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The geometry of certain cocycles associated to derivatives of L-functions

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Abstract. Motivated by Manin's expression relating modular symbols to critical values of L-functions, Goldfeld and the author have constructed group cocycles which are related in an analogous way to values of *derivatives* of L-functions. In this note, we propose a geometric interpretation of these cocycles that is based on the cohomology of the non-compactified modular curve.

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1 Introduction

Notation and terminology

$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}; a, b, c, d \in \mathbb{Z}; ad - Ncd = 1 \right\}$
$\mathfrak{H} = \{x + iy; y > 0\}$: the upper half-plane
$\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{i\infty\}$
$\gamma z = \frac{az+b}{cz+d}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $z \in \mathfrak{H}^*$
$j(\gamma, z) = cz + d$, for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$
$Y_0(N) = \Gamma_0(N) \backslash \mathfrak{H}$: the non-compactified modular curve
$X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}^*$: the (compactified) modular curve
$p : \mathfrak{H}^* \to X_0(N)$: the natural projection map. (Same notation for the projection $\mathfrak{H} \to Y_0(N)$ when there is no danger of confusion.)
$P_m = \{\text{polynomials of degree} \le m\}$
$(F|_m\gamma)(w) = j(\gamma, w)^{-m} F(\gamma w)$ for $F : \mathfrak{H} \to \mathbb{C}$ and $\gamma \in \Gamma_0(N)$
$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$, the Dedekind η-function
$C^i(\Gamma_0(N), M) = \{g : \underbrace{\Gamma_0(N) \times \cdots \Gamma_0(N)}_{i\text{-times}} \to M\}$ where M is a $\Gamma_0(N)$-module

Δ_n: standard n-simplex contained in $\mathbb{R}^{n+1}$
$S_n(X) =$ the group generated by singular n-simplices of a space X, i.e. continuous $T : \Delta_n \to H$
$S(X) := \bigoplus_n S_n(X)$: the complex of singular chains
$(a_0, \ldots, a_n)$: a singular n-simplex with vertices $a_0, \ldots, a_n$

Let f be a cusp form of weight 2 for $\Gamma_0(N)$ which is an eigenfunction for the Hecke operators and let $L_f(s)$ be its L-function. In [M], Manin gave a formula for the critical value $L_f(1)$ in terms of modular symbols

$$\langle f, \gamma \rangle := \int_{i\infty}^{\gamma i\infty} f(z)\, dz, \quad \gamma \in \Gamma_0(N).$$

That formula, among other things, agrees with the form of $L_f(1)$ anticipated by the conjecture of Birch and Swinnerton-Dyer. A key fact is that the modular symbols can be thought of as integrals over $X_0(N)$. This is almost equivalent to the fact that the map $\gamma \to \langle f, \gamma \rangle$ ($\gamma \in \Gamma_0(N)$) is a 1-cocycle with coefficients in $\mathbb{C}$ acted upon by $\Gamma_0(N)$ via $|_0$, i.e. trivially.

More recently, Goldfeld proved a similar formula for $L_f'(1)$ in the case that f is a newform of weight 2 for $\Gamma_0(N)$ and $L_f(1) = 0$. To each $\gamma \in \Gamma_0(N)$ he associates the function $v_f(\gamma) : \mathfrak{H}^* \to \mathbb{C}$ defined by

$$v_f(\gamma)(z) = \int_z^{\gamma z} f(w) u(w)\, dw \quad \text{for all } z \in \mathfrak{H}^* \tag{1}$$

where $u(z) := \log(\eta(z)) + \log(\eta(Nz))$. In [G] it is shown that, if p is a prime not dividing N and $a(p)$ the p-th Fourier coefficient of f, then we can find $\gamma_j \in \Gamma_0(N)$ such that

$$(p + 1 - a(p)) L_f'(1) = 4\pi i \sum_{j=1}^{p-1} v_f(\gamma_j)(0) + A \tag{2}$$

where A is an explicit linear combination of the periods $\int_0^{j/p} f(z)\, dz$ ($j = 0, \ldots, p-1$). In contrast to Manin's formula, the integrals $v_f(\gamma_j)(0)$ on the right-hand side of (2) cannot be automatically considered as closed integrals on the modular curve and this difference reflects the difficulty of extending results about the algebraic structure of values of L-functions to their derivatives. This is one of the motivating problems of this project.

Our approach relies on the observation that, although the integrals in Goldfeld's formula are not defined over the modular curve, they can still be considered as 1-cocycles with coefficients in a $\Gamma_0(N)$-module of functions on $\mathfrak{H}^*$ (cf. Section 2). However, the coefficient module in [G] is different from the one in Manin's work. For this reason, in [D] we established a correspondence between derivatives of L-functions and cocycles in terms of a natural generalization of the representation in Manin's theorem. Specifically, if f is a cusp form of general weight k and level $N = 1$ we define the map $\sigma_f : SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \to P_{k-2}$ by

$$\sigma_f(\gamma_1, \gamma_2) = \int_{i\infty}^{\gamma_1 i\infty} f(z)(z - \gamma_1 X)^{k-2} j(\gamma_1, X)^{k-2} (u(\gamma_2 z) - u(z))\, dz, \quad \gamma_1, \gamma_2 \in SL_2(\mathbb{Z}).$$

We show that σ_f is a 2-cocycle with coefficients in P_{k-2} acted upon by $SL_2(\mathbb{Z})$ via $|_{2-k}$. We also show that it is connected with derivatives of L-functions through the identity

$$(3)\qquad \sigma_f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$
$$= -\sum_{j=0}^{k-2} \frac{(k-2)!}{(k-2-j)!}(i/2\pi)^{j+1} X^j (L_f'(j+1) + \lambda_j L_f(j+1))$$

where $\lambda_j := \Gamma'(j+1)/\Gamma(j+1) - \log(2\pi)$.

Since $SL_2(\mathbb{Z})$ has virtual cohomological dimension 1 over $\mathbb{C}$-modules, σ_f is a coboundary and this prevents an immediate geometric interpretation of it. In [D] we dealt with this difficulty by reducing σ_f to a non-vanishing compactly supported 1-cocycle in $N = C^1(SL_2(\mathbb{Z}), P_{k-2})/C^0(SL_2(\mathbb{Z}), P_{k-2})$ with the action on N induced by $|_{2-k}$. (Details are given in Section 4.) In principle then a geometric interpretation of the cocycle associated to L_f' is provided by the isomorphism between the compact 1-cohomology group $H_c^1(SL_2(\mathbb{Z}), N)$ and the compact 1-cohomology group with coefficients in the vector bundle $\mathfrak{N}$ associated to N (see Sections 3 and 4). In fact, we could go one step further to use de Rham's theorem to associate integrals on $Y_0(1)$ to σ_f and thus to values of derivatives of $L_f'(s)$ which, as mentioned above, was one of our goals.

