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Virtual Element Method for Quasilinear Elliptic Problems

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A Virtual Element Method (VEM) for the quasilinear equation $-\mathrm{div}(\boldsymbol{\kappa}(u)\mathrm{grad}u) = f$ using general polygonal and polyhedral meshes is presented and analysed. The nonlinear coefficient is evaluated with the piecewise polynomial projection of the virtual element ansatz. Well-posedness of the discrete problem and optimal order a priori error estimates in the H^1- and L^2-norm are proven. In addition, the convergence of fixed point iterations for the resulting nonlinear system is established. Numerical tests confirm the optimal convergence properties of the method on general meshes.

Keywords: virtual element method; quasilinear elliptic equations

1. Introduction

In this work we present an arbitrary-order conforming Virtual Element Method (VEM) for the numerical treatment of quasilinear diffusion problems. Both two and three dimensional problems are considered and the method is analysed under the same mesh regularity assumption used in the linear setting (Beirão da Veiga *et al.*, 2013; Cangiani *et al.*, 2017a), allowing for very general polygonal and polyhedral meshes.

Virtual element methods for general linear elliptic problems are now well-established, see e.g., (Beirão da Veiga *et al.*, 2013; Beirão da Veiga & Manzini, 2014; Ahmad *et al.*, 2013; Beirão da Veiga *et al.*, 2016; Ayuso de Dios *et al.*, 2016; Cangiani *et al.*, 2017a; Brenner *et al.*, 2017) and (Sutton, 2017b) for a simple implementation. See also (Beirão Da Veiga *et al.*, 2017; Brenner & Sung, 2018) for an extension to meshes with arbitrarily small edges and (Cangiani *et al.*, 2017b; Mora *et al.*, 2017) where the mesh generality is exploited within an adaptive algorithm driven by rigorous *a posteriori* error estimates. The VEM framework has been concurrently extended to a number of different problems and applications, and, in particular, the literature on VEM for nonlinear problems is growing, the same being true for other approaches to polygonal and polyhedral meshes. Virtual Element methods are developed for semilinear parabolic problem in (Adak *et al.*, 2019), Cahn-Hilliard in (Antonietti *et al.*, 2016), stationary Navier-Stokes in (Beirão da Veiga *et al.*, 2018; Gatica *et al.*, 2018), nonlinear Birkman and quasi-Newtonian Stokes flow in (Gatica *et al.*, 2018; Cáceres *et al.*, 2018), computational mechanics in (Beirão da Veiga *et al.*, 2015; Artioli *et al.*, 2017; Wriggers & Hudobivnik, 2017; Hudobivnik *et al.*, 2019; Wriggers *et al.*, 2018; Taylor & Artioli, 2018; Artioli *et al.*, 2018) and fracture problems in (Aldakheel *et al.*, 2018). The related nodal Mimetic Finite Difference method is analysed in (Antonietti *et al.*, 2015) for elliptic quasilinear problems whereby the nonlinear coefficient depends on the gradient of the solution, however only low-order discretisations are considered. We also mention the arbitrary order Hybrid High-Order method on polygonal meshes for the general class of Leray-Lions elliptic equations (Di Pietro & Droniou, 2017), including the problems considered here. The HHO method belongs to the class of nonconforming/discontinuous discretisations and is, in fact, related to the Hybrid Mixed Mimetic approach and to the nonconforming VEM (Droniou *et al.*, 2010; Cockburn *et al.*, 2016). In (Di Pietro & Droniou, 2017), the convergence of HHO is proven under minimal regularity assumptions, but the rate of convergence of the method is not analysed.

The VEM presented here is based on the C^0-conforming virtual element spaces of (Ahmad *et al.*, 2013) whereby the local L^2-projection of virtual element functions onto polynomials is available and the VEM proposed in (Beirão da Veiga *et al.*, 2016; Cangiani *et al.*, 2017a) for the discretisation of linear elliptic problems with non-constant coefficients. In particular, to obtain a practical (computable) formulation, the nonlinear diffusion coefficient is evaluated with the element-wise polynomial projection of the virtual element ansatz. This results in nonlinear inconsistency errors which have to be additionally controlled.

We present an a priori analysis of the VEM which builds upon and extends the classical framework introduced by Douglas and Dupont (Douglas & Dupont, 1975) for standard conforming finite element methods. The analysis relies on the assumption that the nonlinear diffusion coefficient is bounded and Lipschitz continuous and is based on a bootstrapping argument: 1. existence of solutions for the numerical scheme is shown by a fixed point argument, 2. the H^1-norm error is bounded by optimal order terms plus the L^2-norm error, 3. using a standard duality argument and assuming that the discretisation parameter is small enough, the L^2-norm error is bounded by optimal order terms plus potentially higher-order terms, 4. based on the existence result, L^2-convergence is shown by a compactness argument, and now H^1-convergence follows from step 2. Within this approach, we also obtain optimal order a priori error estimates in the H^1- and L^2-norms, albeit under the (higher) regularity assumptions needed by the duality argument. To the best of our knowledge, this work provides the first optimal order error estimate for a conforming discretisation of quasilinear problems on general polygonal and polyhedral meshes.

To simplify the presentation, we consider homogeneous Dirichlet boundary value problems only. To this end, we introduce the model quasilinear elliptic problem

$$-\nabla\cdot(\boldsymbol{\kappa}(u)\nabla u) = f(\boldsymbol{x}) \text{ in } \Omega, \quad \text{with} \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a convex polygonal or polyhedral domain for $d = 2$ or $d = 3$, respectively. The diffusion coefficient is a twice differentiable function $\boldsymbol{\kappa} : \mathbb{R} \to [\boldsymbol{\kappa}_*, \boldsymbol{\kappa}^*]$ such that $0 < \boldsymbol{\kappa}_* \leqslant \boldsymbol{\kappa}^* < +\infty$, and with bounded derivatives up to second order. Therefore $\boldsymbol{\kappa}$ is Lipschitz continuous, namely there exists a positive constant L such that

$$|\boldsymbol{\kappa}(t) - \boldsymbol{\kappa}(s)| \leqslant L|t-s|, \qquad \text{for a.e } t,s \in \mathbb{R}. \tag{1.2}$$

Writing (1.1) in variational form, we seek $u \in H^1_0(\Omega)$ such that

$$a(u;u,v) := (\boldsymbol{\kappa}(u)\nabla u, \nabla v) = (f,v), \qquad \forall v \in H^1_0(\Omega), \tag{1.3}$$

with $(\cdot,\cdot)$ denoting the standard L^2 inner-product. It is well known that for sufficiently smooth f, problem (1.1) possesses a unique solution u, see eg. (Douglas *et al.*, 1971).

The remainder of this work is structured as follows. We introduce the virtual element method in Section 2. The method is then analysed in Section 3, where the well-posedness and a priori analysis are presented. In Section 4 we establish the convergence of fixed point iterations for the solution of the nonlinear system resulting from the VEM discretisation. We present a numerical test in Section 5 and, finally, we provide some conclusions in Section 6.

We use standard notation for the relevant function spaces. For a Lipschitz domain $\omega \subset \mathbb{R}^d$, $d = 2,3$, we denote by $|\omega|$ its d–dimensional Hausdorff measure. Further, we denote by $H^s(\omega)$ the Hilbert space of index $s \geqslant 0$ of real–valued functions defined on ω, endowed with the seminorm $|\cdot|_{s,\omega}$ and norm $\|\cdot\|_{s,\omega}$; further $(\cdot,\cdot)_\omega$ stands for the standard L^2-inner-product. The domain of definition will be omitted when this coincides with Ω, eg. $|\cdot|_s := |\cdot|_{s,\Omega}$ and so on. Finally, for $\ell \in \mathbb{N}\cup\{0\}$, we denote by $\mathbb{P}_\ell(\omega)$ the space of all polynomials of degree up to ℓ.

2. The Virtual Element Method

We introduce the virtual element method for the discretisation of problem (1.3), using general polygonal and polyhedral decompositions of Ω in two and three dimensions, respectively. We start by recalling the definition of the virtual element spaces from (Ahmad *et al.*, 2013; Cangiani *et al.*, 2017a).

2.1 *The Discrete Spaces*

The definition of the virtual element method relies on the availability of certain local projector operators based on accessing the degrees of freedom. The choice of degrees of freedom for the virtual element spaces is thus important.

DEFINITION 2.1 (Degrees of freedom) Let $\omega \subset \mathbb{R}^d$, $1 \leqslant d \leqslant 3$, be a d-dimensional polytope, that is, a line segment, polygon, or polyhedron, respectively. For any regular enough function v on ω, we define the following sets of *degrees of freedom*:

- *Nodal values*. For a vertex $\mathbf{z}$ of ω, $\mathscr{N}_{\mathbf{z}}^{\omega}(v) := v(\mathbf{z})$ and $\mathscr{N}^{\omega} := \{\mathscr{N}_{\mathbf{z}}^{\omega} : \mathbf{z} \text{ is a vertex}\}$;
- *Polynomial moments*. For $l \geqslant 0$,

$$\mathscr{M}_{\boldsymbol{\alpha}}^{\omega}(v) = \frac{1}{|\omega|}(v, m_{\alpha})_{\omega} \quad \text{with} \quad m_{\boldsymbol{\alpha}} := \left(\frac{\boldsymbol{x} - \boldsymbol{x}_{\omega}}{h_{\omega}}\right)^{\boldsymbol{\alpha}} \text{ and } \quad |\boldsymbol{\alpha}| \leqslant l,$$

  where $\boldsymbol{\alpha}$ is a multi-index with $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_d$ and $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ in a local coordinate system, and $\boldsymbol{x}_{\omega}$ denoting the barycentre of ω. Further, $\mathscr{M}_l^{\omega} := \{\mathscr{M}_{\boldsymbol{\alpha}}^{\omega} : |\boldsymbol{\alpha}| \leqslant l\}$. The definition is extended to $l = -1$ by setting $\mathscr{M}_{-1}^{\omega} := \emptyset$.

