Extendibility of Bosonic Gaussian States

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Extendibility of bosonic Gaussian states is a key issue in continuous-variable quantum information. We show that a bosonic Gaussian state is k-extendible if and only if it has a Gaussian k-extension, and we derive a simple semidefinite program, whose size scales linearly with the number of local modes, to efficiently decide k-extendibility of any given bosonic Gaussian state. When the system to be extended comprises one mode only, we provide a closed-form solution. Implications of these results for the steerability of quantum states and for the extendibility of bosonic Gaussian channels are discussed. We then derive upper bounds on the distance of a k-extendible bosonic Gaussian state to the set of all separable states, in terms of trace norm and Rényi relative entropies. These bounds, which can be seen as "Gaussian de Finetti theorems," exhibit a universal scaling in the total number of modes, independently of the mean energy of the state. Finally, we establish an upper bound on the entanglement of formation of Gaussian k-extendible states, which has no analogue in the finite-dimensional setting.

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Entanglement is the mainspring of modern quantum technologies. To tally the performance of such technologies, a comprehensive characterization and quantification of entanglement is needed. One of the defining features of entanglement is its *monogamy* [1–7], the fact that entangled states cannot be shared among arbitrarily many subsystems. Exploring the middle ground of partially shareable states or, precisely, partially *extendible* states, offers a rich and practically meaningful lookout into the virtues of entanglement as a resource.

A bipartite quantum state ρ_{AB} of systems A and B is called *k-extendible* on B if there exists a quantum state $\tilde{\rho}_{AB_1\ldots B_k}$ on A and k copies $B_1, \ldots, B_k$ of B that is permutation-invariant with respect to the systems B_i and satisfies $\mathrm{Tr}_{B_2\ldots B_k}[\tilde{\rho}_{AB_1\ldots B_k}] = \rho_{AB}$, where $B_1 \equiv B$. It is well known that a state ρ_{AB} is separable if and only if it is k-extendible for all $k \geq 2$ [3–6]. The nested sets of k-extendible states can thus be used to approximate the set of separable states, which has resulted in work on quantum de Finetti theorems [8–14] and other studies of entanglement [15,16]. Extendibility also arises in the contexts of security of quantum key distribution [17–19], capacities of quantum channels [20–22], Bell's inequalities [23,24], and other information-theoretic scenarios [25,26]. More broadly, the extendibility problem is a special case of the QMA-complete quantum marginal problem [27–33], which has been referred to in quantum chemistry as the N-representability problem [34–36]. For fixed k, the extendibility problem can be formulated as a semidefinite program (SDP), making it efficiently solvable for low-dimensional systems A and B [5,6]. Analytic conditions for k-extendibility in finite-dimensional systems are known only for particular values of k and/or for special classes of states [24,37–40].

In the infinite-dimensional case, of central relevance for quantum-optical realizations, the theory of Gaussian entanglement has been explored thoroughly in the past two decades [41–43]. However, more general extendibility questions have been approached sparingly. The only work that we are aware of is Ref. [44], where it was shown that a Gaussian state is separable if and only if it is Gaussian k-extendible for all k.

Here we study and characterize the full hierarchy of extendibility for quantum Gaussian states. After showing that any Gaussian state is k-extendible if and only if it is Gaussian k-extendible, we derive a simple SDP in terms of the state's covariance matrix in order to decide its k-extendibility. The size of our SDP scales linearly with the number of local modes. We also provide an analytic condition that completely characterizes the set of k-extendible states in the case of the extended system containing one mode only, generalizing the well-known positive partial transpose (PPT) criterion [45–47]. We then discuss several applications of this result, deriving the following along the way: (i) analytic conditions for k-extendibility for all single-mode Gaussian channels, (ii) a tight de Finetti-type theorem bounding the distance between any k-extendible

Gaussian state and the set of separable states, tight upper bounds on (iii) Rényi relative entropy of entanglement, and (iv) Rényi entanglement of formation for any k-extendible Gaussian state. Our results reach unexplored depths in the ocean of continuous-variable quantum information.

