega)$ is compactly embedded into $H^1(\Omega)$; thus we can extract a further subsequence (not indicated), such that $u_{m_k} \to u_\infty$ strongly in $H^1(\Omega)$. Now,

$$\|\nabla u_{m_k}\|^2_{L^2(\Omega_{m_k})} = \|\nabla u_{m_k}\|^2_{L^2(\Omega)} + \|\nabla u_{m_k}\|^2_{L^2(\Omega_{m_k}\setminus\Omega)}. \tag{56}$$

Focusing on the second term on the right-hand side of (56), by Hölder's inequality with conjugate exponents $\alpha = p/2$ and $\alpha' = \frac{\alpha}{\alpha-1} = p/(p-2)$, $2 < p < 2n/(n-2)$ (where $2n/(n-2)$ is the critical Sobolev index), and (54), we have that

$$\begin{aligned} \|\nabla u_{m_k}\|_{L^2(\Omega_{m_k}\setminus\Omega)} &\leq \|\nabla u_{m_k}\|_{L^p(\Omega_{m_k}\setminus\Omega)}\,|\Omega_{m_k}\setminus\Omega|^{\frac{p-2}{2p}} \\ &\leq \|\nabla u_{m_k}\|_{L^p(\Omega_{m_k})}\,|\Omega_{m_k}\setminus\Omega|^{\frac{p-2}{2p}} \\ &\leq C\big(|\Omega_{m_k}|, \mathrm{diam}(\Omega_{m_k}), n, p\big)\,\|u_{m_k}\|_{H^2(\Omega_{m_k})}\,|\Omega_{m_k}\setminus\Omega|^{\frac{p-2}{2p}} \\ &\leq C\big(n,p\big)\,\|u_{m_k}\|_{H^2(\Omega_{m_k})}\,|\Omega_{m_k}\setminus\Omega|^{\frac{p-2}{2p}} \\ &\leq C\big(n,p\big)\,C_0\,|\Omega_{m_k}\setminus\Omega|^{\frac{p-2}{2p}} \to 0 \qquad \text{as } k\to\infty. \end{aligned} \tag{57}$$

Passing to the limit $k \to \infty$ in (56) we therefore have that

$$\lim_{k\to\infty} \|\nabla u_{m_k}\|^2_{L^2(\Omega_{m_k})} = \lim_{k\to\infty} \|\nabla u_{m_k}\|^2_{L^2(\Omega)} = \|\nabla u_\infty\|^2_{L^2(\Omega)}, \tag{58}$$

where the last equality follows from the strong convergence $u_{m_k} \to u_\infty$ in $H^1(\Omega)$.

Recalling from (53) that

$$\|u_{m_k}\|_{H^2(\Omega)} \le 2\sqrt{6}\,\big(\|f\|_{L^2(\Omega)} + \frac{1}{\pi}\mathrm{diam}(\Omega_{m_k})\|\nabla u_{m_k}\|_{L^2(\Omega_{m_k})}\big), \tag{59}$$

passing to the limit $k \to \infty$ in inequality (59) using (55), (57), and (58), together with $\lim_{m\to\infty} \mathrm{diam}(\Omega_m) = \mathrm{diam}(\Omega)$, yields that

$$\|u_\infty\|_{H^2(\Omega)} \le 2\sqrt{6}\,\big(\|f\|_{L^2(\Omega)} + \frac{1}{\pi}\mathrm{diam}(\Omega)\|\nabla u_\infty\|_{L^2(\Omega)}\big). \tag{60}$$

Step 4. [Identification of u_∞] It remains to show that $u_\infty = u$, the weak solution of the original Neumann problem on Ω. To this end, we consider the weak formulation of the Neumann problem satisfied by u_m:

$$\int_{\Omega_m} \nabla u_m \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega_m} f_m v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_m)/\mathbb{R}.$$

Thanks to the definition of f_m, this weak formulation is equivalent to

$$\int_{\Omega_m} \nabla u_m \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_m)/\mathbb{R},$$

and therefore

$$\int_{\Omega} \nabla u_m \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_{\Omega_m\setminus\Omega} \nabla u_m \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_m)/\mathbb{R},$$

whereby

$$\int_{\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_{\Omega_{m_k}\setminus\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_{m_k})/\mathbb{R}.$$

