Two-Grid hp-DGFEMs on Agglomerated Coarse Meshes

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We generalise the *a priori* error analysis of two-grid hp-version discontinuous Galerkin finite element methods for strongly monotone second-order quasilinear elliptic partial differential equations to the case when coarse meshes consisting of general agglomerated polytopic elements are employed.

1 Introduction

We study the hp-version of the two-grid incomplete interior penalty (IIP) discontinuous Galerkin finite element method (DGFEM) using an agglomerated coarse mesh, for the numerical approximation of the following problem: find u such that

$$-\nabla \cdot (\mu(\boldsymbol{x}, |\nabla u|)\nabla u) = f(\boldsymbol{x}) \qquad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \Gamma, \tag{1}$$

where Ω is a bounded polygonal/polyhedral Lipschitz domain in $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma := \partial\Omega$ and $f \in L^2(\Omega)$. We assume that $\mu \in C^0(\overline{\Omega} \times [0, \infty))$, and there exists positive constants m_μ and M_μ such that $m_\mu(t - s) \leq \mu(\boldsymbol{x}, t)t - \mu(\boldsymbol{x}, s)s \leq M_\mu(t - s)$, $t \geq s \geq 0$, $\boldsymbol{x} \in \overline{\Omega}$. For ease of notation write $\mu(t)$ instead of $\mu(\boldsymbol{x}, t)$.

The two-grid method was originally introduced by Xu [1, 2]. The key idea of this approach, in the context of numerically approximating nonlinear partial differential equations (PDEs), is to first compute a numerical approximation of the nonlinear PDE on a coarse mesh/approximation space, and subsequently employ this solution to linearize the underlying problem on the fine mesh/approximation space; in this way only a linear solve is required on the fine mesh/approximation space. In the context of hp-version DGFEMs, in [3] and [4] we have considered the application of the two-grid approach to both scalar strongly monotone second-order quasilinear PDEs of the form (1) and non-Newtonian fluids, respectively; in both cases the coarse and fine spaces employ standard meshes employing simplices/tensor-product elements. In this article, we generalize this to the case when general polytopic coarse elements, generated by agglomerating fine mesh elements, are employed.

2 Two-grid hp-version IIP DGFEM

We write $\mathcal{T}_h = \{\kappa\}$ to denote the fine mesh consisting of simplices/tensor-product elements of local mesh size $h_\kappa = \mathrm{diam}(\kappa)$, $\kappa \in \mathcal{T}_h$. Similarly, $\mathcal{T}_H = \{K\}$ denotes the coarse mesh consisting of polytopic elements K constructed by agglomerating elements $\kappa \in \mathcal{T}_h$; $H_K = \mathrm{diam}(K)$, $K \in \mathcal{T}_H$. We assume that $\mathcal{T}_h$ is of bounded local variation. Writing $\boldsymbol{p} = \{p_\kappa : \kappa \in \mathcal{T}_h\}$ and $\boldsymbol{P} = \{P_K : K \in \mathcal{T}_H\}$ to denote the polynomial orders defined over $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, ($\boldsymbol{p}$ is assumed to be of bounded local variation) we write $V_{hp} = \{v \in L^2(\Omega) : v|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa), \kappa \in \mathcal{T}_h\}$ and $V_{HP} = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{P_K}(K), K \in \mathcal{T}_H\}$, where $\mathcal{P}_p(\kappa)$ denotes the space of all polynomials of total degree p on κ.

We write $\mathcal{F}_h$ and $\mathcal{F}_H$ to denote the set of all faces in the meshes $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Furthermore, we write $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]$ to denote suitable average and jump operators, respectively, which are defined on either $\mathcal{F}_h$ or $\mathcal{F}_H$; see [3] for details. With this notation, we first introduce the following *standard* IIP DGFEM on the fine mesh $\mathcal{T}_h$, for the numerical approximation of the problem (1): find $u_{hp} \in V_{hp}$ such that $A_{hp}(u_{hp}; u_{hp}, v_{hp}) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v_{hp} \,\mathrm{d}\boldsymbol{x}$ for all $v_{hp} \in V_{hp}$, where

$$A_{hp}(\phi; u, v) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \mu(|\nabla\phi|)\nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\mu(|\nabla_h\phi|)\nabla_h u\}\!\} \cdot [\![v]\!] \,\mathrm{d}s + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{hp} [\![u]\!] \cdot [\![v]\!] \,\mathrm{d}s$$

and ∇_h is used to denote the broken gradient operator, defined elementwise. Given a face polynomial degree function p_F and a face mesh size function h_F, $F \in \mathcal{F}_h$, the interior penalty parameter σ_{hp} is given by $\sigma_{hp}|_F = \gamma_{hp} p_F^2 h_F^{-1}$, $F \in \mathcal{F}_h$, where $\gamma_{hp} > 0$ is a sufficiently large constant, cf. [3]. The two-grid IIP DGFEM is given by:

1. Compute $u_{HP} \in V_{HP}$ such that $A_{HP}(u_{HP}; u_{HP}, v_{HP}) = \sum_{K \in \mathcal{T}_H} \int_K f v_{HP} \,\mathrm{d}\boldsymbol{x}$ for all $v_{HP} \in V_{HP}$.
2. Find $u_{2G} \in V_{hp}$ such that $A_{hp}(u_{HP}; u_{2G}, v_{hp}) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa f v_{hp} \,\mathrm{d}\boldsymbol{x}$ for all $v_{hp} \in V_{hp}$.

Here, $A_{HP}(u; u, v)$ is defined analogously to $A_{hp}(u; u, v)$, but with a modified interior penalty parameter σ_{HP}, cf. [5].

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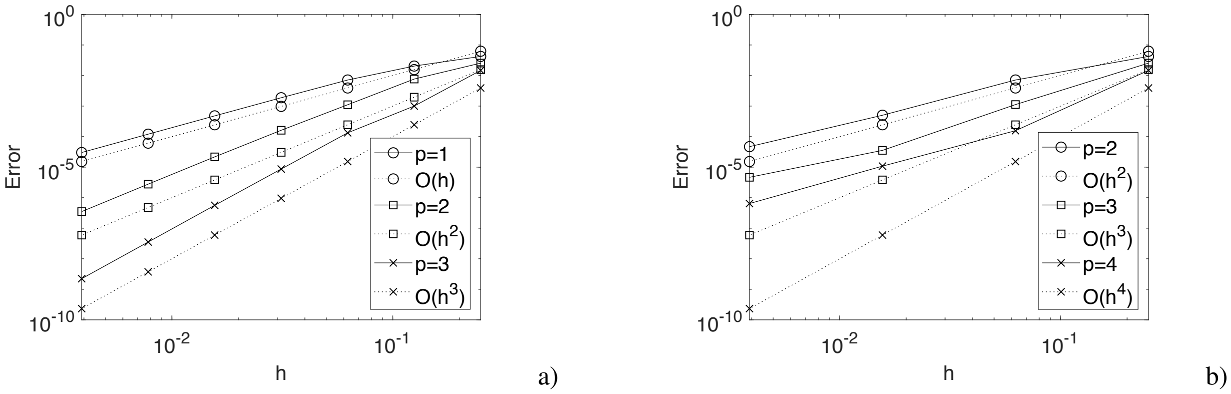

Fig. 1: Plot of $\|u - u_{2G}\|_{hp}$ against h for uniform fine mesh refinement with: **a)** $H \approx h/2$; **b)** $H \approx h^{1/2}$.

3 Error analysis

For the proceeding error analysis, we require the following definitions and assumptions, cf. [5].

Definition 3.1 For $K \in \mathcal{T}_H$ we write $\mathcal{F}_\flat^K$ to be the set of all possible d-simplices contained in K and having at least one face in common with K; we write $K_\flat^F$ to denote a simplex belonging to $\mathcal{F}_\flat^K$ which shares with $K \in \mathcal{T}_H$ the face $F \subset \partial K$.

Assumption 3.2 For any $K \in \mathcal{T}_H$, there exists a set of non-overlapping d-dimensional simplices $\{K_\flat^F\} \subset \mathcal{F}_\flat^K$ contained within K, such that for all $F \subset \partial K$, the condition $H_K \leq C_s d|K_\flat^F||F|^{-1}$ holds, where C_s is a positive constant, which is independent of the discretization parameters, the number of faces that the element possesses, and the measure of F.

Definition 3.3 The *covering* $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ related to $\mathcal{T}_H$ is a set of open shape-regular d-simplices $\mathcal{K}$, such that, for each $K \in \mathcal{T}_H$, there exists a $\mathcal{K} \in \mathcal{T}_H^\sharp$, such that $K \subset \mathcal{K}$. Given $\mathcal{T}_H^\sharp$ we denote by $\Omega_\sharp$ the *covering domain* given by $\overline{\Omega}_\sharp = \bigcup_{\mathcal{K} \in \mathcal{T}_H^\sharp} \overline{\mathcal{K}}$.