Gaussian states.—We recall the basic theory of quantum Gaussian states [41,42,48,49]. Let x_j and p_j $(1 \leq j \leq n)$ denote the canonical operators of a system of n harmonic oscillators (modes), arranged as a vector $r := (x_1, p_1, \ldots, x_n, p_n)^{\mathsf{T}}$. The canonical commutation relations can be compactly written as $[r, r^{\mathsf{T}}] = i\Omega$, where $\Omega := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{\oplus n}$ is the standard symplectic form. Given any (not necessarily Gaussian) n-mode state ρ, its mean or displacement vector is $s := \mathrm{Tr}[r\rho] \in \mathbb{R}^{2n}$, while its quantum covariance matrix (QCM) is the $2n \times 2n$ real symmetric matrix $V := \mathrm{Tr}[\{r - s, (r - s)^{\mathsf{T}}\}\rho]$. Gaussian states ρ^G are (limits of) thermal states of quadratic Hamiltonians and are uniquely identified by their displacement vector s and QCM V. We shall often assume $s = 0$, since the mean can be adjusted by local displacement unitaries that do not affect k-extendibility. Physically legitimate QCMs V satisfy the Robertson-Schrödinger uncertainty principle $V \geq i\Omega$, hereafter referred to as the bona fide condition [50]. Any matrix obeying this condition can be the QCM of a Gaussian state.

Extendibility of Gaussian states.—Let ρ_{AB} be a (not necessarily Gaussian) state of a bipartite system of $n = n_A + n_B$ modes. We assume that ρ_{AB} has vanishing first moments and finite second moments, so that we can construct its QCM

$$V_{AB} = \begin{pmatrix} V_A & X \\ X^{\mathsf{T}} & V_B \end{pmatrix}. \tag{1}$$

It can be shown [51] that every k-extension $\tilde{\rho}_{AB_1\cdots B_k}$ of ρ_{AB} also has (i) vanishing first moments and (ii) finite second moments, arranged in a QCM of the form

$$\tilde{V}_{AB_1\cdots B_k} = \begin{pmatrix} V_A & X & X & \ldots & X \\ X^{\mathsf{T}} & V_B & Y & \ldots & Y \\ X^{\mathsf{T}} & Y & V_B & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Y \\ X^{\mathsf{T}} & Y & \ldots & Y & V_B \end{pmatrix}, \tag{2}$$

where Y is a symmetric matrix. A similar structure had already been identified in Ref. [44]; however, there the crucial fact that Y needs to be symmetric was not observed. We are now concerned with the k-extendibility of Gaussian states. Our first result indicates that Gaussian states are in some sense a closed set under k-extensions:

Theorem 1. A Gaussian state ρ^G_{AB} is k-extendible if and only if it has a Gaussian k-extension.

Proof.—Let $\tilde{\rho}_{AB_1\cdots B_k}$ be a (not necessarily Gaussian) k-extension of ρ^G_{AB}. Consider m identical copies of it across the systems $A_\ell B_{\ell 1}, \ldots, B_{\ell k}$, where $1 \leq \ell \leq m$. For $1 \leq j \leq k$, let U_j be a passive unitary that acts on the annihilation operators $b_{\ell j}$ of the systems $B_{\ell j}$ so that $U_j^\dagger b_{1j} U_j = (b_{1j} + \cdots + b_{mj})/\sqrt{m}$. Set

$$\omega^{(m)}_{A_1 B_{11}\cdots B_{mk}} := (U_1 \otimes \cdots \otimes U_k) \times \left(\bigotimes_{\ell=1}^{m} \tilde{\rho}_{A_\ell B_{\ell 1}\cdots B_{\ell k}}\right)(U_1 \otimes \cdots \otimes U_k)^\dagger. \tag{3}$$

By the quantum central limit theorem [65,66], the reduced state $\omega^{(m)}_{A_1 B_{11}\cdots B_{1k}}$ satisfies $\lim_{m\to\infty} \| \omega^{(m)}_{A_1 B_{11}\cdots B_{1k}} - \tilde{\rho}^G_{AB_1\cdots B_k} \|_1 = 0$, where $\tilde{\rho}^G_{AB_1\cdots B_k}$ is the Gaussian state with the same first and second moments as $\tilde{\rho}_{AB_1\cdots B_k}$, and $A_1 \equiv A$, $B_{1j} \equiv B_j$ [51].