However, in order for this correspondence to be useful for further applications it is important that it be described in a way as explicit possible, a task which is not obvious especially because of the infinite-dimensionality of $\mathfrak{N}$. In this note we show that the isomorphisms defining the correspondence can be described in a surprisingly concrete way that highlights the precise relation between the structure of σ_f and the geometry of the base space.

We begin in Section 2 with the cocycles v_f associated to forms of weight 2 by (1). Even though the representation at work does not generalize the one in Manin's formula, several of the technical difficulties of [D] can be avoided resulting in a clearer geometric description. It does not seem possible to work in the same way in higher weights.

We define a $\Gamma_0(N)$-module M_f with respect to which v_f is a non-trivial 1-cocycle and we construct the associated vector bundle $\mathfrak{M}_f$. The geometric interpretation of this cohomological setting can be summarized by

Theorem 1. *Let $\Gamma_0(N)$ act freely on $\mathfrak{H}$ and let $\mathfrak{F}$ be a fundamental domain of $\Gamma_0(N)$ in $\mathfrak{H}$. There is an isomorphism from $H^1(\Gamma_0(N), M_f)$ to $H^1(Y_0(N), \mathfrak{M}_f)$ such that a representative of the image of v_f in $H^1(Y_0(N), \mathfrak{M}_f)$ sends a simplex T in $Y_0(N)$ to the function*

$$z \to \int_z^{\gamma^{-1}z} f(w)u(w)\,dw$$

where $(z_1, \gamma z_2)$, $(z_1, z_2 \in \mathfrak{F})$ is a lift of T in $\mathfrak{H}$ under p.

In Section 3 we define and analyze a simplicial complex K on a suitable subspace H of $\mathfrak{H}$ needed in order to define the compactly supported cohomology associated to σ_f. We then successively describe the isomorphisms between group cohomology, the cohomology of K, the singular cohomology of H and the compactly supported cohomology on a modular curve. It should be stressed that the emphasis is not on the existence of the isomorphisms between the cohomology groups, which, in most cases, is standard (cf. [H1]), but rather on the choice of explicit isomorphisms that are appropriate in our setting. For this reason, we have opted for old-fashioned presentations of the maps involved, adapted for instance, from the original papers of Eilenberg and we have tried to keep to details not directly connected with our purposes to a minimum.

The maps defined in Section 3 are then used in Section 4 to describe the image of the 1-cocycle induced by σ_f and the values of derivatives of a L-function. It is possible to formulate an analogue of Theorem 1 in this case, but because of the technicalities involved, the statement would be too lengthy to state precisely. However, the basic structure of the construction is similar to the one of case of weight 2.

2 The case of weight 2

In this section we provide a geometric interpretation of the cocycle v_f associated to the derivative of the L-function of a weight 2 cusp form f.

For simplicity we work with groups without elliptic elements (e.g. if $36|N$). We first show that the map v_f defined by (1) can be considered as a non-trivial 1-cocycle.

Fix a newform $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ of weight 2 for $\Gamma_0(N)$. For each $\gamma \in \Gamma_0(N)$, we define a function $v_f(\gamma) : \mathfrak{H}^* \to \mathbb{C}$ with

$$v_f(\gamma)(z) = \int_z^{\gamma z} f(w)u(w)\,dw \quad \text{for all } z \in \mathfrak{H}^*$$

where $u(z) := \log(\eta(z)) + \log(\eta(Nz))$. In [G], it is proved that v_f is a 1-cocycle with respect to the action $|_0$ of $\Gamma_0(N)$ on the vector space of functions from $\mathfrak{H}^*$ to $\mathbb{C}$ and a connection with derivatives of L-functions is given by (2).

We consider the set M_f of functions from $\mathfrak{H}^*$ to $\mathbb{C}$ which can be expressed as linear combinations of functions (in z) of the form $\int_z^{\gamma_i z} f(w)u(w)\,dw$, ($\gamma_i \in \Gamma_0(N)$). It is clear that M_f defines a $\Gamma_0(N)$-module with respect to the action $|_0$ of $\Gamma_0(N)$. In fact, it is the smallest $\Gamma_0(N)$-module containing all $v_f(\gamma)$ with $\gamma \in \Gamma_0(N)$ and a fixed f.

Lemma 2. *The cohomology class of v_f in $H^1(\Gamma_0(N), M_f)$ does not vanish.*

Proof. Suppose that there exists $F \in M_f$ such that $v_f(\gamma) = F|_0(\gamma - 1)$. Let $\gamma_1, \dots, \gamma_n \in \Gamma_0(N)$ and $c_1, \dots, c_n \in \mathbb{C}$ be such that $F(z) = \sum_{i=1}^n c_i \int_z^{\gamma_i z} f(w)u(w)\,dw$, for $z \in \mathfrak{H}^*$. Then,

$$\int_z^{\gamma z} f(w)u(w)\,dw = \sum_{i=1}^n c_i\Big(\int_{\gamma z}^{\gamma_i\gamma z} f(w)u(w)\,dw - \int_z^{\gamma_i z} f(w)u(w)\,dw\Big)$$

and the last sum (as proved in [G], Section 4) equals

$$\sum_{i=1}^{n} c_i \int_z^{\gamma z} f(w)(u(\gamma_i w) - u(w))\, dw.$$

Hence,

$$\int_z^{\gamma z} f(w)[u(w) - \sum_{i=1}^{n} c_i(u(\gamma_i w) - u(w))]\, dw = 0 \quad \text{for all } \gamma \in \Gamma_0(N),\ z \in \mathfrak{H}^*. \tag{4}$$

In particular, upon differentation we deduce that the integrand is invariant under the action $|_2$ of $\Gamma_0(N)$ and therefore (in combination with the growth conditions at the cusps for f and u) the integrand is a cusp form of weight 2. According to the Eichler-Shimura isomorphism and the transformation formula for u, (4) then implies that

$$u(w) = \sum_{i=1}^{n} c_i(u(\gamma_i w) - u(w)).$$

On the other hand, u satisfies the transformation formula

$$u(\gamma z) = u(z) + \log(j(\gamma, z)) + \kappa_\gamma \quad \text{for all } \gamma \in SL_2(\mathbb{Z}),\ z \in \mathfrak{H}$$

where κ_γ is a constant depending only on γ. Therefore

$$u(z) = \sum_{i=1}^{n} c_i(\log(j(\gamma_i, w)) + \kappa_{\gamma_i})$$

and this gives the desired contradiction. □

For a geometric translation of this cocycle we first define an isomorphism between $H^1(\Gamma_0(N), M_f)$ and the equivariant 1-cohomology group of $\mathfrak{H}$ with coefficients in M_f. We then describe the vector bundle associated to M_f in an explicit way and describe the isomorphism between the equivariant cohomology and the cohomology of $Y_0(N)$ with coefficients in this vector bundle.

