OPTIMAL CONSUMPTION AND INVESTMENT WITH INDEPENDENT STOCHASTIC LABOR INCOME*

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Abstract

We develop a new dynamic continuous-time model of optimal consumption and investment to include independent stochastic labor income. We reduce the problem of solving the Bellman equation to a problem of solving an integral equation. We then explicitly characterize the optimal consumption and investment strategy as a function of income-to-wealth ratio. We provide some analytical comparative statics associated with the value function and optimal strategies. We also develop a quite general numerical algorithm for control iteration and solve the Bellman equation as a sequence of solutions to ordinary differential equations. This numerical algorithm can be readily applied to many other optimal consumption and investment problems especially with extra nondiversifiable Brownian risks, resulting in nonlinear Bellman equations. Finally, our numerical analysis illustrates how the presence of stochastic labor income affects the optimal consumption and investment strategy.

Keywords: optimal consumption and investment, stochastic income, Bellman equation, dynamic programming

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1 INTRODUCTION

How should labor income risk affect the demand for consumption and investment? Standard portfolio choice models assume complete markets in which labor income is spanned by the financial market so that an individual can fully diversify her income risk by dynamically trading financial assets. Yet, in reality, the correlation between labor income and stock market returns is close to zero, i.e., labor income is unspanned by the stock market.[1] Therefore, many individuals face nondiversifiable labor income risk.

In this paper, we develop a new dynamic continuous-time model of optimal consumption and investment to include independent stochastic labor income. We consider the optimal consumption and investment strategy when an additional unspanned labor income is available. More specifically, we introduce the extra Brownian motion in labor income dynamics that is independent of the Brownian motion in the stock market, which captures the essence of unspanned income. We focus on a risk averse individual who exhibits a logarithmic utility function, and the individual can invest in one riskless bond and multiple risky stocks in the financial market.

The value function of the individual is the maximized payoff with two controls consumption and investment of a stochastic control problem with two states financial wealth and labor income. By standard invariant embedding arguments of dynamic programming, we can derive the associated Bellman equation of the value function. The two-dimensional Bellman equation of wealth and income can be reduced to the one-dimensional Bellman equation of income-to-wealth ratio. With the probabilistic approach, we can show that the solution of the reduced Bellman equation is the value function of another stochastic control problem with two controls consumption and investment and one state income-to-wealth ratio.

Even though the nonlinear Bellman equation seems to be almost impossible to be solved analytically, remarkably we are able to reduce the problem of solving the Bellman equation to a problem of solving an integral equation. We then explicitly characterize the optimal consumption and investment strategy as a function of income-to-wealth ratio. We provide some analytical comparative statics associated with the value function and optimal strategies. We also develop a quite general numerical algorithm for control iteration and solve the value function as a sequence of solutions to ordinary differential equations. The convergence of this numerical algorithm is guaranteed by showing that the sequence is monotone increasing and bounded above. This numerical algorithm can be readily applied to many other optimal consumption and investment problems especially with extra nondiversifiable Brownian risks, resulting in nonlinear Bellman equations. Finally, our numerical analysis illustrates how the presence of unspanned income risk affects the optimal

[1] There is a large body of empirical and anecdotal evidence for supporting this zero correlation (e.g., Campbell *et al.*, 2001; Cocco *et al.*, 2005; Davis and Willen, 2013; Gomes and Smirnova, 2021). Also, various studies have found that the correlation is positive in the long run (Storesletten *et al.*, 2007; Benzoni *et al.*, 2007; Lynch and Tan, 2011). In particular, a cointegration between the stock and labor market affects the optimal asset allocation (Benzoni *et al.*, 2007).

consumption and investment strategy.

Interestingly, we find a significant discontinuity and dramatic change in the individual's optimal consumption and portfolio choice with a change in the extent of unspanned income risk. Both the optimal consumption and the optimal investment fall sharply as the unspanned income risk rises even when the income risk is quite small. Further, the impact of unspanned income risk on the individual's optimal choice is different with respect to levels of income-to-wealth ratio. That is, the individual's optimal decision is more likely to be affected negatively by unspanned income risk when the income-to-wealth ratio is high than when it is low.

The reduced consumption with unspanned income risk is consistent with the impact of the individual's precautionary savings motive on the optimal consumption/savings decision. To accumulate enough wealth that plays an important role in smoothing out unspanned income risk, the individual's optimal decision is to consume less with unspanned income risk for precautionary reasons. The decreased investment with unspanned income risk is consistent with the impact of the individual's risk diversification motive. Given that labor income with its unspanned risk should be treated as a risky asset resulting in a nondiversifiable risk source, the individual would, thus, invest less in the stock market for risk diversification purposes in an attempt to strike an optimal balance between risk-free and risky assets. The differing effects of unspanned income risk on the individual's optimal decision with respect to levels of income-to-wealth ratio are caused by the inverse relation between the income-to-wealth ratio and wealth. Labor income itself is the staple income of poor individuals, so they should concern themselves with substantial precautionary savings and risk diversification motives. Therefore, the individual's optimal decision is significantly affected by unspanned income risk when the income-to-wealth ratio is high.

Closely related studies include the literature on continuous-time optimal consumption and portfolio choice with either constant income or spanned income (e.g., Merton, 1971; Bodie *et al.*, 1992; Koo, 1998; Farhi and Panageas, 2007; Yang and Koo, 2018). The closest studies to ours include the literature on continuous-time optimal consumption or portfolio choice with labor income risk: Duffie *et al.* (1997), Dybvig and Liu (2010), Jang *et al.* (2013), Bensoussan *et al.* (2016), and Wang *et al.* (2016). Duffie *et al.* (1997) study a very similar model to ours in that they consider stochastic income with its unspanned risk, with different techniques. They develop the so-called viscosity solutions techniques with which the value function can be approximated by a sequence of smooth functions and the unique limit of the sequence (i.e., the unique viscosity solution) becomes the value function. Compared to Duffie *et al.* (1997) with neither explicit characterization nor numerical analysis, our innovation is the development of a way of explicitly characterizing the value function as an integral equation and efficiently computing the value function and optimal strategies in an incomplete market. Dybvig and Liu (2010) examine the impact of constrained borrowing on the optimal consumption and investment strategy. They mostly consider the case in which labor income is spanned by the stock market and provide numerical results only with the unspanned labor income case.[2] They do not provide theoretical analysis on the value function and optimal decision or numerical algorithm on the consumption and portfolio choice problem in the presence of unspanned labor income. Jang *et al.* (2013) investigate the optimal consumption and investment strategy with the risk of forced unemployment, resulting in a downward jump of constant labor income. However, they examine the optimal strategy under complete markets with (private) unemployment insurance so that labor income risk is diversifiable. Bensoussan *et al.* (2016) examine the optimal strategy with unemployment risks as in Jang *et al.* (2013) but in an incomplete market. However, they do not study the effect of general nondiversifiable stochastic fluctuations of labor income, which are important features for understanding the optimal strategy with labor income risk. Thus, our model with independent stochastic labor income differs from Bensoussan *et al.* (2016) with constant income subject to forced unemployment risk. The extra stochastic source of income risk complicates the theoretical and numerical analysis, and the methods provided by Bensoussan *et al.* (2016) no longer apply to the problem with independent stochastic labor income. Wang *et al.* (2016) explore the model of optimal consumption/savings with stochastic income and recursive utility. However, they do not consider the individual's optimal investment decision that is crucial to understand the interactions between optimal portfolio choice and consumption/savings.

Although the problem studied in this paper is similar to the one studied by Duffie *et al.* (1997), there are several major differences. First, contrary to viscosity solutions of Duffie *et al.* (1997) that are not explicit, we first solve the Bellman equation explicitly with the PDE-approach taken, without resorting to the so-called martingale duality approach developed by Cox and Huang (1989), He and Pearson (1991), and Karatzas *et al.* (1991).[3] Notably, we reduce the problem's dimension with the income-to-wealth ratio, whereas Duffie *et al.* (1997) with the wealth-to-income ratio. As a result of different dimension reduction approaches taken, the derived Bellman equations are quite different, but having a nonlinear linkage.[4] To our best knowledge, the Bellman equation considered in this paper has not been studied yet analytically and even numerically. Second, we propose an efficient numerical algorithm for computing the value function and optimal strategies in an incomplete market, whereas Duffie *et al.* (1997) do not.[5] It has long been

[2]The verification theorems developed by Dybvig and Liu (2011) are applicable to the spanned labor income case only.

[3]Our PDE-approach can be also applied to the Bellman equation of Duffie *et al.* (1997) to obtain explicit characterization of the value function and optimal strategies, which is the distinct feature of the paper compared to viscosity solutions of Duffie *et al.* (1997). For the details, refer to Section 6.1.

[4]Merely using a reciprocal change of variable from the wealth-to-income ratio of Duffie *et al.* (1997) to the income-to-wealth ratio of ours does not directly lead to our Bellman equation. One needs to apply a nonlinear transformation to the Bellman equation of Duffie *et al.* (1997) to obtain our Bellman equation. For more details, refer to Section 6.1.

[5]The fact obtained from Duffie *et al.* (1997) that the viscosity solution is the unique solution of the Bellman equation (considered in Duffie *et al.* (1997)) is particularly useful in ensuring that numerical solutions converge to the correct limit, at least under conditions.

known that solving the incomplete-market problems numerically involves some unwanted computational complexities caused by nonlinear features of the Bellman equation.[6] Contrary to Duffie *et al.* (1997) with no numerical analysis, we can provide some interesting economic implications based on numerical analysis. Finally, we specialize the labor income to the independent stochastic income and the utility function to the logarithmic utility resulting in the different case of logarithmic utility, thus requiring a special treatment.[7] Although the logarithmic utility function can be generally regarded as the limiting case of power utility functions when the coefficient of relative risk aversion approaches one, the logarithmic utility function case is not straightforward to be treated in Duffie *et al.* (1997) because it leads to a different derivation of the Bellman equation and hence, a different characterization of the value function and optimal strategies.[8] With the logarithmic utility function, analysis of the general stochastic income case considered in Duffie *et al.* (1997) is therefore an interesting future study.

The rest of this paper is organized as follows. In Section 2, we develop the continuous-time model of optimal consumption and investment with the additional nondiversifiable Brownian income risk. In Section 3, we provide the solution of the nonlinear Bellman equation, explicitly characterizing the value function and optimal strategies. In Section 4, we provide analytic comparative statics associated with the value function and optimal strategies. In Section 5, we conduct numerical analysis with a quite general numerical algorithm newly developed and illustrate the impact of stochastic labor income on the value function and optimal strategies. In Section 6, we suggest extensions of our work to further clarify the relation between this paper and Duffie *et al.* (1997) and deal with the power utility case. In Section 7, we conclude the paper.

2 MODEL

The financial market is composed of n risky assets whose price $Y^i(t)$ evolves as a geometric Brownian motion (GBM)

$$dY^i(t) = Y^i(t)\Big(\alpha_i dt + \sum_{j=1}^{n} \sigma_{ij} dw_j(t)\Big), i = 1, ..., n, \tag{1}$$

$$Y^i(0) = Y_0^i,$$

[6]Garlappi and Skoulakis (2010), Jin and Zhang (2012), and Jin *et al.* (2017) have developed numerical methods for solving the optimal consumption and portfolio choice problems in incomplete markets without both nontradable labor income and its nondiversifiable risk. Our proposed numerical procedure is therefore the first attempt to provide a convenient and efficient numerical solution for computing optimal strategies in incomplete markets with unspanned income risk.

[7]As aforementioned at the very beginning of the paper, the independent stochastic income case still can capture the essence of nondiversifiable income risk features. Reflecting the reality in the U.S., notice that the aggregate labor income has been actually extremely stable laying between 59% and 62% of GDP since 1980 over a period of nearly 50 years. Labor income is extremely stable expect for those who are entrepreneurs or working in finance.

[8]The difference between the logarithmic utility function and power utility functions results from multi-dimensional problems in incomplete markets. That is, the approaches taken for dimension reduction are different for the logarithmic utility function and for power utility functions. Such different dimension reduction results lead to different Bellman equations and optimal strategies accordingly.

where α_i is the drift coefficient, the volatility matrix $\sigma = (\sigma_{ij})$ is invertible, and $w_j(t)$, $j = 1, ..., n$, is the standard Brownian motion. In addition, there is a risk-free asset $Y^0(t)$ defined by

$$dY^0(t) = rY^0(t)dt, \tag{2}$$

$$Y^0(0) = Y_0^0,$$

where r is the risk-free interest rate.

An individual has a source of income $y(t)$ that is external to the market (e.g., a salary) with uncertainties independent of the market. It also follows a GBM[9]

$$dy(t) = y(t)\Big(\mu dt + \rho dw_y(t)\Big), \tag{3}$$

$$y(0) = y,$$

where μ and ρ are the drift and volatility coefficient, respectively, and the Brownian motion $w_y(t)$ is independent of $w_j(t)$, $j = 1, ..., n$. For convenience in the sequel, we assume that[10]

$$r > \mu.$$

The individual then builds a portfolio

$$X(t) = \pi_0(t)Y^0(t) + \sum_{i=1}^{n} \pi_i(t)Y^i(t), \tag{4}$$

where $X(t)$ represents the individual's financial wealth, $\pi_0(t)$ and $\pi_i(t)$, $i = 1, ..., n$, represent the dollar

[9]Our GBM income model can be thought of as the simplest possible form among widely adopted empirical specifications. The income process implies that the growth rate of income, $dy(t)/y(t)$, is independently, and identically distributed (i.i.d.). One may also write the dynamics for logarithmic income, $\ln y(t)$, which then follows an arithmetic Brownian motion:

$$d \ln y(t) = \widetilde{\mu} dt + \rho dw_y(t), \quad \widetilde{\mu} = \mu - \frac{\rho^2}{2},$$

where $\widetilde{\mu}$ is the expected change of the logarithmic income. In this baseline model, $\ln y(t)$ is then a unit-root process by following discrete-time specification:

$$\ln\{y(t+1)\} - \ln\{y(t)\} = \widetilde{\mu} + \rho\epsilon(t+1),$$

where $\epsilon(t+1)$ has the time-t standard normal conditional distribution, thus implying that the first difference of $\ln y(t)$ is independently and normally distributed with mean $\widetilde{\mu}$ and volatility ρ. Such a GBM specification for income dynamics has been commonly used in the literature (Dybvig and Liu, 2010, 2011; Wang *et al.*, 2016).

[10]This condition is necessarily required to ensure that the present value of income discounted at the risk-free interest rate (or human wealth) is finite positively. Indeed, human wealth is

$$E\Big[\int_0^\infty e^{-rt} y(t)dt\Big] = \frac{y}{r - \mu},$$

so that the drift of income μ should be lower than the interest rate r for positive finiteness of human wealth.

amount invested in risk-free asset $Y_0(t)$ and risky assets $Y^i(t)$, $i = 1, ..., n$, and compose the portfolio. $\pi_1(t), ..., \pi_n(t)$ are control variables adapted to the n-dimensional standard Brownian motion process $w(t) = (w_1(t), ..., w_n(t))$ used in the risky assets dynamics and the Brownian motion $w_y(t)$ used in the income dynamics. The individual's wealth dynamics are given by

$$dX(t) = \pi_0(t)dY^0(t) + \sum_{i=1}^{n} \pi_i(t)dY^i(t) - C(t)X(t)dt + y(t)dt, \tag{5}$$

$$X(0) = x,$$

where x is the individual's initial wealth, and $C(t) > 0$ is a new control variable that represents the rate of consumption per unit of wealth.

We now perform standard transformations. We first define the vector θ called the market price of risk as the system of linear equations

$$\alpha_i - r = \sum_{j=1}^{n} \sigma_{ij}\theta_j, \; i = 1, ..., n.$$

We combine (4) and (5) using price dynamics given in (1) and (2) to eliminate $\pi_0(t)$ and define the proportion of wealth invested in risky assets $Y^i(t)$, $i = 1, ..., n$, as

$$\varpi_i(t) = \frac{\pi_i(t)Y^i(t)}{X(t)}, \; i = 1, ..., n.$$

We then obtain the evolution of the individual's wealth as follows

$$dX(t) = rX(t)dt + X(t)\sigma^*\varpi(t).\Big(\theta dt + dw(t)\Big) - C(t)X(t)dt + y(t)dt, \tag{6}$$

$$X(0) = x,$$

where . following $\varpi(t)$ is the inner product for two vectors and the superscript $*$ represents the matrix transpose.

