Forecast and Decision Horizons in a Commodity Trading Model

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ABSTRACT
Forecasts of demands or prices become increasingly unreliable as the future becomes more distant. It is, therefore, beneficial to show that optimal decisions during an initial time interval are either partially or wholly independent of the forecasted data from some future time onwards. Using a commodity trading model as an example, we obtain conditions that allow us to make optimal buying and selling decisions for a commodity in some initial time interval without knowing its price forecast beyond some future time. Such an initial time interval is called a *decision horizon* and the time up to which the forecasted data is required to make the optimal decisions during the decision horizon is called a *forecast horizon*. We use the maximum principle to solve the example and show that the decision and forecast horizons in the problem arise from lower and upper bounds imposed on the on-hand inventory of the commodity.

KEYWORDS
Forecast horizon; decision horizon; wheat trading model; optimal control; maximum principle

1. Introduction

In some dynamic optimization problems, it can be shown that the optimal decisions during an initial time interval may have limited or no dependence altogether on the values of problem parameters beyond a certain time. Specifically, say for e.g., for a problem over the time $t \in [0, T]$, the optimal decisions in the interval $[0, t_1]$ are partially or completely independent of problem data beyond the time τ, where $0 \leq t_1 \leq \tau \leq T$. In such cases, a forecast of the future data is required only as far as τ to make optimal decisions in the initial time interval $[0, t_1]$. In the literature on dynamic optimization problems, this initial time interval (t_1) is termed as the *decision horizon*, and the time up to which forecast is needed (τ) to make these optimal decisions during the decision horizon is termed as *forecast horizon*. The readers may refer to Bhaskaran and Sethi (1987), Bes and Sethi (1988), and Chand et al. (1990) for details on these concepts, and Chand et al. (2002) for a classified bibliography of the literature on horizon research. The presence of these horizons in a dynamic optimization problem means that such a problem can then be partitioned into a series of smaller problems, and potentially making it more tractable. A forecast horizon, say τ, is termed as a *strong forecast horizon* if the optimal solution during the decision horizon have no dependence at all

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on the problem parameters (such as for e.g. demand, costs, market price) beyond the time τ. If on the other hand, the optimality of a solution during the decision horizon is conditional upon some restrictions on the model parameters beyond this forecast horizon (τ), then it is termed as a *weak forecast horizon.*

Chand et al. (2002) presented an extensive review of literature on forecast horizons across different types of problems, different methods used, different sources of horizons, etc. and given its comprehensive nature we will not cover the literature prior to that. In more recent times, there have been some articles on research in forecast horizons, but they have focussed predominantly on dynamic lot sizing inventory problems. The papers which study forecast horizons in dynamic lot sizing problems include Dawande et al. (2006), Dawande et al. (2007), Chand et al. (2007), Dawande et al. (2009), Bardhan et al. (2013), Teyarachakul et al. (2016), Jing and Mu (2019), Jing et al. (2020), and Jing and Chao (2022). A few papers which study forecast horizons in other problems/applications are as follows. Cheevaprawatdomrong et al. (2007) consider a non-homogeneous infinite horizon Markov Decision Process and provide sufficient conditions and an algorithm for determining then. Lortz et al. (2015) investigate the existence of forecast horizon for a general class of infinite horizon deterministic optimization problem under a set of assumptions, and apply it to the case of production planning. In this paper, our focus is on a commodity trading problem. Specifically, on the wheat-trading model by Ijiri and Thompson (1970), and the forecast and decision horizons in this classical continuous time optimal control problem. There are relatively few articles in the literature that focus on forecast horizon in the wheat trading model, and in general in commodity trading problems. In this context, some papers that are relevant to us include, Hartl (1986), and Rempala (1989). Hartl (1986) showed the existence of weak forecast horizon in the wheat trading problem with no warehouse constraint. Rempala (1989) studied decision and forecast horizons for a wheat trading problem with no warehousing constraint but with dynamic limits on the control variable, i.e., where the upper and lower limits on the control variable are functions of time. In this conceptual note, we demonstrate the concepts of strong and weak forecast horizons in the context of the wheat trading model proposed by Ijiri and Thompson (1970). To do that, we will specialize their model to some particular cases and impose constraints that would give rise to forecast and decision horizons. Through simple examples, we demonstrate the existence of forecast and decision horizons for problems without a warehousing constraint as well as with a warehousing constraint on the total stock of wheat. We show the presence of strong horizons in a constrained problem with warehousing constraint. In the case of no warehouse constraint, we highlight the notion of a price shield that gives rise to a weak horizon, and for a general case of non-zero interest rate on cash provide a simple proof of the existence and relevance of the shield. We also discuss a simple graphical proof on the existence of decision horizon and a strong forecast horizon in the case when there is a warehouse constraint. This paper builds on the wheat trading model studied in Sethi (2021) and contributes with further analysis and insights. We present new analysis and insights on the existence of forecast and decision horizons, particularly in the case of non-zero interest rate. We also present theoretical proofs on the existence of strong forecast horizons, and the existence and relevance of price shield in the case of weak forecast horizons.

In many dynamic optimization problems, rolling-horizon decision making is often an effective approach to obtain near-optimal solutions. In a typical rolling-horizon procedure, in the beginning of the first period, an initial T_1 period problem is solved based on the current state and forecast information. Then at the beginning of the second period, the state and future forecasts are updated, and then again a subsequent problem over some time periods T_2 is solved. This procedure repeats every period where the 'horizon', i.e. the time periods over which the problem is solved, effectively gets 'rolled over' each period. For further details, the readers may refer to Sethi and Sorger (1991) for a theoretical framework on rolling-horizon decision making, and Chand et al. (2002) for a classified bibliography of the literature in the area of forecast, solution, and rolling horizons. While this approach leads to effective solutions in many settings, it is difficult to prove its optimality in practical problems. However, the existence of forecast horizons in a dynamic optimization problem, by its very definition, provides a basis for implementing the rolling-horizon solutions. For e.g., a rolling horizon solution over an initial time period T_1 will be optimal if the problem has an initial forecast horizon of time length greater than or equal to T_1. As Chand et al. (2002) argue: "in a way, the rolling-horizon practice is a heuristic for implementing the forecast horizon theory." Furthermore, they also state that: "Any rolling-horizon procedure may lead to sub-optimal decisions if the study horizon chosen in the rolling-horizon procedure is smaller than the forecast horizon...". In this regard, this conceptual study on the existence of forecast horizons in a commodity trading model, contributes to the theoretical background literature on implementation of such rolling-horizon procedures in commodity trading problems. It is to be noted that there aren't too many studies that investigate existence of forecast horizons in wheat trading models, or even commodity trading problems in general.

Our paper also has practical implications. The wheat trading model in this paper is closely related to commodity trading models used in the literature for other applications, particularly for commodities such as natural gas and electricity. These papers include: Lai et al. (2010), Wu et al. (2012), Secomandi (2015), and Nadarajah and Secomandi (2018), all of which consider discrete time stochastic commodity trading models. All these papers consider rolling-horizon optimization of appropriate deterministic representations as an approach to solve these problems and conduct numerical analysis using real data pertaining to natural gas commodity market in US. These papers however do not investigate the issue of forecast horizons in these problems. In this context, a study that is more relevant to us is Cruise et al. (2019). Cruise et al. (2019) consider a variation of the wheat trading model, a discrete time dynamic electricity storage and trading problem where the electricity prices are stochastic in nature. In their model they do not consider the holding cost of the commodity (energy) but consider leakage and convex costs. They present an algorithm to obtain initial forecast and decision horizon for the problem and the optimal control policy of buying and selling the commodity. They use actual historical spot-market wholesale electricity price data in Great Britain along with the total demand data to illustrate their results. For the half-hourly time interval spot price data, they find that the forecast horizons vary between 1 and 15 days. Given the practical applications of these similar commodity trading models as highlighted above, particularly with the computation of forecast horizons in Cruise et al. (2019), we can argue that our model can be used as a theoretical basis to motivate similar practical analysis using available data in other commodity markets.

