A 4D gravity theory and G_2-holonomy manifolds

Yannick Herfray[(1),(2)], Kirill Krasnov[(1)], Carlos Scarinci[(1)] and Yuri Shtanov[(3)]

[(1)] *School of Mathematical Sciences, University of Nottingham, NG7 2RD, UK*

[(2)] *Laboratoire de Physique, ENS de Lyon, 46 allée d'Italie, F-69364 Lyon Cedex 07, France*

[(3)] *Bogolyubov Institute for Theoretical Physics, 14-b Metrologichna St., Kiev 03680, Ukraine*

July 18, 2019

Abstract

Bryant and Salamon gave a construction of metrics of G_2 holonomy on the total space of the bundle of anti-self-dual (ASD) 2-forms over a 4-dimensional self-dual Einstein manifold. We generalise it by considering the total space of an SO(3) bundle (with fibers $\mathbb{R}^3$) over a 4-dimensional base, with a connection on this bundle. We make essentially the same ansatz for the calibrating 3-form, but use the curvature 2-forms instead of the ASD ones. We show that the resulting 3-form defines a metric of G_2 holonomy if the connection satisfies a certain second-order PDE. This is exactly the same PDE that arises as the field equation of a certain 4-dimensional gravity theory formulated as a diffeomorphism-invariant theory of SO(3) connections. Thus, every solution of this 4-dimensional gravity theory can be lifted to a G_2-holonomy metric. Unlike all previously known constructions, the theory that we lift to 7 dimensions is not topological. Thus, our construction should give rise to many new metrics of G_2 holonomy. We describe several examples that are of cohomogeneity one on the base.

1 Introduction

The history of G_2-geometry is almost as old as that of the exceptional Lie group G_2 itself, see [1] for a nice exposition. The existence of metrics of G_2 holonomy was proven in [2]. This paper also gave a construction of the first explicit example. Several more examples, among them complete, were constructed in [3]. The first compact examples where obtained in [4]. More local examples can be obtained by evolving 6-dimensional SU(3) structures, see [5]. These examples, as well as many other things, are reviewed in [6]. Metrics of G_2 holonomy are of importance in physics as providing the internal geometries for compactification of M-theory down to 4 space-time dimensions, while preserving supersymmetry. A nice mathematical exposition of this aspect of G_2 geometry is given in [7].

Our interest in G_2 geometry is motivated by the fact that, as we explain in this paper, solutions of certain 4D gravity theory can be lifted to G_2-holonomy metrics. The gravity theory in question is *not* General Relativity, but rather a certain other theory whose existence can be seen by reformulating 4D gravity as a diffeomorphism invariant theory of SO(3) connections, as was described in [9, 10], and explained from a more mathematical perspective in [11]. Once 4-dimensional General Relativity is reformulated in the language of connections, one finds that there is not one, but an infinite parameter family of theories all resembling GR in their properties. The G_2-holonomy lift that we describe in this paper singles out one of them, and it is distinct from GR. We describe this theory in details in the main text.

A suggestion as to the existence of a link between some theory in 7 dimensions (referred to as topological M-theory) and theories of gravity in lower dimensions was made in [12]. That paper reinterpreted the constructions [3] of 7D metrics of G_2 holonomy from constant curvature metrics in 3D and self-dual Einstein metrics in 4D as giving evidence (among other things) for the existence of such a link. The construction of the present paper is similar in spirit, but we present a much stronger evidence linking 4D and 7D structures. Thus, our construction lifts any solution of a certain 4D gravity theory with local degrees of freedom to a G_2 metric. The main difference with the previous examples is that the theory that one is able to lift to 7D is no longer topological. We find this result to be interesting as it interprets the full-fledged 4D gravity as a dimensional reduction of a theory of differential forms in 7 dimensions.

We now formulate the main result of this paper. Let A^i, $i = 1, 2, 3$, be an SO(3) connection on a 4-dimensional manifold M, and let $F^i = dA^i + (1/2)\epsilon^{ijk} A^j \wedge A^k$ be its curvature 2-form. Then, fixing an arbitrary volume form v on M, define a 3×3 symmetric matrix X^{ij} by the relation $F^i \wedge F^j = -2X^{ij} v$. We will call a connection *definite* if X^{ij} is a definite matrix, i.e. all eigenvalues have the same sign. The factor of 2 in the definition of the matrix X^{ij} is introduced for the future convenience. The minus sign in the same definition has to do with our later usage of anti-self-dual two-forms rather than self-dual ones.

Let E be an associated vector bundle over M with 3-dimensional fibers. We consider the following 3-form on the total space of E:

$$\begin{aligned}\Omega = &\frac{1}{6}(1 + \sigma y^2)^{-3/4} \epsilon^{ijk} d_A y^i \wedge d_A y^j \wedge d_A y^k \\ &+ 2\sigma(1 + \sigma y^2)^{1/4} d_A y^i \wedge F^i.\end{aligned} \tag{1}$$

Here y^i are coordinates in the fiber, and d_A is the covariant derivative with respect to A, and $\sigma = \pm 1$ is the sign of the connection to be defined below.

Theorem 1. *If A is a definite connection satisfying the second order PDE:*

$$d_A \left[(\det X)^{1/3} X^{-1} F \right] = 0, \tag{2}$$

then the 3-form (1) is stable, closed ($d\Omega = 0$) and co-closed ($d^*\Omega = 0$), and hence defines a metric of G_2 holonomy. This metric is of Riemannian signature, and is complete (in the fiber direction) for $\sigma = +1$.

Remark. The object X^{-1} in (2) is the symmetric matrix inverse to X, and $\det X$ is the determinant of X. Note that the expression under the covariant derivative in (2) is of homogeneity degree zero in X; therefore, equation (2) does not depend on the particular choice of the orientation form v used to define X. The sign σ of a definite connection is defined in Sec. 3.5 below.

As we shall explain below, equation (2) arises as the Euler–Lagrange equation of a certain diffeomorphism-invariant theory of connections on M. Thus, the theorem states that every solution of this theory can be lifted to a G_2-holonomy metric in the total space of the bundle. As we shall also explain below, when the matrix $X^{ij} \sim \delta^{ij}$, the connection A is the anti-self-dual part of the Levi-Civita connection on a self-dual Einstein manifold. In this case, equation (2) is satisfied automatically as it reduces to the Bianchi identity for the curvature. The construction in the above theorem in this case reduces to that described in [3].

We also note that the metric in the total space E of the bundle defined by the form (1) induces a metric on the base M. This metric turns out to be in the conformal class of

$$g_F(\xi, \eta) \sim \epsilon^{ijk} i_\xi F^i \wedge i_\eta F^j \wedge F^k / v, \tag{3}$$

where ξ and η are vector fields tangent to the base. The volume form of the metric induced on the base is a constant multiple of

$$v_F = (\det X)^{1/3}\, v, \tag{4}$$

where v is the orientation form used to define the matrix X. It is easy to see that the expression on the right-hand-side of (4) does not depend on the particular choice of the orientation form v, hence, is well-defined. The action functional for the theory that gives rise to (2) is just the total volume of the space as computed using the volume form (4).

Another way to describe the relation between a gravity theory in 4D and theory of 3-forms in 7D is to compare their actions. The action principle that entails the relation $d^*\Omega = 0$ as its Euler–Lagrange equation is the total volume of the space as computed using the metric defined by Ω, see [13] and below. When one computes this 7D functional on the ansatz (1), one finds a constant multiple of the volume of the 4D base computed using (4). In other words, on ansatz (1), the 7D action functional reduces to the action of the 4D theory of connections. This relation between the action functionals makes it less surprising that their critical points are related.

We now proceed to describing all constructions in more detail. We start by reviewing some basic facts about 3-forms in 7 dimensions and their relation to G_2 holonomy. We then describe in Section 3 the diffeomorphism-invariant SO(3) gauge theory that gives rise to (2). In Section 4, we review the construction due to Bryant and Salamon. We present our generalisation of this construction in Section 5, and give examples of metrics that arise in this way in Section 6. We conclude with a discussion.

2 3-Forms in 7 dimensions and G_2-holonomy manifolds

The material in this section is standard (see, e.g., [13]) and is reviewed for the convenience of the reader. It was stunning for us to realise that the beautiful geometry reviewed below has been known for more than a century, see [1]. In particular the characterisation of G_2 via 3-forms is a result due to Engel from 1900.

2.1 Stable 3-forms

Let us start with some linear algebra in $\mathbb{R}^7$. A 3-form $\Omega \in \Lambda^3\mathbb{R}^7$ is called *stable* if it lies in a open orbit under the action of GL(7), see [13]. This notion gives a generalisation of non-degeneracy of forms and implies that any nearby form can be reached by a GL(7) transformation. Thus, stable 3-forms can also be called generic or non-degenerate.