Let $\{\mathscr{T}_h\}_h$ be a sequence of decompositions of Ω into non-overlapping and not self-intersecting polygonal/polyhedral elements such that the diameter of any $E \in \mathscr{T}_h$ is bounded by h.

On $\mathscr{T}_h$, we introduce element-wise projectors as follows. We denote by $P_h^{\ell} \equiv P_h^{\ell,E} : L^2(E) \to \mathbb{P}_{\ell}(E)$, $\ell \in \mathbb{N}$, the standard $L^2(E)$-orthogonal projection onto the polynomial space $\mathbb{P}_{\ell}(E)$. With slight abuse of notation, the symbol P_h^{ℓ} will also be used to denote the global operator obtained from the piecewise projections. Similarly, by $\boldsymbol{P}_h^{\ell} \equiv \boldsymbol{P}_h^{\ell,E}$, $\ell \in \mathbb{N}$, we denote the orthogonal projection of $(L^2(E))^d$ onto the space $\widetilde{\mathbb{P}}_{\ell}(E) = (\mathbb{P}_{\ell}(E))^d$, obtained by applying $P_h^{\ell,E}$ component-wise. Further, we consider the projection $R_h^{\ell} \equiv R_h^{\ell,E} : H^1(E) \to \mathbb{P}_{\ell}(E)$, for $\ell \in \mathbb{N}$, associating any $v \in H^1(E)$ with the element in $\mathbb{P}_{\ell}(E)$ such that

$$(\nabla R_h^{\ell} v, \nabla p)_E = (\nabla v, \nabla p)_E, \quad \forall p \in \mathbb{P}_{\ell}(E), \tag{2.1}$$

with, in order to uniquely determine R_h^{ℓ}, the addition of the following condition:

$$\begin{cases} \int_{\partial E} \left(v - R_h^{\ell} v\right) \mathrm{d}s = 0 & \text{if } \ell = 1, \\ \int_{E} \left(v - R_h^{\ell} v\right) \mathrm{d}\boldsymbol{x} = 0 & \text{if } \ell \geqslant 2. \end{cases} \tag{2.2}$$

Let $k \geqslant 1$ be given, characterising the order of the method. We follow the construction of the corresponding C^0-conforming VEM space presented in (Ahmad *et al.*, 2013) to ensure that all of the above projectors, to be utilised in the definition of the method, are computable.

We first introduce the local spaces on each element E of $\mathscr{T}_h$, for $d = 2$. Let $B_k^2(\partial E)$ be the space defined on the boundary of E as

$$B_k^2(\partial E) := \left\{v \in C^0(\partial E) : v|_e \in \mathbb{P}_k(e) \text{ for each edge } e \text{ of } \partial E\right\}.$$

We define the local virtual element space V_h^E by

$$\begin{aligned} V_h^E := \{v_h \in H^1(E) : v_h|_{\partial E} \in B_k^2(\partial E);\ \Delta v_h \in \mathbb{P}_k(E) \\ \text{and } (v_h - R_h^k v_h, p)_E = 0, \forall p \in \mathscr{M}_k(E) \setminus \mathscr{M}_{k-2}(E)\}. \end{aligned}$$

In (Ahmad *et al.*, 2013) it is shown that the following degrees of freedom (DoF) uniquely determine the elements of V_h^E:

$$\mathrm{DoF}(V_h^E) := \mathscr{N}^E \cup \{\mathscr{M}_{k-2}^e : \text{ for each edge } e \in \partial E\} \cup \mathscr{M}_{k-2}^E. \tag{2.3}$$

The global conforming space V_h is obtained from the local spaces V_h^E as

$$V_h := \left\{ v_h \in H_0^1(\Omega) : v_h|_E \in V_h^E, \quad \forall E \in \mathscr{T}_h \right\},$$

with degrees of freedom given in agreement with the local degrees of freedom (2.3).

The construction of the space for $d = 3$ is similar, although now we define the boundary space to be

$$B_k^3(\partial E) := \left\{ v \in C^0(\partial E) : v|_f \in V_h^f \text{ for each face } f \text{ of } \partial E \right\},$$

where V_h^f is the two-dimensional conforming virtual element space of the same degree k on the face f. The local virtual element space is defined to be

$$\begin{aligned} V_h^E := \{ v \in H^1(E) : v|_{\partial E} \in B_k^3(\partial E);\ \Delta v \in \mathbb{P}_k(E); \\ \text{and } (v - R_h^k v, p)_E = 0,\ \forall p \in \mathscr{M}_k(E) \setminus \mathscr{M}_{k-2}(E) \}. \end{aligned}$$

with degrees of freedom

$$\mathrm{DoF}(V_h^E) := \mathscr{N}^E \cup \{ \mathscr{M}_{k-2}^s \text{ for each edge } \textit{and} \text{ face } s \in \partial E \} \cup \mathscr{M}_{k-2}^E. \tag{2.4}$$

Finally, the global space and the set of global degrees of freedom for $d = 3$ are constructed from these in the obvious way, completely analogously to the case for $d = 2$.

The following are well established properties of the virtual element spaces introduced above (Beirão da Veiga *et al.*, 2013; Ahmad *et al.*, 2013; Cangiani *et al.*, 2017a):

- For each $E \in \mathscr{T}_h$, we have $\mathbb{P}_k(E) \subset V_h^E$ as a subspace;
- For each $E \in \mathscr{T}_h$ and $v \in V_h^E$, the H^1-projector $R_h^{k,E} v$ and L^2-projectors $P_h^{k,E} v$ and $\boldsymbol{P}_h^{k-1,E} \nabla v$ are computable just by accessing the local DoFs of v given by (2.3) and (2.4) in the two and three dimensional case, respectively.
- The global virtual element space $V_h \subset H_0^1(\Omega)$ is a finite dimensional subspace.

2.2 *Virtual element method*

The virtual element method of order $k \geqslant 1$ for the discretisation of (1.1) reads: find $u_h \in V_h$ such that

$$a_h(u_h; u_h, v_h) = (P_h^{k-1} f, v_h), \quad \forall v_h \in V_h, \tag{2.5}$$

where $a_h(\cdot;\cdot,\cdot)$ is *any* bilinear form on V_h defined as the sum of elementwise contributions $a_h^E(\cdot;\cdot,\cdot)$ satisfying the following assumption (Beirão da Veiga *et al.*, 2013).

ASSUMPTION 2.2 For every $E \in \mathscr{T}_h$, the form $a_h^E(\cdot;\cdot,\cdot)$ is bilinear and symmetric in its second and third arguments and satisfies the following properties:

- *Polynomial consistency*: For all $p \in \mathbb{P}_k(E)$ and $v_h \in V_h^E$,

$$a_h^E(z; p, v_h) = \int_E \boldsymbol{\kappa}(P_h z) \nabla p \cdot (\boldsymbol{P}_h \nabla v_h) \,\mathrm{d}\boldsymbol{x}, \quad \forall z \in L^2(E), \tag{2.6}$$

  where $P_h = P_h^k$ and $\boldsymbol{P}_h = \boldsymbol{P}_h^{k-1}$.

- *Stability*: There exist positive constants α_*, α^*, independent of h and the mesh element E, but may depend on the polynomial degree k, such that, for all $v_h, z_h \in V_h^E$,

$$\alpha_* a^E(z_h; v_h, v_h) \leqslant a_h^E(z_h; v_h, v_h) \leqslant \alpha^* a^E(z_h; v_h, v_h), \tag{2.7}$$

  with $a^E(z; v, w) = (\boldsymbol{\kappa}(z) \nabla v, \nabla w)_E$, for all $z \in L^\infty(\Omega)$ and $v, w \in H^1(\Omega)$.

REMARK 2.1 The above defining conditions are essentially those introduced in the linear setting (Beirão da Veiga *et al.*, 2013; Beirão da Veiga & Manzini, 2014; Ahmad *et al.*, 2013; Beirão da Veiga *et al.*, 2016; Cangiani *et al.*, 2017a) with, crucially, the nonlinear diffusion coefficient $\boldsymbol{\kappa}$ evaluated with the polynomial projection of the argument. We note also that the symmetry and stability assumptions imply the continuity in V_h of the form $a_h(z;\cdot,\cdot)$, for $z \in V_h$.

REMARK 2.2 The particular choice of local bilinear forms used in the numerical tests is given below in Section 5. We remark, however, that the following error analysis is valid whenever the assumption above is satisfied.

3. Error Analysis

We recall that $k \geqslant 1$ is a fixed natural number representing the order of accuracy of the method (2.5).

The convergence and a priori error analysis of the VEM relies on the availability of the following best approximation results.

3.1 *Approximation Properties*

We recall the optimal approximation properties of the VEM space V_h introduced above. These where established in a series of papers (Beirão da Veiga *et al.*, 2013; Ahmad *et al.*, 2013; Cangiani *et al.*, 2017b) under the following assumption on the regularity of the decomposition $\mathscr{T}_h$.

ASSUMPTION 3.1 (Mesh Regularity). We assume the existence of a constant $\rho > 0$ such that

- for every element E of $\mathscr{T}_h$ and every edge/face e of E, $h_e \geqslant \rho h_E$
- every element E of $\mathscr{T}_h$ is star-shaped with respect to a ball of radius ρh_E
- for $d = 3$, every face $e \in \mathscr{E}_h$ is star-shaped with respect to a ball of radius ρh_e,

were h_e is the diameter of the edge/face e of E and h_E is the diameter of E.