The rest of the paper is organized as follows. Section 2 presents the wheat trading model with no short-selling allowed and obtains its solution using the maximum principle. We identify a decision horizon for this problem which is also a weak forecast horizon. Then in Section 3, we modify the wheat trading model by adding a warehouse constraint that gives rise to a decision horizon and a strong forecast horizon. We identify decision and forecast horizons by illustrating them with two examples involving markedly different forecast data beyond the forecast horizon. We present some insights and concluding remarks in Section 4. Finally, we end the note in Section 5 by dedicating it to the memory of Professor Jean-Marie Proth.

2. Wheat Trading Model

Consider a firm that buys and sells wheat in response to the fluctuations in the market price of wheat. We assume that the price of wheat over time is exogenous in nature and is known with certainty. The firm's assets are its cash balance and the wheat it holds. The firm earns interest on its cash balance and incurs a holding cost for wheat stored in inventory at any time t. The firm's objective is to maximize the total value of its assets at the end of the horizon T. To meet this objective, it determines its optimal buying and selling policy of wheat over time. More specifically, its decision variables are the quantity of wheat it needs to buy or sell at every time t.

We use the following notation in our model.

$$
\begin{aligned}
T &= \text{time horizon} \\
x(t) &= \text{cash balance in dollars at time } t \\
y(t) &= \text{the stock of wheat held by the firm at time } t \\
v(t) &= \text{the rate of purchase of wheat at time } t\text{, a negative value implies sale of wheat} \\
p(t) &= \text{per unit price of wheat at time } t \\
r &= \text{the fixed interest rate earned on the cash balance} \\
h(y) &= \text{the cost of holding } y \text{ units of wheat in inventory per unit time} \\
h_y(v) &= \left.\frac{\partial h(y)}{\partial y}\right|_{y=v}
\end{aligned}
$$

This is an optimal control problem where the two assets, i.e., $x(t)$, and $y(t)$, are state variables. The rate of sale/purchase of wheat $v(t)$ is the control (decision) variable. We now write the state equations denoting the dynamics of the state variables, as follows.

$$\dot{x} = \frac{dx}{dt} = rx(t) - h(y(t)) - p(t)v(t), \quad x(0) = x_0, \tag{1}$$

$$\dot{y} = \frac{dy}{dt} = v(t), \quad y(0) = y_0. \tag{2}$$

Equation (1) denotes that at any time t, the firm's cash balance grows with the interest earned on the cash, depletes with the holding cost incurred on the inventory of wheat, and rises (or falls) as the wheat is sold (or purchased) based on the per unit price at time t. Equation (2) simply states that the stock of wheat at any time increases (decreases) at the rate of purchase (sale) of wheat at time t. We assume limits on the rate of purchase and sale of wheat any time, and accordingly write the following constraint on the control variable.

$$-V_2 \leq v(t) \leq V_1. \tag{3}$$

Here V_1 and V_2 are non-negative constants with V_1 as the maximum rate of purchase and V_2 being the maximum rate of sale of wheat at any time t. We will disallow short-selling of wheat, implying

$$y(t) \geq 0. \tag{4}$$

We now write the firm's objective function, which is to maximize the total value of its assets at the end of the horizon.

$$\max_{v(t)\in[-V_2,V_1]} \{J = x(T) + p(T)y(T)\} \tag{5}$$

subject to (1)–(4). We solve the optimal control problem using the maximum principle. For complete details on the application of the maximum principle, including solving optimal control problems with state constraints, we refer the readers to Sethi (2021). We denote $\lambda_1(t)$, and $\lambda_2(t)$, as the two adjoint variables corresponding to the state equations of $x(t)$, and $y(t)$, respectively, and write the Hamiltonian function as follows.[1]

$$H(x(t), y(t), v(t), \lambda_1(t), \lambda_2(t), t) = \lambda_1(t)[rx(t) - h(y) - p(t)v(t)] + \lambda_2(t)v(t). \tag{6}$$

The two adjoint equations for $\lambda_1(t)$ and $\lambda_2(t)$ are written below.

$$\dot{\lambda}_1(t) = \frac{d\lambda_1}{dt} = -\lambda_1(t)r, \;\; \lambda_1(T) = 1, \tag{7}$$

$$\dot{\lambda}_2(t) = \frac{d\lambda_2}{dt} = h_y(y(t))\lambda_1(t), \;\; \lambda_2(T) = p(T). \tag{8}$$

We solve (7) as

$$\lambda_1(t) = e^{r(T-t)} \tag{9}$$

and (8) as

$$\lambda_2(t) = p(T) - \int_t^T h_y(y(z))e^{r(T-z)}dz. \tag{10}$$

The adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$, also sometimes termed as the shadow prices, can be understood as marginal changes in the value of the optimized objective function with respect to the two state variables. Thus, $\lambda_1(t)$ can be interpreted as the future value at time T of one dollar of cash held from time t to T. On similar arguments $\lambda_2(t)$ can be understood as the change in the objective function due to an additional unit of wheat at time t, which is equal to the price at time T of a unit of wheat minus the total future value at time T of the cumulative holding cost incurred to store this unit of wheat from t to T. We maximize the Hamiltonian in (6) in the control variable $v(t)$, and from (6) and (3)-(4), we can see that the optimal control is of the bang-bang form, written as follows.

$$v^*(t) = \text{bang}[-V_2, V_1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y(t) > 0, \tag{11}$$

[1]For ease of writing and readability, we occasionally suppress the arguments of variables and functions in some places. E.g., we write $\lambda_2(t)$ as λ_2, and $H(x, y, v, \lambda_1, \lambda_2, t)$ as H.

and when $y(t) = 0$, we impose the condition $\dot{y} = v \geq 0$ to ensure that no short-selling occurs. Consequently, we get

$$v^*(t) = \text{bang}[0, V_1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y(t) = 0. \tag{12}$$

Equation (11) means that when $y(t) > 0$, the optimal policy $v^*(t)$ at any time t is $v^*(t) = V_1$, i.e., to buy at the maximum rate if $\lambda_2(t) - \lambda_1(t)p(t) > 0$; and is $v^*(t) = -V_2$, i.e., to sell at the maximum rate if $\lambda_2(t) - \lambda_1(t)p(t) < 0$. Similarly, equation (12) means that whenever $y = 0$, it is optimal to buy at the maximum rate ($v^*(t) = V_1$) if $\lambda_2(t) - \lambda_1(t)p(t) > 0$ and to do nothing ($v^*(t) = 0$) if $\lambda_2(t) - \lambda_1(t)p(t) < 0$.

For simplicity in demonstrating the solution to this problem and the concept of decision and forecast horizons, we consider the following particular case with $h(y) = y/2,\ r = 0,\ x(0) = 10,\ y(0) = 1,\ V_1 = V_2 = 1,\ T = 3$, and

$$p(t) = \begin{cases} 7 - 2t & \text{for } 0 \leq t < 2, \\ 1 + t & \text{for } 2 \leq t \leq 3. \end{cases} \tag{13}$$

The problem can be written as:

$$\begin{cases} \max\{J = x(3) + p(3)y(3) = x(3) + 4y(3)\} \\ \text{subject to} \\ \dot{x} = -\frac{1}{2}y - pv,\ x(0) = 10, \\ \dot{y} = v,\ y(0) = 1, \\ -1 \leq v \leq 1,\ y \geq 0. \end{cases} \tag{14}$$

We can write the Hamiltonian as

$$H = \lambda_1(t)(-y(t)/2 - p(t)v(t)) + \lambda_2(t)v(t). \tag{15}$$

Again, maximizing the Hamiltonian w.r.t. the control v, we can obtain the optimal control as

$$v^*(t) = \text{bang}[-1, 1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y(t) > 0, \tag{16}$$

and

$$v^*(t) = \text{bang}[0, 1; \lambda_2(t) - \lambda_1(t)p(t)] \text{ when } y(t) = 0. \tag{17}$$

Next, given the constraints on the state variable and the control variable, we write the Lagrangian as follows (see Sethi (2021) for further details).