For real 3-forms, there are exactly two distinct open orbits, characterised by the sign of a certain invariant, see below, each of which is related to a real form of $G_2^{\mathbb{C}}$. In this paper we are mostly concerned with the open orbit corresponding to the compact real form G_2. For every such Ω, there exists a set $\theta^1, \ldots, \theta^7$ of 1-forms in which Ω is expanded in the following canonical form:

$$\Omega = \theta^5 \wedge \theta^6 \wedge \theta^7 + \theta^5 \wedge \Sigma^1 + \theta^6 \wedge \Sigma^2 + \theta^7 \wedge \Sigma^3, \tag{5}$$

where

$$\begin{aligned} \Sigma^1 &= \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4, \\ \Sigma^2 &= \theta^1 \wedge \theta^3 - \theta^4 \wedge \theta^2, \\ \Sigma^3 &= \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3. \end{aligned} \tag{6}$$

Here the particular combinations Σ^i are motivated in relation to (anti-)self-duality in 4 dimensions and are related to the embedding of SO(3) into SO(4) $\subset G_2$. The relation to anti-self-dual 2-forms in 4 dimensions will be important in the construction below.

The fact of central importance about stable 3-forms in 7 dimensions is that a stabilizer of such a form in GL(7) is isomoprhic to the exceptional Lie group G_2. This group has dimension 14, and this number arises as the dimension 49 of GL(7) minus the dimension 35 of $\Lambda^3\mathbb{R}^7$. Thus, the space of stable 3-forms is the homogeneous group manifold GL(7)$/G_2$.

We can then generalise the notion of stable forms to 3-forms on a 7-dimensional differentiable manifold M. These are forms that are stable at every point.

2.2 The metric

The basic fact about stable 3-forms on a 7-dimensional manifold M is that they naturally define a metric in M by the relation

$$g_\Omega(\xi, \eta) v_{g_\Omega} = i_\xi \Omega \wedge i_\eta \Omega \wedge \Omega. \tag{7}$$

Here, v_{g_Ω} is the metric volume form, and i_ξ denotes the operation of insertion of a vector into a form. The sign of the metric volume form v_{g_Ω} is uniquely fixed by the requirement that the metric defined by (7) has specific (say, Euclidean) signature. In this way, a 3-form Ω defines both the metric g_Ω and the orientation, corresponding to v_{g_Ω}.

It is then a simple computation that, for a 3-form presented in the canonical form (5), the arising metric is

$$g_\Omega = \sum_{I=1}^{7} \theta^I \otimes \theta^I, \tag{8}$$

and the orientation is given by $\theta^1 \wedge \cdots \wedge \theta^7$. Given that G_2 is the stabilizer of (5), it also stabilizes metric (8). This gives an embedding $G_2 \subset$ SO(7).

What we have reviewed above concerns the compact real form G_2 of $G_2^{\mathbb{C}}$. There is also the orbit of real 3-forms that is related to the non-compact real form of $G_2^{\mathbb{C}}$. Such 3-forms also have a canonical form similar to (5), but with some signs changed. In exactly the same way as (7), they give rise to a metric of signature $(3, 4)$.

The counting of components shows that 3-forms contain more information than just that of a metric. Indeed, to specify a metric in 7 dimensions, we need $7 \times 8/2 = 28$ numbers, while the dimension of the space of 3-forms is 35. Thus, there are 7 more components in a 3-form. These correspond to components of a unit spinor, see [7] for more details.

2.3 A functional

Given a stable 3-form, we construct the metric and the corresponding volume form as above. We can integrate this volume form over the manifold to get the functional

$$S[\Omega] = \int_M v_{g_\Omega}. \tag{9}$$

This functional can also be computed explicitly, without computing the metric, via the following construction. Let $\tilde{\epsilon}^{\alpha_1\cdots\alpha_7}$ be the canonical anti-symmetric tensor density that exists independently of any metric. Then construct

$$\Omega_{\alpha_1\beta_1\gamma_1}\cdots\Omega_{\alpha_4\beta_4\gamma_4}\tilde{\Omega}^{\alpha_1\cdots\alpha_4}\tilde{\Omega}^{\beta_1\cdots\beta_4}\tilde{\Omega}^{\gamma_1\cdots\gamma_4}, \tag{10}$$

where

$$\tilde{\Omega}^{\alpha_1\cdots\alpha_4} := \tilde{\epsilon}^{\alpha_1\cdots\alpha_7}\Omega_{\alpha_5\alpha_6\alpha_7}. \tag{11}$$

Then the object (10) is of homogeneity degree 7 in Ω, and has density weight 3. Its cube root then has the right density to be integrated over the manifold. The functional constructed in this way is a multiple of (9).

It is interesting to note that the invariant (10) has been known already to Engel in 1900, see [1]. This invariant gives a useful stability criterion: a form Ω is stable iff (10) (equivalently (9)) is non-zero. The sign of this invariant then allows to distinguish between the two GL(7) 3-form orbits described above.

2.4 The first variation

As explained in [13], the first variation of the functional (9) in Ω has a simple form

$$\delta S[\Omega] \sim \int_M {}^*\Omega \wedge \delta\Omega. \tag{12}$$

The precise numerical coefficient in this equation is of no importance for us. The 4-form ${}^*\Omega$ is just the Hodge dual of Ω computed with respect to the metric defined by Ω.

2.5 Holonomy reduction

The fundamental result due to Alfred Gray [8] states: Let $\Omega \in \Lambda^3 M$ be a 3-form on a 7-manifold. Then Ω is parallel with respect to the Levi-Civita connection of g_Ω iff $d\Omega = 0$ and $d^*\Omega = 0$. In other words, the condition of Ω being parallel with respect to the metric it defines is equivalent to the conditions of Ω being closed and co-closed, where co-closedness is again with respect to the metric it defines.

The next basic fact is that if a Riemannian manifold (M, g) has a parallel 3-form Ω, then the holonomy group of M is contained in G_2. In this paper, we will not be concerned whether the holonomy group is all of G_2 or is just contained in it, and will simply refer to 7-manifolds M with 3-forms satisfying $d\Omega = 0$ and $d^*\Omega = 0$ as G_2-holonomy manifolds. Techniques for proving that the holonomy equals G_2 can be found in [3].

Combining Gray's result with the formula (12) for the first variation of the functional $S[\Omega]$, we see that G_2-holonomy manifolds are critical points of $S[\Omega]$, provided one varies Ω in a fixed cohomology class $\delta\Omega = dB$, $B \in \Lambda^2 M$. This variational characterisation is explored in depth in [13].

3 Diffeomorphism-invariant SO(3) gauge theories and gravity

In this section, we review how gravity theories in 4D (including General Relativity) can be described as diffeomorphism-invariant theories of SO(3) connections. This material is mainly from [9, 10], see also [11] for a more mathematical exposition.

3.1 Volume functionals from SO(3) connections

As before, let A^i be an SO(3) connection in an associated $\mathbb{R}^3$ bundle over a 4-dimensional manifold M, and let $F^i = dA^i + (1/2)\epsilon^{ijk}A^j \wedge A^k$ be its curvature. Choose an orientation (volume) form v on M, and define the matrix X^{ij} via

$$F^i \wedge F^j = -2X^{ij}v. \tag{13}$$

Since different orientation forms are related by multiplication by a nowhere vanishing function, it is clear that X^{ij} is defined only modulo such multiplication. The factor of 2 here is for future convenience. The choice of minus sign has to do with later identification of F^i's with anti-self-dual forms.

Let $f(X^{ij})$ be a function from symmetric 3×3 matrices to reals satisfying the following two requirements: (i) it is gauge invariant $f(OXO^T) = f(X)$, where $O \in \mathrm{SO}(3)$; (ii) it is homogeneous of degree one, $f(\alpha X) = \alpha f(X)$ for any real α. It is clear that any such function can be applied to the wedge product of curvatures

$$f(F^i \wedge F^j) := -2f(X^{ij})v, \tag{14}$$

and that the result is a well-defined and gauge invariant 4-form on M. Thus, any such function gives rise to a diffeomorphism and gauge invariant functional of connections

$$S_f[A] = -\frac{1}{2}\int_M f(F^i \wedge F^j), \tag{15}$$

where integration is performed with respect to the orientation v. Note that this functional is just the total volume of M computed using the volume form constructed from the curvature of A. Thus, any choice of function f gives rise to a diffeomorphism-invariant theory of SO(3) connections.

It is clear that there are many functions f satisfying the required properties. An easy way to count is to diagonalise the matrix X. The function f is then a homogeneity degree one function of the eigenvalues. There are as many such functions as functions of two variables. We will describe some most interesting choices of f below.

3.2 Euler–Lagrange equations

The extrema of (15) are connections satisfying the following second order PDE's

$$d_A\left(\frac{\partial f}{\partial X^{ij}}F^j\right) = 0. \tag{16}$$

Note that the matrix of derivatives of the function f with respect to X is homogeneity degree zero in X, and is hence well-defined even though X is only defined modulo multiplication by a function. In other words, equations (16) are independent of the choice of the orientation form v in the definition (13) of X.