We define the equivariant 1-cohomology group of $\mathfrak{H}$ with coefficients in M_f as the first derived group of $C^*(\mathfrak{H}, M_f) = \mathrm{Hom}_{\Gamma_0(N)}(S(\mathfrak{H}), M_f)$ with the coboundary operator induced by the chain operators on $S(\mathfrak{H})$. We denote it by $H^1_{\Gamma_0(N)}(\mathfrak{H}, M_f)$.

Fix a fundamental domain $\mathfrak{F}$ of $\Gamma_0(N)$ in $\mathfrak{H}$. Let $\tau \in C^1(\Gamma_0(N), M_f)$. We define a map $e(\tau) : S_1(\mathfrak{H}) \to M_f$ as follows. Suppose that $T \in S_1(\mathfrak{H})$ has starting point $\gamma_1 z_1$ and ending point $\gamma_2 z_2$ for some $\gamma_1, \gamma_2 \in \Gamma_0(N)$ and $z_1, z_2 \in \mathfrak{F}$. Then we set

$$e(\tau)(T) = \tau(\gamma_2^{-1}\gamma_1)|_0\gamma_1^{-1} \tag{5}$$

and extend by linearity. Since $\Gamma_0(N)$ acts freely on $\mathfrak{H}$, this is a well-defined map.

Lemma 3. *Suppose that $\Gamma_0(N)$ acts freely on $\mathfrak{H}$. The map $e : C^1(\Gamma_0(N), M_f) \to C^1(\mathfrak{H}, M_f)$ induces an isomorphism $e^* : H^1(\Gamma_0(N), M_f) \to H^1_{\Gamma_0(N)}(\mathfrak{H}, M_f)$.*

Proof. See [E3]. □

We next define a vector bundle on $Y_0(N)$ setting first of all $\mathfrak{M}_f := \Gamma_0(N)\backslash(\mathfrak{H} \times M_f)$ where $\Gamma_0(N)$ acts on $\mathfrak{H} \times M_f$ by $\gamma.(z, m) = (\gamma z, m|_0\gamma^{-1})$. We view M_f as a discrete topological space and we then endow $\mathfrak{H} \times M_f$ and $\mathfrak{M}_f$ with the product and quotient topology respectively. Then the projection

$$\pi : \Gamma_0(N)\backslash(\mathfrak{H} \times M_f) \to \Gamma_0(N)\backslash\mathfrak{H}$$

gives rise to a vector bundle with fibers $\pi^{-1}(p(z))$ isomorphic to M_f for every $z \in \mathfrak{H}$.

$$\begin{array}{ccc} & & \mathfrak{H} \\ & & \Big\downarrow p \\ \Gamma_0(N)\backslash(\mathfrak{H} \times M_f) & \xrightarrow{\ \pi\ } & \Gamma_0(N)\backslash\mathfrak{H} = Y_0(N) \end{array}$$

We will often identify the fiber $\pi^{-1}(p(z))$, $(z \in \mathfrak{H})$ with M_f via (the restriction of) the trivializing map f defined by:

$$f(\Gamma_0(N)(z, m)) = m_1, \tag{6}$$

where $\Gamma_0(N)(z, m) = \Gamma_0(N)(z_1, m_1)$ for a $z_1 \in \mathfrak{F} \cap \Gamma_0(N)z$.

Further, to each homotopy class of paths between two points of $Y_0(N)$ we associate a vector space isomorphism between the fibers of the points which is compatible with path multiplication ('local coefficient system'). Let $\alpha_{p(z_2),p(z_1)}$ be a path from $p(z_1)$ to $p(z_2)$, with z_1 and z_2 chosen in $\mathfrak{F}$. Since $\mathfrak{H}$ is the universal cover of $Y_0(N)$ under p, there is a (unique) path with initial point z_1 whose image is $\alpha_{p(z_2),p(z_1)}$. If the terminal point of the lifted path is $\gamma_2 z_2$ for some (unique) $\gamma_2 \in \Gamma_0(N)$, we define the map

$$A(\alpha_{p(z_2),p(z_1)}) : \pi^{-1}(p(z_1)) \to \pi^{-1}(p(z_2))$$

setting:

$$\begin{aligned} A(\alpha_{p(z_2),p(z_1)})&(\Gamma_0(N)(z, m)) \\ &= \Gamma_0(N)(z_2, m|_0\gamma_1^{-1}\gamma_2) \quad \text{for } m \in M_f \text{ and } z = \gamma_1 z_1. \end{aligned} \tag{7}$$

If $\alpha'_{p(z_2),p(z_1)}$ is a path homotopic to $\alpha_{p(z_2),p(z_1)}$ then the lifted path with initial point z_1 will have terminal point $\gamma'_2 z_2 = \gamma_2 z_2$ and, therefore, $\gamma'_2 = \gamma_2$. So, A depends only on the homotopy class of the path. One easily checks that $A(\alpha_{p(z_2),p(z_1)})$ is 1-1, onto, linear and that $A(\alpha_{p(z_3),p(z_2)})A(\alpha_{p(z_2),p(z_1)}) = A(\alpha_{p(z_3),p(z_1)})$.

In view of the identification (6), $A(\alpha_{p(z_2),p(z_1)})$ can be simply thought of as an endomorphism of M_f as follows: If the lift of $\alpha_{p(z_2),p(z_1)}$ in $\mathfrak{H}$ with initial point z_1 has terminal point $\gamma_2 z_2$, then we can write

$$(7') \qquad A(\alpha_{p(z_2),p(z_1)})(m) = m|_0 \gamma_2.$$

We can now define the cohomology group $H^1(Y_0(N), \mathfrak{M}_f)$. We let x_T denote the first vertex of the singular simplex T and we let $C^i(Y_0(N), \mathfrak{M}_f)$ denote the singular i-th cochain group of $Y_0(N)$ with coefficients in $\mathfrak{M}_f$. We think of an element in this group as a map sending each i-simplex T in $Y_0(N)$ to an element of $\pi^{-1}(x_T)$. Using the identification (6) we can then define the coboundary operator on C^i by the formula

$$(8) \qquad (df)(T) = A(\alpha_{p_0,p_1})f(T^{(0)}) + \sum_{j=1}^{i+1} (-1)^j f(T^{(j)})$$

for $f \in C^i(Y_0(N), \mathfrak{M}_f)$ and a $(i+1)$-simplex T. Here α_{p_0,p_1} denotes the path corresponding to the first edge of the simplex T and $T^{(j)}$ is the j-th face of T. The cohomology group $H^1(Y_0(N), \mathfrak{M}_f)$ is the first derived group of $C^*(Y_0(N), \mathfrak{M}_f)$ with the coboundary operator (8).

