Ambiguity Premium and Transaction Costs*

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Abstract

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We generalize the optimal investment model of an ambiguity averse investor with transaction costs. Along the lines of Maenhout (2004), we first show that ambiguity (or model uncertainty) leads to an increase in effective risk aversion by ambiguity aversion even with transaction costs. We compute the utility cost associated with suboptimal investment decisions, which is the so-called ambiguity premium. We then find that ignoring ambiguity aversion with and without transaction costs generates large ambiguity premia when ambiguity aversion is moderate, and the cost of ignoring it becomes larger with higher ambiguity aversion. This would, thus, still support the importance of ambiguity aversion channel for portfolio choice, even concerning the friction markets.

Keywords: optimal investment, ambiguity aversion, transaction costs, ambiguity premium

JEl Classification: C61, E21, G11

I. Introduction

Since the seminal references of Merton (1969, 1971), there has been a very active line of research focusing on optimal consumption and portfolio choice by relaxing Merton's restrictive assumptions such as independent and identical distributed of returns, the absence of uncertainty about returns, frictionless markets, and the absence of labor income etc. Our work sits squarely within the bulk of this research particularly by considering ambiguity (or model uncertainty) in accordance with Maenhout (2004) proposing homothetic robustness with wealth independence, and concerning the friction markets with transaction costs (Constantinides (1986); Liu and Loewenstein (2002)).

Maenhout (2004) claims "Robustness amounts therefore to an increase in effective risk aversion, at least within the confines of the environment studied here." This is precisely the direction we take on the paper. In the simplest possible setup in every other dimension, we isolate and very closely investigate the new issues introduced by ambiguity aversion on portfolio choice especially with transaction costs. In the standard Merton (1969, 1971) framework, we generalize the optimal investment model of an ambiguity averse investor with transaction costs. That is, this paper demonstrates the joint presence of ambiguity aversion (Maenhout, 2004) and transaction costs (Liu and Loewenstein, 2002).

Along the lines of Maenhout (2004), we first show that ambiguity leads to an increase in effective risk aversion by ambiguity aversion even with transaction costs. In order to address the importance of ambiguity aversion for portfolio choice in the friction markets, we compute the utility cost (measured in certainty equivalent wealth units) associated with suboptimal investment decisions, which is the so-called ambiguity premium. More specifically, we compute the utility loss incurred by investors who ignore ambiguity aversion with and without transaction costs. Overall, ignoring ambiguity aversion with and without transaction costs generates large ambiguity premia when ambiguity aversion is moderate. The premia are quite substantial, generating as high as 6% of wealth for moderate ambiguity aversion. Further, the cost of ignoring ambiguity aversion becomes larger with higher ambiguity aversion. The premia with transaction costs can be higher than 12% of wealth for high ambiguity aversion. This would, thus, still support the importance of ambiguity aversion for portfolio choice, even concerning the friction markets.

II. The Basic Model

There are two assets in the market: a risk-free bond and a risky stock. The bond has a return $r > 0$ and the stock price S_t follows a geometric Brownian motion with its expected return $\mu > r$ and the volatility $\sigma > 0$. Trading stocks entails transaction costs of which the ask price is $S_t^A = (1+\alpha)S_t$, and the bid price is $S_t^B = (1-\beta)S_t$, where $\alpha \geq 0$ and $0 \leq \beta < 1$ denote proportional transaction costs. We denote by B_t a standard one-dimensional Brownian motion under a well-defined probability measure. Let x_t and y_t be the dollar amounts invested in the bond and the stock, respectively, and their dynamics are governed by

$$\begin{aligned} dx_t &= rx_t dt - (1+\alpha)dL_t + (1-\beta)dU_t, \\ dy_t &= \mu y_t dt + \sigma y_t dB_t + dL_t - dU_t, \end{aligned} \tag{1}$$

where L_t and U_t are the cumulative purchases and sales of the stock.