3.3 Definite connections and the choice of orientation

A preferred orientation of M can be fixed in the case of an important class of *definite* connections, see [11]. A connection A^i is called definite if the corresponding matrix X^{ij} defined via (13) is definite, i.e., all its eigenvalues are of the same sign. Then a preferred orientation of M is represented by an orientation form v for which the matrix X is positive definite.

In what follows, we will always use the orientation provided by the connection. In particular, we use the orientation that makes X a positive definite matrix in defining the action (15). Note that this does not mean that the functional (15) is always positive definite. For example, the function $f(X) = -\mathrm{Tr}\, X$ gives a negatively oriented volume form. In this paper, in order to avoid confusion, we will always use functions f that give volume forms of the same orientation as is provided by the connection. Thus, our action functionals here will always be of one (positive) sign.[1]

3.4 Metrics from definite connections

An SO(3) connection that satisfies a rather weak requirement that it is definite defines a conformal structure of a Riemannian metric on M. This is the conformal class already defined in (3). This is often referred to (especially in the physics literature) as the Urbantke metric, as it was first introduced in [14]. The significance of this (conformal) metric is that it is the unique conformal structure with respect to which the triple of curvature 2-forms is anti-self-dual.

To complete the definition of the metric we need to specify the volume form. As is explained above, any choice of function f (satisfying gauge invariance and homogeneity properties) gives a volume form. Thus, any choice of f defines a metric in the conformal class of (3).

Thus, once a choice of f is made, we have a metric defined by the connection. When the connection satisfies its Euler-Lagrange equations (16), the metric defined by A is constrained. Below we shall see that Einstein metrics can be obtained in exactly this way, for a certain choice of f.

3.5 The natural choice

Even though there exists freedom in choosing the conformal factor in (3), there exists a mathematically natural choice. We shall refer to the mathematically natural choice of the metric as *the Urbantke metric* g_{U}.

The connection provides an orientation (in which X is positive definite), and we choose the metric volume form to be positively oriented. We also require the metric to be of

[1] While the overall sign of the action is not important in the pure gravity theory, its sign relative to the action of other fields will, of course, be important.

Riemannian (all plus) signature. The Urbantke metric is then defined via

$$g_{\rm U}(\xi,\eta)v_{\rm U} = \frac{\sigma}{6}\epsilon^{ijk} i_\xi F^i \wedge i_\eta F^j \wedge F^k, \tag{17}$$

where $v_{\rm U}$ is the metric volume form, and where $\sigma = \pm 1$ is the sign that depends on the connection. This sign σ in (17) is called *the sign of the definite connection* A^i. It is discussed in more detail in Section 2.2 of [11], see also below. This sign is necessary in (17) to give the Urbantke metric the all plus signature.

3.6 A computation

As we know from above, any volume form constructed from the curvature corresponds to some choice of f. Let us see what this choice is for the Urbantke metric (17).

As we already mentioned above, any metric in the conformal class of (3) makes the triple of curvature 2-forms anti-self-dual (ASD). Let us choose *some* metric g in this conformal class, and introduce a canonical orthonormal basis Σ^i in the space of ASD 2-forms for the metric g. Explicitly, given a frame basis, Σ^i's are the forms that are given by (6). They satisfy the following algebraic relations

$$\frac{1}{2}\Sigma^i \wedge \Sigma^j = -\delta^{ij} v_g, \tag{18}$$

$$\Sigma^{i\ \rho}_{\mu}\Sigma^{j\ \nu}_{\rho} = \epsilon^{ijk}\Sigma^{k\ \nu}_{\mu} - \delta^{ij}\delta_\mu{}^\nu, \tag{19}$$

where the space indices are raised by the metric inverse of g. The minus sign in (18) has to do with our usage of ASD forms rather than SD ones. The volume form v_g in (18) is the metric volume form, positively oriented in the orientation provided by the connection.

Then the curvature 2-forms can be expanded in the basis of Σ^i as

$$F^i = \sigma \left(\sqrt{X}\right)^{ij} \Sigma^j, \tag{20}$$

where $\sigma = \pm 1$ is the sign of the definite connection A^i already introduced in the previous subsection, and $\sqrt{X}$ is the positive-definite matrix square root of the positive-definite matrix X. We stress that the relation (20) can be written for an arbitrary choice of metric g in the conformal class of (3). This relation can also be used as an alternative definition of the sign of the definite connection. The decomposition (20) follows using (18). Indeed, we have $F^i \wedge F^j = \sigma^2 \sqrt{X}^{ik}\sqrt{X}^{jl}(-2)\delta^{kl} v_g = -2X^{ij} v_g$.

We now use (20) with Σ^i's being those for the Urbantke metric (17). Thus, we now take $X = X_{\rm U}$ with respect to the volume form of the metric $g_{\rm U}$. Substituting (20) into (17) and using (19), we get the relation $g_{\rm U} = (\det X_{\rm U})^{1/2} g_{\rm U}$, from which we conclude that

$$\det X_{\rm U} = 1. \tag{21}$$

As we already remarked, the sign (and even the overall factor) of the Lagrangian function $f(X)$ in action (15) does not matter in the pure-gravity theory (see, however, footnote 1 on page 7). We can then always take this function to be positive-valued for positive-definite X. We then note that for any function f we can use the volume form $v_f = f(X)v$ to define X. One then has $v_f = f(X_f)v_f$ and hence $f(X_f) = 1$. This immediately allows us to translate the condition (21) into a choice of the function f. Thus, the condition (21) derived above corresponds to a homogeneous degree one function

$$f(X) = (\det X)^{1/3}\,. \tag{22}$$

We then note that for this function

$$\frac{\partial f}{\partial X} = \frac{1}{3}\,(\det X)^{1/3}\,X^{-1}, \tag{23}$$

and so the field equations (16) reduce to (2). As clear from the preceding subsection, a characteristic property of this function is that $f\,(F\wedge F)$ coincides with the volume form of the Urbantke metric $g_{\rm U}$ defined in (17).

3.7 Einstein connections

There is a *different* choice of $f(X)$ that gives rise to Einstein metrics [10]. Let us define

$$f_{\rm GR} = \left(\mathrm{Tr}\,\sqrt{X}\right)^2. \tag{24}$$

We then have

$$\frac{\partial f_{\rm GR}}{\partial X} = \left(\mathrm{Tr}\,\sqrt{X}\right) X^{-1/2}, \tag{25}$$

and

$$\frac{\partial f_{\rm GR}}{\partial X^{ij}} F^j = \sigma\,\mathrm{Tr}\,\sqrt{X}\,\Sigma^i, \tag{26}$$

where we have used (20). This is valid for X and Σ^i's defined as in (20) with respect to some metric in the conformal class of (3).

We can then fix the metric $g_{\rm GR}$ in the conformal class of (3) so that

$$\mathrm{Tr}\,\sqrt{X_{\rm GR}} = 1. \tag{27}$$

Once the metric is fixed in this way, the field equations (16) become $d_A \Sigma^i_{\rm GR} = 0$, where $\Sigma^i_{\rm GR}$ is the basis (6) of ASD forms for the metric $g_{\rm GR}$. This equation is equivalent to the statement that the connection A is the anti-self-dual part of the Levi-Civita connection for the metric with the basis of ASD 2-forms $\Sigma^i_{\rm GR}$. We then have a metric with the curvature of the ASD part of the Levi-Civita connection being ASD as a 2-form. This is known to be equivalent to the Einstein condition. The arising metrics are Einstein with the cosmological constant $\Lambda = 3\sigma$.

For more information about General Relativity in the language of connections the reader is referred to exposition in [11]. The choice of $f(X)$ that leads to GR is not to play any further role in this paper, and is described here just to illustrate the statement that it is the mathematically more natural choice (22) that plays role in the construction of the G_2 holonomy metrics, not (24).

Another way to state that a pure-connection theory with Lagrangian (22) is not GR is to say that connections satisfying (2) give rise to metrics (17) that are not Einstein. It would be interesting to characterise the arising metrics is some way.

3.8 Generality of the volume functionals

Even though this has little to do with the main subject of this paper, let us remark that the parametrisation (20) of the curvature makes it clear that the only gauge-invariant volume form that can be constructed from the curvature of the connection is of the type (14) for some function f. This follows from the fact that the volume form can only be constructed from factors of the curvature and the anti-symmetric tensor $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ that has density weight one and that exists on any manifold. Using (20) to parametrise the curvature, as well as the fact that Σ's are anti-self-dual, one can convince oneself that all contractions of the spacetime indices are taken care of by the algebra (19), and that what remains is some gauge-invariant scalar built from factors of the matrix $\sqrt{X}$. Thus, for SO(3) connections that define a conformal class of metrics, all gauge-invariant functions of the curvature are of the type (14). One can easily generalise the construction (14) to other gauge groups, but in that case there are functions of the curvature that do not reduce to (14). One should keep in mind this special character of the SO(3) theory.