To relate this cohomology to equivariant cohomology we first fix a base point $x_0 \in \mathfrak{F}$. If $p(T)$ denotes the image of a simplex T under p we define a map $\psi : C^1(\mathfrak{H}, M_f) \to C^1(Y_0(N), \mathfrak{M}_f)$ with the formula

$$(9) \qquad \psi(g)(p(T)) = A(\alpha_{p(x_0),p(x_T)})^{-1} g(T),$$

for all $g \in C^1(\mathfrak{H}, M_f)$ and 1-simplices $p(T)$ of $Y_0(N)$, where $\alpha_{p(x_0),p(x_T)}$ is the image of a path from x_T to x_0 under p and M_f is identified with $\pi^{-1}(p(x_0))$ via the trivializing map. The map $\psi(g)$ is well-defined because of the $\Gamma_0(N)$-invariance of g. In [E3] it is proved that ψ induces an isomorphism ψ^* between the corresponding cohomology groups.

Using the formulas (5) and (9) we can determine explicitly a representative of the cohomology class $\psi^*(e^*(v_f))$, where v_f is the cocycle (1) we have associated to derivatives of L-functions. Specifically, let T be a 1-simplex in $Y_0(N)$. Suppose that its initial point is $p(z_1)$, with z_1 in the fundamental domain $\mathfrak{F}$. If the lift $\tilde{T}$ of T in $\mathfrak{H}$ with initial point z_1 has terminal point γz_2, for some $z_2 \in \mathfrak{F}$, then (5) and (9) imply

$$\psi(e(v_f))(T) = A(\alpha_{p(x_0),p(z_1)})^{-1}(e(v_f)(\tilde{T})) = e(v_f)(\tilde{T}) = v_f(\gamma^{-1}).$$

From this we deduce Theorem 1.

Since, as we mentioned in the introduction, our emphasis is on the explicit form of the maps we conclude this section by listing the formulas for the main isomorphisms we established:

- $e^* : H^1(\Gamma_0(N), M_f) \to H^1_{\Gamma_0(N)}(\mathfrak{H}, M_f)$ induced by the map $e : C^1(\Gamma_0(N), M_f) \to C^1(\mathfrak{H}, M_f)$ which is defined by

$$e(\tau)(T) = \tau(\gamma_2^{-1}\gamma_1)|_0\gamma_1^{-1}$$

for a 1-simplex T of $\mathfrak{H}$ with starting point $\gamma_1 z_1$ and ending point $\gamma_2 z_2$ for some $\gamma_1, \gamma_2 \in \Gamma_0(N)$ and $z_1, z_2 \in \mathfrak{F}$.

- Let $\alpha_{p(z_2),p(z_1)}$ be a path from $p(z_1)$ to $p(z_2)$, $(z_1, z_2 \in \mathfrak{F})$.

$$A(\alpha_{p(z_2),p(z_1)}) : M_f \to M_f$$

is defined by

$$A(\alpha_{p(z_2),p(z_1)})(m) = m|_0\gamma_2$$

where the lift of $\alpha_{p(z_2),p(z_1)}$ in $\mathfrak{H}$ with initial point z_1 has terminal point $\gamma_2 z_2$.

- $\psi^* : H^1_{\Gamma_0(N)}(\mathfrak{H}, M_f) \to H^1(Y_0(N), \mathfrak{M}_f)$ induced by the map $\psi : C^1(\mathfrak{H}, M_f) \to C^1(Y_0(N), \mathfrak{M}_f)$ defined by

$$\psi(g)(p(T)) = A(\alpha_{p(x_0),p(x_T)})^{-1} g(T)$$

for all $g \in C^1(\mathfrak{H}, M_f)$ and 1-simplices $p(T)$ of $Y_0(N)$, where $\alpha_{p(x_0),p(x_T)}$ is the image of a path from x_T to x_0 under p.

3 The compact cohomology

For the geometric interpretation of the cohomological setting that we have associated to derivatives of L-functions in [D] we first construct a suitable base space on which the relevant vector bundle will be defined. The construction is based on the simplicial complex K described in [S], pg. 224. (See also, [H2], pg. 348).

Let Γ be a normal torsion-free congruence subgroup of finite index in $SL_2(\mathbb{Z})$. If S is a set of representatives of the group of cusps of Γ, we denote by π_s the generator of the stabilizer of s, for $s \in S$. We denote by Y the open Riemann surface obtained after we remove a small open disc around each point $p(s)$ for each $s \in S$, without overlaps. Let H be the inverse image of Y under p. It is possible to define a simplicial complex K on H so that:

(i) Every $\gamma \in \Gamma$ induces a simplicial map of K onto itself,

(ii) for each $s \in S$ there is a 1-chain t_s of K mapped onto the boundary of the excluded disc around $p(s)$ and

(iii) there is a fundamental domain $\mathfrak{F}$ for Γ in H whose closure consists of finitely many simplices of K.

Such a complex can be constructed by considering a fundamental polygon $\mathfrak{F}$ of Γ in H all of whose cuspidal vertices are Γ-inequivalent, then defining a finite simplicial complex (of the closure of $\mathfrak{F}$) using (ii) and then covering all of H by translating by elements of Γ. We require that $\mathfrak{F}$ itself (as opposed to its closure) contain a (unique) representative of the Γ orbit of each point of H, so it is not open.

We now consider the chain complex $\mathscr{C} = (C_i, d_i)$ over $\mathbb{Z}$ induced by K. Since H is simply connected we have an exact sequence $0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \mathbb{Z} \to 0$ where d_1, d_2 are the usual boundary maps and $d_0(\sum m_x[x]) = \sum m_x$ for $m_x \in \mathbb{Z}$ and 0-complexes $[x]$. By the construction of K, $d_1 t_s = \pi_s[q_s] - [q_s]$, for some $q_s \in \mathfrak{F}$. We then consider the subcomplex

$$\mathscr{C}' : 0 \to C_1' \to C_0' \to \mathbb{Z} \to 0$$

of $\mathscr{C}$, where C_1' (resp. C_0') is generated by the translations (by elements of Γ) of t_s (resp. q_s) for all $s \in S$. This induces a quotient complex $\frac{\mathscr{C}}{\mathscr{C}'} = \left(\frac{C_i}{C_i'}, \bar{d}_i\right)$.

Let now M be a right Γ-module. We define the first *compactly supported cohomology group* $H_c^1(\Gamma, M)$ as the group $\mathrm{Hom}_\Gamma(\mathscr{D}_0, M)$ where $\mathscr{D}_0$ is the group of divisors of degree 0 supported on $\mathbb{Q} \cup \{i\infty\}$ and equipped with the usual left action of $\Gamma \subset SL_2(\mathbb{Z})$. The next lemma expresses $H_c^1(\Gamma, M)$ in terms of the complex $\mathscr{C}$.

