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Torsion Motives

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In this paper, we study Chow motives whose identity map is killed by a natural number. Examples of such objects were constructed by Gorchinskiy and Orlov [10]. We introduce various invariants of torsion motives, in particular, the *p-level*. We show that this invariant bounds from below the dimension of the variety a torsion motive M is a direct summand of and imposes restrictions on motivic and singular cohomology of M. We study in more details the p-torsion motives of surfaces, in particular, the Godeaux torsion motive. We show that such motives are in 1-to-1 correspondence with certain Rost cycle submodules of free modules over H^*_{et}. This description is parallel to that of mod-p reduced motives of curves.

1 Introduction

The purpose of this article is to initiate the investigation of an interesting class of Chow motives that, for some reason, did not attract attention they deserve before. These are the so-called *torsion motives*, that is, Chow motives that disappear with rational coefficients.

For me, the motivation for studying such objects comes from *isotropic realization functors* of [28]. Such realizations $\psi_E : \mathrm{DM}(k;\mathbb{F}_p) \to \mathrm{DM}(\widetilde{E}/\widetilde{E};\mathbb{F}_p)$ are parameterized by the finitely generated field extensions E/k of the ground field and take values in the *isotropic motivic categories*. Here, $\widetilde{E} = E(t_1, t_2, \ldots)$ is the *flexible closure* of E, and isotropic motivic category $\mathrm{DM}(F/F;\mathbb{F}_p)$ is obtained from $\mathrm{DM}(F;\mathbb{F}_p)$ by moding out the motives of p-anisotropic varieties over F. It appears that, for *flexible* fields

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[28, 1.2], the isotropic motivic categories are sufficiently small, with Hom groups between compact objects expected to be finite. This permits to assign to an object of the “global” Voevodsky motivic category with finite coefficients its “local” much simpler versions. At the same time, this collection of isotropic realization functors (together with the, so-called, *thick* versions of them; see [28, Sect. 5]) is not entirely conservative. And the expectation is that the kernel of this family on motives with $\mathbb{Z}_p$-coefficients is generated exactly by *torsion motives*. Thus, understanding them is essential for completing the “isotropic picture”.

It is not a priori obvious that torsion motives should even exist. But examples of such objects were constructed by Gorchinskiy and Orlov [10] as direct summands in the motives of Godeaux, Beauville, and Burniat surfaces (based on earlier works by Alexeev and Orlov [1], Böhning–Graf von Bothmer–Sosna [3], and Galkin and Shinder [8]). Gorchiskiy and Orlov used these motives to construct the 1st known example of the *phantom category*.

We start our investigation by assigning certain invariants to torsion motives, which measure the complexity of such objects. Namely, by the results of [29], any Chow motive M can be lifted to a cobordism motive M^{Ω} and so (by the universality of algebraic cobordism of Levine and Morel [13], we assume $char(k) = 0$), to a motive M^A in the sense of an arbitrary oriented theory A^*. The annihilator of the cobordism version provides an ideal in the Lazard ring, whose radical is the intersection of finitely many ideals $I(p, n)$, where p is prime and $I(p, n) = (p, v_1, \dots, v_{n-1})$ are invariant prime ideals of Landweber [12] (v_i is a v_i-element of dimension $p^i - 1$). This leads to the notion of a *p-level* of a motive—see Definition 2.12—the number n appearing above (∞, if p is not a torsion prime, and zero, if M is not a torsion motive). This invariant can be interpreted in terms of Morava K-theory versions of M. Namely, if M has p-level n, then $M^{K(r)} = 0$, for $0 \leqslant r < n$, and $M^{K(r)} \neq 0$, for $r \geqslant n$; see Proposition 2.13. Since $K(1)$ is just the usual K-theory, localized at p and re-oriented (in the sense of [18] or [17]), it follows that the motives of p-level > 1 are exactly the *K-phantom motives*, that is, such Chow motives, whose K-version is trivial. The level appears to be a rather rigid invariant—it is stable in tensor powers: the p-level of M is the same as that of $M^{\otimes r}$.