Under the currently available *reference* (or benchmark) probability measure, an investor is concerned about any uncertainty (or misspecification) about stock returns, giving rise to a so-called ambiguity (or model uncertainty). Addressing her model concerns, alternatively *perturbed* probability measures can be considered based on expected worst-case scenarios, thus helping her to make a robust investment decision. Along the lines of Maenhout (2004), the ambiguity averse investor decides to perturb expected stock returns by $\sigma^2 y_t h_t$ so that μ is perturbed to $\mu + \sigma^2 y_t h_t$, where $h_t \neq 0$ represents her belief on true stock returns. By Girsanov Theorem, the perturbed probability measures are generated by

$$dB_t^h \equiv dB_t - \sigma y_t h_t dt,$$

as a result, the investment dynamics in (1) are rewritten as

$$\begin{aligned} dx_t &= rx_t dt - (1+\alpha)dL_t + (1-\beta)dU_t, \\ dy_t &= (\mu + \sigma^2 y_t h_t) y_t dt + \sigma y_t dB_t^h + dL_t - dU_t, \end{aligned} \tag{2}$$

where expected stock return μ under the reference probability measure is now changed to $\mu + \sigma^2 y_t h_t$ under the perturbed probability measure.

The ambiguity averse investor's investment problem is then to maximize her terminal wealth as usual, but suspecting her reference model to be misspecified with some penalty terms which are to be minimized. That is, the following max-min optimization problem is given

$$V(x, y, t) \equiv \max_{(L,U)} \min_{h} \mathbb{E}_t^h \left[\int_t^T \frac{\sigma^2 y_s^2 h_s^2}{2\Psi(x, y, s)} ds + \frac{\big(x_T + (1-\beta) y_T\big)^{1-\gamma}}{1-\gamma} \right], \tag{3}$$

where $\mathbb{E}_t^h[\cdot]$ is the expectation taken at time t under the perturbed probability measures generated by the investor's belief h_t, $T > 0$ is the investor's terminal time, and $\Psi(x, y, t) > 0$ denotes the investor's ambiguity aversion which measures the strength of the preference for robustness, which is state dependent, i.e., it varies with bond investment x and stock investment y at time t.

Following Shreve and Soner (1994) and Dai and Yi (2009), we have the following Hamilton-Jacobi-Bellman (HJB) equation for the ambiguity averse investor's investment problem with transaction costs:

$$\begin{cases} \min\Big\{ \max_h \Big\{ -V_t - \mathcal{D}V - \sigma^2 y^2 h V_y - \dfrac{\sigma^2 y^2 h^2}{2\Psi(x, y, t)} \Big\}, -(1-\beta)V_x + V_y, (1+\alpha)V_x - V_y \Big\} = 0, \\ V(x, y, T) = \dfrac{\big(x + (1-\beta) y\big)^{1-\gamma}}{1-\gamma}, \end{cases} \tag{4}$$

where

$$\mathcal{D}V(x, y, t) = rxV_x + \mu y V_y + \frac{1}{2}\sigma^2 y^2 V_{yy},$$

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \frac{\partial V}{\partial x}, \quad V_y = \frac{\partial V}{\partial y}, \quad \text{and} \quad V_{yy} = \frac{\partial^2 V}{\partial y^2}.$$

The first order condition with respect to the investor's belief h in HJB equation (4) shows that

$$\sigma^2 y^2 V_y + \frac{1}{\Psi}\sigma^2 y^2 h = 0,$$

or equivalently,

$$h = -\Psi V_y.$$

If the investor completely trusts the validity of the model ($\Psi = 0$), then $h = 0$, thus requiring no perturbations any more in equation (2). As long as the investor desires robustness ($\Psi > 0$), the perturbations turn out to decrease expected stock return by $\sigma^2 yh$ when $V_y > 0$ which naturally follows from that marginal utility increases at a decreasing rate due to the investor's risk aversion. Hence, the ambiguity averse investor is somewhat pessimistic or conservative when making an investment decision.

III. Solution

We now solve the HJB equation (4) for the ambiguity averse investor's investment problem with transaction costs (3). Following Maenhout (2004), we decide to use homothetic robustness removing wealth effects and allowing for analytical tractability. Specifically, we alter the investor's state-dependent ambiguity aversion $\Psi(x, y, t)$ by replacing it with constant θ divided by $(1 - \gamma)V(x, y, t)$:

$$\Psi(x, y, t) = \frac{\theta}{(1 - \gamma)V(x, y, t)}, \tag{5}$$

where $\theta > 0$ represents the investor's constant ambiguity aversion instead of her state-dependent ambiguity aversion.