The pair $\{X(t), y(t)\}$ represents the state of a stochastic dynamic system, which is controlled by the pair $\{C(t), \varpi(t)\}$. The payoff for the individual to maximize with her logarithmic utility preference over the infinite horizon is given by

$$J_{x,y}(C(.), \varpi(.)) = E\left[\int_0^{+\infty} e^{-\beta t} \ln\{C(t)X(t)\}dt\right], \tag{7}$$

which is subject to

$$C(t) > 0, \tag{8}$$

where $\beta > 0$ is the individual's subjective discount rate. For any processes $C(t)$ and $\varpi(t)$ adapted to the filtration $\mathcal{F}^t$ generated by the Brownian motion processes $w(t)$ and $w_y(t)$, the wealth $X(t)$ should be positive.

3 THE SOLUTION

3.1 DYNAMIC PROGRAMMING

The payoff to be maximized given in (7) is a stochastic control problem with two states $\{X(t), y(t)\}$ and two adapted controls $\{C(t), \varpi(t)\}$. The value function is then defined by

$$V(x,y) = \sup_{C(.),\varpi(.)} J_{x,y}(C(.), \varpi(.)). \tag{9}$$

We first proceed with the value function $V(x,y)$ which is assumed to be sufficiently smooth. By standard invariant embedding arguments of dynamic programming, we can now write the Bellman equation associated with the value function as follows

$$\begin{aligned}\beta V(x,y) =& \ln x + (rx+y)\frac{\partial V}{\partial x} + \mu y\frac{\partial V}{\partial y} + \frac{1}{2}\rho^2 y^2 \frac{\partial^2 V}{\partial y^2} \\ & + \sup_C \Big(\ln C - Cx\frac{\partial V}{\partial x}\Big) + \sup_{\varpi}\Big(x\sigma^*\varpi.\theta\frac{\partial V}{\partial x} + \frac{1}{2}x^2|\sigma^*\varpi|^2\frac{\partial^2 V}{\partial x^2}\Big).\end{aligned} \tag{10}$$

The precise boundary conditions at $x = 0$, $x = +\infty$, $y = 0$, and $y = +\infty$ are to be specifically given later. We impose the following conditions:[11]

$$\frac{\partial V}{\partial x} > 0, \frac{\partial^2 V}{\partial x^2} < 0, \tag{11}$$

which guarantee that the sup in C and ϖ on the right hand side of equation (10) can be attained at finite distance. In this case, the sup can be attained at the following optimal controls

$$\widehat{C} = \frac{1}{x\dfrac{\partial V}{\partial x}}, \ \widehat{\varpi} = -\frac{1}{x}(\sigma^*)^{-1}\theta\frac{\partial V}{\partial x}\Big/\frac{\partial^2 V}{\partial x^2}. \tag{12}$$

We see that the condition $\widehat{C} > 0$ given in (8) is now satisfied. With substitution of the optimal controls stated in (12), we can rewrite the Bellman equation (10) as

$$\beta V(x,y) = \ln x + (rx+y)\frac{\partial V}{\partial x} + \mu y\frac{\partial V}{\partial y} + \frac{1}{2}\rho^2 y^2\frac{\partial^2 V}{\partial y^2} - 1 - \ln\Big(x\frac{\partial V}{\partial x}\Big) - \frac{|\theta|^2}{2}\Big(\frac{\partial V}{\partial x}\Big)^2\Big/\frac{\partial^2 V}{\partial x^2}. \tag{13}$$

[11] With our explicit characterization of the value function provided in Section 3, we can verify that the value function is twice differentiable. In our Section 4 for analytic comparative statics, we also verify that the value is strictly increasing, bounded, and concave.

3.2 DIMENSION REDUCTION OF BELLMAN EQUATION

We first conjecture that a solution of the Bellman equation (13) has the following form:

$$V(x,y) = \frac{\ln x}{\beta} + W(z), \quad z = \frac{y}{x}, \tag{14}$$

where $W(z)$ is a function to be determined. Notice the formulas as follows:

$$x\frac{\partial V}{\partial x} = \frac{1}{\beta} - zW'(z),$$

$$x^2\frac{\partial^2 V}{\partial x^2} = -\frac{1}{\beta} + 2zW'(z) + z^2W''(z),$$

$$y\frac{\partial V}{\partial x} = \frac{z}{\beta} - z^2W'(z),\ y\frac{\partial V}{\partial y} = zW'(z),\ y^2\frac{\partial^2 V}{\partial y^2} = z^2W''(z).$$

Inserting these formulas in the Bellman equation (13), we obtain the new equation for $W(z)$

$$\begin{aligned}\beta W(z) =& \frac{r+z}{\beta}\Big(1-\beta zW'(z)\Big) + \mu zW'(z) + \frac{1}{2}\rho^2 z^2 W''(z)\\ &+ \ln\beta - 1 - \ln\left(1-\beta zW'(z)\right) + \frac{|\theta|^2}{2\beta}\frac{\left(1-\beta zW'(z)\right)^2}{1-2\beta zW'(z)-\beta z^2W''(z)}.\end{aligned} \tag{15}$$

The conditions imposed in (11) imply that

$$1-\beta zW'(z) > 0$$

and

$$1-2\beta zW'(z)-\beta z^2W''(z) > 0.$$

The optimal controls given in (12) are then rewritten as a function of $W(z)$

$$\widehat{C}(z) = \frac{\beta}{1-\beta zW'(z)} \tag{16}$$

and

$$\widehat{\varpi}(z) = (\sigma^*)^{-1}\theta\frac{1-\beta zW'(z)}{1-2\beta zW'(z)-\beta z^2W''(z)}. \tag{17}$$

3.3 PROBABILISTIC APPROACH

With the probabilistic approach, we could actually interpret the function $W(z)$ newly introduced in (14) as the value function of a stochastic control problem. Indeed, turning to the pair $\{X(t), y(t)\}$ with dynamics

(3) and (6), we introduce the new process $z(t) = y(t)/X(t)$ representing the income-to-wealth ratio at time t. Because the Brownian motion processes $w(t)$ and $w_y(t)$ are independent, we have

$$dz(t) = \frac{1}{X(t)} dy(t) + y(t)\, d\Big(\frac{1}{X(t)}\Big),$$

$$d\Big(\frac{1}{X(t)}\Big) = -\frac{1}{X(t)^2} dX(t) + \frac{1}{X(t)^3}(dX(t))^2.$$

Performing direct calculations, we obtain the dynamics of process $z(t)$

$$dz(t) = z(t)\left(-(r-\mu+z(t))dt + \big(-\sigma^*\varpi(t).\theta + |\sigma^*\varpi(t)|^2\big)dt + C(t)dt - \sigma^*\varpi(t).dw(t) + \rho dw_y(t)\right),$$

$$z(0) = \frac{y}{x}.$$

The individual's payoff $J_{x,y}(C(.), \varpi(.))$ to be maximized in (7) can be rewritten as

$$J_{x,y}(C(.), \varpi(.)) = E\Big[\int_0^{+\infty} e^{-\beta t}\big(\ln X(t) + \ln C(t)\big)dt\Big].$$

Using Itô's formula, we know that

$$d\ln X(t) = \frac{dX(t)}{X(t)} - \frac{1}{2X(t)^2}(dX(t))^2.$$

With (6), we get

$$d\ln X(t) = \left(r + z(t) - C(t) + \sigma^*\varpi(t).\theta - \frac{1}{2}|\sigma^*\varpi(t)|^2\right)dt + \sigma^*\varpi(t).dw(t).$$

Therefore, we obtain

$$\ln X(t) = \ln x + \int_0^t \left(r + z(s) - C(s) + \sigma^*\varpi(s).\theta - \frac{1}{2}|\sigma^*\varpi(s)|^2\right)ds + \int_0^t \sigma^*\varpi(s).dw(s).$$

The change of integration then results in that

$$E\Big[\int_0^{+\infty} e^{-\beta t}\ln X(t)dt\Big] = \frac{\ln x}{\beta} + \frac{1}{\beta}E\Big[\int_0^{+\infty} e^{-\beta t}\left(r + z(t) - C(t) + \sigma^*\varpi(t).\theta - \frac{1}{2}|\sigma^*\varpi(t)|^2\right)dt\Big].$$

The value function $V(x,y)$ given in (9) can be, thus, explicitly written as

$$V(x,y) = \frac{\ln x}{\beta} + \sup_{C(.),\varpi(.)} E\left[\int_0^{+\infty} e^{-\beta t}\Big[\ln C(t) + \frac{1}{\beta}\Big(r + z(t) - C(t) + \sigma^*\varpi(t).\theta - \frac{1}{2}|\sigma^*\varpi(t)|^2\Big)\Big]dt\right].$$

We now consider a stochastic control problem by

$$\widetilde{W}(z) = \sup_{C(.),\varpi(.)} K_z(C(.),\varpi(.)) \tag{18}$$

$$K_z(C(.),\varpi(.)) = E\left[\int_0^{+\infty} e^{-\beta t}\Big[\ln C(t) + \frac{1}{\beta}\Big(r + z(t) - C(t) + \sigma^*\varpi(t).\theta - \frac{1}{2}|\sigma^*\varpi(t)|^2\Big)\Big]dt\right], \tag{19}$$

$$dz(t) = z(t)\Big(-(r-\mu+z(t))dt + \big(-\sigma^*\varpi(t).\theta + |\sigma^*\varpi(t)|^2\big)dt + C(t)dt - \sigma^*\varpi(t).dw(t) + \rho dw_y(t)\Big). \tag{20}$$

$$z(0) = z.$$

The Bellman equation associated with the problem (18) with (19) and (20) is then given by

$$\beta\widetilde{W}(z) = \frac{r+z}{\beta}\Big(1 - \beta z\widetilde{W}'(z)\Big) + \mu z\widetilde{W}'(z) + \frac{1}{2}\rho^2 z^2\widetilde{W}''(z) + \sup_C\Big[\ln C - C\Big(\frac{1}{\beta} - z\widetilde{W}'(z)\Big)\Big]$$
$$+ \frac{1}{\beta}\sup_{\varpi}\Big[\sigma^*\varpi.\theta(1-\beta z\widetilde{W}'(z)) - \frac{1}{2}|\sigma^*\varpi|^2(1 - 2\beta z\widetilde{W}'(z) - \beta z^2\widetilde{W}''(z))\Big]. \tag{21}$$

The optimal controls turn out to be exactly the same as given in (16) and (17) and the same Bellman equation (15) can be recovered with substitution of the optimal controls in (21). We therefore verify that $W(z)$ introduced in (14) is the value function $\widetilde{W}(z)$ of the stochastic problem (18), i.e., $W(z) = \widetilde{W}(z)$.

3.4 BOUNDARY CONDITIONS

We now address boundary conditions of the Bellman equation (15) at $z=0$ and $z \to +\infty$.

If we first take $z=0$ in (15), we then easily obtain that

$$\beta W(0) = \frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1. \tag{22}$$

To ensure the positivity of $W(z)$, we need the assumption

$$\frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1 > 0. \tag{23}$$

To pick a boundary condition of $W(z)$ at $z=+\infty$, we use the lemma as follows.

Lemma 3.1. *The function*

$$\overline{W}(z) = \frac{1}{\beta}\ln\Big(1+\frac{z}{r-\mu+\rho^2}\Big) + \frac{1}{\beta}\Big\{\frac{1}{\beta}\Big(r+\frac{|\theta|^2+\rho^2}{2}\Big) + \ln\beta - 1\Big\} \tag{24}$$

satisfies the equation

$$\begin{aligned}&\beta\overline{W}(z) - \frac{r+z}{\beta}(1-\beta z\overline{W}'(z)) - \mu z\overline{W}'(z) - \frac{1}{2}\rho^2 z^2\overline{W}''(z)\\ &-\ln\beta + 1 + \ln(1-\beta z\overline{W}'(z)) - \frac{|\theta|^2}{2\beta}\frac{(1-\beta z\overline{W}'(z))^2}{1-2\beta z\overline{W}'(z)-\beta z^2\overline{W}''(z)} = \frac{\rho^2}{2\beta}\left(\frac{r-\mu+\rho^2}{z+r-\mu+\rho^2}\right)^2.\end{aligned}$$

Therefore, $\overline{W}(z)$ ($W(z)$ at $z=+\infty$) satisfies the Bellman equation (15).

Proof. By direct calculation. ■ □

Lemma 3.1, thus, motivates the boundary condition of $W(z)$ at $z=+\infty$ as follows

$$W(z) - \overline{W}(z) \to 0 \quad \text{as} \quad z \to +\infty. \tag{25}$$

3.5 EXPLICIT CHARACTERIZATION

Even though the Bellman equation (15) with boundary conditions given in (22) and (24) seems to be almost impossible to be solved analytically, remarkably we are able to reduce the problem of solving the Bellman equation to a problem of solving an integral equation. Consequently, we can explicitly characterize the optimal consumption and investment strategy as a function of income-to-wealth ratio.

Theorem 3.1. *We assume that*[12]

$$|\theta| > \rho. \tag{26}$$

The solution of the Bellman equation (15) with boundary conditions (22) and (24) (i.e., the value function) is explicitly characterized by

$$W(z) = \frac{1}{\beta}\Big[\frac{1}{\beta}\Big(\mu - \frac{\rho^2}{2}\Big) + \ln\Psi(z) - 1 + \frac{r-\mu+\rho^2+z+v(z)}{\Psi(z)}\Big],$$

[12] The classical Merton risky share is $(\sigma^*)^{-1}\theta$ and the ratio of income risk to stock market risk is $(\sigma^*)^{-1}\rho$. For the fixed stock market risk σ^*, the condition would then imply that the demand for risky assets is positive and will be affected by the magnitude of income risk ρ. More precisely, the risky assets demand naturally decreases with income risk ρ, thus leading the wedge of the Merton risky share and the ratio of income risk to stock market risk to decrease accordingly. If the condition does not hold, i.e., when income risk ρ is significantly too large, the individual would not be willing to invest in the stock market. In light of such investment, the condition reflects the sensitivity of investment to changing income risk conditions.

where

$$\Psi(z) = \frac{\beta z}{r-\mu+\rho^2}\exp\Big(\int_z^{+\infty}\frac{-v(\zeta)+\sqrt{v(\zeta)^2+\rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)$$

and $v(z)$ is a solution of the following integral equation:

$$\begin{aligned} v(z) =& \frac{|\theta|^2-\rho^2}{2} \\ &+\beta\Big[1-\frac{z}{r-\mu+\rho^2}\exp\Big(\int_z^{+\infty}\frac{-v(\zeta)+\sqrt{v(\zeta)^2+\rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big) \\ &-\int_0^z\frac{1}{\zeta}\Big\{1-\frac{\zeta}{r-\mu+\rho^2}\exp\Big(\int_\zeta^{+\infty}\frac{-v(u)+\sqrt{v(u)^2+\rho^2|\theta|^2}}{\rho^2 u}du\Big)\Big\}d\zeta\Big] \\ &-\int_0^z\frac{-v(\zeta)+\sqrt{v(\zeta)^2+\rho^2|\theta|^2}}{\rho^2}d\zeta \\ &+\int_0^z\frac{v(\zeta)+r-\mu+\rho^2}{\zeta}\Big(1-\frac{-v(\zeta)+\sqrt{v(\zeta)^2+\rho^2|\theta|^2}}{\rho^2}\Big)d\zeta > 0. \end{aligned} \tag{27}$$

Consequently, the optimal consumption is

$$\widehat{C}(z) = \Psi(z)$$

with

$$\widehat{C}(0) = \beta,$$

$$\widehat{C}(z) - \beta\Big(1+\frac{z}{r-\mu+\rho^2}\Big) \to 0 \quad \textit{as} \quad z\to+\infty,$$

and the optimal portfolio is

$$\widehat{\varpi}(z) = (\sigma^*)^{-1}\theta\frac{\Psi(z)}{\Psi(z)-z\Psi'(z)}.$$

Remark 3.1. *The fractions with z on the denominator on the right-hand side of (27) have a numerator that vanishes at $z=0$, so there is no singularity at $z=0$ mathematically. However, it may not be the case that solving the integral equation numerically is straightforward. Instead, we propose the numerical iteration in Section 5 based on the dynamic programming principle that turns out to be robust and convenient regardless of such numerical singularities at $z=0$.*

Our explicit characterization of the value function $W(z)$ in Theorem 3.1 allows us to verify that the value function is twice differentiable.