We now show that $\tilde{\rho}^G_{AB_1\cdots B_k}$ is indeed a Gaussian k-extension of ρ^G_{AB}. First, it is symmetric under the exchange of any two B systems, say $B_1 \leftrightarrow B_2$. In fact, (i) the state in Eq. (3) is invariant under the exchange $(B_{11}, \ldots, B_{m1}) \leftrightarrow (B_{12}, \ldots, B_{m2})$, (ii) consequently, the reduced state $\omega^{(m)}_{A_1 B_{11}\cdots B_{1k}}$ is invariant under the exchange $B_{11} \leftrightarrow B_{12}$, and (iii) symmetry is preserved under limits. Finally, to show that $\tilde{\rho}^G_{AB_1} = \rho^G_{AB}$ under the identification $B_1 \equiv B$, we observe that the QCM of $\tilde{\rho}^G_{AB_1\cdots B_k}$, which is the same as that of $\tilde{\rho}_{AB_1\cdots B_k}$, is as in Eq. (2). Since its upper-left 2×2 corner corresponds to the QCM of ρ^G_{AB}, we conclude that $\tilde{\rho}^G_{AB_1}$ and ρ^G_{AB} have the same first and second moments; being Gaussian, they must coincide. ∎

By virtue of Theorem 1, we can confine the search of k-extensions of Gaussian states to the same Gaussian realm. The next result shows that this reduces to an efficiently solvable SDP feasibility problem, with the size of the SDP scaling linearly in the number of modes of the B system. In the case of B being composed of one mode only, we find an analytic solution in the form of a simple necessary and sufficient condition for k-extendibility.

Theorem 2. Let ρ_{AB} be a k-extendible (not necessarily Gaussian) state of $n_A + n_B$ modes with QCM V_{AB}. Then there exists a $2n_B \times 2n_B$ quantum covariance matrix $\Delta_B \geq i\Omega_B$ such that

$$V_{AB} \geq i\Omega_A \oplus \left[\left(1 - \frac{1}{k}\right)\Delta_B + \frac{1}{k} i\Omega_B\right]. \tag{4}$$

Moreover, the above condition is necessary and sufficient for k-extendibility when $\rho_{AB} = \rho^G_{AB}$ is Gaussian. If in addition $n_B = 1$, then ρ^G_{AB} is k-extendible if and only if

$$V_{AB} \geq i\Omega_A \oplus \left[-\left(1-\frac{2}{k}\right) i\Omega_B\right]. \tag{5}$$

In the proof of Theorem 2, we employ the following characterization of positive semidefiniteness of Hermitian block matrices [67, Theorem 1.12]:

$$M = \begin{pmatrix} P & Z \\ Z^\dagger & Q \end{pmatrix} \geq 0 \Leftrightarrow P \geq 0,$$
$$M/P \coloneqq Q - Z^\dagger P^{-1} Z \geq 0, \tag{6}$$

where the matrix M/P is called the *Schur complement* of M with respect to P. For details concerning the degenerate case of noninvertible P, see the Supplemental Material [51]. Using Eq. (6), for any QCM V_{AB} as in Eq. (1), the inequality in Eq. (4) and the condition $\Delta_B \geq i\Omega_B$ can be written together as

$$i\Omega_B \leq \Delta_B \leq \frac{k}{k-1}[V_B - X^\mathsf{T}(V_A - i\Omega_A)^{-1}X] - \frac{1}{k-1} i\Omega_B. \tag{7}$$

Analogously, Eq. (5) is equivalent to

$$V_B - X^\mathsf{T}(V_A - i\Omega_A)^{-1}X \geq -\left(1-\frac{2}{k}\right) i\Omega_B. \tag{8}$$