3.9 Instanton solutions

For any function $f(X)$, connections satisfying $X^{ij} \sim \delta^{ij}$ give rise to metrics that are self-dual Einstein. Indeed, in this case the field equations (16) reduce to the Bianchi identity for the curvature and are automatically satisfied for any $f(X)$. When $X^{ij} \sim \delta^{ij}$, there exists a metric with respect to whose volume form $F^i = \sigma\Sigma^i$. The Bianchi identity then states that $d_A\Sigma^i = 0$ and, therefore, A is the ASD part of the Levi-Civita connection. It is clear that the corresponding metric is Einstein, as there is no SD part in the curvature 2-form of A. Also, because $F^i = \sigma\Sigma^i$, the ASD part of the Weyl tensor vanishes ($\text{Weyl}_- = 0$), and we have a self-dual Einstein metric of scalar curvature 12σ.

Thus, self-dual Einstein metrics corresponding to connections with $X^{ij} \sim \delta^{ij}$ are solutions of (16) for any $f(X)$. In particular, these solutions are shared by theory (22) and GR (24).

3.10 More general solutions

Even though we are far from understanding all Einstein metrics on 4-manifolds, some intuition as to how many solutions there exist comes from the Lorentzian version of the theory. Indeed, GR with Lorentzian signature is a theory with local degrees of freedom, and so the space of solutions is infinite-dimensional. For example, solutions can be obtained by evolving the initial data.

A similar description is also possible in the Riemannian context, in particular in the setting of asymptotically hyperbolic metrics. Then, as is well known from the work [15], one can solve for asymptotically hyperbolic Einstein metrics in the form of an expansion in powers of the 'radial' coordinate. The free data for this expansion are a conformal class of metric on the boundary (modulo boundary diffeomorphisms), together with a symmetric traceless transverse tensor that appears as free data in some higher order of the expansion. There are $2 + 2$ free functions on the boundary as free data, and this is the Riemannian analog of the statement that GR has 2 propagating degrees of freedom.

A similar expansion in the language of connections was developed in [16]. One outcome of the analysis of this paper is that the expansion is universal for the whole class of theories

(15), whatever the function f is. Only the details of the expansion at sufficiently high order in the radial coordinate start to depend on f. In the first few orders, the expansion is completely independent of f. In particular, the count of free data that seeds the expansion is f-independent. This means that the free data to be prescribed to get an asymptotically hyperbolic solution of theory (22) (locally near the boundary) are $2+2$ free functions on the 3-dimensional asymptotic boundary. This illustrates the statement that the theory (22) has as many solutions as GR.

Some explicit cohomogeneity one asymptotically hyperbolic solutions of theory (22) are described below.

4 Bryant–Salamon construction

We now review the construction of [3] using the notation compatible with our discussion of the SO(3) formulation of gravity.

4.1 Ansatz

Let (M, g) be a self-dual Einstein 4-manifold, and let Σ^i, $i = 1, 2, 3$, be the basis of ASD 2-forms satisfying properties (18) and (19). For example, the 2-forms Σ^i can be constructed from the frame 1-forms via (6). Let A^i be the ASD part of the Levi-Civita connection. This is an SO(3) connection that satisfies

$$d_A \Sigma^i = 0. \tag{28}$$

The self-dual Einstein condition translates into

$$F^i = \sigma \Sigma^i, \qquad \sigma = \pm 1, \tag{29}$$

where we have normalised our metric so that the scalar curvature is 12σ.

An arbitrary ASD 2-form can be written as $\Sigma(y) = \Sigma^i y^i$, and so the quantities y^i are the fiber coordinates in the bundle of ASD 2-forms over M. We make the following ansatz for the calibrating 3-form:

$$\Omega = \frac{1}{6}\alpha^3 \epsilon^{ijk} d_A y^i \wedge d_A y^j \wedge d_A y^k + 2\alpha\beta^2 d_A y^i \wedge \Sigma^i, \tag{30}$$

where $d_A y^i = dy^i + \epsilon^{ijk} A^j y^k$ is the covariant derivative with respect to A, and α and β are functions of y^2.

4.2 Closure

We now require the form Ω to be closed. Because Ω does not have any internal indices we can apply the covariant derivative instead of the exterior one. When differentiating the first term, we only need to differentiate the quantities $d_A y^i$, as differentiating α would lead to exterior products of four one-forms from the triple $\{d_A y^i\}$, which are zero. In the second term, we do not need to apply the derivative to Σ^i because it is covariantly closed. We also do not need to differentiate $d_A y^i$ since this produces a multiple of $\epsilon^{ijk} F^j y^k \wedge \Sigma^i$, which is equal to zero due to (29) and (18). We thus get

$$d\Omega = \frac{1}{2}\alpha^3 \epsilon^{ijk}\epsilon^{ilm} F^l y^m \wedge d_A y^j \wedge d_A y^k \qquad (31)$$
$$+ 2\left(\alpha\beta^2\right)' \left(2y^i d_A y^i\right) \wedge \left(d_A y^j \wedge \Sigma^j\right).$$

We now use (29) and decompose the product of two epsilon tensors into products of Kronecker deltas. We obtain

$$d\Omega = \left[-\sigma\alpha^3 + 4\left(\alpha\beta^2\right)'\right] \left(y^i d_A y^i\right) \wedge \left(d_A y^i \wedge \Sigma^j\right). \qquad (32)$$

Thus, we must have

$$4\left(\alpha\beta^2\right)' = \sigma\alpha^3 \qquad (33)$$

in order for the form to be closed. The quantity $\sigma = \pm 1$ is the sign already encountered above, see (20).

4.3 Canonical form

We now compute the metric defined by Ω, as well as its Hodge dual. The easiest way to do this is to write the 3-form in the canonical form, so that the metric and the dual form are immediately written. Thus, let $\theta^1, \ldots, \theta^7$ be a set of 1-forms such that the 3-form Ω is

$$\Omega = \theta^5 \wedge \theta^6 \wedge \theta^7 + \theta^5 \wedge \left(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4\right) \qquad (34)$$
$$+ \theta^6 \wedge \left(\theta^1 \wedge \theta^3 - \theta^4 \wedge \theta^2\right) + \theta^7 \wedge \left(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3\right).$$

Then the 1-forms θ are an orthonormal frame for the metric determined by Ω

$$g_\Omega = \left(\theta^1\right)^2 + \cdots + \left(\theta^7\right)^2, \qquad (35)$$

and the Hodge dual ${}^*\Omega$ of Ω is given by

$${}^*\Omega = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 - \left(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4\right) \wedge \theta^6 \wedge \theta^7 \qquad (36)$$
$$- \left(\theta^1 \wedge \theta^3 - \theta^4 \wedge \theta^2\right) \wedge \theta^7 \wedge \theta^5 - \left(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3\right) \wedge \theta^5 \wedge \theta^6.$$

4.4 Calculation of the metric and the dual form

We now put ansatz (30) into the canonical form (34), and compute the associated metric and the dual form. The canonical frame is easily seen to be

$$\theta^{4+i} = \alpha d_A y^i, \qquad \theta^I = \beta\sqrt{2} e^I, \qquad I = 1, 2, 3, 4, \qquad (37)$$

where e^I is the orthonormal frame such that the basis of ASD 2-forms is given by (6). The metric is then

$$g_\Omega = \alpha^2 \sum_i \left(d_A y^i\right)^2 + 2\beta^2 \sum_I \left(e^I\right)^2, \qquad (38)$$

and the dual form is

$${}^*\Omega = -\frac{2}{3}\beta^4 \Sigma^i \wedge \Sigma^i - \beta^2\alpha^2 \epsilon^{ijk} \Sigma^i \wedge d_A y^j \wedge d_A y^k. \qquad (39)$$

4.5 Co-closure

We now demand the 4-form (39) also to be closed. The first point to note is that when we apply the covariant derivative to the factor $\beta^2\alpha^2$ in the second term, we generate a 5-form proportional to the volume form of the fiber. There is no such term arising anywhere else, and we must demand

$$\alpha\beta = \text{const} \tag{40}$$

in order for (39) to be closed. Differentiation of the rest of the terms gives

$$\begin{aligned} d^*\Omega = &-\frac{2}{3}\left(\beta^4\right)'\left(2y^i d_A y^i\right) \wedge \Sigma^j \wedge \Sigma^j \\ &- 2\beta^2\alpha^2\epsilon^{ijk}\Sigma^i \wedge \epsilon^{jlm} F^l y^m \wedge d_A y^k. \end{aligned} \tag{41}$$