We denote the value function $W(z)$ corresponding to $\rho=0$ by $W^0(z)$. That is, $W^0(z)$ is the value function in the absence of unspanned income risk.

Corollary 3.1. *The value function without unspanned income risk ($\rho = 0$) is given by*

$$W^0(z) = \frac{1}{\beta}\ln\Big(1+\frac{z}{r-\mu}\Big) + \frac{1}{\beta}\Big\{\frac{1}{\beta}\Big(r+\frac{|\theta|^2}{2}\Big) + \ln\beta - 1\Big\}. \tag{28}$$

Consequently, the optimal consumption is

$$\begin{aligned}\widehat{C}(z) &= \Psi(z)\\ &= \beta\Big(1+\frac{z}{r-\mu}\Big)\end{aligned}$$

with

$$\widehat{C}(0) = \beta,$$

$$\widehat{C}(z) - \beta\Big(1+\frac{z}{r-\mu}\Big) \to 0 \quad as \quad z \to +\infty,$$

and the optimal portfolio is

$$\begin{aligned}\widehat{\varpi}(z) &= (\sigma^*)^{-1}\theta\frac{\Psi(z)}{\Psi(z) - z\Psi'(z)}\\ &= (\sigma^*)^{-1}\theta\Big(1+\frac{z}{r-\mu}\Big).\end{aligned}$$

4 ANALYTIC COMPARATIVE STATICS

Having obtained the explicit characterization of the value function that solves the Bellman equation (15) with its boundary conditions given in (22) and (24) and the optimal consumption and portfolio choice accordingly, we now provide some analytical comparative statics associated with the value function and optimal strategies.

We first obtain that the value function is bounded above.

Proposition 4.1. *We assume (23). The value function $W(z)$ that solves the Bellman equation (15) with its boundary conditions given in (22) and (24) is bounded above as follows*

$$W(z) \le \overline{W}(z), \tag{29}$$

where $\overline{W}(z)$ given in (24) is the boundary condition of $W(z)$ at $z = +\infty$ as stated in (25).

Proposition 4.1 demonstrates that for the same extent of nondiversifiable labor income risk, i.e., for a fixed level of ρ, the individual receiving far enough income must have the larger utility $\overline{W}(z)$ than the utility $W(z)$ that the individual receiving less income has.

We second obtain another upper bound of the value function $W(z)$.

Proposition 4.2. *We assume (23) and*

$$r - \mu + \rho^2 < 2\beta. \tag{30}$$

The value function $W(z)$ that solves the Bellman equation (15) with its boundary conditions given in (22) and (24) is bounded above as follows

$$W(z) \leq W^0(z), \tag{31}$$

where $W^0(z)$ is the value function without unspanned income risk ($\rho = 0$) given in (28).

Proposition 4.2 shows that the individual without unspanned income risk must have the greater utility $W^0(z)$ than $W(z)$ obtained from the optimal choices of the individual with unspanned income risk.

Remark 4.1. *Note that when $z = 0$, $\overline{W}(0) - W^0(0) = \dfrac{\rho^2}{2\beta^2} > 0$. As a result, $\overline{W}(z) - W^0(z)$ decreases from $\dfrac{\rho^2}{2\beta^2}$ to $-\dfrac{1}{\beta}\ln\Big(\dfrac{r-\mu+\rho^2}{r-\mu}\Big) + \dfrac{\rho^2}{2\beta^2} < 0$. Notice that a more accurate majoration for $W(z)$ is*

$$W(z) \leq \min\big(\overline{W}(z), W^0(z)\big). \tag{32}$$

We third show the positivity of the value function $W(z)$.

Proposition 4.3. *We assume (23) and (30). The value function $W(z)$ that solves the Bellman equation (15) with its boundary conditions given in (22) and (24) is bounded below from 0 as follows*

$$0 \leq W(z) \leq \min\big(\overline{W}(z), W^0(z)\big).$$

The comparative statics that we have provided with Proposition 4.1, Proposition 4.2, and Proposition 4.3 so far would give useful bounds on the value function especially for the monotonicity of the value function, which will guarantee the convergence of the iterative numerical scheme suggested in Section 5 for numerical analysis.

We next provide some analytic comparative statics associated with the optimal consumption choice. We first show that the optimal consumption is positive and an increasing function of income-to-wealth ratio z.

Proposition 4.4. *The optimal consumption $\widehat{C}(z)(=\Psi(z))$ is positive and bounded below from β, i.e.,*

$$\widehat{C}(z) \geq \beta$$

and an increasing function of income-to-wealth ratio z, i.e.,

$$\widehat{C}'(z) \geq 0. \tag{33}$$

By (16), Proposition 4.4 implies that

$$0 \leq W'(z) \leq \frac{1}{\beta z}, \tag{34}$$

as a result, the value function $W(z)$ is an increasing function of income-to-wealth ratio z. Given the relation obtained from (14) that

$$x\frac{\partial V}{\partial x} = \frac{1}{\beta} - zW'(z),$$

we can now therefore verify the monotonicity of the value function $V(x)$ imposed by $\partial V/\partial x > 0$ in (11).

We now make a more stringent assumption than (30) as follows.

$$\beta > r - \mu + \rho^2. \tag{35}$$

We can then show that the consumption $\widehat{C}(z)$ is bounded from above. More precisely,

Proposition 4.5. *We assume (35). We then state that the optimal consumption $\widehat{C}(z)(=\Psi(z))$ is bounded from above as follows*

$$\widehat{C}(z) \leq \beta\Big(1 + \frac{z}{r - \mu + \rho^2}\Big). \tag{36}$$

By (16), Proposition 4.5 implies that

$$0 \leq W'(z) \leq \frac{1}{\beta}\frac{1}{z + r - \mu + \rho^2},$$

which is a more tightened boundedness of $W'(z)$ than (34).

Lastly, we show that the optimal portfolio is positive.

Proposition 4.6. *The optimal portfolio $\widehat{\varpi}(z)$ is positive.*

Proposition 4.6 verifies the concavity of the value function $V(x)$ imposed by $\partial^2 V/\partial x^2 < 0$ in (11) because of $\partial V/\partial x > 0$ and the optimal portfolio in (12).

5 NUMERICAL ANALYSIS

For graphical illustration and more detailed discussion of the value function and optimal strategies given in Theorem 3.1, we provide an extensive numerical analysis. We develop a quite general numerical algorithm for control iteration and solve the value function as a sequence of solutions to ordinary differential equations.[13]

5.1 ALGORITHM FOR POLICY ITERATION

We define a sequence $W_n(z)$, $n = 1, 2, ...$ as follows. If $W_n(z)$ is known, we can define $\widehat{C}_n(z)$ and $\widehat{\varpi}_n(z)$ by minimizing

$$\begin{aligned} &\sup_C \Big[\ln C - C\Big(\frac{1}{\beta} - zW_n'(z)\Big)\Big] \quad \text{and} \\ &\sup_\varpi \Big[\sigma^*\varpi.\theta(1-\beta zW_n^{'}(z)) - \frac{1}{2}|\sigma^*\varpi|^2(1-2\beta zW_n^{'}(z) - \beta z^2 W_n^{''}(z))\Big], \end{aligned} \tag{37}$$

respectively. These problems are only meaningful when

$$\begin{aligned} &1-\beta zW_n^{'}(z) > 0, \\ &1-2\beta zW_n^{'}(z) - \beta z^2 W_n^{''}(z) > 0. \end{aligned} \tag{38}$$

The solution is then uniquely defined by

$$\widehat{C}_n(z) = \frac{\beta}{1-\beta zW_n'(z)} \tag{39}$$

and

$$\sigma^*\widehat{\varpi}_n(z) = \frac{1-\beta zW_n'(z)}{1-2\beta zW_n'(z) - \beta z^2 W_n''(z)}\theta. \tag{40}$$

We also assume that the boundary conditions are given by

$$\beta W_n(0) = \frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1,\ W_n(z) - \overline{W}(z) \to 0, \quad \text{as} \quad z \to +\infty.$$

In the sequel, we postulate that the properties (38) are satisfied. We then define $W_{n+1}(z)$ by solving the differential equation

$$\begin{aligned} \beta W_{n+1}(z) =& \frac{r+z}{\beta} - zW_{n+1}^{'}(z)(z+r-\mu) + \frac{1}{2}\rho^2 z^2 W_{n+1}^{''}(z) \\ &+ \ln\widehat{C}_n(z) - \widehat{C}_n(z)\Big(\frac{1}{\beta} - zW_{n+1}'(z)\Big) + \sigma^*\widehat{\varpi}_n(z).\theta\Big(\frac{1}{\beta} - zW_{n+1}'(z)\Big) \\ &- \frac{1}{2}|\sigma^*\widehat{\varpi}_n(z)|^2\Big(\frac{1}{\beta} - 2zW_{n+1}'(z) - z^2 W_{n+1}^{''}(z)\Big) \end{aligned} \tag{41}$$

[13] The resulting HJB equation (15) can be solved by the standard finite difference method (FDM) that can manage the nonlinear terms caused by the uncertainty in the model stemming from Brownian motion processes. The numerical algorithm we offer here, while different, can be thus viewed as complementary to the FDM. In particular, we solve ordinary differential equations in each iteration based on the FDM. We then emphasize the validity of the iterative approach with the convergence of this iterative procedure by having the monotonicity of the value function. Notice also the penalty method of Dai and Zhong (2010) for a convenient and efficient numerical method to solve a wide variety of models with coupled integro-differential equations having free boundaries.

with the boundary conditions

$$\beta W_{n+1}(0) = \frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1,\ W_{n+1}(z) - \overline{W}(z) \to 0, \quad \text{as} \quad z \to +\infty. \tag{42}$$

We can initiate the algorithm by taking $\widehat{C}_0(z)$ and $\widehat{\varpi}_0(z)$ arbitrarily provided that $W_1(z)$ satisfies (38). We will further discuss this point later. We can actually always take

$$\widehat{C}_0(z) = \beta,\ \sigma^* \widehat{\varpi}_0(z) = \theta.$$

5.2 CONVERGENCE

We now show the convergence of the proposed numerical algorithm. We show that the sequence $W_n(z)$ is monotone increasing and bounded above.

Proposition 5.1. *We assume (35) and postulate (38). We then have that*

$$W_n(z) \leq W_{n+1}(z) \leq \min(\overline{W}(z), W^0(z)). \tag{43}$$

Proposition 5.1 then implies that the sequence $W_n(z)$ converges pointwise to $W(z)$. The convergence is uniform on compact subsets of the interior of $z \geq 0$ (refer to Theorem 10.8 in Rockafellar, 1970). The limiting value function $W(z)$ is differentiable on a dense set of $z \geq 0$, which can be restated as the union of open convex cones (refer to Theorem 25.5 in Rockafellar, 1970). The derivatives of the sequence $W_n(z)$ will then converge to those of the limiting value function $W(z)$ on this union (refer to Theorem 25.7 in Rockafellar, 1970). We can observe in numerical analysis that the derivatives $W_n'(z)$ and $W_n''(z)$ converge pointwise to $W'(z)$ and $W''(z)$, respectively. From (39), (40), (41), (42), the limit is therefore the solution of the Bellman equation (15) with boundary conditions (22), (25). The sequence $W_n(z)$ therefore provides an approximation of the value function $W(z)$.

5.3 IMPLEMENTATION

The iteration (41) is written as follows

$$\begin{aligned}\beta W_{n+1}(z) =& \frac{r+z}{\beta} + zW_{n+1}'(z)\left(\widehat{C}_n(z) - \sigma^*\widehat{\varpi}_n(z).\theta + |\sigma^*\widehat{\varpi}_n(z)|^2 - (z + r - \mu)\right) \\ &+ \frac{\rho^2 + |\sigma^*\widehat{\varpi}_n(z)|^2}{2} z^2 W_{n+1}''(z) + \ln\widehat{C}_n(z) - \frac{\widehat{C}_n(z)}{\beta} - \frac{|\sigma^*\widehat{\varpi}_n(z)|^2}{2\beta} + \frac{\sigma^*\widehat{\varpi}_n(z).\theta}{\beta}\end{aligned} \tag{44}$$

with the boundary conditions

$$\beta W_{n+1}(0) = \frac{r + \frac{|\theta|^2}{2}}{\beta} + \ln\beta - 1,\ W_{n+1}(z) - \overline{W}(z) \to 0 \ \text{ as } \ z \to +\infty.$$

5.3.1 FIRST STEP APPROXIMATION

For $n = 1$, we can start with arbitrarily chosen $\widehat{C}_0(z)$ and $\sigma^*\widehat{\varpi}_0(z)$. In order to speed up the convergence, it is natural to choose that

$$\widehat{C}_0(z) = \overline{C}(z) = \frac{\beta}{1 - \beta z\overline{W}'(z)},$$

$$\sigma^*\widehat{\varpi}_0(z) = \sigma^*\overline{\varpi}(z) = \frac{1 - \beta z\overline{W}'(z)}{1 - 2\beta z\overline{W}'(z) - \beta z^2\overline{W}''(z)}\theta,$$

where $\overline{W}(z)$ is given in (24), which means that

$$\overline{C}(z) = \beta\Big(1 + \frac{z}{r - \mu + \rho^2}\Big),$$

$$\sigma^*\overline{\varpi}(z) = \Big(1 + \frac{z}{r - \mu + \rho^2}\Big)\theta.$$

The first step approximation $W_1(z)$ is thus the solution of the equation (see (44) with $n = 0$)

$$\begin{aligned}
&\beta W_1(z) - \frac{r+z}{\beta} - zW_1'(z)\Big[\Big(1 + \frac{z}{r - \mu + \rho^2}\Big)\Big(\beta + \frac{|\theta|^2 z}{r - \mu + \rho^2}\Big) - (z + r - \mu)\Big] \\
&- \frac{\rho^2 + |\theta|^2(1 + \frac{z}{r-\mu+\rho^2})^2}{2} z^2 W_1''(z) - \ln\beta + 1 - \ln\Big(1 + \frac{z}{r - \mu + \rho^2}\Big) + \frac{z}{r - \mu + \rho^2} \\
&- \frac{|\theta|^2}{2\beta}\Big\{1 - \Big(\frac{z}{r - \mu + \rho^2}\Big)^2\Big\} = 0
\end{aligned}$$

with the boundary conditions

$$\beta W_1(0) = \frac{r + \frac{|\theta|^2}{2}}{\beta} + \ln\beta - 1,\ W_1(z) - \overline{W}(z) \to 0 \ \text{ as } \ z \to +\infty.$$

Note that $\overline{W}(z)$ given in Lemma 3.1 satisfies

$$\begin{aligned}
&\beta \overline{W}(z) - \frac{r+z}{\beta} - z\overline{W}'(z)\Big[\Big(1 + \frac{z}{r - \mu + \rho^2}\Big)\Big(\beta + \frac{|\theta|^2 z}{r - \mu + \rho^2}\Big) - (z + r - \mu)\Big] \\
&- \frac{\rho^2 + |\theta|^2(1 + \frac{z}{r-\mu+\rho^2})^2}{2} z^2 \overline{W}''(z) - \ln\beta + 1 - \ln\Big(1 + \frac{z}{r - \mu + \rho^2}\Big) + \frac{z}{r - \mu + \rho^2} \\
&- \frac{|\theta|^2}{2\beta}\Big\{1 - \Big(\frac{z}{r - \mu + \rho^2}\Big)^2\Big\} = \frac{\rho^2}{2\beta}\left(\frac{r - \mu + \rho^2}{z + r - \mu + \rho^2}\right)^2.
\end{aligned}$$

Hence,

$$\beta\overline{W}(z) = \ln\Big(1 + \frac{z}{r-\mu+\rho^2}\Big) + \frac{1}{\beta}\Big(r + \frac{|\theta|^2+\rho^2}{2}\Big) + \ln\beta - 1,$$

which serves as the boundary condition of $W_1(z)$ at $z = +\infty$.