Proof of Theorem 2.—We first establish necessity of Eq. (4) for k-extendibility of an arbitrary state ρ_{AB}. If ρ_{AB} is k-extendible, then there exists a matrix $\tilde{V}_{AB_1\cdots B_k}$ as in Eq. (2) that obeys the bona fide condition $\tilde{V}_{AB_1\cdots B_k} \geq i(\Omega_A \oplus \Omega_{B_1\cdots B_k})$. Using Eq. (6), and noting that $V_A \geq i\Omega_A$ holds because ρ_A is a valid state, we arrive at the inequality $(\tilde{V}_{AB_1\cdots B_k} - i\Omega_A)/(\tilde{V}_A - i\Omega_A) \geq i\Omega_{B_1\cdots B_k}$. Using Eq. (2), and letting $|+\rangle \coloneqq (1/\sqrt{k})\sum_{j=1}^k |j\rangle \in \mathbb{R}^k$, upon elementary manipulations this can be rephrased as

$$(\mathbb{1}_k - |+\rangle\langle+|) \otimes (V_B - Y - i\Omega_B) + |+\rangle\langle+| \otimes [V_B + (k-1)Y - kX^\mathsf{T}(V_A - i\Omega_A)^{-1}X - i\Omega_B] \geq 0.$$

Since the first factors of the above two addends are orthogonal to each other, positive semidefiniteness can be imposed separately on the second factors. Letting $\Delta_B \coloneqq V_B - Y$, we obtain Eq. (7), whose equivalence to Eq. (4) follows by applying Eq. (6). To deduce Eq. (5) from Eq. (4), simply substitute the complex conjugate bona fide condition $\Delta_B \geq -i\Omega_B$ into Eq. (4).

By Theorem 1, the condition $\tilde{V}_{AB_1\cdots B_k} \geq i(\Omega_A \oplus \Omega_{B_1\cdots B_k})$ is also sufficient to ensure k-extendibility when $\rho_{AB} = \rho_{AB}^G$ is Gaussian. By the above reduction, this condition is equivalent to that in Eq. (4).

We now prove that when $n_B = 1$, Eq. (5) implies the existence of a real Δ_B such that Eq. (7) is satisfied. By [43, Lemma 7], we know that Eq. (7) is satisfied for some real Δ_B if and only if

$$\frac{k}{k-1}[V_B - X^\mathsf{T}(V_A - i\Omega_A)^{-1}X] - \frac{1}{k-1} i\Omega_B \geq \pm i\Omega_B, \tag{9}$$

meaning that both inequalities are satisfied. Using Eq. (6), we see that the condition with the $+$ reduces to $V_{AB} \geq i\Omega_{AB}$, which is guaranteed to hold by hypothesis. That with the $-$ yields instead Eq. (8), which is in turn equivalent to Eq. (5). ∎

Although some of the above manipulations formally resemble those in Ref. [44], the two arguments are conceptually different and lead to different conclusions [51]: in fact, in Ref. [44], the question of k-extendibility of Gaussian states is explicitly mentioned as an outstanding problem.

Recall that a bipartite state is separable if and only if it is k-extendible for all k [3–6] and that any k-extendible state is also $(k-1)$-extendible. Thus, taking the limit $k \to \infty$ of Eq. (4) shows that ρ_{AB}^G is separable if and only if there exists a $2n_B \times 2n_B$ matrix $\Delta_B \geq i\Omega_B$ such that $V_{AB} \geq i\Omega_A \oplus \Delta_B$. This reproduces the analytic condition for separability of Gaussian states found in [43, Theorem 5]. In the same limit $k \to \infty$, it is also easy to verify that Eq. (5) reduces to the PPT criterion [43,45–47,68].

It turns out that the necessary condition in Eq. (5) is no longer sufficient when $n_B > 1$. This is demonstrated by the example of the $(2+2)$-mode bound entangled Gaussian state constructed in Ref. [68], which obeys Eq. (5) for all k (because it is PPT) yet it is not even 2-extendible [51].

Theorem 2 also reveals an implication of 2-extendibility for Gaussian *steerability*, i.e., Einstein-Podolsky-Rosen steerability via Gaussian measurements [69–73]. The $k = 2$ case of Eq. (5) shows that any Gaussian state that is 2-extendible on B is necessarily $B \to A$ Gaussian unsteerable, and hence useless for one-sided-device-independent quantum key distribution. When $n_B = 1$, this condition is also sufficient; i.e., 2-extendibility is equivalent to $B \to A$ Gaussian unsteerability.