$$L = H + \mu_1(t)(v + 1) + \mu_2(t)(1 - v) + \eta(t)v(t), \tag{18}$$

where $\mu_1(t),\ \mu_2(t)$, and $\eta(t)$ characterize the optimal solution and satisfy the complementary slackness and non-negativity conditions:

$$\mu_1(t) \geq 0, \quad \mu_1(t)(v(t) + 1) = 0, \tag{19}$$

$$\mu_2(t) \geq 0, \quad \mu_2(t)(1 - v(t)) = 0, \tag{20}$$

$$\eta(t) \geq 0, \quad \eta(t)y(t) = 0. \tag{21}$$

Furthermore, the optimal control must satisfy

$$\frac{\partial L}{\partial v} = \lambda_2(t) - p(t)\lambda_1(t) + \mu_1(t) - \mu_2(t) + \eta(t) = 0, \ \forall t \in [0, T]. \tag{22}$$

Since $r = 0$, using (9), it is clear that $\lambda_1(t) = 1, \forall t$, and

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial y} = 1/2, \quad \lim_{t \to 3^-} \lambda_2(t) = \lambda_2(3^-) = 4 + \gamma, \tag{23}$$

with

$$\gamma \geq 0, \gamma y(3) = 0. \tag{24}$$

Furthermore, the optimal solution must satisfy the jump conditions, accounting for discontinuous marginal valuations of the state variables and hence jumps in the adjoint variables at any entry/contact time τ (i.e. the when state trajectory 'hits' or 'touches' the state constraint). In this case, the state constraint is the non-negativity constraint in (4). In this specific example, the jump conditions at any entry/contact time τ, where $\lambda_2(t)$ is discontinuous, are

$$\lambda_2(\tau^-) = \lambda_2(\tau^+) + \zeta(\tau) \tag{25}$$

and

$$H[x^*(\tau), y^*(\tau), v^*(\tau^-), \lambda_1(\tau^-), \lambda_2(\tau^-), \tau] = H[x^*(\tau), y^*(\tau), v^*(\tau^+), \lambda_1(\tau^+), \lambda_2(\tau^+), \tau], \tag{26}$$

along with

$$\zeta(\tau) \geq 0, \ \zeta(\tau) y^*(\tau) = 0. \tag{27}$$

We obtain the optimal policy by using (16)-(27) and graph it in Fig. 1, along with the trajectory of the adjoint variable $\lambda_2(t)$. Since the price is relatively high in the beginning and is decreasing, the optimal policy calls for selling at the highest rate ($v^* = -1$) until the wheat stock becomes zero at $t = 1$. The wheat stock becomes 0 at $t = 1$, where λ_2 becomes discontinuous and the jump conditions (25)-(27) apply. After that, it is optimal to do nothing ($v^* = 0$) till $t = 1.8$ and buy at the maximum rate ($v^* = 1$) till the end of the horizon to maximize the total value of assets at $t = 3$.

2.1. *Decision Horizon, Weak Forecast Horizon, and Price Shield*

For the problem in this section we argue here that $t = 1$ is a decision horizon and a weak forecast horizon. To demonstrate the existence of this decision and forecast horizon at $t = 1$, we present an illustration in Fig. 2 in which we consider a general price trajectory for $t > 1$. Furthermore, in Fig. 2 we also highlight an extended curve of the initial $\lambda_2(t)$ trajectory, which we term as a *price shield*. We argue that as long as the price $p(t)$ stays below this price shield for $t \geq 1$, then the optimal policy for $t \in [0, 1]$ stays the same, i.e., sell at the maximum rate. Fig. 2 shows an illustration of how this optimality is maintained. Given this condition that for $t \in [1, 3]$, $p(t)$ must be less than the price shield, $t = 1$ is in effect a *weak* forecast horizon. Since the the slope of $\lambda_2(t)$ trajectory has to be 1/2 (from equation (23)), for a general price trajectory $p(t)$ for $t > 1$ as shown in Fig. 2, one can use the boundary condition of $\lambda_2(3^-)$ in (23)-(24) to sketch the $\lambda_2(t)$ trajectory in the interval $t \in [1, 3]$, as shown in Fig. 2. This will remain as the optimal trajectory of $\lambda_2(t)$ in $t \in [1, 3]$ and will yield the optimal control using equations (16)-(17), as long as the non-negativity constraint of inventory $y(t) \geq 0$ is not violated. The $\lambda_2(t)$ trajectory will have a jump in the interval $t \in (1, 3)$ as well if the inventory constraint is violated, which will happen if the sell interval in Fig. 2 at the end is greater than the buy interval. Even then, $t = 1$ will remain the decision horizon and a weak forecast horizon. Furthermore, we can extend this argument for any finite problem horizon T, where $T > 1$. Specifically, we can say that as long as we have $p(t)$ in $t \in [1, T]$ less than the price shield curve of Fig. 2 extended to T, it is optimal to sell at the maximum rate in $t \in [0, 1]$. In other words, $t = 1$ will remain decision and weak forecast horizon for any general problem horizon $T > 1$. Another simple intuition to interpret the price shield and weak forecast horizon can be explained as follows. In the optimal solution we sell at the maximum rate till $t = 1$ and have $y^*(1) = 0$. If we were to suppose consider not selling a small amount of wheat between $t = 0$ and $t = 1$ (thereby making $y^*(1) > 0$), then we can argue that it is sub-optimal to do so if the price trajectory stays below the price shield. Specifically, we can argue that the potential marginal benefit of holding a small positive quantity of wheat at $t = 1$ (which might be earned by selling this quantity at a later time or by holding it till the end of horizon), does not justify the additional holding cost that will be incurred. This can be seen by observing the fact that the slope of the shield trajectory is equal to the marginal value of holding cost w.r.t. quantity, i.e, $h_y(y)$ (equation (8) with $\lambda_1 = 1$), and that the future price at any time does not exceed the shield. In Section 2.2 we present this argument slightly more formally for a general value of interest rate r, for general limits on the control variable (V_1, V_2), and linear holding cost function $h(y)$.

2.2. *The case of positive interest rate on cash* $(r > 0)$

We now extend the problem in section 2 (equation (14)) to consider a more general scenario with positive interest rate r on cash balance. We solved two examples, one with $r = 0.1$ and one with $r = 0.2$. Thus, the two problem statements are written below.
Example 2.1: Consider Problem in section 2 (equation 14) with $r = 0.1$
Example 2.2: Consider Problem in section 2 (equation 14) with $r = 0.2$

Using insights similar to the solution of problem (14) in Section 2, we obtain solutions to these problems that satisfy optimality conditions in equations (15)-(27). We note that for both these problems, in the optimal solution, the adjoint variables take the following form:

$$\lambda_1(t) = e^{r(3-t)}, \text{for } t \in [0, 3]$$
$$\lambda_2(t) = \begin{cases} \frac{e^{2r}(1-e^{r(1-t)}+10r)}{2r} & \text{for } t \in [0, 1), \\ 4 + \frac{1-e^{r(3-t)}}{2r} & \text{for } t \in [1, 3]. \end{cases}$$

However, given different values of r in these two examples, the exact trajectories of adjoint variables are different in these two examples and therefore, from (16)-(17) yield different optimal policies. The solutions to Example 2.1 and Example 2.2 are summarized below and are shown in Fig. 3, and Fig. 4, respectively.

Example 2.1 Solution: The optimal solution when $r = 0.1$ is summarized below, and is shown in Figure 3.

$$\begin{aligned}
\textbf{Interval}\,[0,1): &\lambda_2 = 5e^{0.2}(2 - e^{0.1(1-t)}),\ \mu_1 = p\lambda_1 - \lambda_2 > 0, \mu_2 = 0,\ \eta = 0;\\
&v^* = -1,\ 0 < y^*(t) \le 1\\
\textbf{Interval}\,[1,1.95): &\lambda_2 = 4 + 5(1 - e^{0.1(3-t)}),\ \mu_1 = \mu_2 = 0,\ \eta = p\lambda_1 - \lambda_2 > 0,\ \dot{\eta} < 0;\\
&v^* = 0,\ y^*(t) = 0\\
\textbf{Interval}\,[1.95,3]: &\lambda_2 = 4 + 5(1 - e^{0.1(3-t)}),\ \mu_1 = 0,\ \mu_2 = \lambda_2 - p\lambda_1 > 0,\ \eta = 0;\\
&v^* = 1,\ y^*(t) \ge 0\\
&\gamma(3) = 0
\end{aligned}$$

Example 2.2 Solution: The optimal solution for $r = 0.2$ is summarized below and shown in Figure 4.