5.3.2 FIRST STEP APPROXIMATION ERROR

The approximation error is defined by

$$E_n(z) = W(z) - W_n(z) > 0.$$

We can majorize $E_1(z)$ by $\overline{E}_1(z) = \overline{W}(z) - W_1(z)$, which is the first step approximation error. It is the solution of the differential equation

$$\begin{aligned}\beta\overline{E}_1(z) - z\overline{E}_1'(z)\Big[\Big(1 + \frac{z}{r-\mu+\rho^2}\Big)\Big(\beta + \frac{|\theta|^2 z}{r-\mu+\rho^2}\Big) - (z + r - \mu)\Big]\\ - \frac{\rho^2 + |\theta|^2(1 + \frac{z}{r-\mu+\rho^2})^2}{2} z^2\overline{E}_1''(z) = \frac{\rho^2}{2\beta}\left(\frac{r-\mu+\rho^2}{z+r-\mu+\rho^2}\right)^2\end{aligned} \tag{45}$$

with the boundary conditions

$$\beta\overline{E}_1(0) = \frac{\rho^2}{2\beta},\ \overline{E}_1(z) \to 0 \ \text{ as } \ z \to +\infty. \tag{46}$$

5.4 NUMERICAL RESULTS

We set the following baseline parameter values to obtain numerical results:

$$\beta = 0.01, r = 0.06, \alpha = 0.1, \sigma = 0.25, \mu = 0.05, \rho = 0.02,$$

which satisfy parameter conditions given in (23) and (30).

Under the baseline parameter values, the theoretical first step approximation error $\overline{E}_1(z)$ that is the solution of the differential equation (45) is now numerically illustrated in Figure 1. Notice that the numerically calculated first step error $\overline{E}_1(z)$ at $z = 0$ is 2.0961, which is exactly the same with the theoretical first step error $\rho^2/(2\beta^2) = 2.0961$ at $z = 0$ given in (46), and the first step error turns out to be a decreasing function of income-to-wealth ratio z and approaches zero as z increases as consistent with the boundary condition $\overline{E}_1(z)$ as $z \to +\infty$ given in (46).

For $n \geq 2$, taking the maximum of $W_n(z) - W_{n-1}(z)$ as tolerance, the sequence $W_n(z)$ converges when $n = 6$ with tolerance 0.0000471083. We can numerically confirm that the value function $W(z)$ is an increasing

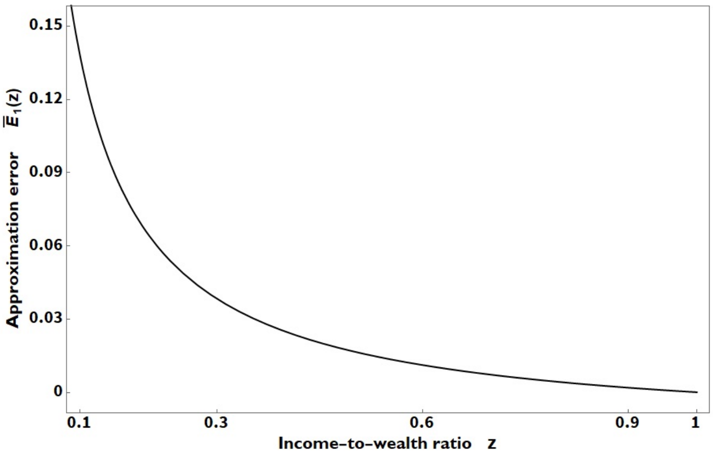

Figure 1: **First step approximation error.**

concave function of income-to-wealth ratio z (Figure 2), thus satisfying the analytic properties of the value function stated in Proposition 4.1, Proposition 4.2, and Proposition 4.3.

The presence of unspanned income risk ($\rho > 0$) negatively affects the value function $W(z)$ and the value function decreases with respect to an increase in the unspanned income risk ρ.

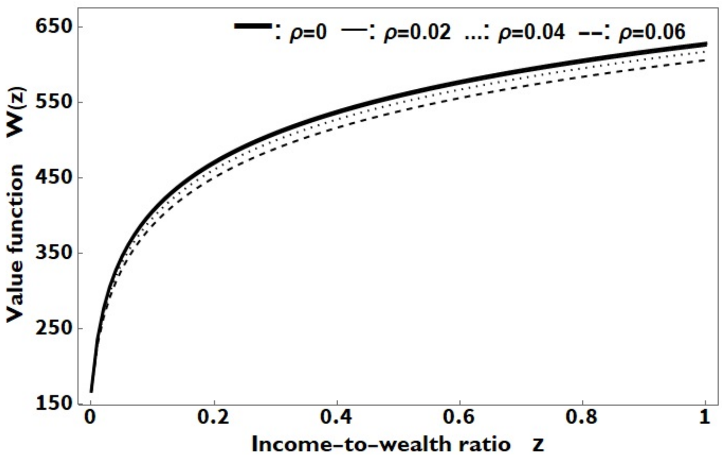

Figure 2: **Value function.**

We numerically demonstrate that the optimal consumption $\widehat{C}(z)$ is positive and an increasing function of income-to-wealth ratio z (Figure 3 Left Panel) as we have theoretically verified in Proposition 4.4. We also numerically confirm that the optimal portfolio $\widehat{\varpi}(z)$ is positive (Figure 3 Right Panel) as we have theoretically proved in Proposition 4.6.

More interestingly, the effects of unspanned income risk ($\rho > 0$) show a significant discontinuity and dramatic change in the optimal consumption and portfolio choice with a change in the unspanned income risk ρ. Both the optimal consumption and the optimal investment fall sharply as the unspanned income risk rises (i.e., as ρ increases) even when the income risk is quite small so that the income volatility is less than just 6% (Figure 3). Further, the impact of unspanned income risk on the individual's optimal choice

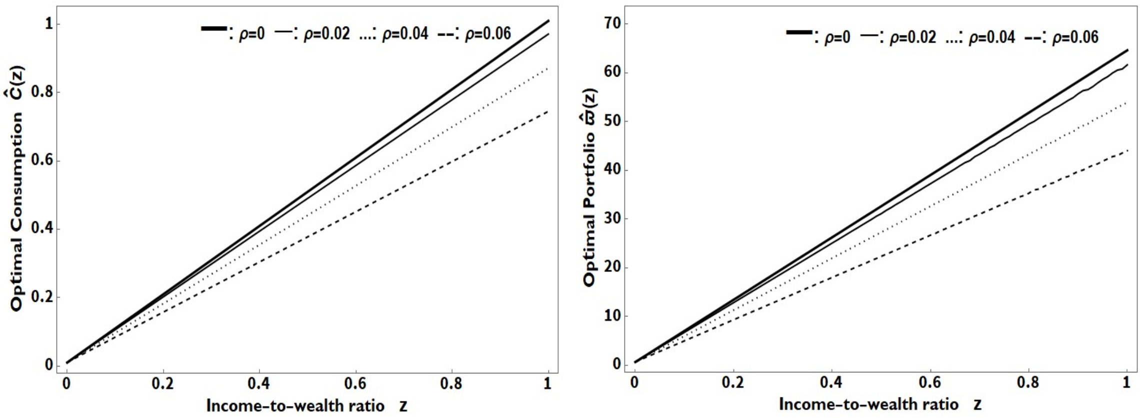

Figure 3: **Optimal consumption and portfolio choice.**

is different with respect to levels of income-to-wealth ratio z. That is, the individual's optimal decision is more likely to be affected negatively by unspanned income risk when the income-to-wealth ratio is high than when it is low.

The reduced consumption with unspanned income risk is consistent with the impact of the individual's precautionary savings motive on the optimal consumption/savings decision. The additional unspanned income risk results in an increase in background risk (Bodie *et al.*, 1992; Heaton and Lucas, 1997; Koo, 1998).[14] More precisely, the increased background risk gives rise to the individual's precaution that causes her to become increasingly conservative, whilst the riskiness of the individual's unspanned labor income rises. Therefore, the individual's optimal choice is to consume less with unspanned income risk for precautionary reasons.

The decreased investment with unspanned income risk is consistent with the impact of the individual's risk diversification motive. In the absence of unspanned income risk, labor income has been traditionally regarded as a substitute for risk-free bonds so that receiving labor income increases the demand for risky assets in the individual's optimal portfolio. In the presence of unspanned income risk, however, labor income should be treated as a risky asset so that obtaining labor income rather exposes the individual to a nondiversifiable income risk source. In an attempt to strike an optimal balance between risk-free and risky assets, the individual would, thus, invest less in the stock market for risk diversification purposes.

The differing effects of unspanned income risk on the individual's optimal consumption and portfolio decision with respect to levels of income-to-wealth ratio are caused by the inverse relation between the income-to-wealth ratio and wealth. Labor income itself is the staple income of poor individuals, so they should concern themselves with substantial precautionary savings and risk diversification motives. Therefore, the individual's optimal decision is significantly affected by unspanned income risk when the income-to-

[14]The background risk results from an undiversifiable risk source affecting the individual's consumption and portfolio choice (e.g., income risk, house ownership risk, etc.).

wealth ratio is high. In contrast, labor income is a relatively smaller income source of the wealthy, so they have greater tolerance for unspanned income risk than the wealth poor do. In other words, the far enough wealth they already accumulate serves as a buffer to smooth out unspanned income risk. Hence, there are only minor quantitative differences between the rich with unspanned income risk and the rich without.

One way that is helpful for analyzing the impact of unspanned income risk on the individual's optimal strategies is to compute the utility cost (measured in certainty equivalent units) associated with suboptimal strategies. We compute the utility loss incurred by individuals who neglect the presence of unspanned income risk as follows:

$$W^0(z - \Delta(z)) = W(z),$$

where $W^0(z)$ is the value function corresponding to the absence of unspanned income risk ($\rho = 0$), $W(z)$ is the value function corresponding to the presence of unspanned income risk ($\rho > 0$), and $\Delta(z)$ is the utility cost (or loss) measured in certainty equivalent.

The utility loss increases with the income-to-wealth ratio z and the extent ρ to which the individual is exposed to unspanned income risk (Figure 4). Intuitively, the ability to self-insure against the income risk improves as wealth increases or equivalently, as z decreases, because the enough wealth the individual accumulates serves as a buffer to smooth out an adverse shock in the market caused by the income risk, thus resulting in small utility losses. Also, the large extent with a high ρ to which the individual is faced with unspanned income risk leads the individual's optimal consumption and portfolio choice to substantially deviate from the individual's suboptimal choice in the absence of unspanned income risk. Such large deviations between the individual's optimal decision with and without unspanned income risk amount to a significant increase in the utility costs the individual is likely to incur. Therefore, ignoring the presence of unspanned income risk or misestimating the magnitude of unspanned income risk can be costly to individuals who aim to make the optimal consumption and portfolio decision.

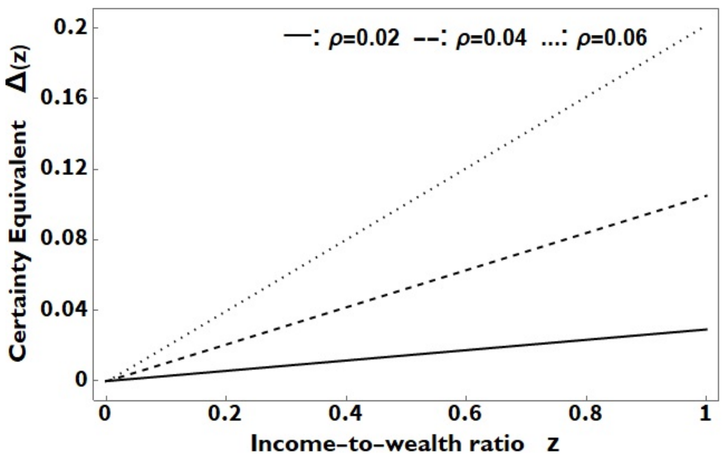

Figure 4: **Utility cost.**

6 EXTENSIONS

6.1 THE RELATION BETWEEN THIS PAPER AND DUFFIE *ET AL.* (1997)

Duffie *et al.* (1997) have studied a very similar model to this paper, so that it is worth investigating their relation to the paper. Applying a transformation suggested by Dai and Yi (2009) to the Bellman equation (13) allows us to obtain the Bellman equation of Duffie *et al.* (1997) when specializing the setting to logarithmic preferences. In particular, one could use the following transformation for a solution of the Bellman equation (13) instead of (14):

$$V(x,y) = \frac{\ln y}{\beta} + u(\xi), \quad \xi = \frac{x}{y}, \tag{47}$$

where the wealth-to-income ratio ξ is the state variable instead of the income-to-wealth ratio z given in (14). Notice the formulas as follows:

$$\begin{aligned}
&x\frac{\partial V}{\partial x} = \xi u'(\xi), \quad x^2\frac{\partial^2 V}{\partial x^2} = \xi^2 u''(\xi),\\
&y\frac{\partial V}{\partial x} = u'(\xi), \quad y\frac{\partial V}{\partial y} = \frac{1}{\beta} - \xi u'(\xi), \quad y^2\frac{\partial^2 V}{\partial y^2} = -\frac{1}{\beta} + 2\xi u'(\xi) + \xi^2 u''(\xi),\\
&xy\frac{\partial^2 V}{\partial x \partial y} = -\xi u'(\xi) - \xi^2 u''(\xi).
\end{aligned}$$

We then obtain by inserting the above formulas in the Bellman equation (13) the following equation for $u(\xi)$:

$$\beta u(\xi) = \{(r-\mu+\rho^2)\xi + 1\}u'(\xi) + \frac{\mu}{\beta} - 1 - \frac{\rho^2}{2\beta} - \ln u'(\xi) + \frac{1}{2}\rho^2\xi^2 u''(\xi) - \frac{|\theta|^2}{2}\frac{u'(\xi)^2}{u''(\xi)}, \tag{48}$$

which is the Bellman equation of Duffie *et al.* (1997) with logarithmic preferences.

Notice the difference between our Bellman equation (15) and the Bellman equation (48) of Duffie *et al.* (1997). The difference is that the income-to-wealth ratio z is the state variable of (15), whereas the wealth-to-income ratio ξ is the variable of (48). Interestingly, the two Bellman equations are quite different, but having a linkage. Using a reciprocal change of variable from ξ to z in (48) does not directly lead to our Bellman equation (15). Instead, we need to use the following relation between $W(z)$ of (15) and $u(\xi)$ of (48) by

$$W(z) = -\frac{\ln \xi}{\beta} + u(\xi), \tag{49}$$

which results from a straightforward comparison between transformations in (14) and (47). Our solution $W(z)$ of the Bellman equation (15) leads to the solution $u(\xi)$ of the Bellman equation (48) of Duffie *et al.* (1997) by the relation (49). One can therefore obtain the solution by either this paper ($W(z)$ of (15)) or

Duffie *et al.* (1997) ($u(\xi)$ of (48)).

Notice also that without $W(z)$ of the Bellman equation (15) in the paper, we can solve the Bellman equation (48) of Duffie *et al.* (1997) explicitly. Different from viscosity solutions by Duffie *et al.* (1997) that are not explicit, the following theorem explicitly characterizes the value function and optimal strategies as a function of wealth-to-income ratio.