Theorem III.1. *As far as Maenhout's homothetic robustness is concerned as in (5), ambiguity (or model uncertainty) leads to an increase in effective risk aversion by ambiguity aversion even with transaction costs, i.e.,*

$$\gamma \quad (\textit{without ambiguity avaersion}) \rightarrow \gamma + \theta \quad (\textit{with ambiguity aversion}),$$

where $\gamma + \theta$ is the increased effective risk aversion by ambiguity aversion θ.

Proof. See Online Appendix. **Q.E.D.**

Theorem III.1 demonstrates that ambiguity (or model uncertainty) leads to an increase in effective risk aversion by ambiguity aversion even with transaction costs.

The ambiguity averse investor's investment problem with transaction costs is solved by characterizing three regions: the sell region (SR), the buy region (BR), and the no-transaction region (NT). The following theorem characterizes the optimal sell and buy boundaries with time-varying functions $z_s(t)$ and $z_b(t)$, respectively.

Theorem III.2. *There are two monotonically increasing functions $z_s(t) : [0, T) \to (\beta - 1, +\infty)$ and $z_b(t) : [0, T) \to (\beta - 1, +\infty)$ such that*

$$z_s(t) < z_b(t) \quad \textit{for all} \quad t \in [0, T)$$

and

$$\begin{aligned} SR &= \{(z,t) : z \le z_s(t), t \in [0,T)\}, \\ BR &= \{(z,t) : z \ge z_b(t), t \in [0,T)\}, \\ NT &= \{(z,t) : z_s(t) < z(t) < z_b(t), t \in [0,T)\}. \end{aligned}$$

Further,

$$z_s(t) \le z_s(T^-) = (1-\beta) z_M^{\theta}, \quad z_b(t) \ge (1+\alpha) z_M^{\theta}, \quad \textit{for all} \quad t \in [0,T), \tag{6}$$

where z_M^{θ} represents the ambiguity-adjusted Merton line and it is given by

$$z_M^{\theta} = -\frac{\mu - r - \sigma^2(\gamma + \theta)}{\mu - r}. \tag{7}$$

Finally, there exists a constant $t_b = T - \dfrac{1}{\mu - r} \ln\left(\dfrac{1+\alpha}{1-\beta}\right)$ such that

$$z_b(t) = +\infty, \quad \textit{if and only if} \quad t \in [t_b, T).$$

Proof. See Online Appendix. **Q.E.D.**

Theorem III.2 allows us to obtain the analytically tractable upper and lower bounds of optimal sell and buy boundaries as in (6). The bounds depend not only on transaction costs α, β, but more interestingly, on ambiguity-adjusted Merton line z_M^{θ} given in (7). The width of no-transaction region (NR) is, at least, larger than or equal to

$$z_b(t) - z_s(t) \geq (\alpha + \beta) z_M^\theta.$$

Since z_M^θ increases as ambiguity aversion θ increases, more ambiguity averse investor is inclined to not frequently trade in the stock market.

Interestingly, the ambiguity averse investor's decision to take leverage or deleverage actually relies on ambiguity aversion, in addition to risk aversion. Letting $\delta_\theta = \mu - r - \sigma^2(\gamma + \theta)$ be the variance-adjusted risk premium with ambiguity aversion and risk aversion, the following theorem details the leverage decisions with respect to δ_θ.

Theorem III.3. *(1) If $\delta_\theta \leq 0$, then $z_s(t) > 0$ for all t, thus requiring no leverage.*
(2) If $\delta_\theta > 0$, then

$$\begin{cases} z_s(t) < 0 & \textit{for all} \;\; t, \\ z_b(t) < 0 & \textit{for} \;\; t < \hat{t}_b, \\ z_b(t) > 0 & \textit{for} \;\; t \geq \hat{t}_b, \end{cases}$$

where

$$\hat{t}_b = T - \frac{1}{\delta_\theta} \ln\left(\frac{1-\beta}{1+\alpha}\right),$$

thus requiring leverage until time $\hat{t}_b$ and deleverage after then.