$$\begin{aligned}
\textbf{Interval}\,[0,1): &\lambda_2 = (5/2)e^{0.4}(3 - e^{0.2(1-t)}),\ \mu_1 = p\lambda_1 - \lambda_2 > 0, \mu_2 = 0,\ \eta = 0;\\
&v^* = -1,\ 0 < y^*(t) \le 1\\
\textbf{Interval}\,[1,3]: &\lambda_2 = 4 + (5/2)(1 - e^{0.2(3-t)}),\ \mu_1 = \mu_2 = 0,\ \eta = p\lambda_1 - \lambda_2 > 0,\ \dot{\eta} < 0;\\
&v^* = 0,\ y^*(t) = 0\\
&\gamma(3) = 0.
\end{aligned}$$

Comparing the optimal solutions in Examples 2.1 and 2.2, we see that a when $r = 0.1$ the optimal trajectory is to first sell at the maximum rate till $t = 1$ when the wheat inventory becomes zero. Then it is optimal to do nothing till $t = 1.95$ followed by purchase of wheat from $t = 1.95$ till the end of the horizon $t = 3$ to take advantage of increase in price towards the end. When $r = 0.2$, the optimal solution stays the same between $t \in [0, 1]$ where it is optimal to sell at maximum rate. However, for $t > 1$, it is then optimal to do nothing and not buy any wheat.

By comparing the solutions in Section 2 (Figure 2) and Examples 2.1 and 2.2 (Figures 3, 4), one can see that while it is optimal to sell all wheat initially given high prices in the beginning, the change in interest rate earned on cash impacts when (if at all) the firm should start purchasing wheat after all the initial stock has been sold ($t = 1$). Since high interest rate favours keeping more cash, tt is intuitive that as interest rate increases it is optimal to spend less cash on buying wheat after $t = 1$, or in other words, to delay the purchase of wheat after $t = 1$. For a high interst rate of $r = 0.2$, it is then optimal to not spend cash on wheat purchase at all.

2.2.1. Weak forecast horizon and shield when $r > 0$

In Figures 3 and 4, we extend the initial $\lambda_2(t)$ trajectory beyond $t = 1$ and term it as *shield*. Suppose we denote the *shield* trajectory as $\bar{\lambda}_2(t)$, $\forall t \in [1, 3]$. Its interpretation is similar to the price shield in Figure 2 when $r = 0$, with a slight difference. Note the slight difference in naming this extended trajectory as *'shield'* instead of *'price shield'* as done in section 2.1. Using similar arguments as in section 2.1, we can conclude that $t = 1$ is a decision horizon as well as a *weak* forecast horizon. Specifically, we argue that as long as $\lambda_1(t) * p(t) \le \bar{\lambda}_2(t)$, $\forall t \in [1, 3]$, the optimal solution in $t \in [0, 1]$ stays the same, i.e., sell at the maximum rate $\implies v^*(t) = -1$. This condition on the price in $(1, 3]$ means that $t = 1$ is a *weak* forecast horizon. We can easily see that when $r = 0$, we have $\lambda_1(t) = 1\,\forall t$ from (9), and therefore the above condition reduces to the price shield condition highlighted in section 2.1 and Figure 2. In a more general setup, the existence of decision horizon, weak forecast horizon and the shield can be argued using a simple intuitive approach as described below.

Consider the wheat trading model in section 2 (equations (1)-(5)) with the holding cost of wheat linear in stock, i.e., $h(y) = c * y$. Suppose the problem parameters, including price forecast $p(t)$, are such that it is optimal to sell of all initial stock of wheat y_0 at the maximum rate starting $t = 0$, as in Figures 1 - 4. Suppose $t = t_0$ is the first instant at which the stock becomes zero (for e.g. $t_0 = 1$ in problem Figures 1 - 4. Clearly from (11)-(12) we have $\lambda_2(t) < \lambda_1(t) * p(t)\ \forall t \in [0, t_0)$, and $\lambda_2(t_0) = \lambda_1(t_0) * p(t_0)$. We extend this initial $\lambda_2(t)$ trajectory beyond $t = t_0$, term it as *shield*, and denote it as $\bar{\lambda}_2(t)$, $\forall t \in [t_0, T]$. We now argue that as long as $\lambda_1(t) * p(t) \le \bar{\lambda}_2(t)$ $\forall t \in [t_0, T]$, we will always have $v^*(t) = -V_2\ \forall t \in [0, t_0)$.

To prove the above claim we argue that, when $\lambda_1(t) * p(t) \le \bar{\lambda}_2(t)\ \forall t \in [t_0, T]$, it is sub-optimal to have any marginal deviation from the optimal policy of selling at the maximum rate in $t \in [0, t_0)$. Let's assume we deviate slightly from this optimal policy at a time $t = \hat{t}$, where $\hat{t} \in [0, t_0)$, such that $v^*(\hat{t}) > -V_2$. This will result in a small incremental quantity, denoted by δ, that is not sold and gets carried over. Consequently we will have $y(t_0) = \delta > 0$. This small incremental quantity of wheat (δ) will be either sold at a later time, say t_1, where $t_0 \le t_1 \le T$, or will be held until the end of horizon T. Note that it cannot be sold in $t \in (\hat{t}, t_0)$ as we already have wheat being sold at maximum rate in this window. If it sold at t_1, the cash earned will be $\delta * p(t_1)$, at time t_1, and given the interest rate on cash, its future value at T will be $\delta * p(t_1) * e^{r(T-t_1)}$. If δ is held in stock till end of horizon, its value at T will be $\delta * p(T)$. One can see that from the point of view of its impact on the objective function, holding δ in inventory till T is equivalent to a sale at $t_1 = T$.

The expression below shows the net change to the overall objective function as a result of this small additional quantity of wheat not sold at $t = \hat{t}$, $\hat{t} \in [0, t_0)$. which is then sold at $t = t_1$, $t_1 \in [t_0, T]$.

$$\delta * p(t_1) * e^{r(T-t_1)} - \int_{\hat{t}}^{t_1} h(\delta) * e^{r(T-z)} dz - \delta * p(\hat{t}) * e^{r(T-\hat{t})}$$

In the above expression, the first term represents the value at time T of revenue earned by sale of δ quantity at time t_1 whereas the third term refers to the value at time T of revenue lost by not selling it at time $\hat{t}$. The second term indicates the value at time T of the additional holding cost incurred for this quantity between time between $\hat{t}$ and t_1. We define Δ as the marginal increase in the objective function per unit of additional wheat δ not sold at $t = \hat{t}$. We can write Δ as follows.

$$\Delta = p(t_1) * e^{r(T-t_1)} - \int_{\hat{t}}^{t_1} \frac{h(\delta)}{\delta} * e^{r(T-z)} dz - p(\hat{t}) * e^{r(T-\hat{t})}. \quad (28)$$

Using (9) we rewrite (28) as

$$\Delta = p(t_1) * \lambda_1(t_1) - \int_{\hat{t}}^{t_1} \frac{h(\delta)}{\delta} * \lambda_1(z) dz - p(\hat{t}) * \lambda_1(\hat{t}). \quad (29)$$

Furthermore, we note that for holding cost function linear in the stock of wheat y, i.e., $h(y) = c * y$ which implies $h(\delta)/\delta = h_y(y)$. Using this relation along with adjoint equation for $\dot{\lambda}_2$ in (8), we rewrite (29) as

$$\begin{aligned}\Delta &= p(t_1) * \lambda_1(t_1) - \int_{\hat{t}}^{t_1} \dot{\lambda}_2(z)dz - p(\hat{t}) * \lambda_1(\hat{t}) \\ &= p(t_1) * \lambda_1(t_1) - \bar{\lambda}_2(t_1) + \lambda_2(\hat{t}) - p(\hat{t}) * \lambda_1(\hat{t}).\end{aligned} \tag{30}$$

Now recall that given the original ban-bang optimal solution, we must have $\lambda_2(\hat{t}) < p(\hat{t}) * \lambda_1(\hat{t})$ and given the shield condition described earlier, we must have $p(t_1) * \lambda_1(t_1) \leq \bar{\lambda}_2(t_1)$. Combining these two observations in (30), we conclude that

$$\Delta < 0. \tag{31}$$

The above result in (31) essentially implies that as long as the shield condition is satisfied, i.e., $\lambda_1(t) * p(t) \leq \bar{\lambda}_2(t)\ \forall t \in [t_0, T]$, it is not optimal to deviate from the original optimal policy of $v^*(t) = -V_2,\ \forall t \in [0, t_0)$.