Equivalently, because $\int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$,

$$\int_{\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_{\Omega_{m_k}\setminus\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_{m_k}).$$

Consider a fixed bounded domain $\Omega_0 \subset \mathbb{R}^n$ such that $\Omega_0 \Supset \Omega_m \supset \Omega$ for all $m = 1, 2, \ldots$. Then,

$$\int_{\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_{\Omega_{m_k}\setminus\Omega} \nabla u_{m_k} \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_0).$$

By noting (57), the strong convergence $u_{m_k} \to u_\infty$ in $H^1(\Omega)$ as $k \to \infty$, we have that

$$\int_{\Omega} \nabla u_\infty \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega_0),$$

hence also

$$\int_{\Omega} \nabla u_\infty \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega),$$

since any element of $v \in H^1(\Omega_0)$ can be viewed as the extension of a $v \in H^1(\Omega)$ to the superset Ω_0. Therefore, again since $\int_\Omega f(x)\,\mathrm{d}x = 0$, also

$$\int_{\Omega} \nabla u_\infty \cdot \nabla v \,\mathrm{d}\mathbf{x} = \int_{\Omega} f v \,\mathrm{d}\mathbf{x} \qquad \forall\, v \in H^1(\Omega)/\mathbb{R}.$$

Thus we have shown that u_∞ coincides with the unique weak solution u of the homogeneous Neumann problem posed on Ω. Returning with this information to (60), we have that the weak solution of the homogeneous Neumann problem on the bounded, open, convex (and therefore Lipschitz) domain Ω satisfies

$$\|u\|_{H^2(\Omega)} \leq 2\sqrt{6}\left(\|f\|_{L^2(\Omega)} + \frac{1}{\pi}\mathrm{diam}(\Omega)\|\nabla u\|_{L^2(\Omega)}\right),$$

as required.

Appendix B. Condition number estimates for non-nested grids

In this Appendix we provide a bound on the condition number of the additive Schwarz operator P_{ad} introduced in Section 3 when the fine and coarse grids $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, are non-nested. For the sake of simplicity, here we assume that $\rho = 1$ on Ω and consider the massively parallel case, i.e., when $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$. For the purposes of the proceeding analysis, we also assume that Ω is convex; moreover, we make the following additional assumption on $\mathcal{T}_h$.

Assumption B.1. *(Coverability) For every polytopic element $\kappa \in \mathcal{T}_h$, there exists a set of m_κ overlapping shape-regular simplices $\mathcal{K}_i$, $i = 1, \dots, m_\kappa$, such that*

$$\mathrm{dist}(\kappa, \partial\mathcal{K}_i) \lesssim {}^{\mathrm{diam}(\mathcal{K}_i)}/{}_{p^2}, \quad \textit{and} \quad |\mathcal{K}_i| \gtrsim |\kappa|,$$

for all $i = 1, \dots, m_\kappa$, see [30, Chapter 3].

Given that Assumption B.1 holds, we state the following inverse inequality, cf. [10].

Lemma B.2. *Suppose that $\mathcal{T}_h$ satisfies Assumption B.1; then, given $v_h \in V_h$, the following bound holds:*

$$\|\nabla v_h\|^2_{L^2(\kappa)} \lesssim p^4 h_\kappa^{-2} \|v_h\|^2_{L^2(\kappa)} \qquad \forall\, \kappa \in \mathcal{T}_h.$$

Proof. We refer to [30] for the proof of this result. □

We first provide a counterpart of Lemma 5.13, which holds in the non-nested case and allows us to prove the validity of Property 6.3 also for non-nested spaces V_h and V_H. The key aspect of our analysis is the construction of the conforming approximant introduced in Theorem 5.12. In particular, we recall the following result.