Proof. See Online Appendix. **Q.E.D.**

Theorem III.3 demonstrates that it may be the case where the investor with risk aversion only finds it optimal to take leverage until time $\hat{t}_b$ as long as the variance-adjusted risk premium is positive. However, such a decision to take leverage represents an overly simplified situation. It would not be optimal if the investor is concerned about her model uncertainty very much with her high ambiguity aversion so that the variance-adjusted risk premium is negative, thereby requiring no leverage.

One way that is useful for addressing the importance of ambiguity aversion for portfolio choice in the friction markets is to compute the utility cost (measured in certainty equivalent wealth units) associated with suboptimal investment decisions, which is the so-called ambiguity premium. More specifically, we compute the utility loss incurred by investors who ignore ambiguity aversion with and without transaction costs.

Definition. The utility cost associated with suboptimal investment decisions ignoring ambiguity aversion with and without transaction costs is measured in certainty equivalent wealth units as follows:

$$V(x-\Delta, y, 0; \theta = 0) = V(x, y, 0; \theta > 0),$$

where Δ is the so-called ambiguity premium.

In the absence of transaction costs, the ambiguity premium is derived in the analytically tractable form. While in the presence of transaction costs, it is numerically derived and its graphical illustrations are to be shown later. For purpose of comparison between our model and Maenhout (2004) with and without transaction costs, we normalize initial wealth $w = x + y$ as $1 + z$ by dividing the both sides by y and setting $y = 1$ (Davis and Norman (1990); Jang et al. (2007)). The percentage ambiguity premium then corresponds to $\Delta/(1+z)$ for our model.

Theorem III.4. *The percentage ambiguity premium $\Delta/(1 + z)$ is given in the following analytically tractable form:*

$$\frac{\Delta}{1+z} = \begin{cases} 1 - \exp\Big(\dfrac{a_{\gamma+\theta} - a_\gamma}{1-\gamma} T\Big), & \textit{without transaction costs}, \\ 1 - \exp\Big(\dfrac{\ln\{(1-\gamma)\varphi_{\gamma+\theta}(z)\} - a_\gamma T}{1-\gamma} - \ln(1+z)\Big), & \textit{with transaction costs}, \end{cases}$$

where

$$a_l = (1-\gamma)\Big(r + \frac{1}{2l}\Big(\frac{\mu - r}{\sigma}\Big)^2\Big),$$

$\varphi(z)$ is the dimension reduced version of value function $V(x, y, t)$ and z is the bond-to-stock ratio.

Proof. See Online Appendix. **Q.E.D.**

[Insert Table 1 Here]

For our numerical analysis, we use parameter values similar to those by Maenhout (2004) calibrating the model to match time series of two different lengths: a long annual data from 1891 to 1994 (the century-long sample) and a quarterly postwar data from 1947.2 to 1996.3 (the postwar sample). The estimated parameter values are summarized in Table 1. The equity premium ($\mu - r$) has been observed as 6.258% and 7.852%, the risk-free rate (r) as 1.955% and 0.7852%, the stock volatility (σ) as 18.534% and 15.218% for the century-long sample and the postwar sample, respectively. Keeping risk aversion γ relatively low at 7 and 10, ambiguity aversion $\theta = 14$ and $\theta = 237$ explain the equity premium and the risk-free rate for the century-long sample and the postwar sample, respectively. Following Jang *et al.* (2007), we set default transaction cost parameter values at $\alpha = \beta = 1\%$, which is consistent with Hasbrouck (2009) demonstrating that the Gibbs estimates of effective transaction cost are around 1%.

[Insert Figure 1 here.]

[Insert Figure 2 here.]