Lemma 4. *If* Γ *is torsion-free we have:*

$$H_c^1(\Gamma, M) \cong H^1(\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M)).$$

Proof. For $i = 0, 1$ we let $\bar{d}_i'$ be the map in $\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M)$ induced by $\bar{d}_i$. We define a map $\alpha^* : \mathrm{Ker}(\bar{d}_1') \to \mathrm{Hom}_\Gamma(\mathscr{D}_0, M)$ setting:

$$(10) \qquad \alpha^*(f)((x) - (\gamma y)) = f([q_x, \gamma q_y]), \quad \text{for } f \in \mathrm{Ker}(\bar{d}_1'), \text{ and } x, y \in S$$

where $[q_x, \gamma q_y]$ denotes a 1-chain such that $d_1([q_x, \gamma q_y]) = [\gamma q_y] - [q_x]$. We extend by $\mathbb{Z}[\Gamma]$-linearity. Conversely, let $g \in \mathrm{Hom}_\Gamma(\mathscr{D}_0, M)$. After we fix a $x_0 \in S$, we set

$$(11) \qquad \beta^*(g)([q_x, \gamma q_y]) = g((x) - (\gamma y))$$

$$\beta^*(g)([z, \gamma q_x]) = g((x_0) - (\gamma x)) \quad \text{for } x, y \in S \text{ and } z \in \mathfrak{F} - \{q_x; x \in S\}.$$

We extend $\beta^*(g)$ by linearity. One can check that (10), (11) give well-defined inverse maps. □

To pass to the (relative) *singular* cohomology, we use the explicit chain equivalences of [E1] and [E2]. We first need to review the definitions of some preliminary notions.

For each $x \in H$ we define a map on the set of 0-simplices in H as follows. If x does not belong to any simplex in K with vertex B, we set $x(B) = 0$.

If x belongs to a simplex which is the homeomorphic image of Δ_n under a map t, then we define $x(B) =$ the coordinate of $t^{-1}(x)$ in terms of (the basis vector of $\mathbb{R}^{n+1}$) $t^{-1}(B)$.

If s is a simplex of K, we consider the set $N(s)$ of points $x \in H$ for which there exists a vertex A of s such that $x(A) > x(B)$, for all 0-simplices B not in s. The singular simplices of H such that $T(\Delta_n) \subset N(s)$ for some simplex s of K form a subcomplex $S_N(H)$ of $S(H)$.

For each $x \in \mathfrak{F}$, we also select a vertex $n(x)$ of K such that $x(n(x)) \geq x(B)$ for any vertex B of K. (In some sense, $n(x)$ is the vertex "nearest to x" and there may be more than one such $n(x)$ for a given x). We can extend $n(x)$ to all $x \in H$ by setting $n(gx) = gn(x)$ for $x \in \mathfrak{F}$ and $g \in \Gamma$. This is well-defined since Γ has no fixed points in H. On the other hand, it is easy to see that $(gx)(gn(x)) = x(n(x))$, since g induces a simplicial homeomorphism of H to itself and hence $gn(x)$ will still be a vertex nearest to gx.

Finally, we define the "subdivision" operator Sd on the group of singular 1-chains. If T is a singular 1-simplex in H,

$$\mathrm{Sd}(T) := T_1 - T_0,$$

where T_1 (resp. T_0) is a 1-simplex whose image is the half of $T(\Delta_1)$ from its midpoint to its last point (resp. to its first point). (The formal details of the definition of T_1 and T_0 which is based on the 'cone construction' are omitted.) The operator Sd is then extended to all of $S_1(H)$ by linearity.

With this notation, the desired isomorphisms between singular and 'simplicial' cohomology groups are induced by the composition of two chain maps, $R : S(H) \to S_N(H)$ and $G : S_N(H) \to \mathscr{C}$. In the case of 0-simplices, they are given by

$$R(c) = c \quad \text{for a 0-simplex } c \text{ in } S(H).$$

$$G(c') = [n(c')] \quad \text{for a 0-simplex } c' \text{ in } S_N(H).$$

Let now T denote a 1-simplex in $S(H)$. Let $m(T)$ be the least non-negative integer such that the $m(T)$-th iteration of Sd (denoted by $\mathrm{Sd}^{m(T)}$) belongs to $S_N(H)$. Then

$$R(T) = \mathrm{Sd}^{m(T)}(T).$$

If T' is a 1-simplex in $S_N(H)$ with vertices T'_0 and T'_1 then

$$G(T') = [n(T'_0), n(T'_1)].$$

We extend G to the whole $S_N(H)$ by linearity.

The maps in the opposite direction required to prove the homotopy equivalence are simply the injection

$$I : S_N(H) \hookrightarrow S(H)$$

and the natural map $L : \mathscr{C} \to S_N(H)$ defined by

$$L([q_0, \dots, q_p] = \text{the singular } p\text{-simplex with vertices } q_i\text{'s } (i = 0, \dots, p)$$

for every p-simplex $[q_0, \dots q_p]$ of K.

The fact that these maps are well-defined is proved in [E1], [E2], [E3], where the following proposition is proved too.

Proposition 5. *The maps $R \circ I$ and $G \circ L$ are the identity maps on $S_N(H)$ and $\mathscr{C}$ respectively. The maps $I \circ R$ and $L \circ G$ are homotopic to the identity of $S(H)$ and $S_N(H)$ respectively. Furthermore, all these maps commute with the action of the group Γ acting* freely *on H and the chain homotopy operators can be chosen so that they commute with the Γ-action too. Finally, the chain maps and the chain homotopy operators are compatible with the inclusion map of a subspace A of H into H.* □

Proposition 5 allows us to connect the group $H^1(\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M))$ with a (relative) equivariant cohomology group.

Specifically, if A is a subspace of H, we define the equivariant cohomology group $H^1_\Gamma(H, A, M)$ of H with coefficients in M relative to A as the first derived group of $\mathrm{Hom}_\Gamma(\frac{S(H)}{S(A)}, M)$ with the coboundary operator induced by the chain operators on $S(H)$. Then, Proposition 5 applied to

$$A = \bigcup_{\gamma \in \Gamma} \bigcup_{s \in S} \gamma t_s$$

with the simplicial decomposition induced on it by K, implies

Proposition 6. *The map*

$$\rho^* : H^1(\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M)) \to H^1_\Gamma(H, A, M)$$

induced by the map $G \circ R : S(H) \to \mathscr{C}$ is an isomorphism. □

We now consider the complex vector bundle on Y associated to M. As in Section 2 we set $\mathfrak{M} := \Gamma \backslash (H \times M)$ where Γ acts on $H \times M$ by $\gamma.(z, m) = (\gamma z, m\gamma^{-1})$. We maintain the notation of Section 2 for the trivialization and identification maps as well as for the isomorphism (7) (or (7′)) between the fibers. We can then describe the isomorphism between the relative equivariant cohomology on H and the relative cohomology on Y with coefficients in M.