Theorem B.3. *Let Ω be a bounded convex polygonal/polyhedral domain and suppose that $\mathcal{T}_h$ satisfies Assumptions 2.2 and 2.3. Let $\mathcal{G}_h(v_h) := \nabla_h v_h + \mathcal{R}_1([\![v_h]\!])$ be the discrete gradient operator of $v_h \in V_h$ defined as in Definition 5.9 and let $\widetilde{v}_h \in H^1_0(\Omega)$ be such that*

$$\int_\Omega \nabla\widetilde{v}_h \cdot \nabla w \;\mathrm{d}\mathbf{x} = \int_\Omega \mathcal{G}_h(v_h) \cdot \nabla w \;\mathrm{d}\mathbf{x} \quad \forall\, w \in H^1_0(\Omega). \tag{61}$$

Then, the following approximation and stability results hold:

$$\|v_h - \widetilde{v}_h\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|\sigma_{h,1}^{1/2}[\![v_h]\!]\|_{L^2(\mathcal{F}_h)}, \tag{62}$$

$$|\widetilde{v}_h|_{H^1(\Omega)} \lesssim \|v_h\|_{h,1}. \tag{63}$$

Proof. We follow [11, proof of Theorem 3.1], cf., also, Theorem 5.12. Let $z \in H_0^1(\Omega)$ be the solution of the following Dirichlet boundary value problem:

$$-\Delta z = v_h - \widetilde{v}_h \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega.$$

Since Ω is convex, we have that $z \in H^2(\Omega)$ and it holds $\|z\|_{H^2(\Omega)} \lesssim \|v_h - \widetilde{v}_h\|_{L^2(\Omega)}$. Employing integration by parts and the definition of $\widetilde{v}_h$ we get

$$\|v_h - \widetilde{v}_h\|_{L^2(\Omega)}^2 = -\int_\Omega \mathcal{R}_1([\![v_h]\!]) \cdot \nabla z \,\mathrm{d}\mathbf{x} \; - \; \int_{\mathcal{F}_h} \nabla z \cdot [\![v_h]\!] \,\mathrm{d}s.$$

Given that $z \in H^2(\Omega)$, we deduce that $[\![z]\!]|_F = \mathbf{0}$ and $\{\!\{\nabla z\}\!\}|_F = \nabla z|_F$ for any $F \in \mathcal{F}_h$. Then, for any $z_h \in V_h$ we get

$$\begin{aligned}\|v_h - \widetilde{v}_h\|_{L^2(\Omega)} &= -\int_\Omega \mathcal{R}_1([\![v_h]\!]) \cdot (\nabla z \; - \nabla_h z_h)\mathrm{d}\mathbf{x} \; - \; \int_{\mathcal{F}_h} \{\!\{\nabla z - \nabla_h z_h\}\!\} \cdot [\![v_h]\!] \,\mathrm{d}s \\ &\lesssim \|\mathcal{R}_1([\![v_h]\!])\|_{L^2(\Omega)} \|\nabla z - \nabla_h z_h\|_{L^2(\Omega)} \\ &\quad + \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|_{L^2(\mathcal{F}_h)} \|\sigma_{h,1}^{-1/2} \{\!\{\nabla z - \nabla_h z_h\}\!\}\|_{L^2(\mathcal{F}_h)}. \end{aligned} \tag{64}$$

Using the definition of $\mathcal{R}_1$, the first term on the right hand side of (64) can be bounded as

$$\begin{aligned}\|\mathcal{R}_1([\![v_h]\!])\|_{L^2(\Omega)}^2 &= \int_{\mathcal{F}_h} [\![v_h]\!] \{\!\{\mathcal{R}_1([\![v_h]\!])\}\!\} \,\mathrm{d}\sigma \\ &\lesssim \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|_{L^2(\mathcal{F}_h)} \; \|\sigma_{h,1}^{-1/2} \{\!\{\mathcal{R}_1([\![v_h]\!])\}\!\}\|_{L^2(\mathcal{F}_h)}. \end{aligned} \tag{65}$$

Moreover, by invoking Lemma 5.4 we get

$$\begin{aligned}\|\sigma_{h,1}^{-1/2} \{\!\{\mathcal{R}_1([\![v_h]\!])\}\!\}_{1/2}\|_{L^2(\mathcal{F}_h)}^2 &\leq C_\sigma^{-1} \sum_{\kappa \in \mathcal{T}_h} \frac{\langle h_\kappa \rangle}{p^2} \|\mathcal{R}_1([\![v_h]\!])\|_{L^2(\partial\kappa)}^2 \\ &\lesssim \sum_{\kappa \in \mathcal{T}_h} \|\mathcal{R}_1([\![v_h]\!])\|_{L^2(\kappa)}^2. \end{aligned} \tag{66}$$

By inserting (66) into (65) we obtain

$$\|\mathcal{R}_1([\![v_h]\!])\|_{L^2(\Omega)} \lesssim \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|_{L^2(\mathcal{F}_h)}. \tag{67}$$