Figure 1 shows the percentage ambiguity premium with respect to changes in cash (or bond investment). Overall, ignoring ambiguity aversion with and without transaction costs generates large ambiguity premia when ambiguity aversion is moderate ($\theta = 14$). The premia are quite substantial, generating as high as 6% of wealth for moderate ambiguity aversion (Figure 1 (a) $\theta = 14$). It turns out that an assumption of moderate ambiguity aversion is enough for the investor to have a dramatic change in her investment decisions with transaction costs (Figure 2 (a) $\theta = 14$), reflecting a significant increase in effective risk aversion as Theorem III.1 predicts. Further, the cost of ignoring ambiguity aversion becomes larger with higher ambiguity aversion. The premia with transaction costs can be higher than 12% of wealth for high ambiguity aversion (Figure 1 (b) $\theta = 237$). Even though transaction costs are known to have a second-order effect on portfolio choice with risk aversion only (Constantinides, 1986; Liu and Loewenstein, 2002), such substantial effects of ambiguity aversion with transaction costs can be understood as a result of an accurate reflection of the investor's strong pessimism about stock returns, and thereby relatively large incurred transaction costs for reducing equity demand amount to a significant increase in ambiguity premium (Figure 2 (b) $\theta = 237$). This would, thus, still support importance of ambiguity aversion channel for portfolio choice, even concerning the friction markets.

IV. Conclusion

In this paper, we develop a tractable workhorse investment model of an ambiguity averse investor with transaction costs. Our analysis demonstrates that ambiguity aversion does matter for portfolio choice, even concerning the friction markets. We show the economic significance of correctly considering ambiguity aversion in the optimal investment with transaction costs. If an investor neglects ambiguity aversion in her routine investment decision with and without transaction costs, the utility cost measured as the certainty equivalent wealth loss can be as high as 6% of wealth for moderate ambiguity aversion and around 12% of wealth for high ambiguity aversion. This finding indicates the economic importance of correctly taking ambiguity aversion into account the optimal investment decision.

We hope this paper will lend itself to the future study of researchers by extending our methods to multi-country economies and exploring both currencies and international flows, especially based on Colacito and Croce (2013) and Colacito et al. (2018).

References

Colacito, R., and M. M. Croce. "International asset pricing with recursive preferences." *Journal of Finance*, 68 (2013), 2651–2686.

Colacito, R., M. M. Croce, F. Gavazzoni, and R. Ready. "Currency risk factors in a recursive multicountry economy." *Journal of Finance*, 73 (2018), 2719–2756.

Constantinides, G. M.. "Capital market equilibrium with transaction costs." *Journal of Political Economy*, 94 (1986), 842–862.

Dai, M., and F. Yi. "Finite-horizon optimal investment with transaction costs: A parabolic double obstacle problem." *Journal of Differential Equations*, 246 (2009), 1445–1469.

Davis, M. H. A., and A. R. Norman. "Portfolio selection with transaction costs." *Mathematics of Operations Research*, 15 (1990), 676–713.

Hasbrouck, J.. "Trading costs and returns for us equities: Estimating effective costs from daily data." *Journal of Finance*, 64 (2009), 1445–1477.

Jang, B. G., H. Keun Koo, H. Liu, and M. Loewenstein. "Liquidity premia and transaction costs." *Journal of Finance*, 62 (2007), 2329–2366.

Liu, H., and M. Loewenstein. "Optimal portfolio selection with transaction costs and finite horizons." *Review of Financial Studies*, 15 (2002), 805–835.

Maenhout, P. J.. "Robust portfolio rules and asset pricing." *Review of Financial Studies*, 17 (2004), 951–983.

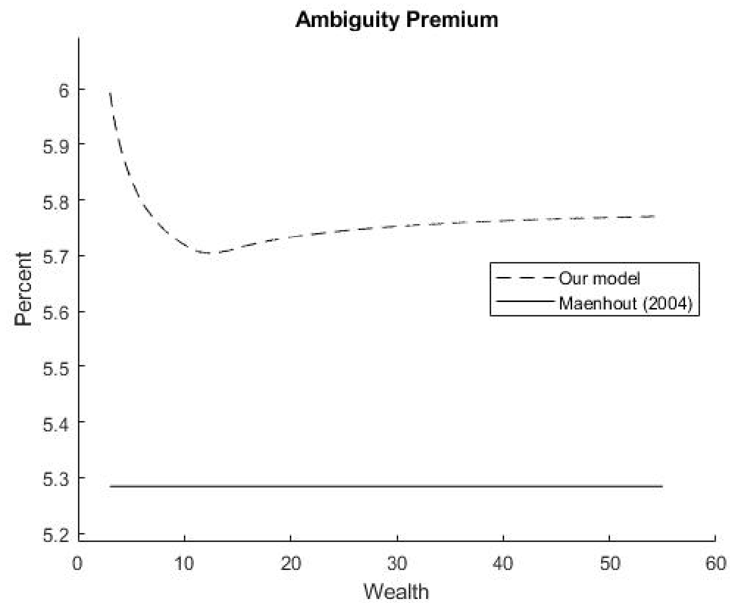
Merton, R. C.. "Lifetime portfolio selection under uncertainty: The continuous-time case." *Review of Economics and Statistics*, (1969), 247–257.