The above star-shapedness assumption can be relaxed by including elements which are union of star-shaped domains (Beirão da Veiga *et al.*, 2013). In particular, the following polynomial approximation result (Brenner & Scott, 2008) is extended to more general shaped elements in (Dupont & Scott, 1980) and the interpolation error bound below can be generalised by modifying the proof in (Cangiani *et al.*, 2017b), see also (Sutton, 2017a).

THEOREM 3.2 (Approximation using polynomials) Suppose that Assumption 3.1 is satisfied and let s be a positive integer such that $1 \leqslant s \leqslant k+1$. Then, for any $w \in H^s(E)$ there exists a polynomial $w_\pi \in \mathbb{P}_k(E)$ such that

$$\|w - w_\pi\|_{0,E} + h_E \|\nabla(w - w_\pi)\|_{0,E} \leqslant C h_E^s |w|_{s,E}.$$

Moreover, we have

$$\|\nabla(w - w_\pi)\|_{L^6(E)} \leqslant C |w|_{W^{1,6}(E)}.$$

In the above bounds, C are positive constants depending only on k and on ρ.

The approximation properties of the virtual element space are characterised by the following interpolation error bound, whose proof can be found in (Cangiani *et al.*, 2017b).

THEOREM 3.3 (Approximation using virtual element functions) Suppose that Assumption 3.1 is satisfied and let s be a positive integer such that $1 \leqslant s \leqslant k+1$. Then, for any $w \in H^s(\Omega)$, there exists an element $w_I \in V_h$ such that

$$\|w - w_I\| + h\|\nabla(w - w_I)\| \leqslant C h^s |w|_s$$

where C is a positive constant which depends only on k and ρ.

Let $\varepsilon_h : L^2(\Omega) \times V_h \to \mathbb{R}$ denote the bilinear form

$$\varepsilon_h(f, v_h) = (P_h^{k-1} f - f, v_h), \quad \forall v_h \in V_h. \tag{3.1}$$

Then, using the fact that $P_h^{k-1} f$ is the L^2 projection on $\mathbb{P}_{k-1}(E)$, we can show the following lemma.

LEMMA 3.1 For $f \in H^s(\Omega)$, $0 \leqslant s \leqslant k$, there exists a positive constant C, independent of h and of f, such that

$$|\varepsilon_h(f, v_h)| \leqslant Ch^{s+j} \|f\|_s \|\nabla^j v_h\|, \quad \forall v_h \in V_h,\ j = 0, 1. \tag{3.2}$$

Proof. For $j = 0$, the desired estimate immediately follows from the Cauchy-Schwarz inequality and standard approximation estimates (Brenner & Scott, 2008). For $j = 1$, we employ the identity

$$\int_E (f - P_h^{k-1} f) v_h = \int_E (f - P_h^{k-1} f)(v_h - P_h^0 v_h),$$

and the desired result follows similarly as before. □

3.2 *Existence*

We first show the existence of a solution u_h of (2.5) using a fixed point argument. To this end, for $M > 0$, we let $\mathscr{B}_M = \{v_h \in V_h : \|\nabla v_h\| \leqslant M\}$.

THEOREM 3.4 Let $f \in L^2(\Omega)$ be given and assume that (1.2) holds. Choose $M > 0$ such that $\|f\| \leqslant Mc_*$, $c_* = \boldsymbol{\kappa}_* \alpha_*$ where α_* is the lower bound constant in (2.7). Then, there exists a solution $u_h \in \mathscr{B}_M \subset V_h$ of (2.5).

Proof. We devise a fixed point iteration for (2.5): for a fixed $f \in L^2(\Omega)$, consider an iteration map $T_h : V_h \to V_h$ given by

$$a_h(v_h; T_h v_h, w_h) = (P_h^{k-1} f, w_h), \quad \forall w_h \in V_h. \tag{3.3}$$

It is easy to see that there exists $h_M > 0$, such that for $h < h_M$, $T_h v_h$ is well defined, see for example (Cangiani *et al.*, 2017a). For $v_h \in \mathscr{B}_M$ and $w_h = T_h v_h$, in view of the stability assumption (2.7) and (3.3), we have

$$c_\star \|\nabla T_h v_h\|^2 \leqslant \alpha_* a(v_h; T_h v_h, w_h) \leqslant a_h(v_h; T_h v_h, w_h) = (P_h^{k-1} f, w_h) \leqslant \|f\| \, \|w_h\|. \tag{3.4}$$

Thus, choosing M sufficiently large, so that $\|f\| \leqslant Mc_\star$, we get

$$\|\nabla T_h v_h\| \leqslant c_*^{-1} \|f\| \leqslant M. \tag{3.5}$$

Therefore, the operator T_h maps the ball $v_h \in \mathscr{B}_M$ into itself. By the Brouwer fixed point theorem, we know that T_h has a fixed point, which implies that (2.5) has a solution $u_h \in \mathscr{B}_M$. □

3.3 *Error bounds*

In our a priori error analysis, we follow a similar-in-spirit approach to the classical work of Douglas and Dupont (Douglas & Dupont, 1975) where standard conforming finite element methods were analysed in the same context.

We start with the following preliminary H^1–norm error bound.

THEOREM 3.5 Let $u \in H_0^1(\Omega)$ be the solution of (1.1) and suppose that $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$, $s \geqslant 2$, assuming that $f \in H^{s-2}(\Omega)$ and $\boldsymbol{\kappa}(u) \in W_\infty^{s-1}(\Omega)$. Then, for $u_h \in V_h$ solution of (2.5) the following bound holds

$$\|\nabla(u - u_h)\| \leqslant C(h^{r-1} + \|u - u_h\|), \tag{3.6}$$

with $r = \min\{s, k+1\}$ and C a positive constant independent of h.

Proof. From Theorem 3.3, there exists a function $u_I \in V_h$, such that $u - u_I$ is bounded as desired. Thus, to show (3.6) it suffices to bound $\|\nabla(u_h - u_I)\|$. Let $\psi = u_h - u_I$, then using the stability Assumption 2.2 with $c_* = \boldsymbol{\kappa}_* \alpha_*$, we have

$$\begin{aligned} c_* \|\nabla(u_h - u_I)\|^2 &\leqslant a_h(u_h; u_h - u_I, \psi) \\ &= \varepsilon_h(f, \psi) + a(u; u, \psi) - a_h(u_h; u_I, \psi) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_h(f,\psi) + ((\boldsymbol{\kappa}(u) - \boldsymbol{\kappa}(P_h u_h))\nabla u, \nabla\psi) + \sum_{E\in\mathscr{T}_h} a^E(P_h u_h; u-u_\pi, \psi) \\
&\quad + \left\{ \sum_{E\in\mathscr{T}_h} a^E(P_h u_h; u_\pi, \psi) - a_h^E(u_h; u_\pi, \psi) \right\} + \sum_{E\in\mathscr{T}_h} a_h^E(u_h; u_\pi - u_I, \psi) \\
&= I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \tag{3.7}
$$

where u_π is, on every element $E \in \mathscr{T}_h$, the polynomial approximation of u given by Theorem 3.2. Next, we will bound the various terms I_i, $i = 1,\ldots,5$. We start with I_1. Using Lemma 3.1, and the fact that $r \leqslant s$, we have

$$
|I_1| \leqslant Ch^{r-1}\|f\|_{r-2}\|\nabla\psi\|. \tag{3.8}
$$

To bound I_2, in view of (1.2), we get

$$
|I_2| \leqslant L\|\nabla u\|_{L_\infty}\|u - P_h u_h\|\,\|\nabla\psi\|. \tag{3.9}
$$

Also, using the fact that $\boldsymbol{\kappa}$ is bounded along with Theorem 3.2, we obtain

$$
|I_3| \leqslant C\sum_E \|\nabla(u-u_\pi)\|_E\|\nabla\psi\|_E \leqslant Ch^{r-1}\|u\|_r\|\nabla\psi\|. \tag{3.10}
$$

Using the fact that $\nabla u_\pi \in \widetilde{\mathbb{P}}_{k-1}(E)$ and Assumption 2.2, we have

$$
\begin{aligned}
I_4 &= \sum_{E\in\mathscr{T}_h}\int_E \boldsymbol{\kappa}(P_h u_h)\nabla u_\pi \cdot (\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi \\
&= \sum_{E\in\mathscr{T}_h}\int_E \boldsymbol{\kappa}(P_h u_h)\nabla(u_\pi - u)\cdot(\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi + \int_E \boldsymbol{\kappa}(P_h u_h)\nabla u\cdot(\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi \\
&= \sum_{E\in\mathscr{T}_h}\int_E (\boldsymbol{\kappa}(P_h u_h) - \boldsymbol{\kappa}(u))\nabla(u_\pi - u)\cdot(\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi + \int_E \boldsymbol{\kappa}(u)\nabla(u_\pi - u)\cdot(\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi \\
&\quad + \sum_{E\in\mathscr{T}_h}\int_E (\boldsymbol{\kappa}(P_h u_h) - \boldsymbol{\kappa}(u))\nabla u\cdot(\boldsymbol{I} - \boldsymbol{P}_h)\nabla\psi + \int_E (\boldsymbol{I} - \boldsymbol{P}_h)(\boldsymbol{\kappa}(u)\nabla u)\cdot\nabla\psi;
\end{aligned}
$$

thus, in view of the stability of $\boldsymbol{P}_h$, the fact that $\boldsymbol{\kappa}$ is Lipschitz continuous, $u \in W_\infty^1(\Omega)$, Theorem 3.2 and the hypothesis $\boldsymbol{\kappa}(u) \in W_\infty^{r-1}(\Omega)$, we deduce

$$
\begin{aligned}
|I_4| &\leqslant C\sum_{E\in\mathscr{T}_h}(\|\nabla(u-u_\pi)\|_E + \|P_h u_h - u\|_E)\|\nabla\psi\|_E + \|(\boldsymbol{I} - \boldsymbol{P}_h)(\boldsymbol{\kappa}(u)\nabla u)\|_E\|\nabla\psi\|_E \\
&\leqslant C(h^{r-1}\|u\|_r + \|P_h u_h - u\|)\|\nabla\psi\|.
\end{aligned} \tag{3.11}
$$