3. Wheat Trading Model with a Warehouse Constraint

In this section, we consider the wheat trading model with a warehousing constraint that limits the maximum inventory of wheat that can be stored at any time. We give examples which demonstrate the existence of strong forecast horizons, arising due to this additional constraint. To demonstrate this, we modify the problem of Section 2 by adding a simple constraint on the upper limit of inventory stock of wheat, as follows.

$$y(t) \leq 1,\ \forall t. \tag{32}$$

We consider a planning horizon of $T = 4$, and the following price trajectory of wheat

$$p(t) = \begin{cases} 7 - 2t & \text{for } 0 \leq t < 2, \\ 1 + t & \text{for } 2 \leq t \leq 4. \end{cases} \tag{33}$$

The expression for Hamiltonian for this problem is same in (15). Furthermore, same as in Section 2, we get $\lambda_1 = 1$. Following similar procedure in Section 2, the optimal control for this problem can be expressed as follows:

$$v^*(t) = \begin{cases} \text{bang}[-1, 1; \lambda_2(t) - p(t)] & \text{when } y \in (0, 1), \\ \text{bang}\ [0, 1; \lambda_2(t) - p(t)] & \text{when } y = 0, \\ \text{bang}[-1, 0; \lambda_2(t) - p(t)] & \text{when } y = 1. \end{cases} \tag{34}$$

Similar to (18), we define a Lagrange multiplier $\eta_1(t)$ for the derivative of the warehouse constraint (32), i.e., for $-\dot{y} = -v \geq 0$, to include it in the Lagrangian. We then write the Lagrangian as follows

$$L = H + \mu_1(t)(v(t) + 1) + \mu_2(t)(1 - v(t)) + \eta(t)v(t) + \eta_1(t)(-v(t)), \tag{35}$$

where $\mu_1(t), \mu_2(t)$, and $\eta(t)$ satisfy (19)–(21) and $\eta_1(t)$ satisfies

$$\eta_1(t) \geq 0,\ \eta_1(t)(1 - y(t)) = 0,\ \forall t. \tag{36}$$

Furthermore, the optimal solution must satisfy

$$\frac{\partial L}{\partial v} = \lambda_2(t) - p(t) + \mu_1(t) - \mu_2(t) + \eta(t) - \eta_1(t) = 0,\ \forall t. \tag{37}$$

Similar to the results in previous section, we get, $\lambda_1(t) = 1\ \forall t \in [0, 4]$, and $\lambda_2(t)$ satisfies

$$\dot{\lambda}_2 = 1/2,\ \lambda_2(4^-) = p(4) + \gamma_1 - \gamma_2 = 5 + \gamma_1 - \gamma_2, \tag{38}$$

where

$$\gamma_1 \geq 0,\ \gamma_1 y(4) = 0,\ \gamma_2 \geq 0,\ \gamma_2(1 - y(4)) = 0. \tag{39}$$

We work backwards to obtain the solution. We first try $\gamma_1 = \gamma_2 = 0$. We assume $\hat{t}$ to be the time of the last jump of the adjoint variable $\lambda_2(t)$, i.e., the last time the inventory hits a state constraint before the end of horizon at $T = 4$. Then, we can write $\lambda_2(t)$ as

$$\lambda_2(t) = t/2 + 3 \text{ for } t \in [\hat{t}, 4.) \tag{40}$$

It can be easily seen that the trajectory of $\lambda_2(t)$ in (40) stays above the price trajectory for $8/5 < t < 4$, implying that the firm would be purchasing wheat in this time period (from (34)). However, that will violate the warehouse storage constraint. Thus, we must have $\hat{t} > 8/5$. Keeping in mind that $\dot{\lambda}_2 = 1/2$, and that the $\lambda_2(t)$ firm can only purchase wheat for a maximum time of 1 unit at a stretch given the warehouse constraint, we can see that the $\lambda_2(t)$ trajectory will hit the price trajectory at $t = 11/6$ and $t = 17/6$, thus making $\hat{t} = 17/6$.

Like Section 2, we apply the jump conditions (25)-(27), and obtain the optimal trajectory λ_2 as

$$\lambda_2(t) = \begin{cases} t/2 + 9/2 & \text{for } 0 \leq t < 1, \\ t/2 + 29/12 & \text{for } 1 \leq t < 17/6, \\ t/2 + 3 & \text{for } 17/6 \leq t \leq 4. \end{cases} \tag{41}$$

The trajectory in (41) validates the initial assumption of $\gamma_1 = \gamma_2 = 0$.

Equation (34) along with (41) characterize the optimal policy for the firm, which is then depicted in Fig. 5. To complete the solution according to the maximum principle, we compute the Lagrangian in equation (35), verify the complimentary slackness conditions in (19)–(21) and (36), as well as the condition in (35) for all four time intervals shown in Fig. 5. We summarize the results below.

$t \in [0,1)$ $\quad \mu_2(t) = \eta(t) = \eta_1(t) = 0, \mu_1(t) = p(t) - \lambda_2(t) > 0, v^*(t) = -1, 0 < y^*(t) < 1.$

$t \in [1, \frac{11}{6})$ $\quad \mu_1(t) = \mu_2(t) = \eta_1(t) = 0, \eta(t) = p(t) - \lambda_2(t) > 0, v^*(t) = 0, y^*(t) = 0.$

$t \in [\frac{11}{6}, \frac{17}{6})$ $\quad \mu_1(t) = \eta(t) = \eta_1(t) = 0, \mu_2(t) = \lambda_2(t) - p(t) > 0, v^*(t) = 1, 0 < y^*(t) < 1.$

$t \in [\frac{17}{6}, 4]$ $\quad \mu_1(t) = \mu_2(t) = \eta(t) = 0, \eta_1(t) = \lambda_2(t) - p(t) > 0, \gamma_1 = \gamma_2 = 0, v^*(t) = 0, y^*(t) = 1.$

For this example, as shown in Fig. 5 we identify $t = 1$ as a decision horizon and $\hat{t} = 17/6$ as a strong forecast horizon. It implies that given the problem data in $t \in [0, 17/6]$, the optimal policy is to sell wheat at the maximum rate in $t \in [0, 1]$, and it does not depend on the price trajectory $p(t)$ beyond $t = 17/6$. Since this is a strong forecast horizon, a price shield as calculated in the case of a weak forecast horizon is not really relevant in this case. To further illustrate the nature of a strong forecast horizon, we present two examples of price changes after $t = 17/6$ and show that they have no impact on the optimal policy during the decision horizon.

Example 3.1. Consider the problem in Section 3 with the following price trajectory

$$p(t) = \begin{cases} 7 - 2t & \text{for } 0 \le t < 2, \\ 1 + t & \text{for } 2 \le t < 6, \\ \frac{25t-44}{7} & \text{for } 17/6 \le t \le 4, \end{cases}$$

as plotted in Fig. 6. Note that while the price in $t \in [0, 17/6]$ is same as before, it goes above the previously computed price shield in Fig. 2 and also depicted in Fig. 5

Solution The optimal policy $v^*(t)$ and the trajectory of adjoint variable $\lambda_2(t)$ is depicted in Fig. 6. The optimal trajectory of $\lambda_2(t)$ is same as that in Fig. 5 for $t \in [0, 17/6]$. For $t > 17/6$, we have $\lambda_2^*(t) = t/2 + 6$. The optimal policy in $t \in [0, 1)$ is same as in Fig. 5.

Example 3.2. Assume the price trajectory to be

$$p(t) = \begin{cases} 7 - 2t & \text{for } 0 \le t < 2, \\ 1 + t & \text{for } 2 \le t < 6, \\ \frac{21}{4} - \frac{t}{2} & \text{for } 17/6 \le t \le 4, \end{cases}$$

sketched in Fig. 7.