Merton, R. C.. "Optimum consumption and portfolio rules in a continuous-time model." *Journal of Economic Theory*, 3 (1971), 373–413.

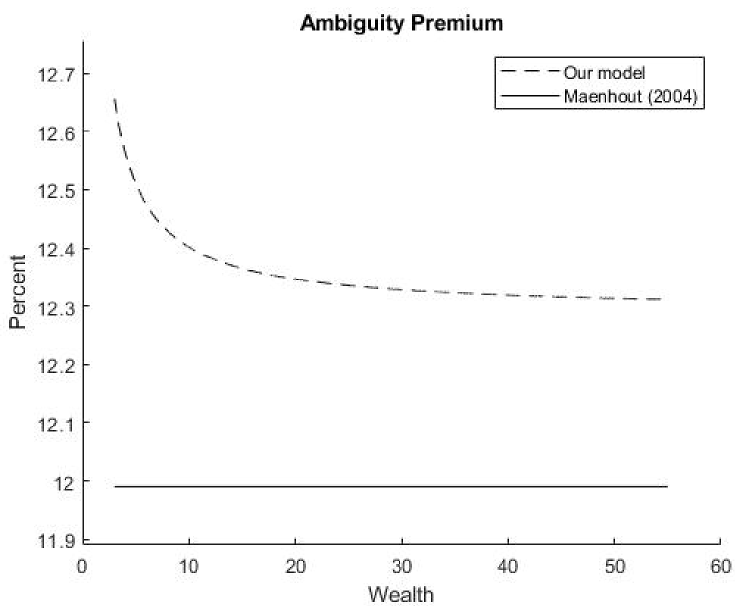
Shreve, S. E., and H. M. Soner. "Optimal investment and consumption with transaction costs." *Annals of Applied Probability*, 4 (1994), 609–692.

Data Horizon	1891-1994	1947.2-1996.3
μ - r	6.258%	7.852%
r	1.955%	0.7852%
σ	18.534%	15.218%
γ	7	10
θ	14	237
$\alpha = \beta$	1%	1%

Table 1: Parameters. The equity premium ($\mu - r$), the risk-free rate (r), the stock volatility (σ) are estimated to match time series of two different lengths: a long annual data from 1891 to 1994 (the century-long sample) and a quarterly postwar data from 1947.2 to 1996.3 (the postwar sample). Keeping risk aversion γ relatively low at 7 and 10, ambiguity aversion $\theta = 14$ and $\theta = 237$ explain the equity premium and the risk-free rate for the century-long sample and the postwar sample, respectively. Following Jang *et al.* (2007), we set default transaction cost parameter values at $\alpha = \beta = 1\%$.