Theorem 6.1. *With the standing assumption (26), the solution of the Bellman equation (48) (i.e., the value function) is explicitly characterized by*

$$u(\xi) = \frac{1}{\beta}\Big[\frac{\mu}{\beta} - 1 - \frac{\rho^2}{2\beta} + \ln\left(\xi H(\xi)\right) + \frac{r - \mu + \rho^2 + 1/\xi + m(\xi)}{H(\xi)}\Big],$$

where

$$H(\xi) = \beta\Big(1 + \frac{1}{\xi(r-\mu+\rho^2)}\Big)\exp\Big(\int_0^{\xi}\frac{-m(\zeta) + \sqrt{m(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)$$

and $m(\xi)$ is a solution of the following integral equation:

$$\begin{aligned}\xi m(\xi) =& \frac{|\theta|^2}{2}\frac{1}{r-\mu+\rho^2}\\ &+ \beta\xi - \int_0^{\xi}\Big(\frac{-m(\eta) + \sqrt{m(\eta)^2 + \rho^2|\theta|^2}}{\rho^2}\Big)\Big\{\beta\Big(1 + \frac{1}{\eta(r-\mu+\rho^2)}\Big)\exp\Big(\int_0^{\eta}\frac{-m(\zeta) + \sqrt{m(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big) - \frac{1}{\eta}\Big\}d\eta\\ &+ \int_0^{\xi} m(\eta)\Big(\frac{-m(\eta) + \sqrt{m(\eta)^2 + \rho^2|\theta|^2}}{\rho^2}\Big)d\eta.\end{aligned}$$

Consequently, the optimal consumption is

$$\widehat{C}(\xi) = H(\xi)$$

with

$$\widehat{C}(\xi) - \beta\Big(1 + \frac{1}{\xi(r-\mu+\rho^2)}\Big) \to 0 \quad as \quad \xi \to 0,$$

$$\widehat{C}(\xi) - \beta \to 0 \quad as \quad \xi \to +\infty,$$

and the optimal portfolio is

$$\widehat{\varpi}(z) = (\sigma^*)^{-1}\theta\frac{H(\xi)}{H(\xi) + \xi H'(\xi)}.$$

6.2 POWER UTILITY CASE

The optimal consumption and investment problem we have considered so far is restrictive by specializing the setting to logarithmic preferences. Provided that the individual's optimal strategies would vary over the levels of risk aversion, it is therefore more realistic to consider the power utility case to consider different levels of risk aversion.

We consider the power utility $C(t)^{\gamma}/\gamma$ over intermediate consumption $C(t)$, where $1 - \gamma > 0$ ($\gamma \neq 0$)

is the constant coefficient of the individual's relative risk aversion. This power utility satisfies the usual conditions that are twice continuously differentiable, strictly increasing, and strictly concave.

The payoff for the individual to maximize with the power utility over the infinite horizon is given by

$$J_{x,y}(C(.), \varpi(.)) = E\left[\int_0^{+\infty} e^{-\beta t}\frac{\{C(t)X(t)\}^\gamma - 1}{\gamma}dt\right],$$

which is subject to

$$C(t) > 0.$$

The specification for the power utility case allows us to obtain the logarithmic utility case by letting γ approach 0. For any adapted processes $C(t)$ and $\varpi(t)$, the wealth $X(t)$ should be positive.

The value function is the maximized payoff with two states $\{X(t), y(t)\}$ and two controls $\{C(t), \varpi(t)\}$ that are adapted to the filtration $\mathcal{F}^t$ generated by the Brownian motion processes $w(t)$ and $w_y(t)$. The value function is then defined by

$$V(x, y) = \sup_{C(.),\varpi(.)} J_{x,y}(C(.), \varpi(.)). \tag{50}$$

6.2.1 BELLMAN EQUATION

We can obtain the Bellman equation associated with the value function (50) as follows

$$\begin{aligned}\beta V(x, y) =& (rx + y)\frac{\partial V}{\partial x} + \mu y\frac{\partial V}{\partial y} + \frac{1}{2}\rho^2 y^2\frac{\partial^2 V}{\partial y^2}\\ &+ \sup_C\Big(\frac{\{Cx\}^\gamma - 1}{\gamma} - Cx\frac{\partial V}{\partial x}\Big) + \sup_\varpi\Big(x\sigma^*\varpi.\theta\frac{\partial V}{\partial x} + \frac{1}{2}x^2\left|\sigma^*\varpi\right|^2\frac{\partial^2 V}{\partial x^2}\Big).\end{aligned} \tag{51}$$

The sup can be attained at the following optimal controls

$$\widehat{C} = \Big(\frac{\partial V}{\partial x}\Big)^{\frac{1}{\gamma-1}}\Big/x, \quad \widehat{\varpi} = -\frac{1}{x}(\sigma^*)^{-1}\frac{\partial V}{\partial x}\Big/\frac{\partial^2 V}{\partial x^2}.$$

We can now rewrite the Bellman equation (51) with the optimal controls stated above as

$$\begin{aligned}\beta V(x, y) =& (rx + y)\frac{\partial V}{\partial x} + \mu y\frac{\partial V}{\partial y} + \frac{1}{2}\rho^2 y^2\frac{\partial^2 V}{\partial y^2}\\ &+ \frac{1-\gamma}{\gamma}\Big(\frac{\partial V}{\partial x}\Big)^{\frac{\gamma}{\gamma-1}} - \frac{1}{\gamma} - \frac{|\theta|^2}{2}\Big(\frac{\partial V}{\partial x}\Big)^2\Big/\frac{\partial^2 V}{\partial x^2}.\end{aligned} \tag{52}$$

Following Dai and Yi (2009), we now perform the following transformation to reduce the dimension of the Bellman equation (52):

$$V(x, y) = y^\gamma v(\xi) - \frac{1}{\beta\gamma}, \quad \xi = \frac{x}{y}.$$

Notice the formulas as follows:

$$
\begin{aligned}
&\frac{\partial V}{\partial x} = y^{\gamma-1} v'(\xi), \quad \frac{\partial^2 V}{\partial x^2} = y^{\gamma-2} v''(\xi), \\
&\frac{\partial V}{\partial y} = \gamma y^{\gamma-1} v(\xi) - y^{\gamma-1} \xi v'(\xi), \\
&\frac{\partial^2 V}{\partial y^2} = \gamma(\gamma-1) y^{\gamma-2} v(\xi) - 2(\gamma-1) y^{\gamma-2} \xi v'(\xi) + y^{\gamma-2} \xi^2 v''(\xi).
\end{aligned}
$$

Inserting these formulas in the Bellman equation (52), we obtain the new equation for $v(z)$

$$
\frac{1}{2}\rho^2 \xi^2 v''(\xi) + \{(r-\mu+\rho^2)\xi + 1\} v'(\xi) - \Big(\beta - \gamma\mu - \frac{1}{2}\gamma(\gamma-1)\rho^2\Big) v(\xi) + \frac{1-\gamma}{\gamma} v'(\xi)^{\frac{\gamma}{\gamma-1}} - \frac{|\theta|^2}{2} \frac{v'(\xi)^2}{v''(\xi)}. \quad (53)
$$

6.2.2 CONVEX-DUALITY APPROACH

We now adopt the convex-duality approach of Bensoussan *et al.* (2016). We first introduce the dual variable $\lambda(\xi)$ as the first derivative of $v(\xi)$:

$$
\lambda(\xi) = v'(\xi). \quad (54)
$$

We then introduce the convex-dual function $G(\lambda(\xi))$ as

$$
G(\lambda(\xi)) = \xi + \frac{1}{r-\mu+\rho^2}, \quad (55)
$$

implying that

$$
yG(\lambda(\xi)) = x + \frac{y}{r-\mu+\rho^2}, \quad (56)
$$

which is the total wealth that is the sum of financial wealth x and present value $y/(r-\mu+\rho^2)$ of future income. The dual function $G(\lambda(\xi))$ satisfies the following relations:

$$
G'(\lambda(\xi))\lambda'(\xi) = 1, \quad G''(\lambda(\xi))\lambda'(\xi)^2 + G'(\lambda(\xi))\lambda''(\xi) = 0. \quad (57)
$$

We simply write $G(\lambda(\xi))$ as $G(\lambda)$ and $\lambda(\xi)$ as λ unless there is any confusion. Differentiating the both sides of the Bellman equation (53) with respect to ξ and using the dual function $G(\lambda)$ in (55) with the relations (54) and (57), we obtain the new equation for $G(\lambda)$: for any $\lambda > 0$,

$$
-\frac{1}{2}|\theta|^2 \lambda^2 G''(\lambda) - (|\theta|^2 + r_2 - r_1)\lambda G'(\lambda) + r_1 G(\lambda) - \lambda^{\frac{1}{\gamma-1}} + \frac{1}{2}\rho^2 \frac{d}{d\lambda}\Big(\frac{\big(G(\lambda) - 1/r_1\big)^2}{G'(\lambda)}\Big) = 0, \quad (58)
$$

where

$$
r_1 = r - \mu + \rho^2, \quad r_2 = \beta - \gamma\mu - \frac{1}{2}\gamma(\gamma-1)\rho^2.
$$

We address the boundary conditions associated with the Bellman equation (58). We first impose the following boundary condition of $G(\lambda)$ at $\lambda = +\infty$:

$$G(\infty) = 0, \tag{59}$$

which implies that the dual variable λ goes up infinity as the financial wealth x goes down to its lower bound $-y/(r - \mu + \rho^2)$ so that the boundary condition (59) is obtained by (56). We then add one more condition of $G(\lambda)$ at $\lambda = 0$ as

$$G(\lambda) = \frac{1}{K}\lambda^{\frac{1}{\gamma-1}}, \tag{60}$$

where

$$K = \frac{\gamma}{\gamma - 1}\Big(r + \frac{|\theta|^2}{2(1-\gamma)}\Big) + \frac{\beta}{1-\gamma},$$

which is known as the Merton constant. The boundary condition (60) is naturally obtained. Notice that the financial wealth x goes up infinity as the dual variable λ goes down to zero so that unspanned income risk can be safely ignored. Therefore, when $\lambda = 0$ (or equivalently, when $x = +\infty$), the problem is to solve the Bellman (58) $\rho = 0$, thus leading to the boundary condition (60).

6.2.3 NUMERICAL RESULTS

Applying the numerical algorithm developed in Section 5 to the Bellman equation (58), we obtain under the same baseline parameter values considered in Section 5 numerical results for optimal consumption and portfolio choice with two different levels of relative risk aversion (1-γ=2 and 1-γ=3) in Figure 5 and Figure 6. Changes in relative risk aversion $1-\gamma$ affect the optimal consumption and portfolio decision. Increased risk aversion leads the individual to reduce both consumption and risky portfolio, regardless of the extent ρ of unspanned income risk. We can confirm that the effects of unspanned income risk on the optimal strategies are not altered with levels of risk aversion. That is, an increase in unspanned income risk still negatively affects both consumption and risky investment of risk averse individuals with substantial reductions.

7 CONCLUSION

In this paper, we have developed a new dynamic programming approach for solving the optimal consumption and investment problem especially with independent stochastic labor income. The challenge with the additional nondiversifiable Brownian risk source of labor income is to solve the derived two-dimensional nonlinear Bellman equation. Remarkably, we address the challenge by reducing the two-dimensional equation of wealth and income to the one-dimensional equation of income-to-wealth ratio. We are able to explic-

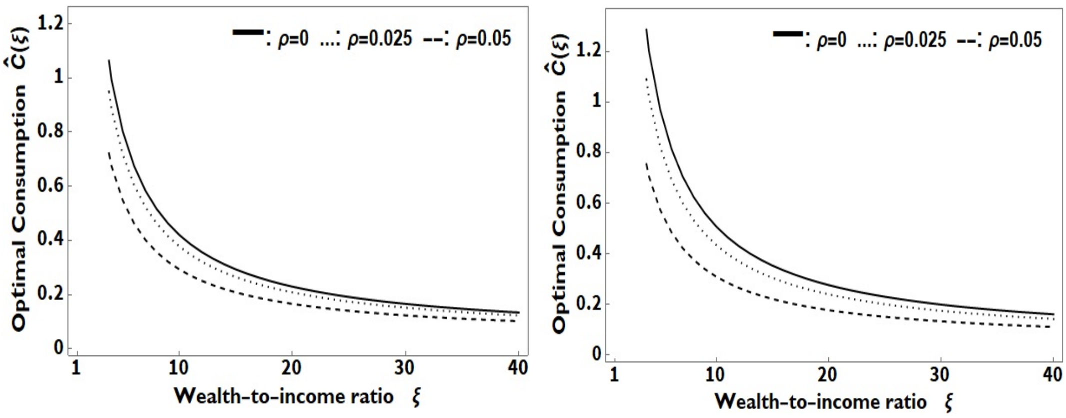

Figure 5: **Optimal consumption choice.** Left (1-γ=2), Right (1-γ=3).

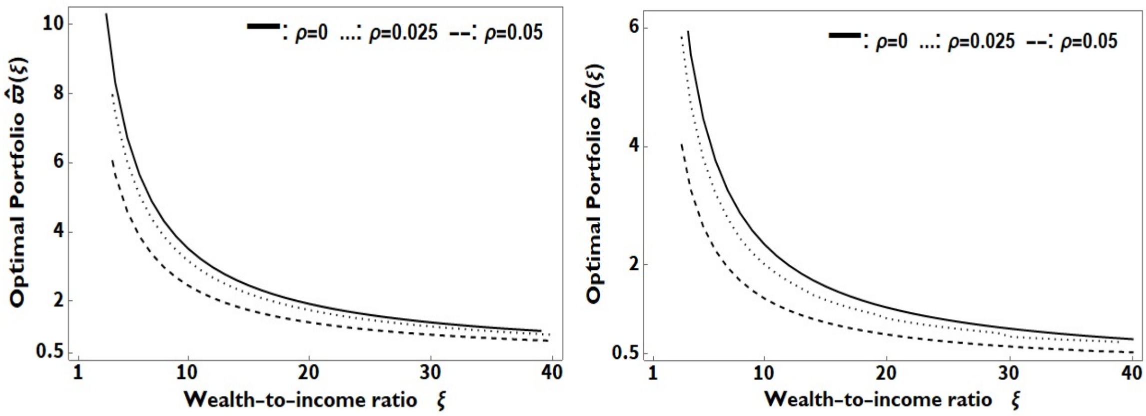

Figure 6: **Optimal portfolio choice.** Left (1-γ=2), Right (1-γ=3).

itly characterize the value function and the optimal consumption and investment strategy as a function of income-to-wealth ratio. We provide the useful analytic comparative statics associated with the value function and optimal strategies.

Importantly, we have developed a quite general numerical algorithm for control iteration and solve the Bellman equation as a sequence of solutions to ordinary differential equations. This numerical algorithm can be readily applied to many other optimal stochastic control problems especially with additional nondiversifiable Brownian risks, giving rise to nonlinear Bellman equations. Finally, our numerical analysis offers new insights on the individual's optimal consumption and portfolio choice with stochastic labor income.

The several interesting extensions to this model should prove relatively straightforward. A first important extension on the model would be to investigate the general case of stochastic labor income in which labor income and stock market returns are partially correlated (not independent only as we have assumed in the paper). Not only the extent of unspanned income risk itself, but interestingly, also the extent of the correlation between labor income and stock market returns would alter quantitative and qualitative features of the model result. Further, incorporating a cointegration between labor income and stock price into the model can reflect the fact in reality that the correlation is actually positive in the long run, not in the short run (Benzoni *et al.*, 2007). We hope our paper will serve as a stepping stone for such a future work on the general correlation case.

A second important extension on the model would be to consider for individual preference the more general recursive utility than the present logarithmic utility. The individual's optimal decisions would change a lot with a change in the levels of elasticity of intertemporal substitution.

A third extension of the model would be to consider retirement flexibility in the baseline consumption/savings and investment model with stochastic labor income. The extension suggested can be viewed as a crucial complement to our current understanding of retirement models without stochastic labor income.

A fourth extension of the model is to study its general equilibrium asset pricing implications. It would be of particular interest how the features of returns are likely to change according to the differing consumption/savings and investment decisions with stochastic labor income.

Declarations

Funding and/or Conflicts of interests/Competing interests. This paper is not under consideration for publication by another journal or similar publication. The authors do not have financial or non-financial interests that are directly or indirectly related to the work submitted for publication. The authors certify that there are no conflicts of interests/competing interests.