Finally, we easily get

$$
|I_5| \leqslant C(\|\nabla(u-u_\pi)\| + \|\nabla(u-u_I)\|)\|\nabla\psi\| \leqslant Ch^{r-1}\|u\|_r\|\nabla\psi\|. \tag{3.12}
$$

Therefore, combining the above estimates (3.8)–(3.12) with (3.7) we obtain

$$
c_\star\|\nabla(u_h - u_I)\| \leqslant C(h^{r-1} + \|u - P_h u_h\|).
$$

Then, in view of Theorem 3.2 and the stability of P_h in L^2–norm, we obtain the estimate

$$
\|\nabla(u_h - u_I)\| \leqslant C(h^{r-1} + \|u - u_h\|).
$$

□

Next we show two auxiliary lemmas in view of proving an L^2-error bound.

LEMMA 3.2 Let $u \in H_0^1(\Omega)$ be the solution of (1.1) and assume that $u \in H^s(\Omega)\cap W_\infty^1(\Omega)$, $s \geqslant 2$, $f \in H^{s-1}(\Omega)$, $\boldsymbol{\kappa}(u) \in W_\infty^{s-1}(\Omega)$ and $\phi \in H^2 \cap H_0^1$. Then, there exists a constant C independent of h such that,

$$
|a_h(u_h; u_h, \phi_\pi^1) - a(u_h; u_h, \phi_\pi^1)| \leqslant C(\|\nabla(u-u_h)\| + \|u-u_h\|^{1/2}\|\nabla(u-u_h)\|^{3/2} + h^r\|u\|_r)\|\phi\|_2,
$$

where $\phi_\pi^1 \in \mathbb{P}_1(E)$ for all $E \in \mathscr{T}_h$, is given by Theorem 3.2, and $r = \min\{s, k+1\}$.

Proof. Let $\bar{\boldsymbol{\kappa}}_u$ be such that

$$\boldsymbol{\kappa}(u)-\boldsymbol{\kappa}(u_h)=(u-u_h)\int_0^1 \boldsymbol{\kappa}_u(u-t(u-u_h))\,dt=\bar{\boldsymbol{\kappa}}_u(u-u_h). \tag{3.13}$$

Using polynomial consistency (2.6), the fact that $\boldsymbol{P}_h\nabla u_\pi=\nabla u_\pi$, with $u_\pi\in\mathbb{P}_k(E)$ given by Theorem 3.2 and the definition of $\bar{\boldsymbol{\kappa}}_u$ given by (3.13), we have for all $E\in\mathscr{T}_h$

$$\begin{aligned}
&a_h^E(u_h;u_h,\phi_\pi^1)-a^E(u_h;u_h,\phi_\pi^1)=\int_E \boldsymbol{\kappa}(P_h u_h)(\boldsymbol{P}_h\nabla u_h)\cdot\nabla\phi_\pi^1-\boldsymbol{\kappa}(u_h)\nabla u_h\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&=\int_E \boldsymbol{\kappa}(P_h u_h)(\boldsymbol{P}_h-\boldsymbol{I})\nabla u_h\cdot\nabla\phi_\pi^1+(\boldsymbol{\kappa}(P_h u_h)-\boldsymbol{\kappa}(u_h))\nabla u_h\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&=\int_E \boldsymbol{\kappa}(P_h u_h)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}+\int_E \bar{\boldsymbol{\kappa}}_u(P_h u_h-u_h)\nabla u_h\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&=\int_E (\boldsymbol{\kappa}(P_h u_h)-\boldsymbol{\kappa}(u))(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}+\int_E \boldsymbol{\kappa}(u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&+\int_E \bar{\boldsymbol{\kappa}}_u(P_h u_h-u_h)\nabla u_h\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&=\int_E \bar{\boldsymbol{\kappa}}_u(P_h u_h-u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}+\int_E \boldsymbol{\kappa}(u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}\\
&+\int_E \bar{\boldsymbol{\kappa}}_u(P_h u_h-u_h)\nabla u_h\cdot\nabla\phi_\pi^1\,\mathrm{d}\boldsymbol{x}=I_E+II_E+III_E.
\end{aligned}$$

Let $I=\sum_E I_E$, then we easily get

$$|I|\leqslant C\|P_h u_h-u\|_{L_3}\|\nabla\phi_\pi^1\|_{L_6}\|\nabla(u_h-u_\pi)\|.$$

Using Theorem 3.2, we have $\|\nabla\phi_\pi^1\|_{L_6}\leqslant C\|\nabla\phi\|_{W^{1,6}}$ and, hence, using a Sobolev imbedding,

$$\|\nabla\phi_\pi^1\|_{L_6}\leqslant C|\phi|_2. \tag{3.14}$$

Now, using Theorem 3.2 once again, we get

$$|I|\leqslant C(\|u_\pi-u_h\|^{1/2}\|\nabla(u_\pi-u_h)\|^{3/2}+h^{r-1/2}\|\nabla(u_\pi-u_h)\|)\|\phi\|_2.$$

To bound II_E, we rewrite this term as

$$\begin{aligned}
II&=\int_E \boldsymbol{\kappa}(u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla(\phi_\pi^1-\phi)\,\mathrm{d}\boldsymbol{x}+\int_E \boldsymbol{\kappa}(u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x}\\
&=\int_E \boldsymbol{\kappa}(u)(\boldsymbol{P}_h-\boldsymbol{I})\nabla(u_h-u_\pi)\cdot\nabla(\phi_\pi^1-\phi)\,\mathrm{d}\boldsymbol{x}+\int_E (\boldsymbol{P}_h-\boldsymbol{I})(\boldsymbol{\kappa}(u)\nabla\phi)\nabla(u_h-u_\pi)\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

Then for $II=\sum_E II_E$, using Theorem 3.2, it immediately follows that

$$|II|\leqslant Ch\|\nabla(u_h-u_\pi)\|\|\phi\|_2.$$

Next, we consider the term III_E, which can be rewritten as

$$III_E=\int_E (P_h u_h-u_h)\bar{\boldsymbol{\kappa}}_u[\nabla(u_h-u_\pi)\cdot\nabla\phi_\pi^1+\nabla u_\pi\cdot\nabla\phi_\pi^1]\,\mathrm{d}\boldsymbol{x}=III_{E,1}+III_{E,2}.$$

Then using the Hölder inequality

$$\|vw\|\leqslant\|v\|_{L_3}\|w\|_{L_6}, \tag{3.15}$$

we obtain for $III_1=\sum_E III_{E,1}$

$$|III_1|\leqslant C\|P_h u_h-u_h\|_{L_3}\|\nabla\phi_\pi^1\|_{L_6}\|\nabla(u_h-u_\pi)\|.$$

Further, using the stability property of P_h, namely $\|P_h\phi_I\|_{L_3(E)} \leqslant \tilde{C}\|\phi_I\|_{L_3(E)}$, with $\tilde{C} > 0$ independent of E and the Gagliardo–Nirenberg–Sobolev inequality

$$\|v\|_{L_3} \leqslant C\|v\|^{1/2}\,\|\nabla v\|^{1/2}, \tag{3.16}$$

we obtain

$$\|P_h u_h - u_h\|_{L_3} \leqslant C\|u_\pi - u_h\|^{1/2}\|\nabla(u_\pi - u_h)\|^{1/2}. \tag{3.17}$$

Then, in view of (3.14), we get

$$|III_1| \leqslant C\|u_\pi - u_h\|^{1/2}\|\nabla(u_\pi - u_h)\|^{3/2}\|\phi\|_2.$$

Next, in view of the fact that $\nabla u_\pi \cdot \nabla\phi_\pi^1 \in \mathbb{P}_k(E)$, we have

$$III_{E,2} = \int_E (P_h u_h - u_h)(\bar{\boldsymbol{\kappa}}_u - c)\nabla u_\pi \cdot \nabla\phi_\pi^1 \,\mathrm{d}\boldsymbol{x}, \quad \forall c \in \mathbb{R}. \tag{3.18}$$

Thus, for $III_2 = \sum_E III_{E,2}$, we get

$$|III_2| \leqslant Ch\|u_h - P_h u_h\|_{L_3}\|\nabla\phi_\pi^1\|_{L_6}\|\nabla u_\pi\|.$$

Therefore, Theorem 3.2, and the Sobolev inequalities (3.16), (3.14), give

$$|III_2| \leqslant Ch\|u_h - u_\pi\|^{1/2}\|\nabla(u_h - u_\pi)\|^{1/2}\|\phi\|_2.$$

Collecting the above bounds, yields for $III = III_1 + III_2$

$$|III| \leqslant C(h\|u_h - u_\pi\|^{1/2}\|\nabla(u_h - u_\pi)\|^{1/2} + \|u_h - u_\pi\|^{1/2}\|\nabla(u_h - u_\pi)\|^{3/2})\|\phi\|_2.$$

Therefore

$$|a_h(u_h;u_h,\phi_\pi^1) - a(u_h;u_h,\phi_\pi^1)| \leqslant C(h\|\nabla(u_h - u_\pi)\| + \|u_h - u_\pi\|^{1/2}\|\nabla(u_h - u_\pi)\|^{3/2})\|\phi\|_2,$$

from which the desired bound follows using once again Theorem 3.2. □

LEMMA 3.3 Let $u \in H_0^1(\Omega)$ be the solution of (1.1) and assume that $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$, $s \geqslant 2$, $f \in H^{s-1}(\Omega)$, $\boldsymbol{\kappa}(u) \in W_\infty^{s-1}(\Omega)$ and $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$. Then there exists a positive constant C independent of h such that

$$|a(u;u,\phi) - a(u_h;u_h,\phi)| \leqslant C(h\|\nabla(u - u_h)\| + \|u - u_h\|^{1/2}\|\nabla(u - u_h)\|^{3/2} + h^r\|u\|_r + h^r\|f\|_{r-1})\|\phi\|_2,$$

where $r = \min\{s, k+1\}$.