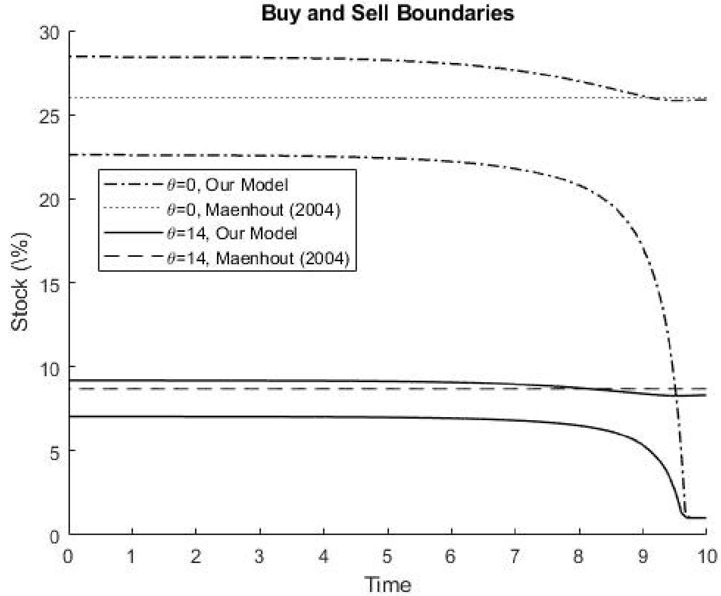


(a) $\theta = 14$

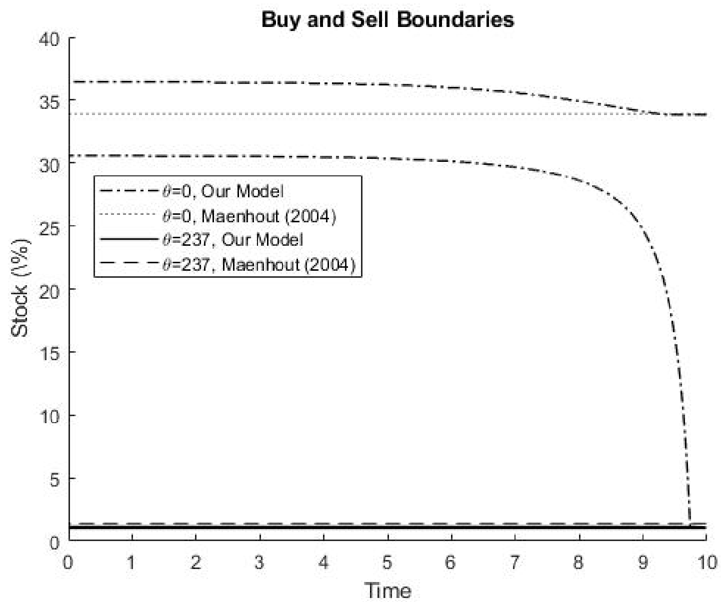


(b) $\theta = 237$

Figure 1: Ambiguity Premium. The ambiguity premium reported here is the percentage, i.e., $\Delta/(1+z)$ (Our model) and Δ/w (Maenhout, 2004), where we normalize initial wealth $w = x + y$ as $1 + z$ by dividing the both sides by y and setting $y = 1$ as in Davis and Norman (1990) and Jang *et al.* (2007). The transaction costs are set $\alpha = \beta = 0.01$. The other baseline parameter values are chosen for two sample date periods in exactly the same manner as Maenhout (2004) does: (a) a long annual sample from 1891 to 1994 and (b) a quarterly postwar sample from 1947.2 to 1996.3. (a) $\mu - r = 0.06258$, $r = 0.01955$, $\sigma = 0.18534$, $\gamma = 7$, and $\theta = 14$ and (b) $\mu - r = 0.07852$, $r = 0.007852$, $\sigma = 0.15218$, $\gamma = 10$, and $\theta = 237$.



(a) $\theta = 14$



(b) $\theta = 237$

Figure 2: Portfolio Share. The portfolio share plotted here is the proportion of wealth invested in the stock market when the entire investment horizon is 10 years ($T = 10$). As Theorem III.2 implies, there are two functions of time corresponding to the sell region (SR) and the buy region (BR) in our model with transaction costs. While there exists the so-called constant Merton line without transaction costs (Maenhout, 2004). The transaction costs are set $\alpha = \beta = 0.01$. The other baseline parameter values are chosen for two sample date periods in exactly the same manner as Maenhout (2004). (a) a long annual sample from 1891 to 1994 and (b) a quarterly postwar sample from 1947.2 to 1996.3. (a) $\mu - r = 0.06258$, $r = 0.01955$, $\sigma = 0.18534$, $\gamma = 7$, and $\theta = 14$ and (b) $\mu - r = 0.07852$, $r = 0.007852$, $\sigma = 0.15218$, $\gamma = 10$, and $\theta = 237$.