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APPENDIX

A The Proof of Theorem 3.1

Recall from (16) that the optimal consumption is given by

$$\widehat{C}(z) = \frac{\beta}{1 - \beta z W'(z)}.$$

For notational simplicity, we define

$$\Psi(z) = \frac{\beta}{1 - \beta z W'(z)}. \tag{A-1}$$

We can rewrite (A-1) as

$$W'(z) = \frac{1}{z}\Big(\frac{1}{\beta} - \frac{1}{\Psi(z)}\Big)$$

and hence,

$$W''(z) = -\frac{1}{z^2}\Big(\frac{1}{\beta} - \frac{1}{\Psi(z)} - \frac{z\Psi'(z)}{\Psi^2(z)}\Big).$$

Therefore,

$$1 - \beta z W'(z) = \frac{\beta}{\Psi(z)} \tag{A-2}$$

and

$$1 - 2\beta z W'(z) - \beta z^2 W''(z) = \frac{\beta}{\Psi^2(z)}\Big(\Psi(z) - z\Psi'(z)\Big). \tag{A-3}$$

Thus,

$$\frac{(1 - \beta z W'(z))^2}{1 - 2\beta z W'(z) - \beta z^2 W''(z)} = \frac{\beta}{\Psi(z) - z\Psi'(z)}. \tag{A-4}$$

We can then rewrite the Bellman equation (15) as

$$\begin{aligned}\beta W(z) =& \frac{1}{\beta}\Big(\mu - \frac{\rho^2}{2}\Big) + \frac{r - \mu + z + \frac{\rho^2}{2}}{\Psi(z)} \\ &+ \frac{\rho^2}{2}\frac{z\Psi'(z)}{\Psi^2(z)} + \ln\Psi(z) - 1 + \frac{|\theta|^2}{2}\frac{1}{\Psi(z) - z\Psi'(z)}\end{aligned} \tag{A-5}$$

or equivalently,

$$\begin{aligned}\beta W(z) =& \frac{1}{\beta}\Big(\mu - \frac{\rho^2}{2}\Big) + \ln\Psi(z) - 1 + \frac{r - \mu + \rho^2 + z}{\Psi(z)} \\ &+ \frac{1}{\Psi(z)}\Big[\frac{|\theta|^2}{2}\frac{1}{1 - z\frac{\Psi'(z)}{\Psi(z)}} - \frac{\rho^2}{2}\Big(1 - z\frac{\Psi'(z)}{\Psi(z)}\Big)\Big]\end{aligned}$$

with the boundary conditions

$$\Psi(0) = \beta,\ \Psi(z) - \beta\Big(1 + \frac{z}{r - \mu + \rho^2}\Big) \to 0, \quad \text{as} \quad z \to +\infty. \tag{A-6}$$

We claim the following property:

$$0 < 1 - \frac{z\Psi'(z)}{\Psi(z)} < 1. \tag{A-7}$$

Notice that $1 - \dfrac{z\Psi'(z)}{\Psi(z)} = 1$ at $z = 0$ and $1 - \dfrac{z\Psi'(z)}{\Psi(z)} = 0$ as $z \to +\infty$. Because

$$1 - \frac{z\Psi'(z)}{\Psi(z)} = z\frac{d}{dz}\Big(\ln\frac{z}{\Psi(z)}\Big),$$

the property (A-7) is, thus, a consequence that $\dfrac{z}{\Psi(z)}$ is an increasing function of z as stated in the following proposition:

Proposition A.1. *We assume (35). We then have*

$$z \to \frac{z}{\Psi(z)} \textit{ is increasing.}$$

We introduce the function

$$v(z) = \frac{|\theta|^2}{2}\frac{1}{1 - z\frac{\Psi'(z)}{\Psi(z)}} - \frac{\rho^2}{2}\Big(1 - z\frac{\Psi'(z)}{\Psi(z)}\Big). \tag{A-8}$$

We know from (A-6) that

$$v(0) = \frac{|\theta|^2 - \rho^2}{2},\ v(z) - \frac{|\theta|^2}{2}\Big(1 + \frac{z}{r - \mu + \rho^2}\Big) \to 0 \quad \text{as} \quad z \to +\infty.$$

$v(z)$ is therefore positive at $z = 0$ and at $z = +\infty$. We claim that

$$v(z) > 0.$$

If $v(z)$ becomes negative for $z > 0$, there then exists a point z^* such that $v(z^*) = 0$. In this case, however, by (A-8) we obtain with (A-7) that

$$\frac{|\theta|^2}{2} = \frac{\rho^2}{2}\Big(1 - z\frac{\Psi'(z)}{\Psi(z)}\Big)^2 \leq \frac{\rho^2}{2},$$

which contradicts to the assumption (26) in the theorem.

Notice that the rewritten Bellman equation (A-5) can be restated as

$$\beta W(z) = \frac{1}{\beta}\Big(\mu - \frac{\rho^2}{2}\Big) + \ln\Psi(z) - 1 + \frac{r - \mu + \rho^2 + z + v(z)}{\Psi(z)},$$

which is the value function as stated in the theorem.

We will then explicitly characterize $\Psi(z)$ and $v(z)$. By (A-8), we clearly see that $1 - z\frac{\Psi'(z)}{\Psi(z)}$ satisfies the following second order equation:

$$\frac{\rho^2}{2}\Big(1 - z\frac{\Psi'(z)}{\Psi(z)}\Big)^2 + v(z)\Big(1 - z\frac{\Psi'(z)}{\Psi(z)}\Big) - \frac{|\theta|^2}{2} = 0. \tag{A-9}$$

Due to the property (A-7), $1 - z\frac{\Psi'(z)}{\Psi(z)}$ is the positive root of the equation (A-9) and hence,

$$1 - z\frac{\Psi'(z)}{\Psi(z)} = \frac{-v(z) + \sqrt{v(z)^2 + \rho^2|\theta|^2}}{\rho^2}. \tag{A-10}$$

We have from (A-10) that

$$\frac{1}{z} - \frac{\Psi'(z)}{\Psi(z)} = \frac{-v(z) + \sqrt{v(z)^2 + \rho^2|\theta|^2}}{\rho^2 z},$$

which implies that

$$-\frac{d}{dz}\ln\Big(\frac{\Psi(z)}{z}\Big) = \frac{-v(z) + \sqrt{v(z)^2 + \rho^2|\theta|^2}}{\rho^2 z}.$$

By (A-6), we know that $\frac{\Psi(z)}{z} \to \frac{\beta}{r - \mu + \rho^2}$ as $z \to +\infty$. By integrating between z and $+\infty$, we now obtain the analytic expression of function $\Psi(z)$ as follows:

$$\Psi(z) = \frac{\beta z}{r - \mu + \rho^2}\exp\Big(\int_z^{+\infty}\frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big). \tag{A-11}$$

It remains to obtain the analytic expression of function $v(z)$. The following proposition shows that $v(z)$ is actually a solution of the differential equation.

Proposition A.2. *The function $v(z)$ is a solution of the differential equation*

$$\begin{aligned}
-v'(z) =& \frac{\beta}{r - \mu + \rho^2}\frac{d}{dz}\Big\{z\exp\Big(\int_z^{+\infty}\frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)\Big\} \\
&+ \frac{\beta}{z}\Big\{1 - \frac{z}{r - \mu + \rho^2}\exp\Big(\int_z^{+\infty}\frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)\Big\} \\
&+ \frac{-v(z) + \sqrt{v(z)^2 + \rho^2|\theta|^2}}{\rho^2} + \frac{v(z) + r - \mu + \rho^2}{z}\Big(\frac{-v(z) + \sqrt{v(z)^2 + \rho^2|\theta|^2}}{\rho^2} - 1\Big).
\end{aligned} \tag{A-12}$$

We can finally derive an integral equation for $v(z)$ with $v(0) = \frac{|\theta|^2 - \rho^2}{2}$ as follows

$$\begin{aligned}
v(z) =& \frac{|\theta|^2 - \rho^2}{2} \\
&+ \beta\Big[1 - \frac{z}{r - \mu + \rho^2}\exp\Big(\int_z^{+\infty}\frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big) \\
&- \int_0^z\frac{1}{\zeta}\Big\{1 - \frac{\zeta}{r - \mu + \rho^2}\exp\Big(\int_\zeta^{+\infty}\frac{-v(u) + \sqrt{v(u)^2 + \rho^2|\theta|^2}}{\rho^2 u}du\Big)\Big\}d\zeta\Big] \\
&- \int_0^z\frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2}d\zeta \\
&+ \int_0^z\frac{v(\zeta) + r - \mu + \rho^2}{\zeta}\Big(1 - \frac{-v(\zeta) + \sqrt{v(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2}\Big)d\zeta. \quad \blacksquare
\end{aligned}$$

B The Proof of Proposition A.1

We set $\Gamma(z) = \frac{z}{\Psi(z)}$. We know that $\Gamma(0) = 0$ and $\Gamma(z) \to \frac{r - \mu + \rho^2}{\beta}$ as $z \to +\infty$.

We will find a differential equation for $\Gamma(z)$. We have that

$$\Psi(z) = \frac{z}{\Gamma(z)},\ \Psi'(z) = \frac{1}{\Gamma(z)} - \frac{z\Gamma'(z)}{\Gamma^2(z)},$$

$$\Psi''(z) = -\frac{z\Gamma''(z)}{\Gamma^2(z)} + 2z\frac{(\Gamma'(z))^2}{\Gamma^3(z)} - 2\frac{\Gamma'(z)}{\Gamma^2(z)}.$$

As a result,

$$1 - \frac{z\Psi'(z)}{\Psi(z)} = z\frac{\Gamma'(z)}{\Gamma(z)},$$

$$z^2\frac{\Psi''(z)}{\Psi(z)} = -z^2\frac{\Gamma''(z)}{\Gamma(z)} + 2\Big(z\frac{\Gamma'(z)}{\Gamma(z)}\Big)^2 - 2z\frac{\Gamma'(z)}{\Gamma(z)}.$$

Differentiating the equation (A-5) with respect to z allows us to obtain that

$$\begin{aligned}\beta W'(z) = &-\Big(r-\mu+z\Big)\frac{\Psi'(z)}{\Psi^2(z)} + \frac{1}{\Psi(z)} + \frac{\Psi'(z)}{\Psi(z)} - \rho^2 z\frac{(\Psi'(z))^2}{\Psi^3(z)}\\ &+ \frac{1}{2}z\Psi''(z)\Big(\frac{\rho^2}{\Psi^2(z)} + \frac{|\theta|^2}{(\Psi(z)-z\Psi'(z))^2}\Big).\end{aligned}$$

Using (A-2), we obtain the equation

$$\begin{aligned}\beta =&(\Psi(z)-z)\Big(1-\frac{z\Psi'(z)}{\Psi(z)}\Big) + (r-\mu)\frac{z\Psi'(z)}{\Psi(z)} + \rho^2\Big(\frac{z\Psi'(z)}{\Psi(z)}\Big)^2\\ &-\frac{z^2}{2}\frac{\Psi''(z)}{\Psi(z)}\Big\{\rho^2 + |\theta|^2\Big/\Big(1-\frac{z\Psi'(z)}{\Psi(z)}\Big)^2\Big\}.\end{aligned} \tag{A-13}$$

Therefore, the equation (A-13) becomes

$$\begin{aligned}\beta =&z\Big(\frac{1}{\Gamma(z)}-1\Big)z\frac{\Gamma'(z)}{\Gamma(z)}\\ &+(r-\mu)\Big(1-z\frac{\Gamma'(z)}{\Gamma(z)}\Big) + \rho^2\Big(1-z\frac{\Gamma'(z)}{\Gamma(z)}\Big)^2\\ &+\frac{1}{2}\Big[z^2\frac{\Gamma''(z)}{\Gamma(z)} - 2\Big(z\frac{\Gamma'(z)}{\Gamma(z)}\Big)^2 + 2z\frac{\Gamma'(z)}{\Gamma(z)}\Big]\Big(\rho^2 + |\theta|^2\Big/\Big(z\frac{\Gamma'(z)}{\Gamma(z)}\Big)^2\Big).\end{aligned} \tag{A-14}$$

Because $\Gamma(z) > 0$ and $\Gamma(0) = 0$, the function $\Gamma(z)$ increases for $z > 0$. Otherwise, there exists a local minimum of $\Gamma(z)$ at $z = z^*$ with $\Gamma'(z^*) = 0$ and $\Gamma''(z^*) < 0$. In this case, however, by the equation (A-14) we obtain that

$$\beta = r - \mu + \rho^2 + \frac{1}{2}(z^*)^2\frac{\Gamma''(z^*)}{\Gamma(z^*)} < r - \mu + \rho^2,$$

which contradicts to the assumption (35). Therefore, $\Gamma(z)$ is an increasing function of $z > 0$ and the proof is now complete. ■

C The Proof of Proposition A.2

By differentiating (A-8) with respect to z with rearrangements, we obtain that

$$\begin{aligned}-zv'(z) =& - z^2\frac{\Psi''(z)}{\Psi(z)}\Big(\frac{\rho^2}{2}+\frac{|\theta|^2}{2(1-z\frac{\Psi'(z)}{\Psi(z)})^2}\Big)\\ &+\frac{\rho^2}{2}\Big(z\frac{\Psi'(z)}{\Psi(z)}\Big)^2-\frac{\rho^2}{2}z\frac{\Psi'(z)}{\Psi(z)}-\frac{|\theta|^2}{2}\frac{z\frac{\Psi'(z)}{\Psi(z)}}{1-z\frac{\Psi'(z)}{\Psi(z)}}.\end{aligned}$$

We can then easily get that

$$\begin{aligned}-zv'(z)+v(z)z\frac{\Psi'(z)}{\Psi(z)} =& - z^2\frac{\Psi''(z)}{\Psi(z)}\Big(\frac{\rho^2}{2}+\frac{|\theta|^2}{2(1-z\frac{\Psi'(z)}{\Psi(z)})^2}\Big)\\ &+\rho^2\Big(z\frac{\Psi'(z)}{\Psi(z)}\Big)^2-\rho^2 z\frac{\Psi'(z)}{\Psi(z)}.\end{aligned}$$

It follows from (A-13) that

$$\begin{aligned}\beta =&(\Psi(z)-z)\Big(1-\frac{z\Psi'(z)}{\Psi(z)}\Big)+(r-\mu)\frac{z\Psi'(z)}{\Psi(z)}\\ &-zv'(z)+v(z)z\frac{\Psi'(z)}{\Psi(z)}+\rho^2 z\frac{\Psi'(z)}{\Psi(z)}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\beta-\big(r-\mu+\rho^2+v(z)\big)}{z} =& - v'(z)\\ &+\Big(1-z\frac{\Psi'(z)}{\Psi(z)}\Big)\Big(\frac{\Psi(z)-\big(r-\mu+z+\rho^2+v(z)\big)}{z}\Big)\end{aligned}$$

or equivalently,

$$\begin{aligned}-v'(z) =&\Psi'(z)+\frac{\beta-\Psi(z)}{z}+1-z\frac{\Psi'(z)}{\Psi(z)}\\ &-\frac{v(z)+r-\mu+\rho^2}{z}z\frac{\Psi'(z)}{\Psi(z)}.\end{aligned}$$

Using (A-10) and (A-11), we finally obtain (A-12). ■

D The Proof of Corollary 3.1

When $\rho=0$, the integral equation for $v(z)$ given in (27) in Theorem 3.1 reduces to

$$\begin{aligned}v(z) =&\frac{|\theta|^2}{2}\\ &+\beta\Big[1-\frac{z}{r-\mu}\exp\Big(\int_z^{+\infty}\frac{|\theta|^2}{2\zeta v(\zeta)}d\zeta\Big)\\ &-\int_0^z\frac{1}{\zeta}\Big\{1-\frac{\zeta}{r-\mu}\exp\Big(\int_\zeta^{+\infty}\frac{|\theta|^2}{2uv(u)}du\Big)\Big\}d\zeta\Big]\\ &-\int_0^z\frac{|\theta|^2}{2v(\zeta)}d\zeta+\int_0^z\frac{v(\zeta)+r-\mu}{\zeta}\Big(1-\frac{|\theta|^2}{2v(\zeta)}\Big)d\zeta.\end{aligned} \tag{A-15}$$

The integral equation (A-15) has a closed-form solution as follows

$$v(z) = \frac{|\theta|^2}{2}\Big(1 + \frac{z}{r-\mu}\Big)$$

because of

$$\exp\Big(\int_z^{+\infty} \frac{|\theta|^2}{2\zeta v(\zeta)} d\zeta\Big) = \frac{z + r - \mu}{z}.$$

When $\rho = 0$, the function $\Psi(z)$ given in (A-11) in Theorem 3.1 then reduces to

$$\begin{aligned}
\Psi(z) &= \frac{\beta z}{r-\mu} \exp\Big(\int_z^{+\infty} \frac{|\theta|^2}{2\zeta v(\zeta)} d\zeta\Big) \\
&= \frac{\beta z}{r-\mu} \frac{z + r - \mu}{z} \\
&= \beta\Big(1 + \frac{z}{r-\mu}\Big).
\end{aligned}$$

By Theorem 3.1, the value function when $\rho = 0$ is given by

$$\begin{aligned}
W^0(z) &= \frac{1}{\beta}\Big[\frac{\mu}{\beta} + \ln \Psi(z) - 1 + \frac{r - \mu + z + v(z)}{\Psi(z)}\Big] \\
&= \frac{1}{\beta}\Big[\frac{\mu}{\beta} + \ln\Big\{\beta\Big(1 + \frac{z}{r-\mu}\Big)\Big\} - 1 + \Big\{r - \mu + z + \frac{|\theta|^2}{2}\Big(1 + \frac{z}{r-\mu}\Big)\Big\}\Big/\Big\{\beta\Big(1 + \frac{z}{r-\mu}\Big)\Big\}\Big] \\
&= \frac{1}{\beta}\ln\Big(1 + \frac{z}{r-\mu}\Big) + \frac{1}{\beta}\Big\{\frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1\Big\},
\end{aligned}$$

which is the value function as stated in the corollary.