We now use (29) and (18) to get

$$d^*\Omega = -\frac{2}{3}\left[\left(\beta^4\right)' - \sigma\beta^2\alpha^2\right]\left(2y^i d_A y^i\right) \wedge \Sigma^j \wedge \Sigma^j, \tag{42}$$

and so we must have

$$\left(\beta^4\right)' = \sigma\beta^2\alpha^2. \tag{43}$$

4.6 Determining α and β

The overdetermined system of equations (33), (40) and (43) is nevertheless compatible. Without loss of generality, we can simplify things and rescale y^i (and therefore α) so that

$$\alpha\beta = 1. \tag{44}$$

With this choice, we have only one remaining equation to solve, which gives

$$\beta^4 = k + \sigma y^2,$$

where k is an integration constant. We can then further rescale y and β, keeping $\alpha\beta = 1$, to set $k = \pm 1$ at the expense of multiplying the 3-form Ω by a constant. After all these rescalings, we get the following incomplete solutions:

$$\begin{aligned} &\sigma = 1, \qquad \beta = (y^2 - 1)^{1/4}, \quad y^2 > 1, \\ &\sigma = -1, \quad\ \ \beta = (1 - y^2)^{1/4}, \quad y^2 < 1, \end{aligned} \tag{45}$$

as well as a complete solution for the positive scalar curvature:

$$\sigma = 1, \quad \beta = (1 + y^2)^{1/4}. \tag{46}$$

The two most interesting solutions, the incomplete solution for $\sigma = -1$ and the complete solution for $\sigma = +1$, can be combined together as

$$\beta = (1 + \sigma y^2)^{1/4}. \tag{47}$$

5 Construction in Theorem 1

We now give details of our generalisation of the Bryant–Salamon construction.

5.1 Ansatz and closure

We parametrise the 3-form by an SO(3) connection in an $\mathbb{R}^3$ bundle over M:

$$\Omega = \frac{1}{6}\alpha^3\epsilon^{ijk} d_A y^i \wedge d_A y^j \wedge d_A y^k + 2\sigma\alpha\beta^2 d_A y^i \wedge F^i, \tag{48}$$

where the factor $\sigma = \pm 1$ is the sign of the definite connection. It is introduced in the ansatz so that (48) reduces to (30) for instantons (29). It is then easy to see, using the Bianchi identity $d_A F^i = 0$, that the condition of closure of (48) is unmodified and is still given by (33).

5.2 The canonical form and the metric

We now put (48) into the canonical form (34). To this end, we use the parametrisation (20) of the curvature. It is then clear that the 1-forms θ^{4+i} are some multiples of $\alpha\sqrt{X}^{ij} d_A y^j$. The correct factors are easily found, and we have

$$\theta^{4+i} = (\det X)^{-1/6}\,\alpha\left(\sqrt{X}\right)^{ij} d_A y^j, \qquad \theta^I = \beta\sqrt{2}\,(\det X)^{1/12}\, e^I, \tag{49}$$

where e^I are the frame 1-forms for the metric which makes F^i anti-self-dual and whose volume form is used to define the matrix X^{ij}, see (13).

The metric determined by (48) is then

$$g_\Omega = \alpha^2\,(\det X)^{-1/3}\, d_A y^i X^{ij} d_A y^j + 2\beta^2\,(\det X)^{1/6} \sum_I \left(e^I\right)^2. \tag{50}$$

5.3 The dual form and the co-closure

The dual form reads

$$\begin{aligned} {}^*\Omega = &-\frac{2}{3}\beta^4\,(\det X)^{1/3}\left(X^{-1}F\right)^i \wedge F^i \\ &- \sigma\beta^2\alpha^2\,(\det X)^{1/3}\left(X^{-1}F\right)^i \epsilon^{ijk} \wedge d_A y^j \wedge d_A y^k, \end{aligned} \tag{51}$$

where again we expressed all ASD 2-forms on the base in terms of the curvature 2-forms using (20). Note that, in both terms, the curvature appears either as itself, or in the combination $(\det X)^{1/3}\,(X^{-1}F)^i$. It is now easy to see that the same steps we followed in the Bryant–Salamon case can be repeated provided

$$d_A\left[(\det X)^{1/3}\, X^{-1} F\right] = 0. \tag{52}$$

This is the field equation for theory (22) already quoted in (2). The Theorem stated in the Introduction is proven.

5.4 Complete indefinite G_2 metrics for $\sigma = -1$

We can modify our construction by not putting the sign σ in front of the second term in (48). Then all of the construction goes unchanged except that σ does not appear either in Ω or in ${}^*\Omega$. The differential equations for α and β then give $\beta = (1+y^2)^{1/4}$, and the metric is then complete in the fiber direction for either sign. But the price one pays in this case is that the second term in (50) will appear with a minus sign in front for $\sigma = -1$. This will give a complete (in the fiber direction) metric of G_2 holonomy, but of signature $(3,4)$ rather than a Riemannian metric.

5.5 Metric induced on the base

The 3-form (48) defines the metric (50) on the total space of the bundle. The metric induced on the base is in the conformal class that makes the curvature 2-forms F^i anti-self-dual. The conformal factor can be read off from (50). In particular, the corresponding volume form is

$$v_\Omega = 4\left(1+\sigma y^2\right)(\det X)^{1/3}\,\epsilon, \tag{53}$$

where ϵ is the orientation form used to define the matrix X. Thus, for a constant y^2 the induced metric is a multiple of the metric that we already encountered in the context of diffeomorphism-invariant SO(3) gauge theory defined by the function (22).

We now remark that, in the context of SO(3) gauge theory, the metric interpretation is possible, but nothing forces us to introduce this metric, as the theory itself is about gauge fields, and metric is a secondary object. However, after embedding into 7D, we see that the connection is a field that parametrises the closed 3-form Ω, and the 3-form naturally defines a metric in the total space of the bundle. In the context of 7D theory, the metric arises more naturally and unavoidably. Since this 7D metric induces a metric on the base, the 7D construction provides an explanation why the metric should also be considered in the context of 4D SO(3) gauge theory.

5.6 Relation between the 7D and 4D action functionals

As is well-known (see [13]), the co-closure condition $d^*\Omega$ can be obtained by minimising a certain volume functional of Ω with respect to variations of Ω by an exact form. The functional in question is just the volume of the 7D manifold computed using the metric defined by Ω. For our ansatz (48), the metric is given by (50). The fiber part gives the volume element $\alpha^3(dy)^3$, while the base part gives $4\beta^4(\det X)^{1/3}\,\epsilon$. Thus, the volume functional reduces for our ansatz to

$$S[\Omega] = 4\int d^3y\left(1+\sigma y^2\right)^{1/4}\int_M (\det X)^{1/3}\,\epsilon. \tag{54}$$

This is proportional to the action of the SO(3) gauge theory on the base. In the incomplete case $\sigma = -1$, the integral over the fiber (from $y=0$ to $y=1$) can be taken and is finite, and we get

$$S_{\sigma=-1}[\Omega] = \frac{16\sqrt{\pi}\,\Gamma^2(1/4)}{21\sqrt{2}}\int_M (\det X)^{1/3}\,\epsilon. \tag{55}$$

In either case, the volume functional for the 3-form (48) in 7 dimensions is a multiple of the volume functional for the SO(3) connection in 4D. Thus, there is a relation not only between solutions of the two theories, but also between the action functionals.

Let us note that we can also get relation (55) to work in the case $\sigma = +1$ at the expense of making the 7D metric indefinite of signature $(3, 4)$. This is achieved just by putting the minus sign in front of the second term in (48) also for the $\sigma = +1$ case. The 7D metric is then indefinite, but induces a Riemannian signature metric on the base. In this case, the function $\beta = (1 - y^2)^{1/4}$, and so we get an incomplete metric in the fiber direction, and a finite multiple relation (55) between the volumes.

6 Examples

We now describe some examples of 7D metrics obtained from the above construction. We build 7D metrics from cohomogeneity one metrics on the base. We describe two easy examples in which the base metrics are asymptotically hyperbolic. In our first example, the asymptotic metric on the conformal boundary is that of $\mathbb{R}^3$, while, in the second example, it is $S^1 \times S^2$. Finally, we attempt the Bianchi IX case, but find ourselves unable to solve the arising ODE's even in the bi-axial case. We still describe this example, as it is likely the most interesting one from the mathematical viewpoint.