The map ϕ from $C^i(Y, \mathfrak{M})$ to the equivariant cochains $C^i(H, M) = \mathrm{Hom}_\Gamma(S_i(H), M)$ is given by:

$$\phi(f)(T) := A(\alpha_{p(x_0),p(x_T)})f(p(T))$$

$$\text{for every } i\text{-simplex } T \text{ of } H \text{ and } f \in C^i(Y, \mathfrak{M})$$

where $\alpha_{p(x_0),p(x_T)}$ is the image of a path from x_T to x_0 under p. We extend to all of $C^i(H, M)$ by $\mathbb{C}$-linearity. It is easy to see that $\phi(f)$ is Γ-invariant. Strictly speaking, we should write $C^i(H, \pi^{-1}(x_0))$ instead of $C^i(H, M)$ but we use the identification of $\pi^{-1}(x_0)$ with M.

The inverse map $\psi : C^i(H, M) \to C^i(Y, \mathfrak{M})$ is given by the formula

$$\psi(g)(p(T)) = A(\alpha_{p(x_0),p(x_T)})^{-1}g(T), \tag{12}$$

for all $g \in C^i(\mathfrak{H}, M_f)$ and i-simplices $p(T)$ of $Y_0(N)$, where $\alpha_{p(x_0),p(x_T)}$ is the image of a path from x_T to x_0 under p and M is identified with $\pi^{-1}(p(x_0))$ via the trivializing map. It can be proved that ϕ and ψ are well-defined inverse maps commuting with the coboundary operators (8) (cf. [E3], Ch. 5). We now show that, using ϕ, ψ, we can also obtain the compact cohomology $H^1_c(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{M})$ as the isomorphic image of $H^1_c(\Gamma, M)$.

Theorem 7. *Let j be the injection $Y - p(A) \hookrightarrow Y$, and Φ a homeomorphism from $\Gamma\backslash\mathfrak{H}$ to $Y - p(A)$. If ψ^*, j^*, Φ^* denote the map induced by ψ, j, Φ on the cohomology then the map $\Phi^*j^*\psi^*\rho^*\beta^*$ is an isomorphism from $H^1_c(\Gamma, M)$ to $H^1_c(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{M})$. Here Φ^{-1}, j^{-1} denote the sheaf inverse images induced by Φ and j.*

Proof. Set $p(A) = \bigcup_{s\in S} p(t_s)$. As mentioned above, ψ is an isomorphism from $C^*(H, M)$ to $C^*(Y, \mathfrak{M})$ commuting with the coboundary operator. From (12) we immediately deduce that ψ maps the elements of $C^1(H, M)$ vanishing on simplices whose support is contained in A to elements of $C^1(Y, \mathfrak{M})$ vanishing on simplices contained in $p(A)$. The analogous statement for ϕ is also true. Therefore, ψ induces an isomorphism between $H^1_\Gamma(H, A, M)$ and $H^1(Y, p(A), \mathfrak{M})$. Combined with Lemma 4 and Proposition 6, this proves that $\psi^*\rho^*\beta^*$ is an isomorphism from $H^1_c(\Gamma, M)$ to $H^1(Y, p(A), \mathfrak{M})$.

On the other hand, by 'invariance under relative homeomorphisms', the map $j^* : H^1(Y, p(A), \mathfrak{M}) \to H^1_c(Y - p(A), j^{-1}\mathfrak{M})$, is an isomorphism. From this we deduce Theorem 7. □

We finish this section too with a list of the main maps we have defined.

- $\beta^* : H^1_c(\Gamma, M) \to H^1(\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M))$ defined by

$$\beta^*(g)([q_x, \gamma q_y]) = g((x) - (\gamma y)) \quad \text{and}$$

$$\beta^*(g)([z, \gamma q_x]) = g((x_0) - (\gamma x)) \quad \text{for } x, y \in S \text{ and } z \in \mathfrak{F} - \{q_x; x \in S\}$$

  for $g \in H^1_c(\Gamma, M)$.

- For $x \in H$ set

$$x(B) = 0, \quad \text{if } x \text{ does not belong to any simplex in } K \text{ with vertex } B,$$

$$x(B) = \text{the coordinate of } t^{-1}(x) \text{ in terms of } t^{-1}(B)$$

  if x belongs to a simplex which is the homeomorphic image of Δ_n under a map t.

- For every $x \in \mathfrak{F}$ select a vertex x of K such that $x(n(x)) \geq x(B)$ for any vertex B of K. Extend to all of H by Γ-linearity.

- If T is a singular 1-simplex in H, we define

$$\mathrm{Sd}(T) := T_1 - T_0,$$

  where T_1 (resp. T_0) is a 1-simplex whose image is the half of $T(\Delta_1)$ from its mid-point to its last point (resp. to its first point). We extend Sd to all of $S_1(H)$ by linearity.

- $R : S(H) \to S_N(H)$ is defined for 0- and 1-simplices by

$$R(c) = c, \quad \text{for a 0-simplex } c$$

$$R(T) = \mathrm{Sd}^{m(T)}(T), \quad \text{for a 1-simplex } T$$

  where $m(T)$ is the least non-negative integer such that the $m(T)$-th iteration of Sd belongs to $S_N(H)$.

- $G : S_N(H) \to \mathscr{C}$ is defined for 0- and 1-simplices by

$$G(c) = [n(c)], \quad \text{for a 0-simplex } c$$

$$G(T) = [n(T_0), n(T_1)], \quad \text{for a 1-simplex } T \text{ with vertices } T_0, T_1.$$

- $\rho^* : H^1(\mathrm{Hom}_\Gamma(\frac{\mathscr{C}}{\mathscr{C}'}, M)) \to H^1_\Gamma(H, A, M)$ induced by the map

$$G \circ R : S(H) \to \mathscr{C}.$$

- $\psi^* : H^1_\Gamma(H, A, M) \to H^1(Y, p(A), \mathfrak{M})$ induced by $\psi : C^1(H, M) \to C^1(Y, \mathfrak{M})$ defined by

$$\psi(g)(p(T)) = A(\alpha_{p(x_0), p(x_T)})^{-1} g(T)$$

  for all $g \in C^1(H, M)$ and 1-simplices $p(T)$ of Y, where $\alpha_{p(x_0), p(x_T)}$ is the image of a path from x_T to x_0 under p.