Alternatively, we can solve the Bellman equation (15) directly. With substitution of the value function $W^0(z)$ in the Bellman equation (15), we have that

$$\begin{aligned}
&\beta W^0(z) - \frac{r+z}{\beta}\Big(1 - \beta z W^{0\prime}(z)\Big) - \mu z W^{0\prime}(z) - \frac{1}{2}\rho^2 z^2 W^{0\prime\prime}(z) \\
&- \ln\beta + 1 + \ln\big(1 - \beta z W^{0\prime}(z)\big) - \frac{|\theta|^2}{2\beta}\frac{\big(1 - \beta z W^{0\prime}(z)\big)^2}{1 - 2\beta z W^{0\prime}(z) - \beta z^2 W^{0\prime\prime}(z)} \\
&= -\frac{1}{2}\rho^2 z^2 W^{0\prime\prime}(z) \\
&= \frac{\rho^2}{2\beta}\frac{z^2}{(z + r - \mu)^2} > 0
\end{aligned} \tag{A-16}$$

with the boundary conditions

$$W(0) = \frac{\frac{r + \frac{|\theta|^2}{2}}{\beta} + \ln\beta - 1}{\beta}$$

and

$$W(z) - \frac{1}{\beta}\ln\Big(1 + \frac{z}{r-\mu}\Big) + \frac{1}{\beta}\Big\{\frac{1}{\beta}\Big(r + \frac{|\theta|^2}{2}\Big) + \ln\beta - 1\Big\} \to 0, \quad \text{as} \quad z \to +\infty.$$

The Bellman equation (15) is now solved when $\rho = 0$ with $W^0(z)$. ■

E The Proof of Proposition 4.1

The relation (29) is true at the boundaries $z = 0$ and $z = +\infty$. Next, we can rewrite (21) at $z = +\infty$ as

$$
\begin{aligned}
\beta\overline{W}(z) =& \frac{r+z}{\beta}\Big(1 - \beta z\overline{W}'(z)\Big) + \mu z\overline{W}'(z) + \frac{1}{2}\rho^2 z^2\overline{W}''(z) + \ln\beta \\
&+ \sup_C \Big[\ln C - C\big(1 - \beta z\overline{W}'(z)\big)\Big] \\
&+ \frac{1}{\beta}\sup_{\varpi}\Big[\sigma^*\varpi.\theta\big(1 - \beta z\overline{W}'(z)\big) - \frac{1}{2}|\sigma^*\varpi|^2\big(1 - 2\beta z\overline{W}'(z) - \beta z^2\overline{W}''(z)\big)\Big] \\
&+ \frac{\rho^2}{2\beta}\Big(\frac{r-\mu+\rho^2}{z+r-\mu+\rho^2}\Big)^2.
\end{aligned}
\tag{A-17}
$$

Comparing (A-17) and (21), we obtain that

$$
\begin{aligned}
&\beta\big(\overline{W}(z) - W(z)\big) \\
&\geq -(r-\mu+z)z(\overline{W}'(z) - W'(z)) + \frac{1}{2}\rho^2 z^2(\overline{W}''(z) - W''(z)) \\
&\quad + \sup_C\Big[\ln C - C\big(1 - \beta z\overline{W}'(z)\big)\Big] - \sup_C\Big[\ln C - C\big(1 - \beta zW'(z)\big)\Big] \\
&\quad + \frac{1}{\beta}\sup_{\varpi}\Big[\sigma^*\varpi.\theta\big(1 - \beta z\overline{W}'(z)\big) - \frac{1}{2}|\sigma^*\varpi|^2\big(1 - 2\beta z\overline{W}'(z) - \beta z^2\overline{W}''(z)\big)\Big] \\
&\quad - \frac{1}{\beta}\sup_{\varpi}\Big[\sigma^*\varpi.\theta\big(1 - \beta zW'(z)\big) - \frac{1}{2}|\sigma^*\varpi|^2\big(1 - 2\beta zW'(z) - \beta z^2W''(z)\big)\Big].
\end{aligned}
$$

The optimal controls $\widehat{C}(z)$ and $\widehat{\varpi}(z)$ given in (16) and (17), respectively, then imply that

$$
\begin{aligned}
&\beta\big(\overline{W}(z) - W(z)\big) \\
&\geq -(r-\mu+z)z\big(\overline{W}'(z) - W'(z)\big) + \frac{1}{2}\rho^2 z^2\big(\overline{W}''(z) - W''(z)\big) \\
&\quad + \beta\widehat{C}(z)z\big(\overline{W}'(z) - W'(z)\big) - \sigma^*\widehat{\varpi}(z).\theta z\big(\overline{W}'(z) - W'(z)\big) \\
&\quad + |\sigma^*\widehat{\varpi}(z)|^2 z\big(\overline{W}'(z) - W'(z)\big) + \frac{1}{2}|\sigma^*\widehat{\varpi}(z)|^2 z^2\big(\overline{W}''(z) - W''(z)\big).
\end{aligned}
$$

Suppose that $\overline{W}(z) - W(z) < 0$. At $z = 0$ or $z = +\infty$, $\overline{W}(z) - W(z)$ cannot be negative. Also, any negative local minimum of $\overline{W}(z) - W(z)$ at $z = z^*$ that satisfies $\overline{W}'(z^*) - W'(z^*) = 0$ and $\overline{W}''(z^*) - W''(z^*) > 0$ leads to $\overline{W}(z) - W(z) \geq 0$, which is contradict to $\overline{W}(z) - W(z) < 0$. Further, $\overline{W}(z) - W(z)$ cannot have an infimum at $z = +\infty$ because $\overline{W}(z) - W(z)$ is strictly negative at $z = +\infty$. Therefore, $\overline{W}(z) - W(z) \geq 0$, thus obtaining (29). ■

F The Proof of Proposition 4.2

Because $W(0) = \frac{1}{\beta}\Big(\frac{r + \frac{|\theta|^2}{2}}{\beta} + \ln\beta - 1\Big) = W^0(0)$, the relation (31) is true at $z = 0$. By formulas (24) and (28), we know that

$$
\overline{W}(z) - W^0(z) = \frac{1}{\beta}\ln\Big(\frac{r-\mu+\rho^2+z}{r-\mu+z}\Big) - \frac{1}{\beta}\ln\Big(\frac{r-\mu+\rho^2}{r-\mu}\Big) + \frac{\rho^2}{2\beta^2}.
$$

Therefore,

$$\lim_{z\to+\infty}\left\{\overline{W}(z)-W^0(z)\right\}=-\frac{1}{\beta}\ln\left(\frac{r-\mu+\rho^2}{r-\mu}\right)+\frac{\rho^2}{2\beta^2}<0$$

due to the assumption (30). Because of (25),

$$\lim_{z\to+\infty}\left\{W(z)-W^0(z)\right\}<0.$$

Hence, the relation (31) is also true at $z=+\infty$. With similar arguments in the proof of Proposition (4.1), we can conclude that the relation (31) is true for any z. ■

G The Proof of Proposition 4.3

It suffice to prove that $0\le W(z)$. $0\le W(z)$ is true at $z=0$ and $z\to+\infty$. Next, using (21) we can obtain by taking $C=1$ and $\sigma^*\varpi=\theta$ that

$$\begin{aligned}&\beta W(z)\\&\ge\frac{1}{\beta}\left(r+\frac{|\theta|^2}{2}+z\right)+\ln\beta-1\\&\quad+\big(\beta-(r-\mu)-z\big)zW'(z)+\frac{\rho^2+|\theta|^2}{2}z^2W''(z)\\&>\big(\beta-(r-\mu)-z\big)zW'(z)+\frac{\rho^2+|\theta|^2}{2}z^2W''(z).\end{aligned}$$

If $W(z)<0$, the negative values of $W(z)$ cannot be obtained at $z=0$ and $z=+\infty$. There then must be a local minimum of $W(z)$ at some point z^* such that $W(z^*)<0$ with $W'(z^*)=0$ and $W''(z^*)>0$, which results in $W(z)>0$ that contradicts to $W(z)<0$. Therefore, $0\le W(z)$. ■

H The Proof of Proposition 4.4

The property $\Psi(z)\ge\beta$ is true at $z=0$ and as $z\to+\infty$. If we have points whose value is below β, there is a local minimum of $\Psi(z)$ at $z=z^*$ such that $\Psi(z^*)<\beta$ with $\Psi'(z^*)=0$ and $\Psi''(z^*)>0$. In this case, however, by the equation (A-13) we can obtain that $\Psi(z^*)>\beta$, which is a contradiction. Hence, $\Psi(z)\ge\beta$.

We now prove (33). Because $\Psi(z)\ge\beta$, the boundary condition $\Psi(0)=\beta$ given in (A-6) shows that a minimum of $\Psi(z)$ can be attained at $z=0$. So, $\Psi(z)$ increases as z increases. Otherwise, the result (33) follows immediately. Suppose that $\Psi(z)$ decreases as z increases. There then exists a local maximum of $\Psi(z)$ at $z=z^*$ with $\Psi'(z^*)=0$ and $\Psi''(z^*)<0$. We then have from the equation (A-13) that

$$\beta\ge\Psi(z^*)-z^*. \tag{A-18}$$

Notice that $\Psi(z)(\ge\beta)$ cannot keep decreasing after z^* because $\Psi(z)$ approaches $\beta\Big(1+\dfrac{z}{r-\mu+\rho^2}\Big)(\ge\beta)$ as $z\to+\infty$. There then exists a local minimum of $\Psi(z)$ at $z=z^{**}>z^*$ with $\Psi'(z^{**})=0$ and $\Psi''(z^{**})>0$. In this case, however, by the equation (A-13) we can obtain that

$$\beta \leq \Psi(z^{**}) - z^{**} < \Psi(z^*) - z^*,$$

which contradicts to (A-18). Hence, (33) is now proven. ■

I The Proof of Proposition 4.5

We first begin by computing $\Psi'(0)$. We can rewrite (A-13) when z is close to 0 as

$$\beta \sim \beta + \Psi'(0)z - z + \frac{r-\mu}{\beta}\Psi'(0)z,$$

so necessarily

$$\Psi'(0) = \frac{\beta}{\beta + r - \mu}.$$

By (35) with $r > \mu$, we have in particular $\beta > \rho^2$. Therefore, the tangent of $\Psi(z)$ at $z = 0$ is below the upper bound $\beta\Big(1 + \dfrac{z}{r-\mu+\rho^2}\Big)$ given in (36).

If the function $\Psi(z)$ crosses the upper bound, the function cannot decrease any more because $\Psi'(z) \geq 0$ given in (33) but should approach the upper bound as $z \to +\infty$ due to the boundary condition given in (A-6). There then exists a local maximum of $\Psi(z)$ at $z = z^*$ such that

$$\Psi(z^*) > \beta\Big(1 + \frac{z^*}{r-\mu+\rho^2}\Big)$$

with $\Psi'(z^*) = 0$ and $\Psi''(z^*) < 0$. In this case, by the equation (A-13) we can obtain that

$$\beta \geq \Psi(z^*) - z^*,$$

as a result,

$$\beta \geq \beta + \frac{\beta z^*}{r-\mu+\rho^2} - z^*,$$

or equivalently,

$$\beta \leq r - \mu + \rho^2,$$

which is contradict to the assumption (35). Therefore, $\Psi(z)$ cannot cross the upper bound and the proof is complete.

■

J The Proof of Proposition 4.6

Recall from (17) that the optimal portfolio is given by

$$\widehat{\varpi}(z) = (\sigma^*)^{-1}\theta \frac{1 - \beta z W'(z)}{1 - 2\beta z W'(z) - \beta z^2 W''(z)}.$$

With the relations given in (A-3) and (A-4), the optimal portfolio can be rewritten as a function of consumption $\Psi(z)$ as follows

$$\begin{aligned}\widehat{\varpi}(z) &= (\sigma^*)^{-1}\theta \frac{\Psi(z)}{\Psi(z) - z\Psi'(z)} \\ &= (\sigma^*)^{-1}\theta \frac{1}{1 - z\frac{\Psi'(z)}{\Psi(z)}}.\end{aligned}$$

Due to the property (A-7) with

$$1 - z\frac{\Psi'(z)}{\Psi(z)} = 1 \quad \text{at} \quad z = 0$$

and

$$1 - z\frac{\Psi'(z)}{\Psi(z)} = 0 \quad \text{as} \quad z \to +\infty,$$

the optimal portfolio $\widehat{\varpi}(z)$ is, thus, positive. ■

K The Proof of Proposition 5.1

We can rewrite (41) as follows

$$\begin{aligned}\beta W_{n+1}(z) =& \frac{r+z}{\beta} - zW_{n+1}^{'}(z)(z + r - \mu) + \frac{1}{2}\rho^2 z^2 W_{n+1}^{''}(z) \\ &+ \Big(\widehat{C}_n(z) - \sigma^*\widehat{\varpi}_n(z).\theta + |\sigma^*\widehat{\varpi}_n(z)|^2\Big) z\Big(W_{n+1}^{'}(z) - W_n^{'}(z)\Big) \\ &+ \frac{1}{2}|\sigma^*\widehat{\varpi}_n(z)|^2 z^2\Big(W_{n+1}^{''}(z) - W_n^{''}(z)\Big) + \ln \widehat{C}_n(z) - \widehat{C}_n(z)\Big(\frac{1}{\beta} - zW_n^{'}(z)\Big) \\ &+ \sigma^*\widehat{\varpi}_n(z).\theta\Big(\frac{1}{\beta} - zW_n'(z)\Big) - \frac{1}{2}|\sigma^*\widehat{\varpi}_n(z)|^2\Big(\frac{1}{\beta} - 2zW_n^{'}(z) - z^2W_n^{''}(z)\Big).\end{aligned}$$

From the definition (37) of $\widehat{C}_n(z)$ and $\widehat{\varpi}_n(z)$, we always obtain that

$$\begin{aligned}\beta W_{n+1}(z) \geq& \frac{r+z}{\beta} - zW_{n+1}^{'}(z)(z + r - \mu) + \frac{1}{2}\rho^2 z^2 W_{n+1}^{''}(z) \\ &+ \Big(\widehat{C}_n(z) - \sigma^*\widehat{\varpi}_n(z).\theta + |\sigma^*\widehat{\varpi}_n(z)|^2\Big) z\Big(W_{n+1}^{'}(z) - W_n^{'}(z)\Big) \\ &+ \frac{1}{2}|\sigma^*\widehat{\varpi}_n(z)|^2 z^2\Big(W_{n+1}^{''}(z) - W_n^{''}(z)\Big) + \ln \widehat{C}_n(z) - \widehat{C}_n(z)\Big(\frac{1}{\beta} - zW_n^{'}(z)\Big) \\ &+ \sigma^*\widehat{\varpi}_n(z).\theta\Big(\frac{1}{\beta} - zW_n'(z)\Big) - \frac{1}{2}|\sigma^*\widehat{\varpi}_n(z)|^2\Big(\frac{1}{\beta} - 2zW_n^{'}(z) - z^2W_n^{''}(z)\Big).\end{aligned} \tag{A-19}$$