Proof. Let $\phi_I \in V_h$ be the approximation of ϕ given by Theorem 3.3 and using (1.3) and (2.5) we split the difference $a(u;u,\phi) - a(u_h;u_h,\phi)$ as

$$\begin{aligned} a(u;u,\phi) - a(u_h;u_h,\phi) =& \{a(u;u,\phi - \phi_I) - a(u_h;u_h,\phi - \phi_I)\} + (f - P_h^{k-1}f, \phi_I) \\ &+ \{a_h(u_h;u_h,\phi_I) - a(u_h;u_h,\phi_I)\} = I + II + III. \end{aligned}$$

Then, in view of (3.13), we rewrite term I as

$$\begin{aligned} I &= (\boldsymbol{\kappa}(u_h)\nabla(u - u_h) + (\boldsymbol{\kappa}(u) - \boldsymbol{\kappa}(u_h))\nabla u, \nabla(\phi - \phi_I)) \\ &= (\boldsymbol{\kappa}(u_h)\nabla(u - u_h) + \bar{\boldsymbol{\kappa}}_u(u - u_h)\nabla u, \nabla(\phi - \phi_I)). \end{aligned}$$

Employing Theorem 3.3 and (3.25), we obtain

$$|I| \leqslant Ch(\|\nabla(u - u_h)\| + \|u - u_h\|\,\|\nabla u\|_{L_\infty})\|\phi\|_2 \leqslant Ch\|\nabla(u - u_h)\|\,\|\phi\|_2.$$

As for term II, using Lemma 3.1 we get

$$|II| \leqslant Ch^r \|f\|_{r-1} \|\nabla \phi_I\| \leqslant Ch^r \|f\|_{r-1} \|\phi\|_2. \tag{3.19}$$

In view of bounding term III, we write

$$\begin{aligned} III =& \{a_h(u_h; u_h - u_\pi, \phi_I - \phi_\pi^1) - a(u_h; u_h - u_\pi, \phi_I - \phi_\pi^1)\} \\ &+ \{a_h(u_h; u_\pi, \phi_I - \phi_\pi^1) - a(u_h; u_\pi, \phi_I - \phi_\pi^1)\} + \{a_h(u_h; u_h, \phi_\pi^1) - a(u_h; u_h, \phi_\pi^1)\} \\ =& III_1 + III_2 + III_3, \end{aligned} \tag{3.20}$$

with $\phi_\pi^1|_E \in \mathbb{P}_1(E)$ and $u_\pi|_E \in \mathbb{P}_k(E)$, for any $E \in \mathscr{T}_h$ given by Theorem 3.2. Using Theorems 3.2 and 3.3 we bound the term III_1 in (3.20) as

$$|III_1| \leqslant Ch\|\nabla(u_h - u_\pi)\| \|\phi\|_2 \leqslant Ch(\|\nabla(u - u_h)\| + h^{r-1}\|u\|_r)\|\phi\|_2.$$

Next, to estimate III_2, we split this term as a summation over each $E \in \mathscr{T}_h$ and use the polynomial consistency (2.6) and the definition of $\bar{\boldsymbol{\kappa}}_u$, given by (3.13), to get

$$\begin{aligned} & a_h^E(u_h; u_\pi, \phi_I - \phi_\pi^1) - a^E(u_h; u_\pi, \phi_I - \phi_\pi^1) \\ &= \int_E (\boldsymbol{\kappa}(P_h u_h) \nabla u_\pi \cdot \boldsymbol{P}_h \nabla(\phi_I - \phi_\pi^1) - \boldsymbol{\kappa}(u_h) \nabla u_\pi \cdot \nabla(\phi_I - \phi_\pi^1)\, \mathrm{d}\boldsymbol{x} \\ &= \int_E (\boldsymbol{\kappa}(P_h u_h) \nabla u_\pi \cdot (\boldsymbol{P}_h - \boldsymbol{I}) \nabla(\phi_I - \phi_\pi^1) + (\boldsymbol{\kappa}(P_h u_h) - \boldsymbol{\kappa}(u_h)) \nabla u_\pi \cdot \nabla(\phi_I - \phi_\pi^1))\, \mathrm{d}\boldsymbol{x} \\ &= III_2^1 + III_2^2. \end{aligned}$$

Then, following the steps used in the estimation of I_4 in (3.11) and using Theorems 3.2 and 3.3, we can see that

$$|III_2^1| \leqslant Ch(h^{r-1}\|u\|_{r,E} + \|P_h u_h - u\|_E)\|\phi\|_{2,E}. \tag{3.21}$$

To bound III_2^2, we first note, in view of (3.15), that

$$|III_2^2| \leqslant C\|P_h u_h - u_h\|_{L_3(E)} \|\nabla u_\pi\|_{L_6(E)} \|\nabla(\phi_I - \phi_\pi^1)\|_E. \tag{3.22}$$

Further, using the stability property of P_h, namely $\|P_h \phi_I\|_{L_3(E)} \leqslant \tilde{C}\|\phi_I\|_{L_3(E)}$, and the Gagliardo–Nirenberg–Sobolev inequality (3.16), we obtain

$$\|P_h u_h - u_h\|_{L_3(E)} \leqslant C\|u_\pi - u_h\|_E^{1/2} \|\nabla(u_\pi - u_h)\|_E^{1/2}, \tag{3.23}$$

with $C, \tilde{C} > 0$ independent of E. Using this in (3.22) and summing this new bound of (3.22) and (3.21) over all $E \in \mathscr{T}_h$ and using Theorems 3.2 and 3.3, it follows that

$$|III_2| \leqslant Ch(\|\nabla(u - u_h)\| + \|P_h u_h - u\| + h^{r-1}\|u\|_r)\|\phi\|_2.$$

Finally, as a consequence of Lemma 3.2 below, we have

$$|III_3| \leqslant C(\|u - u_h\|^{1/2} \|\nabla(u - u_h)\|^{3/2} + h^r\|u\|_r)\|\phi\|_2.$$

Combining this with (3.19), the bounds for III_1, and III_2, the desired bound follows. □

We are now in a position to prove the following preliminary L^2–norm, error bound.

THEOREM 3.6 Let $u \in H_0^1(\Omega)$ be the solution of (1.1) and assume that $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$, $s \geqslant 2$, $f \in H^{s-1}(\Omega)$ and $\boldsymbol{\kappa}(u) \in W_\infty^{s-1}(\Omega)$, with Ω convex. Then, for h small enough and $u_h \in V_h$ solution of (2.5) the following bound holds

$$\|u - u_h\| \leqslant C(h^r + \|u - u_h\|^3), \tag{3.24}$$

where $r = \min\{s, k+1\}$ and C is a positive constant independent of h.

Proof. We use a duality argument. Consider the (linear) auxiliary problem: find $\phi \in H_0^1(\Omega)$ such that

$$-\mathrm{div}(\boldsymbol{\kappa}(u)\nabla\phi) + \boldsymbol{\kappa}_u(u)\nabla u\cdot\nabla\phi = u - u_h.$$

Noting that this equates to $\boldsymbol{\kappa}(u)\Delta\phi = u - u_h$ and since we have assumed that Ω is convex, we have $\phi \in H^2(\Omega)$ and

$$\|\phi\|_2 \leqslant C\|u-u_h\|. \tag{3.25}$$

In variational form, the above problem reads

$$(\boldsymbol{\kappa}(u)\nabla\phi,\nabla v) + (\boldsymbol{\kappa}_u(u)\nabla u\cdot\nabla\phi, v) = (u-u_h, v), \quad \forall v \in H_0^1(\Omega), \tag{3.26}$$