Solution We follow a similar approach as applied in Section 3 (Fig. 5) and try $\gamma_1 = \gamma_2 = 0$ to obtain the adjoint variable trajectory in a similar way as in (40). This gives us $\lambda_2(t) = t/2 + 5/4$ for $t \in [\hat{t}_1, 4]$, where once again we assume $\hat{t}$ to be the time of the last jump of the adjoint variable $\lambda_2(t)$. We look to take advantage of the price changes in $t \in [2, 17/6]$ by buying and selling some wheat. However, it is also clear that the $\lambda_2(t)$ trajectory will have to 'jump down' in order to satisfy the boundary condition at $t = 4$, i.e., $\lambda_2(4^-) = p(4)$. This implies that the inventory stock has to deplete to 0 sometime between $t \in [17/6, 4]$. It is also apparent that the firm will have to sell all the stock it buys after $t = 1$. With these observations, we can compute that $\hat{t} = 163/54$, with the optimal $\lambda_2(t)$ along with the times to purchase and sell wheat depicted in Fig. 7. The optimal trajectory for the adjoint variable $\lambda_2(t)$ can be computed as below

$$\lambda_2(t) = \begin{cases} \frac{t}{2} + \frac{9}{2} & \text{for } 0 \le t < 1, \\ \frac{t}{2} + \frac{241}{108} & \text{for } 1 \le t < \frac{163}{54}, \\ \frac{t}{2} + \frac{5}{4} & \text{for } \frac{163}{54} \le t \le 4. \end{cases}$$

It can be easily shown that all other conditions in the maximum principle including the Lagrange multiplier conditions are satisfied with this solution.

We now present a simple graphical proof to demonstrate existence of forecast horizon in an optimal control problem with a state variable $x(t)$ and planning horizon T, where the state variable is constrained between a lower and an upper limit, i.e., $L \le x(t) \le U, \forall t$. Let's say that given a forecast of an exogenous input parameter $p(t)$ that impacts the optimal policy (such as for e.g. price forecast in the case of the wheat trading model) over a horizon 0 to T, the optimal trajectory of the state variable is the path $x_o \to A \to B \to C$. Suppose the optimal policy from 0 to T, given the forecast $p(t), t \in [0, T]$ and other input parameters, is such that the optimal trajectory of $x(t)$ hits the lower limit L as well as the upper limit U at least once. Specifically, we consider the scenario that the state variable $x(t)$ hits the lower limit for the first time at point A and then hits upper limit for the first time at point B. Now suppose the problem is extended to new horizon $\tilde{T}$, where $\tilde{T} > T$, with additional forecast of $p(t)$ for $t \in [T, \tilde{T}]$. We now make the following claim.

Proposition 1: T_A is a decision horizon and T is a strong forecast horizon
Proof: Let $x_o \to D \to E$ be the optimal trajectory on $[0, \tilde{T}]$, given the added forecast of $p(t)$ in $[T, \tilde{T}]$. It is clear that any trajectory, that is different than $x_o \to A \to B$ between $t \in [0, T]$ will have to intersect $x_o \to A \to B$ between the points A and B. Suppose that point is D in a new optimal with an added forecast of $p(t)$ in $[T, \tilde{T}]$. To prove Proposition 1, we first make the following claim.
Claim 1.1: Given the added forecast of $p(t)$ in $[T, \tilde{T}]$, If $x_o \to D \to E$ is an optimal trajectory on $[0, \tilde{T}]$, then $x_o \to A \to D \to E$ is also optimal trajectory in $[0, \tilde{T}]$.
To prove this claim, lets say the above claim is not true and $x_o \to D \to E$ is the only optimal given the added forecast of $p(t)$ in $[T, \tilde{T}]$. In that case we will clearly have $J_{x_o \to D} > J_{x_o \to A \to D}$, where J denotes the profit function corresponding to a trajectory. We then add $J_{D \to B \to C}$ to both sides of the inequality and get

$$J_{x_o \to D} + J_{D \to B \to C} > J_{x_o \to A \to D} + J_{D \to B \to C}.$$

This in turn yields $J_{x_o \to D \to B \to C} > J_{x_o \to A \to D \to B \to C}$, which contradicts the original setting of the problem that $x_o \to A \to B \to C$ is the optimal trajectory given forecast of $p(t) \in [0, T]$. Hence, by using contradiction we are able to prove Claim 1.1. We note the fact that D will be a point between A and B, i.e., $T_A \le T_D \le T_B$. Claim 1.1 then effectively states that regardless of the added forecast of $p(t)$ in $[T, \tilde{T}]$, and hence regardless of the exact position of D between A and B, the path $x_0 \to A$ will always be optimal given the initial forecast of $p(t)$ in $[0, T]$. This effectively proves Proposition 1, that in this problem, T_D is a decision horizon and T is a strong forecast horizon.

Finally, we note that while we have argued this graphical proof in a setting where the optimal state trajectory hits lower limit L before it hits the upper limit U. However, one can easily extend the same arguments to the situation where the optimal state trajectory in $[0, T]$ hits the upper limit first followed by hitting the lower limit.

4. Concluding Remarks

In this paper, we advance the research in the area of forecast horizons by demonstrating the existence of weak and strong forecast horizons in a commodity trading model. In Section 2, in a wheat trading model where short-selling is not allowed, we showed the existence of a decision horizon and a weak forecast horizon. We note that this weak forecast horizon arises as the initial wheat stock depleted to 0, i.e., $y(t) = 0$. We argued that the existence of this weak forecast horizon is contingent upon the price of wheat beyond this forecast horizon staying below a 'price shield' which was easily calculated. In Section 3, we extended the model in Section 2 to include an upper limit on the inventory stock via a warehousing constraint and obtain a decision horizon and a strong forecast horizon. It can be noted that had we considered a planning horizon T equal to 17/6, this would have been the smallest planning horizon for which the optimal state trajectory $y^*(t)$ hits its lower limit ($y^*(t) = 0$ at $t = 1$) and its upper limit and ($y^*(t) = 1$ at $t = 17/6$). This is one approach to find a decision horizon ($t = 1$) and a forecast horizon ($t = 17/6$), and we also presented a simple graphical proof of such an approach. There are other ways as well to find strong forecast horizons, and for a detailed survey of the literature on decision and forecast horizons in general, the readers are referred to Chand et al. (2002).

Finally, we would like to comment that our insights on the existence of forecast horizons, even in a deterministic setting, offer helpful knowledge to researchers. It has been shown in the literature that rolling-horizon optimization of appropriate deterministic representations is an effective and practical approach to solving stochastic dynamic optimization problems (see, for e.g. Lai et al. (2010), Wu et al. (2012), Secomandi (2015), and Nadarajah and Secomandi (2018)).

5. Dedication

We dedicate this paper to the memory of Jean-Marie Proth. Sethi visited him at INRIA - Lorraine, Metz, France, in April 1992 and stayed at his residence in Pouilly, France. Proth came on a short-term research visit to the University of Toronto to work with Sethi. During that visit, they wrote two papers, including Chand, Sethi, and Proth (1990), on the topic of this paper. Sethi remembers him fondly and misses him greatly.

Disclosure Statement

The authors report there are no competing interests to declare.

Data availability Statement

The authors confirm that the data supporting the findings of this study are available within the article.

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Figures

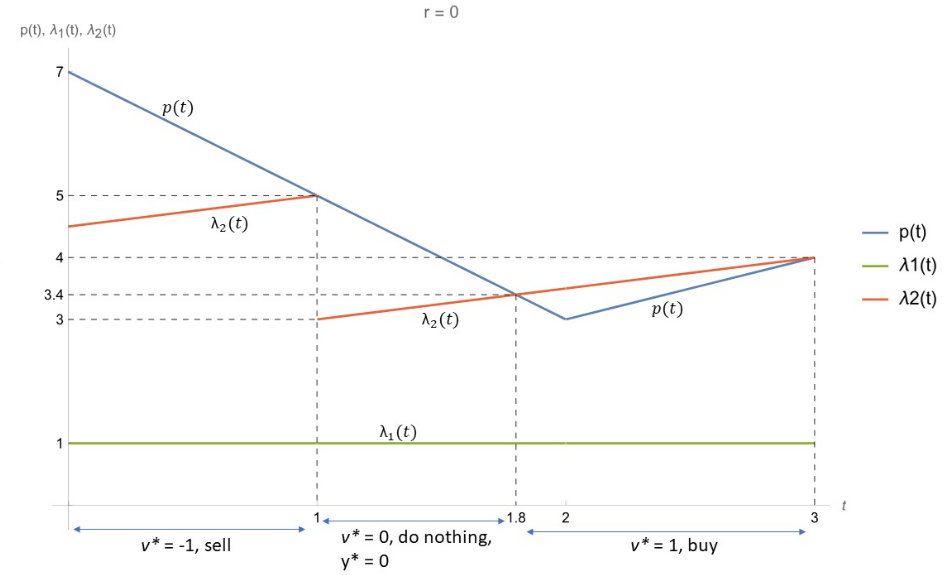

Figure 1. Optimal adjoint variable trajectory and optimal policy for the wheat trading model in Section 2