Extendibility of Gaussian channels.—We now apply Theorem 2 to study k-extendibility of single-sender single-receiver Gaussian quantum channels. A quantum channel $\mathcal{N}_{A\to B}$ is called k-extendible [21,74] if there exists another quantum channel $\tilde{\mathcal{N}}_{A\to B_1\ldots B_k}$ from the sender A to k receivers $B_1, \ldots, B_k$ such that the reduced channel from the sender to any one of the receivers is the same as the original channel $\mathcal{N}_{A\to B}$.

A Gaussian channel $\mathcal{N}_{A\to B}$ with n input modes and m output modes maps Gaussian states to Gaussian states and is uniquely characterized by a real $2m \times 2n$ matrix X, a real symmetric $2m \times 2m$ matrix Y, and a real vector $\delta \in \mathbb{R}^{2m}$, such that $Y + i\Omega \geq iX\Omega X^T$ [42]. Its action can be described directly in terms of the mean vector s and QCM V of the input Gaussian state as follows: $s \mapsto Xs + \delta$, $V \mapsto XVX^\mathsf{T} + Y$. In what follows, we set $\delta = 0$ without loss of generality.

To any channel $\mathcal{N}_{A\to B}$ we can associate its Choi-Jamiołkowski state $\rho_{AB}^{\mathcal{N}}(r) \coloneqq \mathcal{N}_{A'\to B}(|\psi_r\rangle\langle\psi_r|_{AA'}^{\otimes n})$, where

for $r > 0$ the two-mode squeezed vacuum is defined as $|\psi_r\rangle \coloneqq \mathrm{sech}(r)\sum_{j=0}^{\infty}\tanh(r)^j|j,j\rangle$ [75]. It can be seen that $\mathcal{N}_{A\to B}$ is k-extendible if and only if $\rho_{AB}^{\mathcal{N}}(r)$ is k-extendible on B for some (and hence all) $r > 0$ [51]. The same conclusion follows from arguments in Refs. [76–78]. For any Gaussian channel $\mathcal{N}$, the state $\rho_{AB}^{\mathcal{N}}(r)$ is Gaussian. Hence, via Theorem 2, we deduce that a Gaussian channel is k-extendible if and only if there exists a $2m \times 2m$ real matrix Δ such that

$$i\Omega \le \Delta \le \frac{k}{k-1}(Y + iX\Omega X^{\mathsf{T}}) - \frac{1}{k-1}i\Omega. \tag{10}$$

When $m = 1$, this is equivalent to $Y + iX\Omega X^{\mathsf{T}} + (1-2/k)i\Omega \ge 0$. If also $n = 1 = m$, a simplified equivalent condition that incorporates also the complete positivity requirements is

$$\sqrt{\det Y} \ge 1 - \frac{1}{k} + \left|\det X - \frac{1}{k}\right|. \tag{11}$$

By applying Eq. (11), we find necessary and sufficient conditions for the k-extendibility of all possible single-mode Gaussian channels, which play a prominent role in modeling optical quantum communication [42,79,80]. By the results of Ref. [79], the following characterization of k-extendibility for three fundamental single-mode Gaussian channels suffices to solve the problem for *all* single-mode Gaussian channels [51]. (i) The thermal channel of transmissivity $\eta \in (0,1)$ and environment thermal photon number $N_B \ge 0$ is defined by $X = \sqrt{\eta}\mathbb{1}$ and $Y = (1-\eta)(2N_B+1)\mathbb{1}$. It is k-extendible if and only if $\eta \le (N_B + 1/k)/(N_B+1)$. For the case $N_B = 0$, corresponding to a pure-loss channel, this reduces to $\eta \le 1/k$. (ii) The amplifier channel of gain $G > 1$ and environment thermal photon number $N_B \ge 0$ is defined by $X = \sqrt{G}\mathbb{1}$ and $Y = (G-1)(2N_B+1)\mathbb{1}$. This channel is k-extendible if and only if $N_B > 0$ and $G \ge (N_B + 1 - 1/k)/N_B$. (iii) The additive noise channel with noise parameter $\xi > 0$ is defined by $X = \mathbb{1}$ and $Y = \xi\mathbb{1}$. This channel is k-extendible if and only if $\xi \ge 2(1-1/k)$.

As expected, the above conditions reduce to their entanglement-breaking counterparts from Ref. [81] for $k \to \infty$.