Now, by selecting $z_h = \Pi_h^p z$ in (64), exploiting (67) and the bounds of Lemma 5.5, cf., also, Remark 5.6, we deduce that

$$\|v_h - \widetilde{v}_h\|_{L^2(\Omega)}^2 \lesssim {}^h/_p \|z\|_{H^2(\Omega)} \|\sigma_{h,1}^{1/2} [\![v_h]\!]\|_{L^2(\mathcal{F}_h)}. \tag{68}$$

The bound (62) immediately follows from (68) together with $\|z\|_{H^2(\Omega)} \lesssim \|v_h - \widetilde{v}_h\|_{L^2(\Omega)}$. To prove (63) we first observe that, by selecting $w = \widetilde{v}_h$ in (61) and using Cauchy-Schwarz inequality, we get

$$|\widetilde{v}_h|_{H^1(\Omega)} \leq \|\mathcal{G}_h(v_h)\|_{L^2(\Omega)}. \tag{69}$$

Then, the bound (63) immediately follows from (69) employing the definition of $\mathcal{G}_h$ together with (67). □

Remark B.4. Theorem B.3 provides global bounds for $v_h \in V_h$ in the L^2-norm. This result is a particular case of Theorem 5.12, where local bounds on each coarse element $\mathcal{D}_j \in \mathcal{T}_H$ are provided.

On the basis of the previous result, Lemma 5.13 can be generalized to non-nested spaces as follows.

Lemma B.5. *Let Ω be a bounded convex polygonal/polyhedral domain and suppose that $\mathcal{T}_h$ satisfies Assumptions 2.2, 2.3, and B.1, and $\mathcal{T}_H$ satisfies Assumption 2.3. Then, for any $v_h \in V_h$ there exists a coarse function $v_H \in V_H$ such that*

$$\|v_h - R_0^\top v_H\|_{L^2(\Omega)} \lesssim \frac{H}{q}\|v_h\|_{h,1}, \tag{70}$$

$$\|v_h - R_0^\top v_H\|_{h,1} \lesssim \frac{p^2}{q}\frac{H}{h}\|v_h\|_{h,1}. \tag{71}$$

Proof. Let $v_h \in V_h$ and let $v_H \in V_H$ be defined as $v_H = \Pi_H^q \widetilde{v}_h$, with $\widetilde{v}_h$ as defined in Theorem B.3 and where Π_H^q is the hp-approximant introduced in Lemma 5.5. Then, by employing the triangle inequality we have

$$\|v_h - R_0^\top v_H\|_{L^2(\Omega)} \lesssim \|v_h - \widetilde{v}_h\|_{L^2(\Omega)} + \|\widetilde{v}_h - \Pi_2\widetilde{v}_h\|_{L^2(\Omega)} + \|\Pi_2\widetilde{v}_h - R_0^\top(\Pi_H^q\widetilde{v}_h)\|_{L^2(\Omega)},$$

where $\Pi_2 : L^2(\Omega) \to V_h$ is the L^2-projection operator onto V_h. From the definition of $R_0^\top$, we note that $\Pi_2 v_H = R_0^\top v_H$ for all $v_H \in V_H$. Hence, exploiting Lemma 5.5 together with Assumption 2.3, cf. Remark 5.6, gives

$$\begin{aligned}\|v_h - R_0^\top v_H\|_{L^2(\Omega)} &\lesssim \|v_h - \widetilde{v}_h\|_{L^2(\Omega)} + \|\widetilde{v}_h - \Pi_2\widetilde{v}_h\|_{L^2(\Omega)} + \|\Pi_2(\widetilde{v}_h - \Pi_H^q\widetilde{v}_h)\|_{L^2(\Omega)}\\ &\leq \|v_h - \widetilde{v}_h\|_{L^2(\Omega)} + \|\widetilde{v}_h - \Pi_h^p\widetilde{v}_h\|_{L^2(\Omega)} + \|\widetilde{v}_h - \Pi_H^q\widetilde{v}_h\|_{L^2(\Omega)}\\ &\lesssim \|v_h - \widetilde{v}_h\|_{L^2(\Omega)} + {}^{h}/_{p}\,\|\widetilde{v}_h\|_{H^1(\Omega)} + {}^{H}/_{q}\,\|\widetilde{v}_h\|_{H^1(\Omega)};\end{aligned}$$