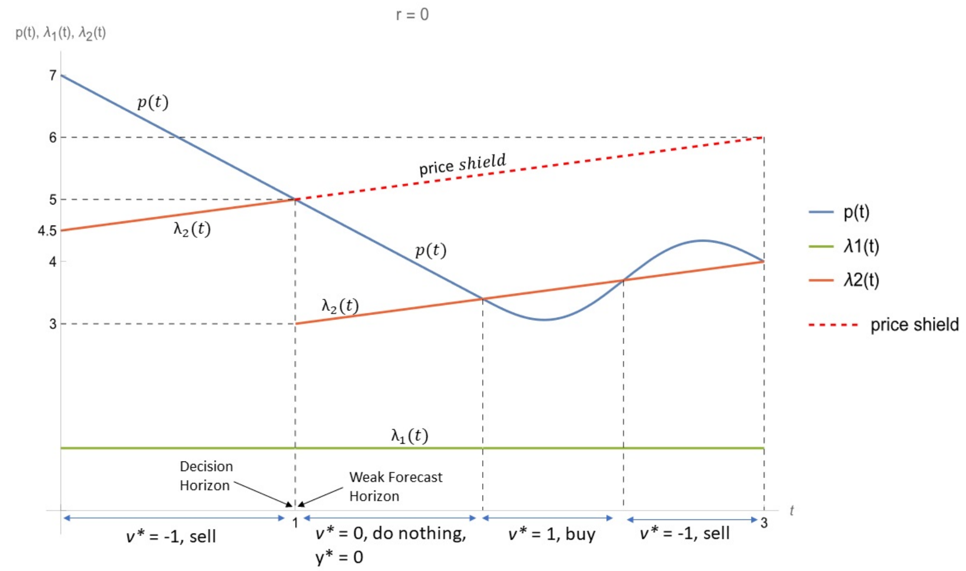

Figure 2. Decision horizon, weak forecast horizon, and optimal policy for the wheat trading model in Section 2

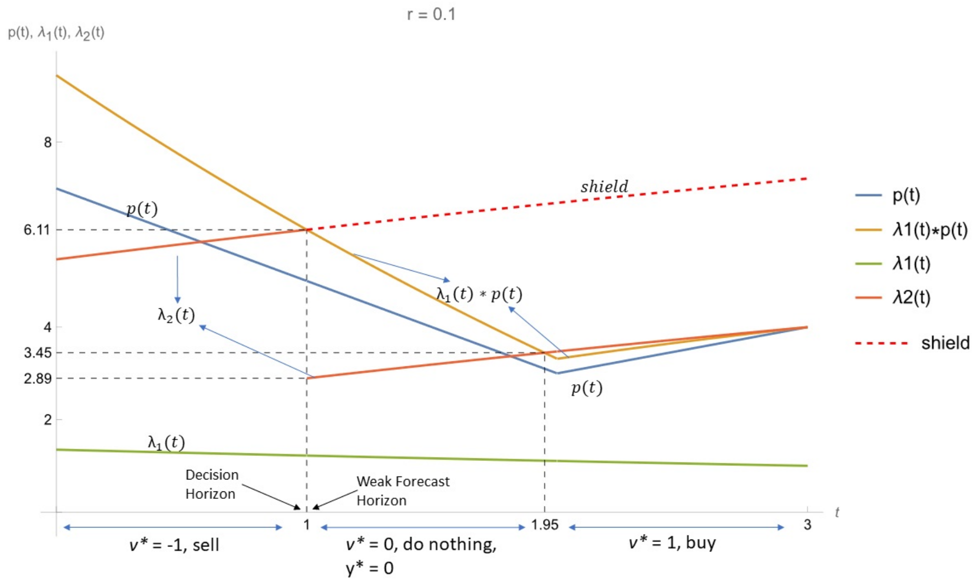

Figure 3. Optimal adjoint variable trajectory, optimal policy, and horizons for Example 2.1 with $r = 0.1$

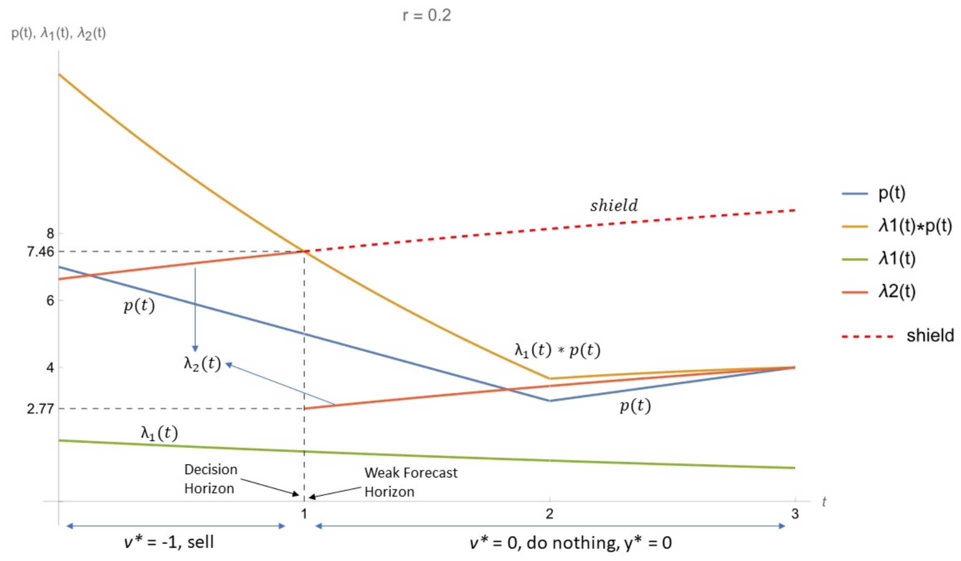

Figure 4. Optimal adjoint variable trajectory, optimal policy, and horizons for Example 2.2 with $r = 0.2$

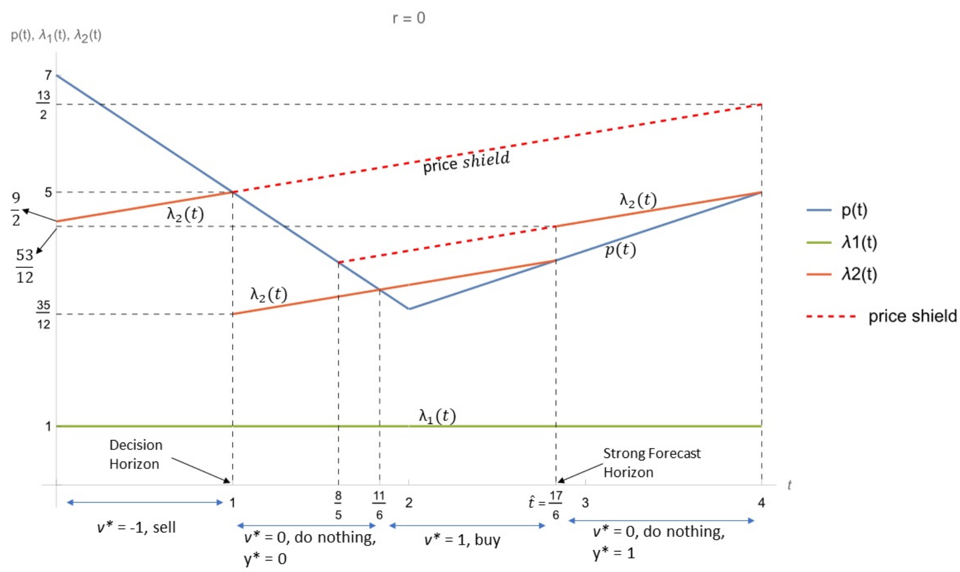

Figure 5. Optimal adjoint variable trajectory, optimal policy, and horizons under a warehouse constraint