Then choosing $v = u - u_h$ in (3.26)

$$\begin{aligned}
\|u-u_h\|^2 &= (\boldsymbol{\kappa}(u)\nabla\phi,\nabla(u-u_h)) + (\boldsymbol{\kappa}_u(u)(u-u_h)\nabla u,\nabla\phi)\\
&= (\boldsymbol{\kappa}(u)\nabla u,\nabla\phi) - (\boldsymbol{\kappa}(u_h)\nabla u_h,\nabla\phi) - ((\boldsymbol{\kappa}(u)-\boldsymbol{\kappa}(u_h))\nabla u_h,\nabla\phi)\\
&+ (\boldsymbol{\kappa}_u(u)(u-u_h)\nabla u,\nabla\phi)\\
&= (\boldsymbol{\kappa}(u)\nabla u,\nabla\phi) - (\boldsymbol{\kappa}(u_h)\nabla u_h,\nabla\phi) + ((\boldsymbol{\kappa}(u)-\boldsymbol{\kappa}(u_h))\nabla(u-u_h),\nabla\phi)\\
&- ((\boldsymbol{\kappa}(u)-\boldsymbol{\kappa}(u_h))\nabla u - \boldsymbol{\kappa}_u(u)(u-u_h)\nabla u,\nabla\phi)\\
&= \Big(a(u;u,\phi) - a(u_h;u_h,\phi)\Big)\\
&+ \Big(((\bar{\boldsymbol{\kappa}}_u(u-u_h)\nabla(u-u_h),\nabla\phi) - ((\bar{\boldsymbol{\kappa}}_{uu}(u-u_h)^2\nabla u,\nabla\phi)\Big) =: I + II,
\end{aligned} \tag{3.27}$$

with $\bar{\boldsymbol{\kappa}}_u$ given by (3.13) and $\bar{\boldsymbol{\kappa}}_{uu}$ such that

$$\boldsymbol{\kappa}(u) - \boldsymbol{\kappa}(u_h) - \boldsymbol{\kappa}_u(u)(u-u_h) = (u-u_h)^2\int_0^1 \boldsymbol{\kappa}_{uu}(u-t(u-u_h))\,dt = \bar{\boldsymbol{\kappa}}_{uu}(u-u_h)^2. \tag{3.28}$$

In the sequel we will show Lemma 3.3, which in view of (3.25), gives

$$|I| \leqslant C(h\|\nabla(u-u_h)\| + \|u-u_h\|^{1/2}\|\nabla(u-u_h)\|^{3/2} + h^r\|u\|_r + h^r\|f\|_{r-1})\|u-u_h\|. \tag{3.29}$$

For II in (3.27), using the Hölder inequality (3.15) and the fact that $\bar{\boldsymbol{\kappa}}_u, \bar{\boldsymbol{\kappa}}_{uu}$ are bounded uniformly on $\mathbb{R}$, we get

$$\begin{aligned}
|II| &\leqslant C\|\nabla(u-u_h)\|\,\|(u-u_h)\nabla\phi\| + C\|(u-u_h)\nabla u\|\,\|(u-u_h)\nabla\phi\|\\
&\leqslant C\|\nabla(u-u_h)\|\,\|u-u_h\|_{L_3}\|\nabla\phi\|_{L_6} + C\|u-u_h\|_{L_3}^2\|\nabla u\|_{L_6}\,\|\nabla\phi\|_{L_6}.
\end{aligned}$$

Next, in view of the Gagliardo–Nirenberg–Sobolev inequality (3.16), the Sobolev Imbedding Theorem and the elliptic regularity (3.25), we have

$$\begin{aligned}
|II| &\leqslant C\|\nabla(u-u_h)\|^{3/2}\|u-u_h\|^{1/2}\|u-u_h\| + C\|\nabla(u-u_h)\|\,\|u-u_h\|\|u-u_h\|\\
&\leqslant C\|\nabla(u-u_h)\|^{3/2}\|u-u_h\|^{1/2}\|u-u_h\|.
\end{aligned} \tag{3.30}$$

Combining the previous estimates for terms I and II, we obtain

$$\|u-u_h\| \leqslant Ch\|\nabla(u-u_h)\| + Ch^r(\|u\|_r + \|f\|_{r-1}) + C\|\nabla(u-u_h)\|^3 + \frac{1}{2}\|u-u_h\|,$$

from which, in view of Theorem 3.5, we conclude that

$$\|u-u_h\| \leqslant Ch\|u-u_h\| + Ch^r(\|u\|_r + \|f\|_{r-1}) + C\|u-u_h\|^3.$$

The desired bound now follows for h sufficiently small. □

Having concluded the proof of Theorem 3.6, in order to show optimal convergence rate of the error in H^1 and L^2-norms, it remains to demonstrate that u_h converge to u.

THEOREM 3.7 Under the same assumptions as in Theorems 3.5 and 3.6, the VEM solution u_h converges to the exact solution u in $H_0^1(\Omega)$.

Proof. From Theorem 3.4 it follows that $\|\nabla u_h\|$ is bounded from above. Therefore, we can choose a subsequence u_{h_k} such that for some $z \in H_0^1(\Omega)$, $u_{h_k} \to z$, weakly in $H_0^1(\Omega)$, as $h_k \to 0$ and, thus, strongly in $L^2(\Omega)$. Also, for arbitrary $v \in C_0^\infty(\Omega)$ let v_{h_k} be a sequence in V_{h_k} such that

$$\|\nabla(v - v_{h_k})\| \to 0, \quad h_k \to 0. \tag{3.31}$$

Then

$$\begin{aligned}|a(z;z,v) - (f,v)| &\leqslant |(\boldsymbol{\kappa}(z)\nabla z, \nabla(v - v_{h_k})| \\ &+ |(\boldsymbol{\kappa}(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| + |(P_h^{k-1} f, v_{h_k} - v)| + |\varepsilon_h(f,v)| \\ &\leqslant C\|\nabla(v - v_{h_k})\| + |(\boldsymbol{\kappa}(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| + Ch_k\|f\|_1\|v\|.\end{aligned}$$

Thus, if

$$|(\boldsymbol{\kappa}(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \to 0, \quad h_k \to 0, \tag{3.32}$$

then z is the weak solution of (1.1). To show (3.32), we rewrite its left-hand side as

$$\begin{aligned}&|(\boldsymbol{\kappa}(z)\nabla z, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \\ &\leqslant |(\boldsymbol{\kappa}(z)\nabla z - \boldsymbol{\kappa}(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k})| + |(\boldsymbol{\kappa}(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \\ &\leqslant C\|\nabla(v - v_{h_k})\| + |(\boldsymbol{\kappa}(z)\nabla(z - u_{h_k}), \nabla v)| + |((\boldsymbol{\kappa}(z) - \boldsymbol{\kappa}(u_{h_k}))\nabla u_{h_k}, \nabla v)| \\ &+ |(\boldsymbol{\kappa}(u_{h_k})\nabla u_{h_k}, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})|\end{aligned}$$

Using the fact that $u_{h_k} \to z$, and $v_{h_k} \to v$, we see that (3.32) holds. Hence $a(z;z,v) = (f,v)$, and thus $u = z$, since u is the unique solution of (1.1). Then, it follows that $u_h \to u$ in $L^2(\Omega)$. Hence, $\|u - u_h\| \to 0$ and the result follows from Theorems 3.6, and 3.5. □

In view of Theorems 3.5, 3.6 and 3.7, the following a priori error estimates now readily follows.

THEOREM 3.8 Let $u \in H_0^1(\Omega)$ be the solution of (1.1) and suppose that $u \in H^s(\Omega) \cap W_\infty^1(\Omega)$, $s \geqslant 2$, assuming that $f \in H^{s-1}(\Omega)$ and $\boldsymbol{\kappa}(u) \in W_\infty^{s-1}(\Omega)$, with Ω convex. Let also $u_h \in V_h$ be the solution of (2.5). Then, there exists a constant C independent of h such that, for h sufficiently small,

$$\|u - u_h\| + h\|\nabla(u - u_h)\| \leqslant Ch^r, \tag{3.33}$$

where $r = \min\{k+1, s\}$.

4. Iteration method

In this section we show that, given a virtual element space V_h, the sequence of solutions we obtain using fixed point iterations to solve the VEM problem (2.5) converges to the true solution $u_h \in V_h$ of (2.5).

Starting with a given $u_h^0 \in V_h$ we construct a sequence u_h^n, $n \geqslant 0$, such that

$$a_h(u_h^n; u_h^{n+1}, v_h) = (P_h^{k-1} f, v_h), \quad \forall v_h \in V_h. \tag{4.1}$$

The convergence in H^1 of the sequence u_h^n as $n \to \infty$ to a fixed point of (4.1), and hence a solution of (2.5), is an immediate consequence of the following result.

THEOREM 4.1 Let $\{u_h^n\} \subset V_h$ be the sequence produced in (4.1), then

$$\|\nabla(u_h^n - u_h^{n+1})\| \to 0, \quad \text{as } n \to \infty. \tag{4.2}$$

Proof. In view of Assumption 2.2 and the fact that $a_h(u_h^n; \cdot, \cdot)$ is symmetric, we have

$$\begin{aligned}c_\star\|\nabla(u_h^n - u_h^{n+1})\|^2 &\leqslant a_h(u_h^n; u_h^n - u_h^{n+1}, u_h^n - u_h^{n+1}) \\ &= a_h(u_h^n; u_h^n, u_h^n) - 2a_h(u_h^n; u_h^{n+1}, u_h^n) + a_h(u_h^n; u_h^{n+1}, u_h^{n+1}),\end{aligned} \tag{4.3}$$

with $c_\star = \boldsymbol{\kappa}_* \alpha_*$. Then using (4.1), we obtain

$$a_h(u_h^n; u_h^{n+1}, u_h^n) = (P_h^{k-1} f, u_h^n - u_h^{n+1}) + a_h(u_h^n; u_h^{n+1}, u_h^{n+1}),$$

giving

$$\begin{aligned} c_\star \|\nabla(u_h^n - u_h^{n+1})\|^2 &\leqslant a_h(u_h^n; u_h^n, u_h^n) - 2(P_h^{k-1} f, u_h^n - u_h^{n+1}) - a_h(u_h^n; u_h^{n+1}, u_h^{n+1}) \\ &= \mathscr{F}(u_h^n) - \mathscr{F}(u_h^{n+1}), \end{aligned} \tag{4.4}$$

where $\mathscr{F}(v) = a_h(u_h^n; v, v) - 2(P_h^{k-1} f, v)$. Therefore, $\mathscr{F}(u_h^n)$ is a decreasing sequence and, in view of the fact that

$$\mathscr{F}(v) = a_h(u_h^n; v, v) - 2(P_h^{k-1} f, v) \geqslant \boldsymbol{\kappa}_* \|\nabla v\|^2 - 2\|f\| \|\nabla v\| \geqslant -\frac{\|f\|^2}{\boldsymbol{\kappa}_*}, \tag{4.5}$$