The p-level imposes various restrictions on the motive. In particular, on the dimension of the respective smooth projective variety X (whose motive M is a direct summand of). In Theorem 2.19, we show that, for a motive of p-level n, such dimension is $\geqslant \frac{p^n-1}{p-1}$. Moreover, over any field extension F/k, our motive has no Chow groups in co-dimensions smaller than the specified bound; see Proposition 2.20. These results are obtained using *symmetric operations* of [25]. We also show that if, for a motive of

p-level n, the $\dim(X) \leqslant \frac{(2p-1)(p^{n-1}-1)}{p-1}$, then the associated ideal $\overline{J}(M)$ (Definition 2.1) is radical. This implies—Corollary 3.10 that Milnor's operations Q_i, $0 \leqslant i < n$ of Voevodsky act as exact differentials on the motivic cohomology of M (i.e., $\mathrm{Im}(Q_i) = \mathrm{Ker}(Q_i)$). This shows, in particular, that over an algebraically closed field, the etale cohomology of M (with finite coefficients) are absent in dimensions and co-dimensions $< n$. That is, the motive M cannot reside "too close to the surface" of the motive of the respective variety, which suggests that the bound on the possible dimension of X may be improved. This is indeed done in Corollary 3.19: for $n \leqslant 4$ and algebraically closed ground field, $\dim(X) \geqslant \lceil \frac{n}{2} \rceil + \frac{p^n-1}{p-1}$. We conjecture that the same holds in general.

In the 2nd part of the paper, we take a more detailed look at the torsion motives of surfaces. We start with the torsion direct summand in the motive of the Godeaux surface. We show that (as for any torsion motive of a surface), such a motive M is completely described by the zero-slice of it in the homotopy t-structure [7]. This slice corresponds to the Rost cycle module A, which is naturally a submodule of a free module over $\mathrm{H}^*_{et} = \mathrm{H}^*_{et}(-, \mathbb{Z}/5)$ of rank two with generators in degrees -1 and -2. The component of degree zero of A is exactly $\mathrm{CH}_0(M)$—the group of zero-cycles on M (considered over all field extensions $F/\mathbb{C}$). The latter group can be described as the group of $\mathrm{H}^*_{et}(F)$-relations between two unramified elements $u \in \mathrm{H}^1_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/5)$ and $v \in \mathrm{H}^2_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/5)$ in $\mathrm{H}^3_{et}(F(X), \mathbb{Z}/5)$. Among such relations, there is the generic one: (v, u), corresponding to $F = \mathbb{C}(X)$. We prove that this element of A^0 generates A as a Rost cycle module—Proposition 4.6. The mentioned free module $\mathrm{H}^*_{et}\langle -1\rangle \oplus \mathrm{H}^*_{et}\langle -2\rangle$ possesses a natural bilinear form (which is an extension of the standard hyperbolic form on $\mathbb{F}_5\langle -1\rangle \oplus \mathbb{F}_5\langle -2\rangle$ into the skew-commutative realm of H^*_{et}). We show that our submodule A is *Lagrangian* with respect to this form—Proposition 4.7. Finally, we express the diagonal class of the Godeaux surface in terms of the classes u and v.

We prove that torsion motives of surfaces satisfy the Krull–Schmidt principle; see Theorem 4.10. We look specifically at p-torsion motives over an algebraically closed field. Assigning to M the zero-th t-homotopic slices of M and $M^\vee$ we obtain a pair A and B of Rost cycle modules that are naturally submodules of free H^*_{et} cycle modules H_A and H_B with generators in degrees -1 and -2. There is a natural (H^*_{et}-valued) pairing between H_A and H_B, such that A and B are orthogonal to each other with respect to it. As in the case of the Godeaux motive above, we can describe the generators of A and B; Proposition 4.18. In particular, this gives the description of $\mathrm{CH}_0(M)$. The above assignment provides a 1-to-1 correspondence between p-torsion direct summands in the motives of surfaces and pairs of submodules of dual free H^*_{et}-modules with generators in degrees -1 and -2, satisfying certain conditions; see Theorem 4.13. This description is completely parallel to the description of direct summands of the reduced motives of curves with $\mathbb{Z}/p$-coefficients (Theorem 4.20). The only difference is that, in the latter case, the respective dual free modules have generators in degree -1 only (and duality is shifted accordingly). The abundance of mod-p motives of curves suggests that a similar situation should take place in the case of p-torsion motives of surfaces. We show that any nonzero p-torsion motive of surfaces has a nontrivial Picard group $\mathrm{CH}^1(M_{\overline{k}})$ over an algebraic closure; Corollary 4.22. Finally, we describe the automorphism group of a torsion motive in terms of the respective embedding $A \subset H_A$; Proposition 4.24.