Distance between k-extendible and separable states.—A problem of central interest in quantum information theory is determining how close k-extendible states are to the set of separable states. In Ref. [10] [Theorem II.7'], it was found that a finite-dimensional k-extendible state is $4d^2/k$ close to the set of separable states in trace norm, where d is the dimension of the extended system. Moreover, it was also shown [10 Corollary III.9] that the error term in the approximation necessarily depends on d at least linearly. One can instead obtain a $\ln d$ dependence by resorting to different norms [82].

Can similar estimates be provided in the Gaussian case? Results in this setting have been obtained in Ref. [12] for fully symmetric systems of the form $B_1, \ldots, B_k$. Here we extend these de Finetti theorems to the case where the symmetry is relative to a fixed reference system A. We are interested in the distance of a given Gaussian state ρ_{AB}^G to the set $\mathrm{SEP}(A:B)$ of bipartite separable states on systems A and B, as measured by either (i) the trace norm, yielding the quantity $\|\rho_{AB}^G - \mathrm{SEP}(A:B)\|_1 \coloneqq \inf_{\sigma_{AB}\in\mathrm{SEP}(A:B)}\|\rho_{AB}-\sigma_{AB}\|_1$, or (ii) the quantum Petz-Rényi relative entropy $D_\alpha(\rho\|\sigma) \coloneqq [1/(\alpha-1)]\ln \mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$ for $\alpha > 0$ [83], which leads to the measure $E_{R,\alpha}(\rho_{AB}^G) \coloneqq \inf_{\sigma_{AB}\in\mathrm{SEP}(A:B)} D_\alpha(\rho_{AB}\|\sigma_{AB})$. For $\alpha = 1$ the Petz-Rényi relative entropy reduces to the Umegaki relative entropy [84], and we obtain the standard relative entropy of entanglement [85]. We find the following:

Theorem 3. Let ρ_{AB}^G be a k-extendible Gaussian state of $n \coloneqq n_A + n_B$ modes. Then,

$$\|\rho_{AB}^G - \mathrm{SEP}(A:B)\|_1 \le \frac{2n}{k}, \tag{12}$$

$$E_{R,\alpha}(\rho_{AB}^G) \le n\ln\left(1 + \frac{\eta_{k,\alpha}}{k-1}\right) \le \frac{n\eta_{k,\alpha}}{k-1}, \tag{13}$$

where $\eta_{k,\alpha} = 1$ if $\alpha \le k+1$, and $\eta_{k,\alpha} = 2$ otherwise.

The proof is in the Supplemental Material [51]. Remarkably, the upper bounds in Eqs. (12)–(13) hold universally for all Gaussian states, independently, e.g., of their mean photon number. This is in analogy with the main results of Ref. [12], and constitutes a *quantitative* improvement over the finite-dimensional case, where—as we mentioned before—the bound has to depend on the underlying dimension. Furthermore, for two-mode states, the bounds in Eqs. (12)–(13) can be shown to be tight up to a constant for all k and all $\alpha \ge 1$. Namely, for all $k \ge 2$ there exists a k-extendible two-mode Gaussian state ρ_{AB}^G such that $\|\rho_{AB}^G - \mathrm{SEP}(A:B)\|_1 \ge 1/(2k-1)$ and $E_{R,1}(\rho_{AB}^G) \ge E_D(\rho_{AB}^G) \ge \ln[k/(k-1)] - o(1)$ as $r \to \infty$, where E_D denotes the distillable entanglement [51].

Entanglement of formation of Gaussian k-extendible states.—We now show that one can also obtain an upper bound on the entanglement of formation of Gaussian k-extendible states. This is a *qualitative* improvement over the finite-dimensional case, as a result of this kind has no analogue in that setting. We employ the recently developed theory of Rényi-2 Gaussian correlation quantifiers [70,73,86,87], and especially the monogamy of the Gaussian Rényi-2 version of the entanglement of formation [73], which stems in turn from the equality between this measure and the Gaussian Rényi-2 squashed entanglement [87].

For a bipartite state ρ_{AB} and for some $\alpha \ge 1$, the *Rényi-α entanglement of formation* $E_{F,\alpha}(\rho_{AB})$ is defined as the infimum of $\sum_i p_i S_\alpha(\psi_A^{(i)})$ over all pure-state

decompositions $\sum_i p_i \psi_{AB}^{(i)} = \rho_{AB}$ of ρ_{AB} [88]. Here, $S_\alpha(\sigma) \coloneqq [1/(1-\alpha)] \ln \mathrm{Tr}[\sigma^\alpha]$ is the Rényi-α entropy.