Applying (41) with $n-1$, we have that

$$\begin{aligned}\beta W_n(z) =&\frac{r+z}{\beta} - zW_n^{'}(z)(z+r-\mu) + \frac{1}{2}\rho^2 z^2 W_n^{''}(z)\\ &+ \ln \widehat{C}_{n-1}(z) - \widehat{C}_{n-1}(z)\Big(\frac{1}{\beta} - zW_n'(z)\Big) + \sigma^* \widehat{\varpi}_{n-1}(z).\theta\Big(\frac{1}{\beta} - zW_n'(z)\Big)\\ &- \frac{1}{2}|\sigma^* \widehat{\varpi}_{n-1}(z)|^2\Big(\frac{1}{\beta} - 2zW_n'(z) - z^2 W_n^{''}(z)\Big).\end{aligned} \tag{A-20}$$

Subtracting (A-20) from (A-19), we obtain that

$$\begin{aligned}&\beta\Big(W_{n+1}(z) - W_n(z)\Big)\\ &\geq \Big(\widehat{C}_n(z) - \sigma^* \widehat{\varpi}_n(z).\theta + |\sigma^* \widehat{\varpi}_n(z)|^2 - (z+r-\mu)\Big) z\Big(W_{n+1}^{'}(z) - W_n'(z)\Big)\\ &+ \frac{1}{2}\Big(\rho^2 + |\sigma^* \widehat{\varpi}_n(z)|^2\Big) z^2\Big(W_{n+1}^{''}(z) - W_n^{''}(z)\Big).\end{aligned} \tag{A-21}$$

On the boundary we have that

$$W_{n+1}(0) - W_n(0) = 0,\ W_{n+1}(z) - W_n(z) \to 0 \ \text{ as } \ z \to +\infty. \tag{A-22}$$

We claim that $W_{n+1}(z) - W_n(z) \geq 0$. If $W_{n+1}(z) - W_n(z) < 0$, there then exists a local minimum of $W_{n+1}(z) - W_n(z)$ at $z = z^*$ due to the boundary conditions given in (A-22) such that $W_{n+1}'(z^*) - W_n'(z^*) = 0$ and $W_{n+1}''(z^*) - W_n''(z^*) > 0$. In this case, however, (A-21) results in that

$$\begin{aligned}&\beta\Big(W_{n+1}(z^*) - W_n(z^*)\Big)\\ &\geq \frac{1}{2}\Big(\rho^2 + |\sigma^* \widehat{\varpi}_n(z)|^2\Big) z^2\Big(W_{n+1}''(z^*) - W_n''(z^*)\Big) > 0,\end{aligned}$$

as a result, $W_{n+1}(z^*) - W_n(z^*) > 0$, which is a contradiction. We, thus, obtain the first inequality of (43) in the proposition.

We turn to the second inequality of (43). We first rewrite (A-17) as

$$\begin{aligned}\beta \overline{W}(z) =&\frac{r+z}{\beta} - z\overline{W}'(z)(z+r-\mu) + \frac{1}{2}\rho^2 z^2 \overline{W}''(z)\\ &+ \sup_C \Big[\ln C - C\Big(\frac{1}{\beta} - z\overline{W}'(z)\Big)\Big] + \frac{\rho^2}{2\beta}\left(\frac{r-\mu+\rho^2}{z+r-\mu+\rho^2}\right)^2\\ &+ \sup_{\varpi} \Big[\sigma^* \varpi.\theta\Big(\frac{1}{\beta} - z\overline{W}'(z)\Big) - \frac{1}{2}|\sigma^* \varpi|^2\Big(\frac{1}{\beta} - 2z\overline{W}'(z) - z^2\overline{W}''(z)\Big)\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\beta \overline{W}(z) \geq&\frac{r+z}{\beta} - z\overline{W}'(z)(z+r-\mu) + \frac{1}{2}\rho^2 z^2 \overline{W}''(z)\\ &+ \ln \widehat{C}_n(z) - \widehat{C}_n(z)\Big(\frac{1}{\beta} - z\overline{W}'(z)\Big) + \sigma^* \widehat{\varpi}_n(z).\theta\Big(\frac{1}{\beta} - z\overline{W}'(z)\Big)\\ &- \frac{1}{2}|\sigma^* \widehat{\varpi}_n(z)|^2\Big(\frac{1}{\beta} - 2z\overline{W}'(z) - z^2\overline{W}''(z)\Big).\end{aligned} \tag{A-23}$$

Next, comparing (A-23) with (41), we obtain that

$$
\begin{aligned}
&\beta(\overline{W}(z) - W_{n+1}(z)) \\
&\geq \Big(\widehat{C}_n(z) - \sigma^*\widehat{\varpi}_n(z).\theta + |\sigma^*\widehat{\varpi}_n(z)|^2 - (z + r - \mu)\Big) z \Big(\overline{W}'(z) - W_{n+1}'(z)\Big) \\
&+ \frac{1}{2}\Big(\rho^2 + |\sigma^*\widehat{\varpi}_n(z)|^2\Big) z^2 \Big(\overline{W}''(z) - W_{n+1}''(z)\Big).
\end{aligned}
$$

On the boundary we have that

$$\overline{W}(0) - W_{n+1}(0) = \frac{\rho^2}{2\beta^2},\ \overline{W}(z) - W_{n+1}(z) \to 0 \ \text{ as } \ z \to +\infty.$$

This, thus, proves that

$$\overline{W}(z) - W_{n+1}(z) > 0.$$

A similar proof holds for $W^0(z)$. From (A-16), we also have that

$$
\begin{aligned}
\beta W^0(z) \geq &\frac{r+z}{\beta} - z(W^0)'(z)(z + r - \mu) + \frac{1}{2}\rho^2 z^2 (W^0)''(z) \\
&+ \sup_C \Big[\ln C - C\Big(\frac{1}{\beta} - z(W^0)'(z)\Big)\Big] \\
&+ \sup_{\varpi} \Big[\sigma^*\varpi.\theta\Big(\frac{1}{\beta} - z(W^0)'(z)\Big) - \frac{1}{2}|\sigma^*\varpi|^2\Big(\frac{1}{\beta} - 2z(W^0)'(z) - z^2(W^0)''(z)\Big)\Big].
\end{aligned}
$$

We note also that at the boundaries we have that

$$W^0(0) - W_{n+1}(0) = 0,$$

$$\lim_{z \to +\infty} W^0(z) - W_{n+1}(z) = \frac{1}{\beta}\ln\frac{r - \mu + \rho^2}{r - \mu} - \frac{\rho^2}{2\beta^2} > 0.$$

This, therefore, proves that

$$W^0(z) - W_{n+1}(z) > 0,$$

and the proof has been completed. ■

L Proof of Theorem 6.1

Recall from (12) that the optimal consumption is given by

$$\widehat{C} = \frac{1}{x\dfrac{\partial V}{\partial x}},$$

which can be rewritten with the transformation (47) as

$$\widehat{C} = \frac{1}{\xi u'(\xi)}.$$

For notational simplicity, we define

$$H(\xi)=\frac{1}{\xi u'(\xi)}. \tag{A-24}$$

We can rewrite (A-24) as

$$u'(\xi)=\frac{1}{\xi H(\xi)}$$

and thus,

$$u''(\xi)=-\frac{H(\xi)+\xi H'(\xi)}{\big(\xi H(\xi)\big)^2}.$$

We can now rewrite the Bellman equation (48) as

$$\begin{aligned}\beta u(\xi)=&\frac{\mu}{\beta}-1-\frac{\rho^2}{2\beta}+\ln\big(\xi H(\xi)\big)+\frac{r-\mu+\rho^2+1/\xi}{H(\xi)}\\&+\frac{1}{H(\xi)}\Big[\frac{|\theta|^2}{2}\frac{1}{1+\xi\frac{H'(\xi)}{H(\xi)}}-\frac{\rho^2}{2}\Big(1+\xi\frac{H'(\xi)}{H(\xi)}\Big)\Big]\end{aligned} \tag{A-25}$$

with boundary conditions

$$H(\xi)-\beta\Big(1+\frac{1}{\xi(r-\mu+\rho^2)}\Big)\to 0 \ \text{ as } \ \xi\to 0 \tag{A-26}$$

and

$$H(\xi)-\beta\to 0 \ \text{ as } \ \xi\to+\infty.$$

Notice that

$$1+\xi\frac{H'(\xi)}{H(\xi)}\to 0 \ \text{ as } \ \xi\to 0$$

and

$$1+\xi\frac{H'(\xi)}{H(\xi)}\to 1 \ \text{ as } \ \xi\to+\infty.$$

Because

$$1+\xi\frac{H'(\xi)}{H(\xi)}=\xi\frac{d}{d\xi}\ln\big(\xi H(\xi)\big)$$

and $\xi H(\xi)$ is an increasing function similar to Proposition A.1, we therefore obtain that

$$0<1+\xi\frac{H'(\xi)}{H(\xi)}<1. \tag{A-27}$$

We then introduce the function

$$m(\xi)=\frac{|\theta|^2}{2}\frac{1}{1+\xi\frac{H'(\xi)}{H(\xi)}}-\frac{\rho^2}{2}\Big(1+\xi\frac{H'(\xi)}{H(\xi)}\Big). \tag{A-28}$$

We know from (A-26) that

$$m(\xi)-\frac{|\theta|^2}{2}\Big(1+\frac{1}{\xi(r-\mu+\rho^2)}\Big)\to 0 \ \text{ as } \ \xi\to 0$$

and

$$m(\xi) - \frac{|\theta|^2 - \rho^2}{2} \to 0 \quad \text{as} \quad \xi \to +\infty.$$

$m(\xi)$ is thus positive at $\xi = 0$ and at $\xi = +\infty$. We claim that

$$m(\xi) > 0.$$

Otherwise, there exists a point ξ^* such that $m(\xi^*) = 0$. In this case, however, by (A-28) with (A-27) that

$$\frac{|\theta|^2}{2} = \frac{\rho^2}{2}\Big(1 + \xi\frac{H'(\xi)}{H(\xi)}\Big)^2 \leq \frac{\rho^2}{2},$$

which contradicts to the standing assumption (26).

We now restate the Bellman equation (A-26) as

$$\beta u(\xi) = \frac{\mu}{\beta} - 1 - \frac{\rho^2}{2\beta} + \ln\big(\xi H(\xi)\big) + \frac{r - \mu + \rho^2 + 1/\xi + m(\xi)}{H(\xi)},$$

which results in the value function as stated in the theorem.

It remains to explicitly characterize $\Gamma(\xi)$ and $m(\xi)$. By (A-28), we clearly see that $1 + \xi\frac{H'(\xi)}{H(\xi)}$ satisfies the follower second order equation:

$$\frac{\rho^2}{2}\Big(1 + \xi\frac{H'(\xi)}{H(\xi)}\Big)^2 + m(\xi)\Big(1 + \xi\frac{H'(\xi)}{H(\xi)}\Big) - \frac{|\theta|^2}{2} = 0. \tag{A-29}$$

Due to the property (A-27), $1 + \xi\frac{H'(\xi)}{H(\xi)}$ is the positive root of the equation (A-29) and thus,

$$1 + \xi\frac{H'(\xi)}{H(\xi)} = \frac{-m(\xi) + \sqrt{m(\xi)^2 + \rho^2|\theta|^2}}{\rho^2} \tag{A-30}$$

or equivalently,

$$\frac{d}{d\xi}\ln\big(\xi H(\xi)\big) = \frac{-m(\xi) + \sqrt{m(\xi)^2 + \rho^2|\theta|^2}}{\rho^2\xi}.$$

By integrating between 0 and z with (A-26), we now obtain the analytic expression of function $H(\xi)$ as follows:

$$H(\xi) = \beta\Big(1 + \frac{1}{\xi(r - \mu + \rho^2)}\Big)\exp\Big(\int_0^\xi \frac{-m(\zeta) + \sqrt{m(\zeta)^2 + \rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big), \tag{A-31}$$

which is the same as given in the theorem.

By differentiating (A-28) with respect to ξ with rearrangements, we obtain that

$$\begin{aligned}
\xi m'(\xi) = &- \xi^2\frac{H''(\xi)}{H(\xi)}\Big(\frac{\rho^2}{2} + \frac{|\theta|^2}{2\big(1 + \xi\frac{H'(\xi)}{H(\xi)}\big)^2}\Big) \\
&+ \frac{\rho^2}{2}\Big(\xi\frac{H'(\xi)}{H(\xi)}\Big)^2 - \frac{\rho^2}{2}\xi\frac{H'(\xi)}{H(\xi)} - \frac{|\theta|^2}{2}\frac{H'(\xi)}{H(\xi)}\frac{1 - \xi\frac{H'(\xi)}{H(\xi)}}{\big(1 + \xi\frac{H'(\xi)}{H(\xi)}\big)^2}.
\end{aligned}$$

We can then easily get that

$$\begin{aligned}\xi m'(\xi)-m(\xi)\xi\frac{H'(\xi)}{H(\xi)}=&-\xi^2\frac{H''(\xi)}{H(\xi)}\Big(\frac{\rho^2}{2}+\frac{|\theta|^2}{2\big(1+\xi\frac{H'(\xi)}{H(\xi)}\big)^2}\Big)\\&+\rho^2\Big(\xi\frac{H'(\xi)}{H(\xi)}\Big)^2-|\theta|^2\xi\frac{H'(\xi)}{H(\xi)}\frac{1}{\big(1+\xi\frac{H'(\xi)}{H(\xi)}\big)^2}.\end{aligned}$$

Differentiating the equation (A-25) with respect to ξ therefore allows us to obtain that

$$\beta=\Big(H(\xi)-\frac{1}{\xi}\Big)\Big(1+\frac{\xi H'(\xi)}{H(\xi)}\Big)+\xi m'(\xi)-m(\xi)\xi\frac{H'(\xi)}{H(\xi)}.$$

Hence,

$$\xi m'(\xi)=\beta-\Big(H(\xi)-\frac{1}{\xi}\Big)\Big(1+\frac{\xi H'(\xi)}{H(\xi)}\Big)+m(\xi)\xi\frac{H'(\xi)}{H(\xi)}.$$

Using (A-30) and (A-31), we obtain the following differential equation

$$\begin{aligned}&m(\xi)+\xi m'(\xi)\\&=\beta-\Big(\frac{-m(\xi)+\sqrt{m(\xi)^2+\rho^2|\theta|^2}}{\rho^2}\Big)\Big\{\beta\Big(1+\frac{1}{\xi(r-\mu+\rho^2)}\Big)\exp\Big(\int_0^\xi\frac{-m(\zeta)+\sqrt{m(\zeta)^2+\rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)-\frac{1}{\xi}\Big\}\\&\quad+m(\xi)\Big(\frac{-m(\xi)+\sqrt{m(\xi)^2+\rho^2|\theta|^2}}{\rho^2}\Big).\end{aligned}$$

Because

$$m(\xi)+\xi m'(\xi)=\frac{d}{d\xi}\big(\xi m(\xi)\big),$$

we therefore obtain the analytic expression of function $m(\xi)$ as follows:

$$\begin{aligned}\xi m(\xi)=&\frac{|\theta|^2}{2}\frac{1}{r-\mu+\rho^2}\\&+\beta\xi-\int_0^\xi\Big(\frac{-m(\eta)+\sqrt{m(\eta)^2+\rho^2|\theta|^2}}{\rho^2}\Big)\Big\{\beta\Big(1+\frac{1}{\eta(r-\mu+\rho^2)}\Big)\exp\Big(\int_0^\eta\frac{-m(\zeta)+\sqrt{m(\zeta)^2+\rho^2|\theta|^2}}{\rho^2\zeta}d\zeta\Big)-\frac{1}{\eta}\Big\}d\eta\\&+\int_0^\xi m(\eta)\Big(\frac{-m(\eta)+\sqrt{m(\eta)^2+\rho^2|\theta|^2}}{\rho^2}\Big)d\eta\end{aligned}$$

with

$$\xi m(\xi)-\frac{|\theta|^2}{2}\frac{1}{r-\mu+\rho^2}\to 0\quad\text{as}\quad\xi\to 0.$$