- $j^* : H^1(H, p(A), \mathfrak{M}) \to H^1_c(Y - p(A), j^{-1}M)$ with $p(A) = \bigcup_{s \in S} p(t_s)$ is induced by the injection $j : Y - p(A) \hookrightarrow Y$. j^{-1} is the sheaf inverse image induced by j.

- $\Phi^* : H^1_c(Y - p(A), j^{-1}M) \to H^1_c(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{M})$ is induced by a homeomorphism $\Phi : \Gamma\backslash\mathfrak{H} \to Y - p(A)$. Φ^{-1} is the sheaf inverse image induced by Φ.

4 Cocycles associated to *L*-functions

In this section we apply the results of Section 3 to the cohomological setting of [D].

Let f be a cusp form of (even) weight k for $SL_2(\mathbb{Z})$. Then we define a map σ_f from $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ to P_{k-2} by

$$\sigma_f(\gamma_1, \gamma_2) = \int_{i\infty}^{\gamma_1 i\infty} f(z)(z - \gamma_1 X)^{k-2} j(\gamma_1, X)^{k-2}(u(\gamma_2 z) - u(z))\, dz$$

for all $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$, where X is the polynomial variable. This map satisfies a 2-cocycle condition in terms of the right action $|_{2-k}$. Moreover the map σ_f is intimately connected with the values of derivatives of $L_f(s)$ inside the critical strip (i.e. $s = 1, \ldots, k-1$), since each such value appears in the coefficients of $\sigma_f(T, T)$, where $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (see (3)).

On the other hand, σ_f induces a compactly supported 1-cocycle of $SL_2(\mathbb{Z})$ with coefficients in a certain $SL_2(\mathbb{Z})$-module. Specifically, we consider the map $b_f : SL_2(\mathbb{Z}) \to P_{k-2}$ given by

$$b_f(\gamma) = \sigma_f(T, \gamma)|_{2-k} T.$$

We also define an action $\|$ of $SL_2(\mathbb{Z})$ on $C^1(SL_2(\mathbb{Z}), P_{k-2})$ such that

$$(b\|\gamma)(\delta) = b(\delta\gamma^{-1})|_{2-k}\gamma \quad \text{for all } \gamma, \delta \in SL_2(\mathbb{Z}),$$

where $b \in C^1(SL_2(\mathbb{Z}), P_{k-2})$. We extend $\|$ to $\mathbb{Z}[SL_2(\mathbb{Z})]$ by linearity. Then we have

Proposition 8 ([D]). *If* $U := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *both* $b_f \| (T+1)$ *and* $b_f \| (U^2 + U + 1)$ *are constant.* □

For the proof see [D], Prop. 2. Although in [D] we work with functions of $\Gamma_\infty\backslash SL_2(\mathbb{Z})$ to a quotient of P_{k-2} (where Γ_∞ is generated by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) the crucial element in the proof is the 2-cocycle relation which holds in both cases.

We now consider the map β_f on $SL_2(\mathbb{Z})$ sending U and T to the image $\bar{b}_f$ of b_f under the projection $C^1(SL_2(\mathbb{Z}), P_{k-2}) \to C^1(SL_2(\mathbb{Z}), P_{k-2})/C^0(SL_2(\mathbb{Z}), P_{k-2})$. Because of Proposition 8, β_f defines a 1-cocycle of $SL_2(\mathbb{Z})$ with coefficients in

$$N := C^1(SL_2(\mathbb{Z}), P_{k-2})/C^0(SL_2(\mathbb{Z}), P_{k-2})$$

(the action of $SL_2(\mathbb{Z})$ being induced by $\|$ and denoted by $\|$ too). So, for example, we have

$$(13) \qquad \beta_f(UTU^{-1}T) = \bar{b}_f \parallel (TU^{-1}T + U^{-1}T - T + 1).$$

In fact, β_f is a compactly supported cohomology class. This is a consequence of

Lemma 9. *Let M be a right $SL_2(\mathbb{Z})$-module. Then $H_c^1(SL_2(\mathbb{Z}), M)$ is isomorphic to*

$$\{f : SL_2(\mathbb{Z}) \to M; f(g_2 g_1) = f(g_2)g_1 + f(g_1)$$

$$\textit{for all } g_1, g_2 \in SL_2(\mathbb{Z}) \textit{ and } f(S) = 0\}.$$

Proof. One easily checks that the map given by

$$(14) \qquad \chi(g)(\gamma) = g((i\infty) - (\gamma^{-1}i\infty)) \quad \text{for } g \in H_c^1(SL_2(\mathbb{Z}), M) \text{ and } \gamma \in SL_2(\mathbb{Z})$$

is an isomorphism. □

Since $SL_2(\mathbb{Z})$ does have elliptic points we cannot apply the results of Section 3 directly. However, we can use the fact that $SL_2(\mathbb{Z})$ has a normal torsion-free subgroup of finite index, which we denote by Γ. Indeed, by definition, $H_c^1(SL_2(\mathbb{Z}), N) = \mathrm{Hom}(\mathscr{D}_0, N)^{SL_2(\mathbb{Z})}$, where the right-hand side is the group of invariant elements of $\mathrm{Hom}(\mathscr{D}_0, N)$ with respect to the action of $SL_2(\mathbb{Z})$ defined by

$$(f.\gamma)((x) - (y)) = f((\gamma x) - (\gamma y))\|\gamma, \quad \text{for } \gamma \in SL_2(\mathbb{Z}).$$

Trivially, $\mathrm{Hom}(\mathscr{D}_0, N)^{SL_2(\mathbb{Z})} = (\mathrm{Hom}(\mathscr{D}_0, N)^{\Gamma})^{SL_2(\mathbb{Z})/\Gamma} = \mathrm{Hom}_{\Gamma}(\mathscr{D}_0, N)^{SL_2(\mathbb{Z})/\Gamma}$. Therefore, the injection

$$\mathrm{Hom}_{SL_2(\mathbb{Z})}(\mathscr{D}_0, N) \hookrightarrow \mathrm{Hom}_{\Gamma}(\mathscr{D}_0, N)$$

induces an isomorphism

$$H_c^1(SL_2(\mathbb{Z}), N) \cong H_c^1(\Gamma, N)^{SL_2(\mathbb{Z})}.$$

At the other end, let pr denote the natural holomorphic covering $\Gamma\backslash\mathfrak{H} \to SL_2(\mathbb{Z})\backslash\mathfrak{H}$ and let $\mathfrak{N}$ be the vector bundle on $\Gamma\backslash\mathfrak{H}$ associated to N. The transfer map induces an isomorphism

$$(15) \qquad H_c^1(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{N})^{SL_2(\mathbb{Z})/\Gamma} \cong H_c^1(SL_2(\mathbb{Z})\backslash\mathfrak{H}, pr_*\Phi^{-1}j^{-1}\mathfrak{N}),$$

where $pr_*\Phi^{-1}j^{-1}\mathfrak{N}$ denotes the direct image of $\Phi^{-1}j^{-1}\mathfrak{N}$ by pr. Specifically, for all $x \in SL_2(\mathbb{Z})\backslash\mathfrak{H}$,

$$(pr_*\Phi^{-1}j^{-1}\mathfrak{N})_x \cong \bigoplus(\Phi^{-1}j^{-1}\mathfrak{N})_{\tilde{x}},$$

where the sum ranges over the pre-images $\tilde{x}$ of x under pr. (15) is induced by the map sending $f \in C_c^1(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{N})$ to the map $tr(f)$ such that