For a Gaussian state ρ_{AB}^G with QCM V_{AB}, we can derive an upper bound on $E_{F,\alpha}(\rho_{AB}^G)$ by restricting the decompositions to include pure Gaussian states only. This leads to the *Gaussian Rényi-α entanglement of formation*, given by [87,89]

$$E_{F,\alpha}^G(\rho_{AB}^G) = \inf\{S_\alpha(\gamma_A)\colon \gamma_{AB} \text{ pure QCM and } \gamma_{AB} \le V_{AB}\}, \tag{14}$$

where we denote by $S_\alpha(W)$ the Rényi-α entropy of a Gaussian state with QCM W, and "pure" QCMs are those that correspond to pure Gaussian states. While the typical choice $\alpha = 1$ yields the standard entanglement of formation, Rényi-2 quantifiers arise naturally in the Gaussian setting, as they reproduce Shannon entropies of measurement outcomes [86,87]. For $\alpha = 2$, Eq. (14) becomes

$$E_{F,2}^G(\rho_{AB}) = \min\{M(\gamma_A)\colon \gamma_{AB} \text{ pure QCM and } \gamma_{AB} \le V_{AB}\}, \tag{15}$$

where for a positive definite matrix V we set $M(V) \coloneqq S_2(V) = \frac{1}{2} \ln \det V$. We then find the following:

Theorem 4. The Rényi-2 Gaussian entanglement of formation of a k-extendible Gaussian state ρ_{AB}^G of $n_A + n_B$ modes with QCM V_{AB} is bounded from above as $E_{F,2}^G(\rho_{AB}^G) \le M(V_A)/k$. Consequently, the standard entanglement of formation of ρ_{AB}^G satisfies $E_{F,1}(\rho_{AB}^G) \le E_{F,1}^G(\rho_{AB}^G) \le n_A \varphi(M(V_A)/(n_A k))$, where $\varphi(x) \coloneqq [(\mathrm{e}^x+1)/2] \ln[(\mathrm{e}^x+1)/2] - [(\mathrm{e}^x-1)/2] \ln[(\mathrm{e}^x-1)/2]$.

Observe that the function M plays the role of some "effective dimension" in the bounds above. It is related to other quantities conventionally thought of as infinite-dimensional substitutes for the dimension, such as the mean photon number, defined for a state ρ of n modes as $\langle N \rangle = \langle N \rangle_\rho \coloneqq \mathrm{Tr}[(\sum_j a_j^\dagger a_j)\rho]$. When ρ is zero-mean Gaussian and has QCM V, one has $\langle N \rangle = \frac{1}{4}(\mathrm{Tr} V - 2n)$. By applying the arithmetic-geometric mean inequality, one can show that $M(V) \le n \ln[(2\langle N \rangle/n) + 1]$, which can be further relaxed to $M(V) \le 2\langle N \rangle$.

Summary and outlook.—We accomplished a comprehensive analysis of the k-extendibility of Gaussian quantum states. We determined that a Gaussian state is k-extendible if and only if it is Gaussian k-extendible, which allowed us to derive a simple semidefinite program that solves the problem completely in a computationally efficient way. When the extended system contains one mode only, we fully characterized the set of k-extendible Gaussian states by a simple analytic condition reminiscent of the PPT criterion. We demonstrated further applications to Gaussian state steerability, k-extendiblity of Gaussian channels, bounding the distance between k-extendible and separable states, and the Rényi entanglement of formation for Gaussian states. Our results also yield necessary criteria for k-extendibility of non-Gaussian states based on second moments. This Letter sheds novel light onto the fine structure of entanglement and its uses in continuous-variable systems.

It remains an intriguing open problem to find an analytic condition for k-extendibility of arbitrary Gaussian states. Another topic for future work is to explore applications of Theorem 2 to the nonasymptotic capacities of Gaussian channels, in light of recent work [21,22] exploiting k-extendibility to bound the performance of quantum processors.

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