Assumption 3.4 We assume a covering $\mathcal{T}_H^\sharp$ of $\mathcal{T}_H$ and positive constant $\mathcal{O}_\Omega$ exists, independent of the mesh, such that $\max_{K \in \mathcal{T}_H} \mathrm{card}\{K' \in \mathcal{T}_H : K' \cap \mathcal{K} \neq \emptyset, \mathcal{K} \in \mathcal{T}_H^\sharp \text{ such that } K \subset \mathcal{K}\} \leq \mathcal{O}_\Omega$, and $h_\mathcal{K} := \mathrm{diam}(\mathcal{K}) \leq C_D H_K$, for each pair $K \in \mathcal{T}_H, \mathcal{K} \in \mathcal{T}_H^\sharp$ with $K \subset \mathcal{K}$, for a constant $C_D > 0$, uniformly with respect to the mesh size.

We now state the main result of this article; see [6] for details.

Theorem 3.5 *Let $\mathcal{T}_H$ be a coarse agglomerated mesh satisfying Assumptions 3.2 and 3.4, with $\mathcal{T}_H^\sharp = \{\mathcal{K}\}$ an associated covering of $\mathcal{T}_H$ consisting of d-simplices; cf. Definition 3.3. If the analytical solution $u \in H^1(\Omega)$ to (1) satisfies $u|_\kappa \in H^{l_\kappa}(\kappa)$, $l_\kappa \geq 2$, and $u|_K \in H^{L_K}(K)$, $L_K \geq 3/2$, for $K \in \mathcal{T}_H$, such that $\mathfrak{E}u|_\mathcal{K} \in H^{L_K}(\mathcal{K})$, where $\mathcal{K} \in \mathcal{T}_H^\sharp$ with $K \subset \mathcal{K}$; then, writing $\|v\|_{hp}^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_h} \int_F \sigma_{hp} |[\![v]\!]|^2 \,\mathrm{d}s$, the solution $u_{2G} \in V_{hp}$ satisfies the error bound*

$$\|u - u_{2G}\|_{hp}^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \frac{h_\kappa^{2s_\kappa - 2}}{p_\kappa^{2l_\kappa - 3}} \|u\|_{H^{l_\kappa}(\kappa)}^2 + C \sum_{K \in \mathcal{T}_H} \frac{H_\kappa^{2S_K - 2}}{P_K^{2L_K - 2}} (1 + \mathcal{G}_K(H_K, P_K)) \|\mathfrak{E}u\|_{H^{L_K}(\mathcal{K})}^2,$$

where $\mathcal{G}_K(H_K, P_K) := (P_K + P_K^2) H_K^{-1} \max_{F \subset \partial\kappa} \sigma_{HP}^{-1}|_F + H_K P_K^{-1} \max_{F \subset \partial K} \sigma_{HP}|_F$, $S_K = \min(P_K + 1, L_K)$, for $K \in \mathcal{T}_H$, $s_\kappa = \min(p_\kappa + 1, l_\kappa)$, for $\kappa \in \mathcal{T}_h$, and C is a positive constant independent of u, h, H, $\mathbf{p}$, and $\boldsymbol{P}$, but depends on the constants m_μ, M_μ from the monotonicity properties of $\mu(\cdot)$. Finally, $\mathfrak{E}$ denotes the extension operator defined in [7].

To confirm Theorem 3.5, we set $\Omega = (0,1)^2 \subset \mathbb{R}^2$, $\mu(|\nabla u|) = 2 + (1 + |\nabla u|^2)^{-1}$, and select f so that $u(x,y) = x(1-x)y(1-y)(1-2y)\mathrm{e}^{-20(2x-1)^2}$. Firstly, we consider a sequence of uniform fine meshes consisting of $n \times n$, $n = 4, 8, 16, 32, 64, 128, 256$, square elements with a coarse mesh containing elements constructed by agglomerating 4 fine elements, i.e., $H \approx \mathcal{O}(h)$ for all meshes; see Figure 1(a) which confirms the optimal convergence rate of $\mathcal{O}(h^p)$, for p fixed. By selecting $H \approx \mathcal{O}(h^{1/2})$, cf. Figure 1(b), we observe a deterioration in the order of convergence to $\mathcal{O}(h^{p/2})$, as expected.

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