6.1 Bianchi I

The simplest, but still non-trivial example to consider is that of cohomogeneity one on the base, with the base manifold having the structure $\mathbb{R}^3 \times \mathbb{R}$. Thus, let $dx^{1,2,3}$ be the one-forms in the $\mathbb{R}^3$ directions, and let r be the coordinate in the remaining direction, which we call 'radial.' We make the following ansatz of the connection:

$$A^1 = a_1(r) dx^1, \qquad \text{etc.} \tag{56}$$

The curvature forms read

$$F^1 = a_1' dr \wedge dx^1 + a_2 a_3 \, dx^2 \wedge dx^3, \qquad \text{etc.} \tag{57}$$

We get the Euler–Lagrange equations by first evaluating the action of the full theory on this ansatz, and then performing the variation. Due to the symmetry of the problem, one gets the same equations as would follow by substituting the ansatz into (2). The volume action evaluated on the above ansatz is then a multiple of

$$S \sim \int dr \left(a_1' a_2' a_3' a_1^2 a_2^2 a_3^2 \right)^{1/3}. \tag{58}$$

The arising Euler–Lagrange equations read

$$-\left(\frac{L}{a_1'} \right)' + \frac{2L}{a_1} = 0, \qquad \text{etc,} \tag{59}$$

where L is the Lagrangian in (58). They can easily be solved by choosing the radial coordinate so that

$$L \equiv \left(a_1' a_2' a_3' a_1^2 a_2^2 a_3^2 \right)^{1/3} = 1. \tag{60}$$

This gives

$$a_1 = c_1(r - r_1)^{1/3}, \qquad \text{etc}, \tag{61}$$

where c_i and r_i are integration constants.

The corresponding matrix X, calculated with respect to the orientation form $\epsilon = dr \wedge d^3x$, is given by

$$X = \frac{1}{3} c_1 c_2 c_3 \left[(r - r_1)(r - r_2)(r - r_3)\right]^{1/3} \operatorname{diag}\left(\frac{1}{r - r_1}, \frac{1}{r - r_2}, \frac{1}{r - r_3}\right). \tag{62}$$

We want our connection to be definite. Let us now assume that $c_1 c_2 c_3 = 3$, which is always possible for $c_1 c_2 c_3 > 0$ by rescaling the metric. We can always order the integration constants so that $r_3 < r_2 < r_1$. Then we get a definite connection for $r > r_1$ and $r < r_3$.

Under these assumptions, we can easily compute the associated basis of ASD 2-forms $\Sigma = \sigma X^{-1/2} F$. In order for the basis of Σ's to take the canonical form in the orientation of $dr \wedge d^3x$, we need to take

$$\sigma = -1. \tag{63}$$

We get

$$\begin{aligned} \Sigma^1 = &- \left[(r - r_2)(r - r_2)\right])^{-1/6} \\ &\times \left[\frac{c_1 dr \wedge dx^1}{3(r - r_1)^{1/3}} + c_2 c_3 \left(\prod_i (r - r_i)\right)^{1/3} dx^2 \wedge dx^3\right], \end{aligned} \tag{64}$$

etc. We can then easily determine the orthonormal basis for the metric:

$$\begin{aligned} \theta^r &= -\frac{1}{3}\left[\prod_i (r - r_i)\right]^{-1/3} dr, \\ \theta^1 &= c_1 \left[(r - r_2)(r - r_2)\right]^{1/6} dx^1, \qquad \text{etc}, \end{aligned} \tag{65}$$

so that the metric is

$$\begin{aligned} ds_4^2 = &\frac{1}{9}\left[\prod_i (r - r_i)\right]^{-2/3} dr^2 \\ &+ \left[\prod_i (r - r_i)\right]^{1/3} \left[\frac{c_1^2 (dx^1)^2}{(r - r_1)^{1/3}} + \frac{c_2^2 (dx^2)^2}{(r - r_2)^{1/3}} + \frac{c_3^2 (dx^3)^2}{(r - r_3)^{1/3}}\right]. \end{aligned}$$

For large r, the matrix X approaches the identity matrix, so we have an asymptotically self-dual Einstein solution. We can introduce the new coordinate $r^{1/3} = \exp \rho$, in terms of which the asymptotic metric takes the form $ds^2 = d\rho^2 + e^{2\rho} \sum_i c_i^2 (dx^i)^2$. This is the metric of the 4D hyperbolic space with $\Lambda = -3$. The 7D lift of the full metric is given by (50):

$$\begin{aligned} ds_7^2 = &\frac{\left[(r - r_1)(r - r_2)(r - r_3)\right]^{1/3}}{(1 - y^2)^{1/2}} \\ &\times \sum_i \frac{1}{r - r_i}\left[dy^i + \sum_{j,k} \epsilon^{ijk} c_j (r - r_j)^{1/3} dx^j y^k\right]^2 \\ &+ 2\left(1 - y^2\right)^{1/2} ds_4^2. \end{aligned} \tag{66}$$

6.2 Spherically symmetric solution

We take the following spherically symmetric ansatz:

$$A^1 = a(R)dt + \cos\theta\, d\phi, \qquad A^2 = -b(R)\sin\theta\, d\phi, \qquad A^3 = b(R)d\theta, \tag{67}$$

where R is some radial coordinate. The curvatures are

$$\begin{aligned} F^1 &= -a' dt \wedge dR + \left(b^2 - 1\right)\sin\theta\, d\theta \wedge d\phi, \\ F^2 &= ab\, d\theta \wedge dt - b' \sin\theta\, dR \wedge d\phi, \\ F^3 &= -ab \sin\theta\, dt \wedge d\phi + b' dR \wedge d\theta. \end{aligned} \tag{68}$$

The action evaluated on the ansatz reads

$$S \sim \int dR \left[a' \left(b^2 - 1\right) a^2 \left(\left(b^2 - 1\right)' \right)^2 \right]^{1/3}. \tag{69}$$

Minimizing it with respect to a and $b^2 - 1$, and again choosing the radial coordinate in which $L = 1$, we get

$$-\left(\frac{1}{a'}\right)' + \frac{2}{a} = 0, \qquad -\left[\frac{2}{\left(b^2 - 1\right)'}\right]' + \frac{1}{b^2 - 1} = 0. \tag{70}$$

This integrates to

$$a = C_1 \left(R - R_1\right)^{1/3}, \qquad b^2 - 1 = C_2 \left(R - R_2\right)^{2/3}, \tag{71}$$

where $C_{1,2}$ and $R_{1,2}$ are integration constants. The corresponding matrix X^{ij}, determined with respect to the orientation form $\epsilon = -dt \wedge dR \wedge \sin\theta\, d\theta \wedge d\phi$, is

$$X = \frac{C_1 C_2}{3} \mathrm{diag} \left[\left(\frac{R - R_2}{R - R_1}\right)^{2/3}, \left(\frac{R - R_1}{R - R_2}\right)^{1/3}, \left(\frac{R - R_1}{R - R_2}\right)^{1/3} \right]. \tag{72}$$

Let us now set $C_1 C_2 = 3$. The components of the metric are determined by computing $\Sigma = -X^{-1/2} F$, where we need to choose $\sigma = -1$ to get the canonical expressions for the ASD 2-forms Σ^i. We then get the frame fields

$$\begin{aligned} \theta^t &= \frac{C_1}{\sqrt{C_2}} \sqrt{1 + C_2 (R - R_2)^{2/3}}\, dt, \\ \theta^R &= \frac{\sqrt{C_2}\, dR}{3\sqrt{1 + C_2 (R - R_2)^{2/3}} (R - R_1)^{1/3} (R - R_2)^{1/3}}, \\ \theta^\theta &= \sqrt{C_2} (R - R_1)^{1/6} (R - R_2)^{1/6} d\theta, \\ \theta^\phi &= \sqrt{C_2} (R - R_1)^{1/6} (R - R_2)^{1/6} \sin\theta\, d\phi, \end{aligned}$$

and so the metric reads

$$\begin{aligned} ds_4^2 = &\frac{C_1^2}{C_2} (1 + C_2 (R - R_2)^{2/3}) dt^2 \\ &+ \frac{C_2 dR^2}{9 (1 + C_2 (R - R_2)^{2/3}) (R - R_1)^{2/3} (R - R_2)^{2/3}} \\ &+ C_2 (R - R_1)^{1/3} (R - R_2)^{1/3} d\Omega^2, \end{aligned} \tag{73}$$

where, as usual, $d\Omega^2$ is the unit sphere metric. Asymptotically for large R, introducing $\sqrt{C_2}R^{1/3} = r$ we get the following metric: $ds_4^2 = r^2[(C_1^2/C_2)dt^2 + d\Omega^2] + dr^2/r^2$. This is the hyperbolic space metric with the conformal structure of the boundary being that of $S^1 \times S^2$, provided we identify the 'time' coordinate t periodically. It is now easy to write the lift (50) of the metric (73). We get

$$
\begin{aligned}
ds_7^2 = \left(1-y^2\right)^{-1/2} \Bigg[& \left(\frac{R-R_2}{R-R_1}\right)^{2/3} \left[dy^1 - b(R)\sin\theta\, d\phi\, y^3 - b(R) d\theta\, y^2\right]^2 \\
& + \left(\frac{R-R_1}{R-R_2}\right)^{1/3} \left[dy^2 + b(R) d\theta\, y^1 - (a(R)dt + \cos\theta\, d\phi) y^3\right]^2 \\
& + \left(\frac{R-R_1}{R-R_2}\right)^{1/3} \left[dy^3 + (a(R)dt + \cos\theta\, d\phi) y^2 + b(R)\sin\theta\, d\phi\, y^1\right]^2 \Bigg] \\
& + 2\left(1-y^2\right)^{1/2} ds_4^2, \qquad (74)
\end{aligned}
$$

with $a(R)$ and $b(R)$ given by (71).