here we have used that $\|\Pi_2 v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ for all $v \in L^2(\Omega)$ and $\|v - \Pi_2 v\|_{L^2(\Omega)} \leq \|v - w\|_{L^2(\Omega)}$ for all $w \in V_h$. By applying Poincaré's inequality to $\widetilde{v}_h \in H_0^1(\Omega)$ and noting Theorem B.3, inequality (70) immediately follows by observing that $h \leq H$ and $q \leq p$. In order to obtain (71) we proceed as follows:

$$\|v_h - R_0^\top v_H\|_{h,1}^2 = \|\nabla_h(v_h - R_0^\top v_H)\|_{L^2(\mathcal{T}_h)}^2 + \|\sigma_{h,1}^{1/2}[\![v_h - R_0^\top v_H]\!]\|_{L^2(\mathcal{F}_h)}^2. \tag{72}$$

We bound the first term on the right-hand side of (72) by means of Lemma B.2 and (70) as follows:

$$\begin{aligned}\|\nabla_h(v_h - R_0^\top v_H)\|_{L^2(\mathcal{T}_h)}^2 &= \sum_{\kappa\in\mathcal{T}_h}\|\nabla_h(v_h - R_0^\top v_H)\|_{L^2(\kappa)}^2\\ &\lesssim \Big(\frac{p^2}{h}\Big)^2\|v_h - R_0^\top v_H\|_{L^2(\Omega)}^2 \lesssim \frac{p^4}{q^2}\frac{H^2}{h^2}\|v_h\|_{h,1}^2.\end{aligned} \tag{73}$$

The second term on the right-hand side of (72) can be bounded by recalling the definition of $\sigma_{h,1}$, Lemma 5.4 and (70) as follows:

$$\begin{aligned}\|\sigma_{h,1}^{1/2}[\![v_h - R_0^\top v_H]\!]\|_{L^2(\mathcal{F}_h)}^2 &\lesssim \frac{p^2}{h}\sum_{\kappa\in\mathcal{T}_h}\|v_h - R_0^\top v_H\|_{L^2(\partial\kappa)}^2\\ &\lesssim \Big(\frac{p^2}{h}\Big)^2\|v_h - R_0^\top v_H\|_{L^2(\Omega)}^2 \lesssim \frac{p^4}{q^2}\frac{H^2}{h^2}\|v_h\|_{h,1}^2.\end{aligned} \tag{74}$$

Inserting (73) and (74) into (72) we obtain (71). □

With Lemma B.5 in hand, we can prove the following result.

Theorem B.6. *Let Ω be a bounded convex polygonal/polyhedral domain and suppose that $\mathcal{T}_h$ satisfies Assumptions 2.2, 2.3, and B.1, $\mathcal{T}_H$ satisfies Assumption 2.3, and $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$. Then, for $\rho = 1$, Property 6.3 holds with*

$$C_\sharp^2 \eqsim \Big(\frac{p^4}{q^2}\frac{H^2}{h^2}\Big).$$

Proof. Let $v_h \in V_h$. Proceeding as in the proof of Theorem 6.5, by selecting $v_0 = v_H$ as in Lemma B.5, v_h can be decomposed as $v_h = \sum_{i=0}^{N_h} R_i^\top v_i$, with $v_i = R_i(v_h - R_0^\top v_0) \in V_i$, $i = 1, \ldots, N_h$, so that

$$\Big|\sum_{i=0}^{N_h} \mathcal{A}_i(v_i, v_i)\Big| \lesssim \|v_h - R_0^\top v_0\|_{h,1}^2 + \mathcal{A}_h(v_h, v_h),$$

where we have used (38), (40) and (43) with the hypothesis $\mathcal{T}_{\mathbb{H}} = \mathcal{T}_h$. The result then immediately follows by noting (71) together with the coercivity of $\mathcal{A}_h$, cf. Lemma 5.3. □

Remark B.7. Based on Theorem B.6, for non-nested coarse and fine spaces V_H and V_h, respectively, the condition number $K(P_{ad})$ of the additive Schwarz operator P_{ad} can be bounded as follows:

$$K(P_{ad}) \lesssim \Big(\frac{p^4}{q^2}\frac{H^2}{h^2}\Big)(N_{\mathbb{S}} + 2).$$

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