2 Torsion Motives and their Invariants

Throughout the article, we will consider only fields of characteristic zero (the exception is Subsection 3.1, where results hold over an arbitrary field). In Sections 2 and 3 (aside from Subsection 3.1) and Section 5 our ground field will be any field of characteristic zero. In Section 4, we will work mostly over an algebraically closed field of characteristic zero, respectively $\mathbb{C}$, but some results will still be proven for an arbitrary field of characteristic zero.

Recall that to any oriented cohomology theory, one can assign a formal group law. Among such theories, there is a universal one—the algebraic cobordism of Levine–Morel Ω^* [13, Theorem 1.2.6]. The formal group law, in this case, is also universal. In particular, $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L}$—the Lazard ring. To any oriented cohomology theory A^*, one can assign its graded version GrA^* corresponding to the co-dimension of support filtration F_r on A^*, where an element belongs to F_r, if it vanishes on a complement to some closed subscheme of co-dimension r. In particular, below we will use such a graded version $Gr\Omega^*$ for the algebraic cobordism. We have the natural surjection of $\mathbb{L}$-algebras $\phi : \mathrm{CH}^* \otimes_{\mathbb{Z}} \mathbb{L} \twoheadrightarrow Gr\Omega^*$—[13, Corollary 4.5.8], which is an isomorphism rationally.

To any oriented cohomology theory A^*, one can assign the category of A^*-Chow motives $Chow^A(k)$; see, for example, [29, Sect. 2]. Any morphism of oriented cohomology theories $A^* \to B^*$ gives the functor $Chow^A(k) \to Chow^B(k)$. In particular, the natural projection $\Omega^* \to \mathrm{CH}^*$ from the algebraic cobordism of Levine–Morel to Chow groups leads to the functor $Chow^{\Omega}(k) \to Chow^{CH}(k) = Chow(k)$.

By [29, Corollary 2.8], the mentioned functor defines a 1-to-1 correspondence between isomorphism classes of Chow-motives and Ω^*-motives. Let M be an object of $Chow(k)$ (with integral coefficients) and $M^{\Omega} \in Ob(Chow^{\Omega}(k))$ be the respective Ω^*-motive. This permits to assign the following invariant to M.

Definition 2.1. Let $M \in Ob(Chow(k))$. Define $J(M) \lhd \mathbb{L}$ as the annihilator of $id_{M^{\Omega}}$. In other words, it is the annihilator of the projector ρ^{Ω} in $\Omega^*(X^{\times 2})$, where $M^{\Omega} = (X, \rho^{\Omega})$. Define $\overline{J}(M) \lhd \mathbb{L}$ as the annihilator of ρ^{Ω} in $Gr\Omega^*(X^{\times 2})$.

Obviously, $J(M) \subset \overline{J}(M)$. Recall that on the algebraic cobordism of Levine–Morel Ω^*, we have the action of Landweber–Novikov operations [13, Example 4.1.25]. The total Landweber–Novikov operation $S^{Tot}_{L-N} : \Omega^* \to \Omega^*[b_1, b_2, \ldots]$ is multiplicative (respects the product). These operations descend to the graded algebraic cobordism. Moreover, the action on $Gr\Omega^*$ is much simpler. Namely, if $z \in \mathrm{CH}(X)$, $u \in \mathbb{L}$ and we denote $\phi(z \otimes u)$ as $z \cdot u \in Gr\Omega^*(X)$, then $S^{Tot}_{L-N}(z \cdot u) = z \cdot S^{Tot}_{L-N}(u)$. Considering $z = \rho \in \mathrm{CH}_{\dim(X)}(X^{\times 2})$, we obtain the following.

Observation 2.2. The ideal $\overline{J}(M)$ is stable under Landweber–Novikov operations.