6.3 Bianchi IX

As we already mentioned, our treatment of this case is incomplete, because we are unable to solve the arising ODE's. The main result of this subsection is the ODE (101) to which the problem reduces in the bi-axial case. If one can solve this ODE (e.g. numerically) one would obtain cohomogeneity one 4D metrics that asymptote to Bianchi IX bi-axial instantons – the Taub-NUT metrics.

6.3.1 Ansatz

Let e^1, e^2, e^3 be the standard basis of $\mathfrak{su}(2)$ left-invariant one-forms on S^3 with structure equations $de^1 = e^2 \wedge e^3$ etc. Consider the ansatz

$$A^1 = h_1 e^1, \qquad A^2 = h_2 e^2, \qquad A^3 = h_3 e^3, \qquad (75)$$

where h_i are functions of the 'radial' coordinate r. The curvature components are

$$F^1 = h_1' dr \wedge e^1 + (h_1 + h_2 h_3)\, e^2 \wedge e^3, \quad \text{etc.} \qquad (76)$$

6.3.2 The metric

Our first aim is to calculate the appropriately normalized (Euclidean) metric in which the curvature components are anti-self-dual. As before, we do this computation by computing the matrix of curvature wedge products $F^i \wedge F^j = 2X^{ij} dr \wedge e^1 \wedge e^2 \wedge e^3$:

$$X = \text{diag}\left[h_1'(h_1 + h_2 h_3), h_2'(h_2 + h_1 h_3), h_3'(h_3 + h_1 h_2)\right]. \qquad (77)$$

We assume all h_i and their derivatives h_i' to be positive, so that the connection is definite. We then get $\Sigma = -X^{-1/2} F$, and read off the basis of frame 1-forms by comparing the resulting

Σ's with (6). Note that the sign of the connection must be chosen to be $\sigma = -1$ to identify the curvature 2-forms with ASD forms. We get the metric of the form

$$ds^2 = N^2 (dr)^2 + \sum_{i=1}^{3} a_i^2 \left(e^i\right)^2, \tag{78}$$

with

$$N^2 = \frac{\left(h_1' h_2' h_3'\right)^{2/3}}{\left[\left(h_1 + h_2 h_3\right)\left(h_2 + h_3 h_1\right)\left(h_3 + h_1 h_2\right)\right]^{1/3}}, \tag{79}$$

$$a_1^2 = \frac{\left(h_1'\right)^{2/3}}{\left(h_2' h_3'\right)^{1/3}} \frac{\left[\left(h_2 + h_3 h_1\right)\left(h_3 + h_1 h_2\right)\right]^{2/3}}{\left(h_1 + h_2 h_3\right)^{1/3}}, \quad \text{etc.} \tag{80}$$

6.3.3 The action

The action of the theory is

$$\begin{aligned} S &= \int f\left(F \wedge F\right) \\ &= V_{S^3} \int \left[h_1' h_2' h_3' \left(h_1 + h_2 h_3\right)\left(h_2 + h_3 h_1\right)\left(h_3 + h_1 h_2\right)\right]^{1/3} dr, \end{aligned} \tag{81}$$

where $V_{S^3} = \int_{S^3} e^1 \wedge e^2 \wedge e^3$. Variation of this action with respect to h_i, $i = 1, 2, 3$, gives equations of motion. Choosing the 'radial' coordinate r so that the Lagrangian in (81) becomes constant (equal to unity), we obtain the following system of equations:

$$\left(\frac{1}{h_1'}\right)' = \frac{1}{h_1 + h_2 h_3} + \frac{h_3}{h_2 + h_3 h_1} + \frac{h_2}{h_3 + h_1 h_2}, \tag{82}$$

and equations obtained by cyclic transmutation of indices. A particular solution of these equations for all positive and increasing h_i asymptotically tends to the shape-preserving expansion:

$$h_1 \propto h_2 \propto h_3 \propto r^{1/3} \quad \text{as } r \to \infty. \tag{83}$$

6.3.4 Self-dual case

In the particular self-dual case, characterised by the condition $F^i \wedge F^j \propto \delta^{ij}$, we have

$$h_1' \left(h_1 + h_2 h_3\right) = h_2' \left(h_2 + h_3 h_1\right) = h_3' \left(h_3 + h_1 h_2\right) = 1, \tag{84}$$

and equations (82) are satisfied identically. As for the metric components (79) and (80), we have

$$N^2 = h_1' h_2' h_3', \qquad a_1^2 = \frac{h_1'^{\,2}}{N^2}, \quad \text{etc.} \tag{85}$$

Equations (84) imply the relation

$$h_1^2 + h_2^2 + h_3^2 + 2 h_1 h_2 h_3 = 6r, \tag{86}$$

where a particular shift of the variable r was made to absorb the integration constant.

6.3.5 Self-dual bi-axial case

One can find a family of exact solutions with the biaxial ansatz

$$h_1 = a, \qquad h_2 = h_3 = b. \tag{87}$$

In this case, equations (84) read

$$a'(a + b^2) = bb'(1 + a), \tag{88}$$

and have an integral

$$b^2 = k(1 + a)^2 - 2a - 1, \tag{89}$$

where k is the integration constant. Relation (86) then determines the radial coordinate as a function of a:

$$r = \frac{k(1 + a)^3}{3} - \frac{(1 + a)^2}{2} + \frac{1}{6}. \tag{90}$$

The metric components of this solution read

$$N^2 dr^2 = \frac{kx - 1}{x\left[1 + x\left(kx - 2\right)\right]} dx^2, \tag{91}$$

$$a_1^2 = \frac{x\left[1 + x\left(kx - 2\right)\right]}{kx - 1}, \tag{92}$$

$$a_2^2 = a_3^2 = x\left(kx - 1\right), \tag{93}$$

where we have introduced a new radial variable $x = 1 + a$.

By changing the radial variable from x to ρ as

$$x = 2n^2 \left(1 \mp \frac{\rho}{n}\right), \qquad n = \frac{1}{2\sqrt{k}}, \tag{94}$$

we bring the metric into the form presented in [17]:

$$ds^2 = \frac{\rho^2 - n^2}{\Delta} d\rho^2 + \frac{4n^2 \Delta}{\rho^2 - n^2} \left(e^1\right)^2 + \left(\rho^2 - n^2\right) \left[\left(e^2\right)^2 + \left(e^3\right)^2\right], \tag{95}$$

where

$$\Delta = (\rho \mp n)^2 \left[1 + (\rho \pm 3n)(\rho \mp n)\right]. \tag{96}$$

It describes the self-dual (upper sign) or anti-self-dual (lower sign) Taub–NUT–anti-de Sitter metric.

6.3.6 General bi-axial case

In the general case with biaxial ansatz (87), equations of motion (82) read

$$\left(\frac{1}{a'}\right)' = \frac{1}{a + b^2} + \frac{2}{1 + a}, \qquad \left(\frac{1}{b'}\right)' = \frac{1}{b} + \frac{b}{a + b^2}. \tag{97}$$

It is convenient to change the variables to

$$x = 1 + a, \qquad y = b^2 - 1, \tag{98}$$

so that these equations become

$$\left(\frac{1}{x'}\right)' = \frac{1}{y+x} + \frac{2}{x}, \qquad \left(\frac{1}{y'}\right)' = \frac{1}{2(y+x)}. \tag{99}$$

We can obtain a closed differential equation for the trajectory $y(x)$. We have

$$\frac{d^2y}{dx^2} = \frac{1}{x'}\left(\frac{y'}{x'}\right)' = \frac{y''}{x'^2} + \left(\frac{1}{x'}\right)'\frac{y'}{x'} = -\left(\frac{1}{y'}\right)'\left(\frac{y'}{x'}\right)^2 + \left(\frac{1}{x'}\right)'\frac{y'}{x'}. \tag{100}$$

Now, using equations (99), we get

$$\frac{d^2y}{dx^2} + \frac{1}{2(y+x)}\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{y+x} + \frac{2}{x}\right)\frac{dy}{dx} = 0. \tag{101}$$

This equation looks very difficult to solve. Note that its partial solution $y(x) = kx^2 - 2x$ is precisely the self-dual solution (89).

We also have the constraint reflecting our choice of radial variable:

$$x'y'^2(y+x)x^2 = 4. \tag{102}$$

It is only needed to determine the original radial variable r along the trajectory:

$$dr = \left[\frac{1}{4}\left(\frac{dy}{dx}\right)^2 (y+x)x^2\right]^{1/3} dx. \tag{103}$$

While we are unable to solve the general bi-axial case analytically (apart from the already known self-dual case), the ODE (101) can be used for a numerical solution, which can then be lifted to 7D.