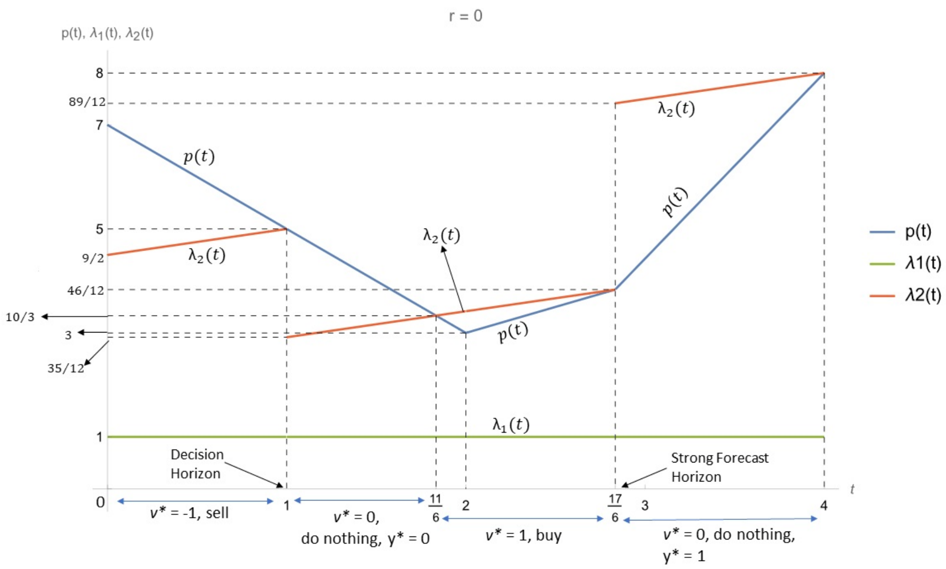

Figure 6. Optimal adjoint variable trajectory, optimal policy, and horizons for Example 3.1

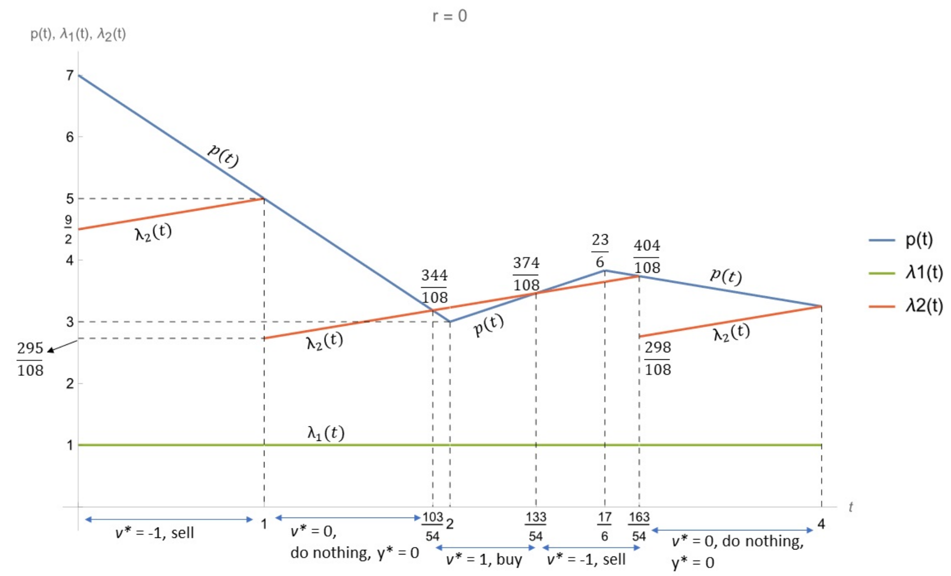

Figure 7. Optimal adjoint variable trajectory, optimal policy, and horizons for Example 3.2

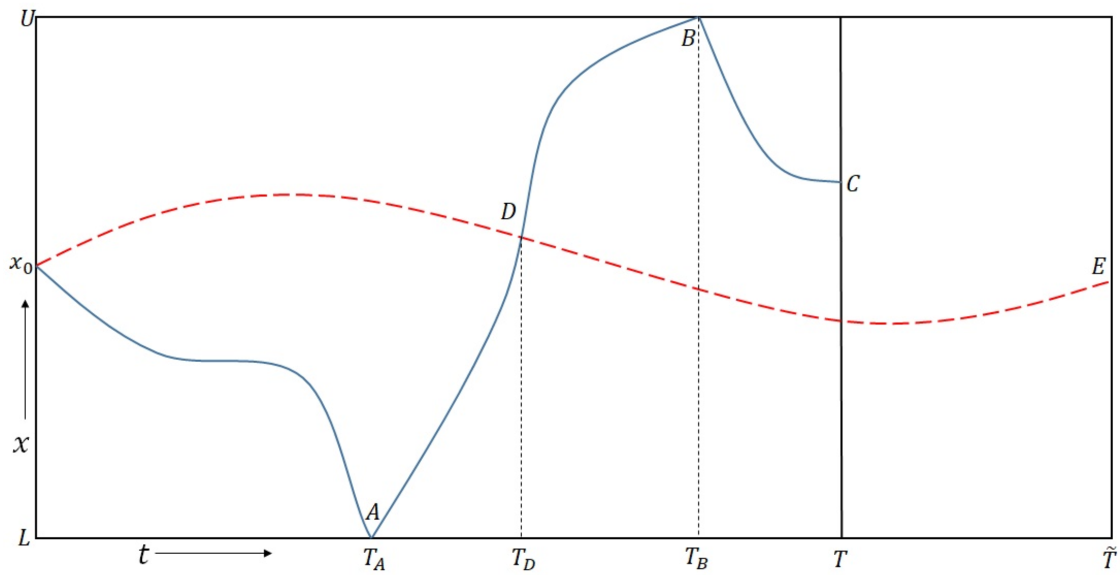

Figure 8. Graphical proof for existence of horizons in problem with warehousing constraint

Figure Captions

Figure 1 Caption: Optimal adjoint variable trajectory and optimal policy for the wheat trading model in Section 2.
Figure 1 Alt Text: Figure 1 shows the optimal policy for the wheat trading model along with the trajectory of the adjoint variable.

Figure 2 Caption: Decision horizon, weak forecast horizon, and optimal policy for the wheat trading model in Section 2.
Figure 2 Alt Text: Figure 2 shows a representative picture of optimal policy for a general price trajectory while still ensuring that the state constraint is not violated.

Figure 3 Caption: Optimal adjoint variable trajectory, optimal policy, and horizons for Example 2.1 with $r = 0.1$
Figure 3 Alt Text: Figure 3 shows optimal policy for an example of wheat trading model with interest rate for cash $r = 0.1$. It plots the optimal policy and decision and forecast horizon for this example.

Figure 4 Caption: Optimal adjoint variable trajectory, optimal policy, and horizons for Example 2.2 with $r = 0.2$
Figure 4 Alt Text: Figure 4 shows optimal policy for an example of wheat trading model with interest rate for cash $r = 0.2$. It plots the optimal policy and decision and forecast horizon for this example.

Figure 5 Caption: Optimal adjoint variable trajectory, optimal policy, and horizons under a warehouse constraint.
Figure 5 Alt Text: Figure 5 shows optimal policy for an example of wheat trading model with warehousing constraint. It plots the optimal policy and decision and forecast horizon for this example.

Figure 6 Caption: Optimal adjoint variable trajectory, optimal policy, and horizons for Example 3.1.
Figure 6 Alt Text: Figure 6 shows optimal policy and decision and forecast horizon for an example with a different price trajectory than Figure 5.

Figure 7 Caption: Optimal adjoint variable trajectory, optimal policy, and horizons for Example 3.2.
Figure 7 Alt Text: Figure 7 shows optimal policy and decision and forecast horizon for an example with a different price trajectory than Figures 5 and 6.

Figure 8 Caption: Graphical proof for existence of horizons in problem with warehousing constraint.
Figure 8 Alt Text: Figure 8 shows a simple graphical proof of existence of forecast and decision horizon for wheat trading problem with warehousing constraint.