Remark 2.3. Clearly, under field extensions, these ideals can only increase. So, these invariants are trivial for geometrically split motives and even for motives containing any split components over some extensions. In particular, for (the whole) motives of varieties. △

Definition 2.4. A Chow motive M is a "torsion motive", if there exists $n \in \mathbb{N}$ such that $n \cdot id_M = 0$.

Proposition 2.5. The following conditions are equivalent:

(1) M is a torsion motive;

(2) $J(M) \neq 0$.

Proof. Suppose $M = (X, \rho)$. (1) → (2): if $n \cdot \rho = 0$, then $n \cdot \rho^{\Omega}$ has support in codimension $> \dim(X)$. Hence, it is $\circ$-nilpotent. But ρ^{Ω} is a projector. Hence, some power of n belongs to $J(M)$.

(2) → (1): Recall that the total Landweber–Novikov operation acts on $Gr\Omega^*(X^{\times 2})$ as follows: for $z \in \mathrm{CH}(X^{\times 2})$ and $u \in \mathbb{L}$, we have $S^{Tot}_{L-N}(z \cdot u) = z \cdot S^{Tot}_{L-N}(u)$. So, if $u \in J(M)$, then for any individual Landweber–Novikov operation $S^{\overline{b}}_{L-N}$, the product $S^{\overline{b}}_{L-N}(u) \cdot \rho^{\Omega}$ has support in co-dimension $> \dim(X)$, so is nilpotent. Thus, some power of $S^{\overline{b}}_{L-N}(u)$ belongs to $J(M)$. But $u \neq 0 \Leftrightarrow (S^{Tot}_{L-N}(u))_{deg=\dim(u)} \neq 0$ and the latter elements reside in $\mathbb{L}_{\dim=0} = \mathbb{Z}$, which has no nilpotents. Hence, if $J(M) \neq 0$, it contains some nonzero integers. ■

We also can bound $J(M)$ by an invariant ideal from below.

Definition 2.6. Let $M \in Ob(Chow(k))$. Define $\widetilde{J}(M) \lhd \mathbb{L}$ as the annihilator of $\mathrm{S}^{Tot}_{L-N}(\rho^{\Omega})$ in $\Omega^*(X^{\times 2})$, where $M^{\Omega} = (X, \rho^{\Omega})$.

Proposition 2.7. The ideal $\widetilde{J}(M)$ is well defined, that is, it does not depend on the presentation of M in the form (X, ρ).

Proof. For an Ω^*-correspondence $\alpha : X \rightsquigarrow Y$, let us introduce $\widetilde{\mathrm{S}}^{Tot}_{L-N}(\alpha) := \mathrm{S}^{Tot}_{L-N}(\alpha) \cdot \mathrm{S}^{L-N}_{Tot}(1_Y)$. It follows from the Riemann–Roch theorem (for multiplicative operations)—Panin [16, Theorem 2.5.4] that it respects composition of correspondences: $\widetilde{\mathrm{S}}^{Tot}_{L-N}(\beta \circ \alpha) = \widetilde{\mathrm{S}}^{Tot}_{L-N}(\beta) \circ \widetilde{\mathrm{S}}^{Tot}_{L-N}(\alpha)$.

Let (X, ρ) and (Y, ε) be two presentations for M^{Ω} (from now on, we will drop the superscript Ω in the projectors). Then there are Ω^*-correspondences $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow X$ such that $\rho = g \circ \varepsilon \circ f$ and $\varepsilon = f \circ \rho \circ g$, and so $\widetilde{\mathrm{S}}^{Tot}_{L-N}(\rho) = \widetilde{\mathrm{S}}^{Tot}_{L-N}(g) \circ \widetilde{\mathrm{S}}^{Tot}_{L-N}(\varepsilon) \circ \widetilde{\mathrm{S}}^{Tot}_{L-N}(f)$, and similarly for ε. Hence, $u \in \mathbb{L}$ annihilates $\widetilde{\mathrm{S}}^{Tot}_{L-N}(\varepsilon)$ if and only if it annihilates $\widetilde{\mathrm{S}}^{Tot}_{L-N}(\rho)$. But $\mathrm{S}^{L-N}_{Tot}(1_Z)$ is invertible, so $u \cdot \mathrm{S}^{Tot}_{L-N}(\varepsilon) = 0 \Leftrightarrow u \cdot \mathrm{S}^{Tot}_{L-N}(\rho) = 0$. ∎

Proposition 2.8. The ideal $\widetilde{J}(M)$ is invariant under Landweber–Novikov operations.