$\mathscr{F}(u_h^n)$ is bounded from below. Therefore $\mathscr{F}(u_h^n) - \mathscr{F}(u_h^{n+1}) \to 0$, as $n \to \infty$, which completes the proof. □

5. Numerical results

In order to test the VEM proposed in Section 2 we need to specify a bilinear form satisfying Assumption 2.2. We fix a_h^E as follows:

$$a_h^E(z_h; v_h, w_h) = \int_E \boldsymbol{\kappa}(P_h z_h)(\boldsymbol{P}_h \nabla v_h) \cdot (\boldsymbol{P}_h \nabla u_h)\,\mathrm{d}\boldsymbol{x} + S^E(z_h; (I - P_h)v_h, (I - P_h)w_h),$$

with the VEM stabilising form S^E given by

$$S^E(z_h; (I - P_h)v_h, (I - P_h)w_h) := \boldsymbol{\kappa}_E(P_h^{0,E} z_h) h_E^{d-2} \overrightarrow{(I - P_h)v_h} \cdot \overrightarrow{(I - P_h)w_h}.$$

here, I denotes the identity operator, $\overrightarrow{v_h}$ is the vector with entries the degrees of freedom of $v_h \in V_h^E$, and $\overrightarrow{v_h} \cdot \overrightarrow{w_h}$ is the euclidean scalar product of the degrees of freedom of $v_h, w_h \in V_h^E$.

The above definition of the local bilinear form extends to the nonlinear setting the one considered in (Cangiani *et al.*, 2017a) and, similarly to the linear case, it is straightforward to show that it satisfies the stability condition (2.7). Following (Beirão da Veiga *et al.*, 2013) instead, the projector R_h^ℓ can be used in place of P_h in the stabilising term. The practical implementation of these projector operators and VEM assembly are discussed in (Beirão da Veiga *et al.*, 2014; Cangiani *et al.*, 2017a).

In the examples below, approximation errors are measured by comparing the piecewise polynomial quantities $P_h^k u_h$ and $P_h^{k-1} \nabla u_h$ with the exact solution u and solution's gradient ∇u, respectively.

The tests are performed using the VEM implementation within the Distributed and Unified Numerics Environment (DUNE) library (Blatt *et al.*, 2016), available from (Cangiani *et al.*, 2019).

We use fixed point iterations analysed in Section 4 to solve the nonlinear system resulting from the VEM discretisation. This is compared below with Newton-Raphson iterations, defined as follows. Given an initial iterate $u_h^0 \in V_h$, we construct a sequence $u_h^{n+1} = u_h^n + \delta^n$, $n \geqslant 0$, by solving at each iteration the linearised problem: find $\delta^n \in V_h$ such that

$$a_h(u_h^n; \delta^n, v_h) + b_h(u_h^n; \delta^n, v_h) = (P_h^{k-1} f, v_h) - a_h(u_h^n; u_h^n, v_h), \quad \forall v_h \in V_h. \tag{5.1}$$

Here, the extra terms stemming from the linearisation of both the consistency and stability terms in a_h are collected in the global form $b_h := \sum_{E \in \mathscr{T}_h} b_h^E$, with the local form b_h^E, $E \in \mathscr{T}_h$, given by

$$\begin{aligned} b_h^E(u_h^n; \delta^n, v_h) = &\int_E \boldsymbol{\kappa}_u(P_h u_h^n) P_h \delta^k (\boldsymbol{P}_h \nabla u_h^n) \cdot (\boldsymbol{P}_h \nabla v_h)\,\mathrm{d}\boldsymbol{x} \\ &+ h_E^{d-2} \boldsymbol{\kappa}_u(P_h^{0,E} u_h^n) P_h^{0,E} \delta^k \overrightarrow{u_h^n - P_h u_h^h} \cdot \overrightarrow{v_h - P_h v_h}. \end{aligned}$$

Numerical test 1. We consider the following test problem from (Chatzipantelidis *et al.*, 2005). We solve (1.1) on $\Omega = (0,1)^2$ with $\boldsymbol{\kappa}(u) = 1/(1+u)^2$ and the function f chosen such that the exact solution is $u = (x - x^2)(y - y^2)$. Note that, although the diffusion coefficient is not even bounded on the whole of $\mathbb{R}$, it is smooth in a neighbourhood of the range of u. As initial guess for the nonlinear solve we use the constant zero function and the conjugate-gradient method is used to solve the linear system at each iteration. The relative errors for the approximation of u and its gradient as a function of the mesh size h are shown in Table 5 for $k = 1$ and a sequence of polygonal meshes generated using (Talischi *et al.*, 2012), cf. the right-most plot in Figure 1. The numerical results confirm the theoretical rate of convergence. The

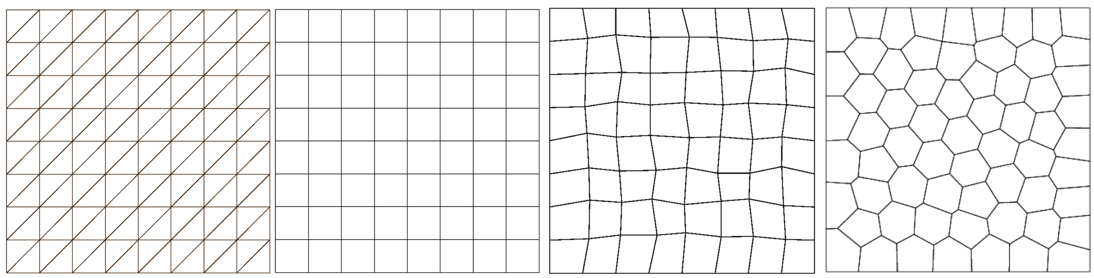

FIG. 1. Numerical Test 1. Sample meshes corresponding to an 8×8 subdivision of the domain: triangles, squares, radom quads, and polygons.

table also displays the number of fixed point and Newton-Raphson iterations performed until the indicated stopping criteria is reached.

DOF	$\|u - P_h^k u_h\|$	EOC	$\|\nabla u - P_h^{k-1} \nabla u_h\|$	EOC	FP	NR
9	1.30E-02	–	9.44E-02	–	6	4
34	3.40E-03	2.018	4.96E-02	0.967	7	4
129	8.16E-04	2.140	2.51E-02	1.022	6	4
510	1.89E-04	2.131	1.25E-02	1.012	6	4
2042	4.49E-05	2.070	6.26E-03	1.001	6	3
8162	1.11E-05	2.011	3.12E-03	1.006	6	3

Table 1. Numerical test 1. Errors and empirical order of convergence (EOC) on a sequence of polygonal meshes. The Fixed Point (FP) and Newton-Raphson (NR) iterations needed to reach the tolerance 10^{-10} are reported in the right-most columns.

The convergence history with respect to all meshes in Figure 1 are reported in the loglog plots of Figure 2 showing that the performance is similar in all cases. Note that, as $k = 1$, in the case of the

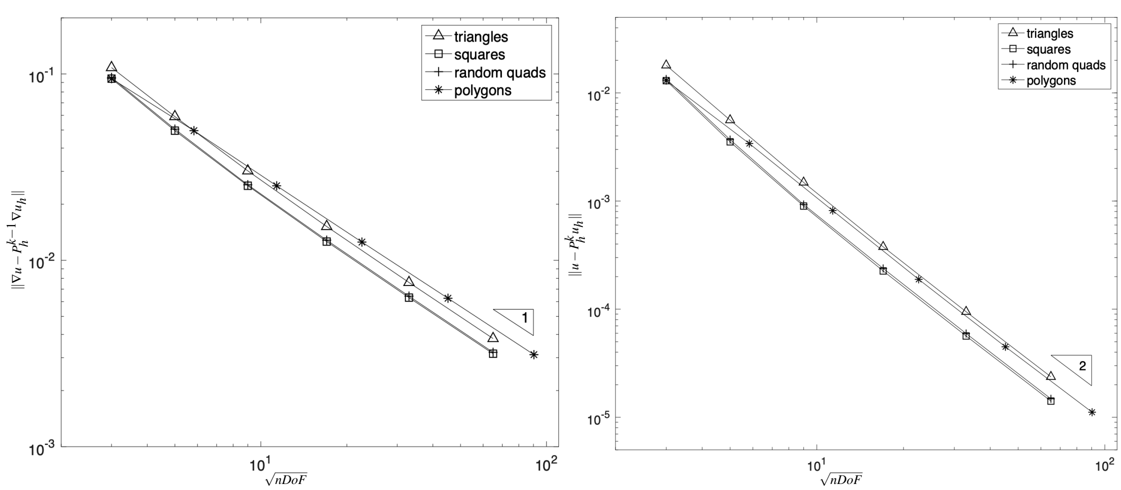

FIG. 2. Numerical test 1. Convergence history for $k = 1$ and the sequences of meshes represented in Figure 1.

sequence of triangular meshes, the VEM coincides with the standard linear finite element method.