7 Discussion

In this paper we generalised the construction of [3] by parametrising the 3-form in 7 dimensions with an SO(3) connection on the 4-dimensional base instead of a self-dual Einstein metric. We get a G_2 holonomy metric in 7 dimensions provided the connection satisfies the Euler-Lagrange equations of the theory (22). Our construction then intersects with that of Bryant-Salamon precisely for self-dual Einstein metrics – instantons. These are also the solutions that are shared by the theory (22) and GR. The solutions of (22) that are not instantons give metrics on the base that are not Einstein. Our work interprets these non-Einstein metrics as restrictions of 7 dimensional Ricci flat metrics to the 4-dimensional base.

Our construction shows that a certain theory of gravity in 4D can be understood as arising via a dimensional reduction of a theory of differential forms in 7 dimensions. This realisation of a 4D gravity theory as coming from a theory of a very different nature appears to be the most interesting aspect of our work. While the proper interpretation of our dimensional reduction is still to be developed, we give some comments in this direction, leaving the final word to future studies.

If we are to interpret a 4D gravity as arising by some sort of Kaluza–Klein reduction from a higher-dimensional theory of differential forms, the must first answer the question what this theory of forms is. As we have already described in the main text, a (stable) 3-form in 7 dimensions naturally defines a metric, and one can compute the volume of the manifold with respect to this metric. This leads to functional (9) whose critical points are 3-forms that are co-closed $d^*\Omega = 0$, provided one varies the 3-form within a fixed cohomology class $\Omega = \Omega_0 + dB$, $B \in \Lambda^2 E$. Thus, one possible interpretation of theory (9) is as a theory of 2-forms B in 7 dimensions, with the action constructed from their field strength $\Omega = dB$. Viewed in this light, it becomes an analog of Maxwell's theory, where the basic field is a 1-form A, and the action is constructed from the field strength $F = dA$. The principal difference between these two cases is that Maxwell's theory requires a metric for its formulation, while the theory of 2-forms in 7 dimensions exists on an arbitrary differentiable manifold without any extra structure. Moreover, the field strength $\Omega = dB$ itself *defines* a metric on E, and this is why such a theory can ultimately be given, as in our construction, some gravitational interpretation. Considering this theory on classes of forms of type $\Omega = \Omega_0 + dB$ with fixed non-trivial Ω_0, one can interpret this as a theory around different vacua, with different cohomology classes to be summed over in the path integral.

Having defined the 7-dimensional theory as a diffeomorphism-invariant analog of electromagnetism in 7 dimensions (but now for 2-forms rather than one-forms), we can discuss the meaning of our ansatz (48) in which we parametrised Ω by a connection 1-form on the 4-dimensional base. Our first remark is that, because our ansatz (48) is closed, it can locally be written as $\Omega = dB$, with some B parametrised by the connection field. This way of writing Ω is of course not unique, because B is only defined modulo $B \to B + d\theta$, $\theta \in \Lambda^1 E$. We can now get some insight into the meaning of our ansatz (48) if we consider the manifold E to have the structure of a product manifold with 3-dimensional fibers and a 4-dimensional base. Then the 21 components of B decompose as follows: we get 3 components of B that is a 2-form in the fiber, these should be interpreted as scalars from the point of view of the base; we get the components of B that are basic 1-forms, as well as 1-forms in the fiber direction, these are interpreted as three 1-forms on the base; finally, there is the component that is a 2-form on the base. Counting the numbers we get $3 + 3 \times 4 + 6 = 21$ components as required.

It is then clear that if we make an assumption that B is invariant under an action of some group in the fiber directions, we get a structure of a fiber bundle with the basic 1-form components of B receiving the interpretation of a connection in this bundle. In our ansatz (48), we have parametrised Ω and thus B just by these connection components, thus setting to zero all other possible fields that could have been present. This led us to a diffeomorphism-invariant SO(3) gauge theory with the Lagrangian $(\det X)^{1/3}$ on the base. It would be very interesting to keep all the components of the 2-form B, and find the resulting theory. This is to be described elsewhere.

It thus appears that the right interpretation of our construction is that we have made a particular Kaluza–Klein ansatz (48) for some 7-dimensional theory, and saw how the 7-dimensional field equations impose some 4D field equations on our ansatz. This is not yet Kaluza–Klein reduction, in which one would instead make an assumption that the 7-dimensional field is invariant with respect to some group action, and determine all the fields and their dynamics that arise in lower dimensions. To put it differently, what we have obtained is analogous to the Kaluza–Klein ansatz that obtains gravity plus Maxwell in 4 dimensions from gravity in 5 dimensions, while setting the other field necessarily present in this dimensional reduction — the scalar field — to zero by hand. It is very interesting to determine the full content and dynamics of the 7D theory (9) dimensionally reduced to 4D.

A potentially more difficult question is to study the dimensional reduction without any symmetry assumptions, and thus take into account the full infinite set of modes that arise by decomposing the field dependence on the 'internal' coordinates into appropriate spherical harmonics. In the usual Kaluza–Klein story, the fields with non-trivial dependence on the internal coordinates get interpreted as massive modes. It would be very interesting to see if the same interpretation persists for the 7-dimensional theory of differential forms.

Finally, the most interesting physics question that arises in this context is whether General Relativity (24) can arise by a similar dimensional reduction from a higher-dimensional theory of differential forms. This could be precisely GR in its form (24), or, perhaps, a theory that only resembles GR in some appropriate range of energies, but differs from it in general. At the moment of writing this, it appears to us that this last possibility is the most likely one.

Whatever the final word on this story will be, it appears clear to us that there is a very interesting class of dynamically non-trivial theories of differential forms in higher dimensions, e.g., 3-forms in 7 and 8 dimensions, see [5] for details of how the action functional is defined in 8D. The construction presented in this paper makes it clear that these theories, when dimensionally reduced, are related to dynamically non-trivial gravity theories in 4 dimensions. The main question now is what gravity theories can arise in this way, and what kind of other fields that accompany gravity arise in such a dimensional reduction. Our paper can be viewed as a first step in answering these questions.

Acknowledgments

KK and CS were supported by ERC Starting Grant 277570-DIGT. YS acknowledges support from the same grant and from the State Fund for Fundamental Research of Ukraine. YH was supported by a grant from ENS Lyon. KK is grateful to Joel Fine for important discussions. The idea that later developed into this paper was born in one of them.

References

[1] I. Agricola, *Old and new on the exceptional group G_2*, Notices of the AMS **55** (2008), 922.

[2] R. L. Bryant, *Metrics with exceptional holonomy*, Annals of Mathematics **126** (1987), 525–576.

[3] R. L. Bryant and S. S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. Journ. **58** (1989), 829–850.

[4] D. Joice, *Compact Riemannian manifolds with holonomy G_2 I & II*, J. Diff. Geom. **43** (1996), 291–328 and 329–375.

[5] N. Hitchin, *Stable forms and special metrics*, `arXiv:math/0107101`.

[6] S. Salamon, *A tour of exceptional geometry*, Milan J. Math. **72** (2002), 1–31.

[7] F. Witt, *Gauge theory in dimension* 7, `arXiv:1012.5912 [math.DG]`.

[8] A. Gray, *Weak holonomy groups*, Math. Z. **123** (1971), 290–300.

[9] K. Krasnov, *Gravity as a diffeomorphism invariant gauge theory*, Phys. Rev. D **84** (2011), 024034.

[10] K. Krasnov, *Pure connection action principle for general relativity*, Phys. Rev. Lett. **106** (2011), 251103.

[11] J. Fine, K. Krasnov, and D. Panov, *A gauge theoretic approach to Einstein* 4*-manifolds*, New York J. Math. **20** (2014), 293.

[12] R. Dijkgraaf, S. Gukov, A. Neitzke, and C. Vafa, *Topological M-theory as unification of form theories of gravity*, Adv. Theor. Math. Phys. **9** (2005), no. 4, 603.

[13] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, `arXiv:math/0010054`.

[14] H. Urbantke, *On integrability properties of* $SU(2)$ *Yang-Mills fields. I. Infinitesimal part*, J. Math. Phys. **25** (1984), 2321.

[15] C. Fefferman and C. R. Graham, *Conformal invariants*, in: The Mathematical Heritage of Élie Cartan (Lyon, 1984), Astérisque (1985), Numero Hors Serie, 95–116.

[16] J. Fine, Y. Herfray, K. Krasnov, and C. Scarinci, *Asymptotically hyperbolic connections*, `arXiv:1512.07109 [gr-qc]`.

[17] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope, *Bianchi IX selfdual Einstein metrics and singular* $G(2)$ *manifolds*, Class. Quant. Grav. **20** (2003), 4239.