Proof. The total Landweber–Novikov operation is multiplicative, so $\mathrm{S}^{Tot}_{L-N}(x \cdot u) = \mathrm{S}^{Tot}_{L-N}(x) \cdot \mathrm{S}^{Tot}_{L-N}(u)$. This implies by induction on the degree of the monomial $\overline{b}^{\overline{r}}$ that $\mathrm{S}^{Tot}_{L-N}(\rho) \cdot \mathrm{S}^{\overline{b}^{\overline{r}}}_{L-N}(u) = 0$, if $\mathrm{S}^{Tot}_{L-N}(\rho) \cdot u = 0$. ∎

The following result shows that the ideals $J(M)$, $\overline{J}(M)$, and $\widetilde{J}(M)$ have the same radical.

Proposition 2.9. $\widetilde{J}(M) \subset J(M) \subset \overline{J}(M) \subset \sqrt{\widetilde{J}(M)}$.

Proof. By definition, $\widetilde{J}(M) \subset J(M) \subset \overline{J}(M)$.

Suppose $u \in J(M)$. Let us show by induction on r that the degree r part $(\mathrm{S}^{Tot}_{L-N}(\rho))_r$ of the total Landweber–Novikov operation applied to ρ is annihilated by some power of u. The base of induction is provided by our condition $\rho \cdot u = 0$ on u. Suppose $(\mathrm{S}^{Tot}_{L-N}(\rho))_{<r} \cdot u^a = 0$. Since $0 = (\mathrm{S}^{Tot}_{L-N}(\rho \cdot u))_r = \big((\mathrm{S}^{Tot}_{L-N}(\rho))_{<r} \cdot (\mathrm{S}^{Tot}_{L-N}(u))_{>0}\big)_r + (\mathrm{S}^{Tot}_{L-N}(\rho))_r \cdot u$, we obtain that $(\mathrm{S}^{Tot}_{L-N}(\rho))_r \cdot u^{a+1} = 0$.

If $v \in \overline{J}(M)$, then $v \cdot \rho$ is nilpotent by co-dimensional considerations. But ρ is an idempotent. Hence, some power of v annihilates ρ already in Ω^* and so belongs to $J(M)$. ∎

Landweber–Novikov operations provide a rich structure on the Lazard ring. In particular, prime ideals of $\mathbb{L}$ invariant under these operations can be classified. This was done by Landweber [12, Theorem 2.7]. Finitely generated nonzero invariant prime ideals of $\mathbb{L}$ are exactly ideals $I(p,n) = (p, v_1, \ldots, v_{n-1})$, where p is prime and v_i is a v_i-*element* of dimension $p^i - 1$. Such ideals are parametrized by pairs (p,n), where p is prime and n is a natural number. In addition, the ideal $I(0) = (0)$ is also prime and invariant.

Proposition 2.10. For a torsion motive M,

$$\sqrt{J(M)} = \sqrt{\overline{J}(M)} = \sqrt{\widetilde{J}(M)} = \bigcap_{(p,n)} I(p,n),$$

where (p,n) runs over finitely many distinct pairs.

Proof. It follows from [24, Proof of Theorem 4.1] that $\overline{J}(M)$ is generated by elements u such that the codimension of $\rho^{\Omega} \cdot u$ is positive. In particular, this ideal is finitely generated. By Observation 2.2, this ideal of $\mathbb{L}$ is also invariant under Landweber–Novikov operations. By the Theorem of Landweber—[12, Proposition 3.4], $\overline{J}(M) = Q_1 \cap \ldots \cap Q_r$, where $P_i = \sqrt{Q_i}$ are invariant finitely generated prime ideals, which by [12, Theorem 2.7] should have the form: $P_i = I(p_i, n_i)$. Then $\sqrt{\overline{J}(M)} = \cap_i P_i = \cap_i I(p_i, n_i)$. Finally, from Proposition 2.9, it follows that $\sqrt{J(M)} = \sqrt{\overline{J}(M)} = \sqrt{\widetilde{J}(M)}$. ■

It follows from [21, Proposition 2.21] that $\widetilde{J}(M)$ is finitely generated too. Thus, Proposition 2.8 and the mentioned result of Landweber imply that $\widetilde{J}(M) = \widetilde{Q}_1 \cap \ldots \cap \widetilde{Q}_r$ for some finitely generated invariant primary ideals with the same radicals $\sqrt{\widetilde{Q}_i} = P_i$.