(16) $$tr(f)(T) = (f(\tilde{T}_1), \dots, f(\tilde{T}_n)) \quad \text{for all 1-simplices } T \text{ in } SL_2(\mathbb{Z})\backslash\mathfrak{H}.$$

Here, $\tilde{T}_1, \dots, \tilde{T}_n$ are the liftings of T to $\Gamma\backslash\mathfrak{H}$ (counting multiplicity).

Before discussing an example we list the main maps on this section.

- For a weight k cusp form f we define $b_f : SL_2(\mathbb{Z}) \to P_{k-2}$ by
$$b_f(\gamma) = \sigma_f(T,\gamma)|_{2-k}T \quad \text{for } \gamma \in SL_2(\mathbb{Z}).$$
- $\beta_f : SL_2(\mathbb{Z}) \to N := C^1(SL_2(\mathbb{Z}), P_{k-2})/C^0(SL_2(\mathbb{Z}), P_{k-2})$ is a map satisfying the 1-cocycle condition and such that
$$\beta_f(U) = \beta_f(T) = \bar{b}_f$$
where $\bar{b}_f$ is the image of b_f under the projection $C^1(SL_2(\mathbb{Z}), P_{k-2}) \to N$.
- $\chi : H_c^1(SL_2(\mathbb{Z}), M) \to \{f : SL_2(\mathbb{Z}) \to M; f(g_2g_1) = f(g_2)g_1 + f(g_1)$ for all $g_1, g_2 \in SL_2(\mathbb{Z})$ and $f(S) = 0\}$ defined by
$$\chi(g)(\gamma) = g((i\infty) - (\gamma^{-1}i\infty)) \quad \text{for } g \in H_c^1(SL_2(\mathbb{Z}), M) \text{ and } \gamma \in SL_2(\mathbb{Z}).$$
- The injection
$$\mathrm{Hom}_{SL_2(\mathbb{Z})}(\mathscr{D}_0, N) \hookrightarrow \mathrm{Hom}_\Gamma(\mathscr{D}_0, N)$$
induces an isomorphism
$$H_c^1(SL_2(\mathbb{Z}), N) \cong H_c^1(\Gamma, N)^{SL_2(\mathbb{Z})}$$
- $pr : \Gamma\backslash\mathfrak{H} \to SL_2(\mathbb{Z})\backslash\mathfrak{H}$: natural holomorphic covering.
- Let tr be the map sending $f \in C_c^1(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{N})$ to the map $tr(f)$ such that
$$tr(f)(T) = (f(\tilde{T}_1), \dots, f(\tilde{T}_n)) \quad \text{for all 1-simplices } T \text{ in } SL_2(\mathbb{Z})\backslash\mathfrak{H}.$$

Here, $\tilde{T}_1, \dots, \tilde{T}_n$ are the liftings of T to $\Gamma\backslash\mathfrak{H}$ (counting multiplicity).

The map tr induces an isomorphism
$$H_c^1(\Gamma\backslash\mathfrak{H}, \Phi^{-1}j^{-1}\mathfrak{N})^{SL_2(\mathbb{Z})/\Gamma} \cong H_c^1(SL_2(\mathbb{Z})\backslash\mathfrak{H}, pr_*\Phi^{-1}j^{-1}\mathfrak{N}).$$

Example. Consider a cusp form of weight k on $SL_2(\mathbb{Z})$. We will describe a representative $\bar{\beta}_f$ of the image of β_f in $H_c^1(SL_2(\mathbb{Z})\backslash\mathfrak{H}, pr_*\Phi^{-1}j^{-1}\mathfrak{N})$ which we obtain after we successively apply the maps we have defined.

As a torsion-free subgroup Γ we will use $\Gamma(2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z}), a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod 2\}$. A set of representatives of $SL_2(\mathbb{Z})/\Gamma(2)$ (where we identify matrices differing by a factor of $-I$) is

$$\{\gamma_i, i = 1, \dots, 6\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let T be a 1-simplex $(SL_2(\mathbb{Z})z_1, SL_2(\mathbb{Z})z_2)$ on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ which does not pass through any elliptic points. By (15), (16) we have

$$\bar{\beta}_f(T) = (\beta'_f(\tilde{T}_1), \beta'_f(\tilde{T}_2), \beta'_f(\tilde{T}_3), \beta'_f(\tilde{T}_4), \beta'_f(\tilde{T}_5), \beta'_f(\tilde{T}_6))$$

where $\tilde{T}_i = (\Gamma\gamma_i z_1, \Gamma\gamma_i z_2)$ and the prime signifies that β'_f corresponds to $\Gamma(2)$ rather than $SL_2(\mathbb{Z})$.

By the definition of the transfer map, for each $i = 1, \dots, 6$ there is a $\delta_i \in \Gamma(2)$ such that $\Phi(\tilde{T}_i)$ is the image (under p) of a 1-simplex in H with end points $\gamma_i z_1 \in \mathfrak{F}$ and $\delta_i\gamma_i z_2$. Assume that this simplex is already small enough for it not to be necessary to be subdivided (see Section 3). Unraveling the definitions of j^*, ρ^*, and ψ^*, if the 0-simplices of K "nearest" to $\gamma_i z_1$ (resp. $\gamma_i z_2$) are $[q_{x_i}]$ (resp. $[q_{y_i}]$), then

$$\beta'_f(\tilde{T}_i) = \beta^*(\beta_f)([q_{x_i}, \delta_i q_{y_i}])$$

where β^* is the map defined by (11). Taking into account (14), this can be written as $\beta_f(\delta_i^{-1})$ and, therefore,

$$\bar{\beta}_f(T) = (\beta_f(\delta_1^{-1}), \beta_f(\delta_2^{-1}), \beta_f(\delta_3^{-1}), \beta_f(\delta_4^{-1}), \beta_f(\delta_5^{-1}), \beta_f(\delta_6^{-1})).$$

Each of the coordinates can be expressed in terms of $\bar{b}_f$ as in (13).

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