Numerical test 2. We consider a test problem with smooth diffusion coefficient proposed in (Bi & Ginting, 2007). Namely, we solve (1.1) on $\Omega = (0,1)^2$ with $\boldsymbol{\kappa}(u) = 1 + 1/(1+u^2)$ and the function f chosen such that the exact solution is $u = \sin(3\pi x)\sin(3\pi y)$. We use the same initial guess and linear solver as in the first test, but only consider Newton-Raphson iterations this time. We test the VEM of order $k = 1$ up to 4 on a sequence of Voronoi meshes generated from random seeds exemplified in Figure 3. The convergence history reported in Figure 4 confirms the theoretical results. The slightly unsettled

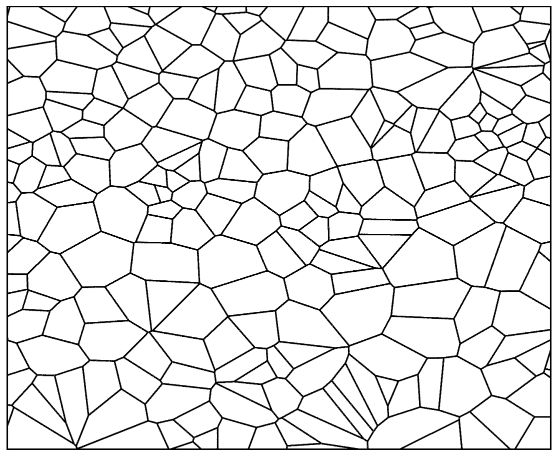

FIG. 3. Sample mesh fromthe Vornonoi sequence used in numerical tests 2 and 3.

behaviour of some of the convergence curves is due to the uneven size of the mesh elements of Voronoi meshes. Another characteristic of Vornonoi meshes is that mesh edges can be very small with respect to the element's diameter. Hence this test confirms, in the quasilinear setting, the well-known robustness of the VEM with respect to mesh quality, even though we do not consider here the refined methods of (Beirão Da Veiga *et al.*, 2017; Brenner & Sung, 2018).

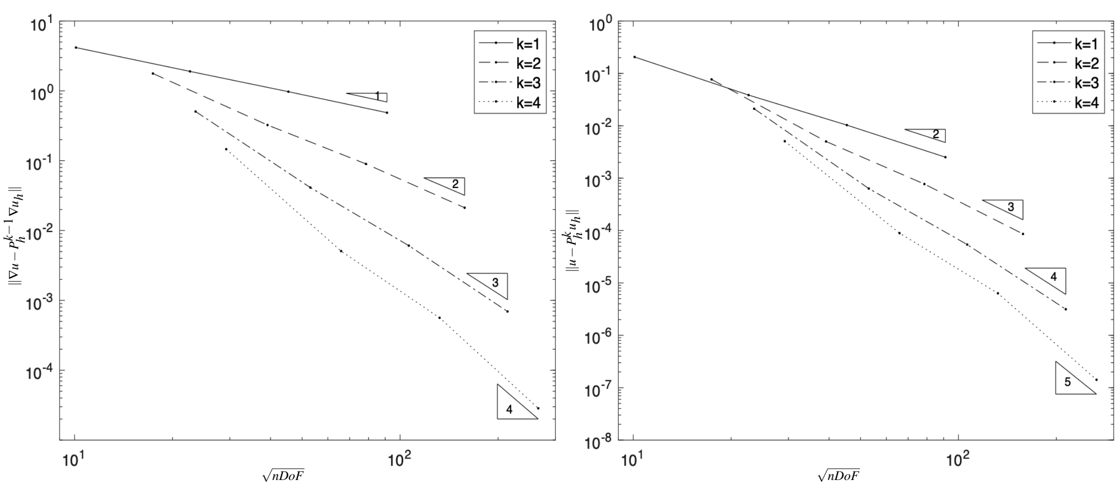

FIG. 4. Numerical test 2. Convergence history for $k = 1,2,3,4$ on a sequences of Voronoi meshes with random seeds.

Numerical test 3. The following test problem was proposed in (Chatzipantelidis *et al.*, 2005). We solve (1.1) on $\Omega = (0,1)^2$ with $\boldsymbol{\kappa}(u) = 1 + u$ and the forcing f chosen such that the exact solution is $u = x^{1.6}$. This solution belongs to $H^2(\Omega)$ but not to $H^3(\Omega)$ and the source term is in $L^2(\Omega)$ only. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and, given the low regularity of the solution, we only consider $k = 1,2$. In all cases, 3 Newton-Raphson iterations were needed to reach the tolerance 10^{-10}. The respective convergence histories are reported in Figure 5. As expected, the rate of convergence does not increase for $k = 2$ for this non-smooth problem.

The results for $k = 1$ can be compared to those obtained with the similar order Finite Volume Element Method of (Chatzipantelidis *et al.*, 2005) on structured triangular meshes. Although we have employed here the more irregular Voronoi meshes, the two methods give very similar results.

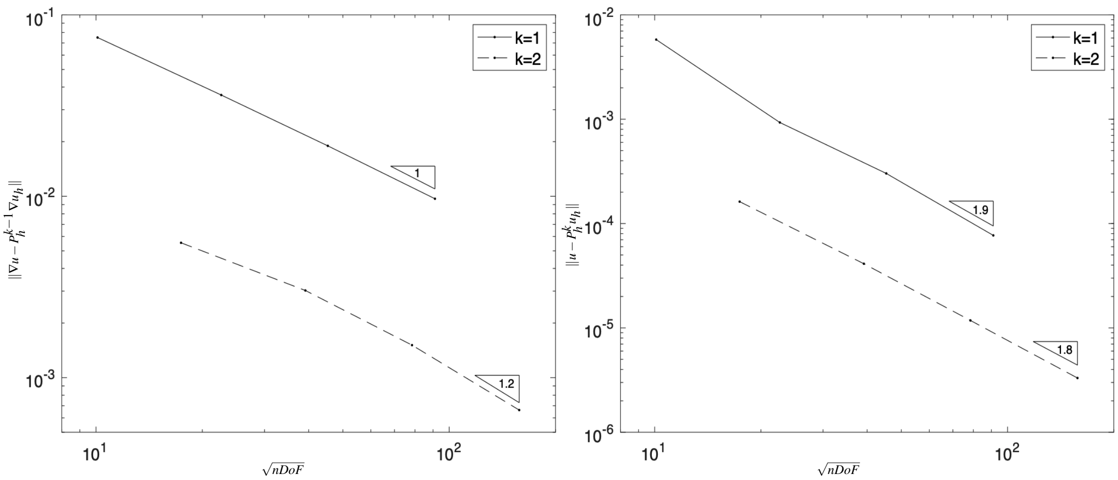

FIG. 5. Numerical test 3. Convergence history for $k = 1, 2$ on a sequences of Voronoi meshes with random seeds.

Numerical test 4. The following test problem is similar to a problem proposed in (Bi & Ginting, 2011). We solve (1.1) on $\Omega = (0,1)^2$ with $\boldsymbol{\kappa}(u) = 1 - 0.9\sin(8\pi u)$ and the forcing f chosen such that, as in numerical test 1, the exact solution is $u = (x - x^2)(y - y^2)$. Note that the diffusion coefficient is characterised by the oscillatory behaviour and may reach close to zero. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and $k = 1, 2, 3, 4$. In all computations, either 4 or 5 Newton-Raphson iterations were necessary to reach the tolerance of 10^{-10} starting from the initial guess $u = 0$. The convergence history is reported in Figure 6, once more confirming the theoretical rate of convergence. And we observe that, given that the solution is a simple polynomial, in the last iteration with $k = 4$ the L^2-norm error convergence is slowed down as the error has reached the Newton-Raphson tolerance.

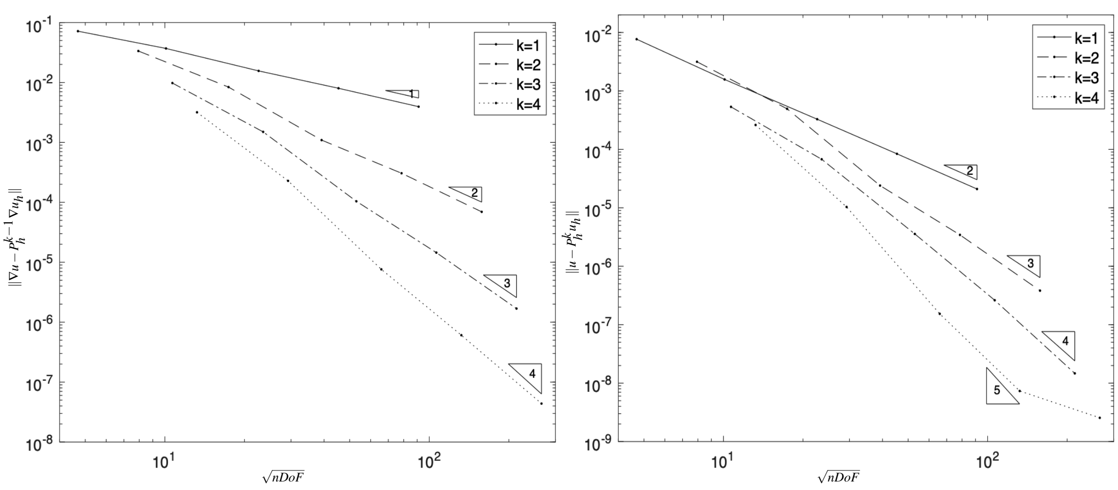

FIG. 6. Numerical test 4. Convergence history for $k = 1, 2, 3, 4$ on a sequences of Voronoi meshes with random seeds.

6. Conlusions

With this paper, we propose a VEM for elliptic quasilinear problems with Lipschitz continuous diffusion in two and three dimensions, showing that it suffices to evaluate the diffusion coefficient with the component of the VEM solution which is readily accessible. We prove optimal order a priori error estimates under the same mesh assumptions used in the linear setting.

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