To simplify the picture, we will move to motives with $\mathbb{Z}_{(p)}$-coefficients. Recall that, as in topology, the p-localized algebraic cobordism $\Omega^*_{\mathbb{Z}_{(p)}}$ naturally splits as a polynomial algebra over the theory BP^* with the coefficient ring $BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where v_i are v_i-elements of dimension $p^i - 1$. We will denote the ideal $(p, v_1, \ldots, v_{n-1})$ of BP as $I(n)$. The Landweber–Novikov operations descend naturally to the BP-theory and the respective prime invariant finitely generated ideals are exactly $I(n)$, for $n \in \mathbb{N}$, and $I(0) = (0)$ [12]. We will denote the respective Landweber–Novikov algebra as BP_*BP as in topology. In the p-localized case, we will use the same notations $J(M)$ and $\overline{J}(M)$ for the similar ideals of BP. We have the graded version $GrBP$ and the natural surjection $\phi : \mathrm{CH}^* \otimes_{\mathbb{Z}} BP \twoheadrightarrow GrBP^*$, which we will denote simply by $\phi(z \otimes u) =: z \cdot u$.

Corollary 2.11. Let $M \in Ob(Chow(k, \mathbb{Z}_{(p)}))$ be nonzero, then $\sqrt{J(M)} = I(n)$, for some n. In particular, $J(M)$ contains some powers of v_r, for all $r < n$, and does not contain any such powers, for $r \geqslant n$.

Definition 2.12. Let $M \in Ob(Chow(k))$. Define the "p-level" of M as the number n, such that $\sqrt{J(M_{\mathbb{Z}_{(p)}})} = I(n)$, and ∞, if $M_{\mathbb{Z}_{(p)}} = 0$.

Let $K(r)$, for $r \geqslant 1$, be the r-th Morava K-theory. It is obtained from $\Omega^*(X)$ by change of coefficients: $K(r)(X) = \Omega^*(X) \otimes_{\mathbb{L}} K(r)$, where the coefficient ring $K(r)$ is $\mathbb{Z}/p[v_r, v_r^{-1}]$. We can also introduce $K(0)(X) = \mathrm{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. As was explained above, any Chow motive corresponds uniquely to a cobordism motive. Having a cobordism motive, we can construct A^*-motive for any oriented theory A^*, by using the *universality* of algebraic cobordism Ω^*—[13, Theorem 1.2.6]. In particular, any Chow motive M produces a sequence of Morava-motives $M^{K(r)}$. We can interpret the p-level of M in terms of these motives.

Proposition 2.13. The p-level n of M is uniquely determined from the condition: $M^{K(r)} = 0$, for $r < n$, and $M^{K(r)} \neq 0$, for $r \geqslant n$.

Proof. The fact that $M^{K(r)} = 0$, for $r < n$, is obvious, as $J(M)$ contains some power of v_r, while this element is inverted in $K(r)$.

Let now $r \geqslant n$. Let $M = (X, \rho)$ and $M^\vee = (X, \rho^\vee)$ be the direct summand given by the dual projector (we swap two factors in $X^{\times 2}$). Then no power of v_r annihilates $BP(M \otimes M^\vee)$ (as the identity map of M is represented by the diagonal class there), and as M is p-primary torsion, no power of v_r annihilates the mod-p reduction $\overline{BP}(M \otimes M^\vee)$, where $\overline{BP} = BP/p$. Then no power of v_r will annihilate the graded version $Gr\overline{BP}(M \otimes M^\vee)$. Consider the graded component $\overline{BP}_{(c)}(M \otimes M^\vee)$ of codimension of support $= c$. We have two cases: (1) $v_r \in \sqrt{Ann(\overline{BP}_{(c)}(M \otimes M^\vee))}$, and (2) $v_r \notin \sqrt{Ann(\overline{BP}_{(c)}(M \otimes M^\vee))}$.

(1) From Sechin [21, Proposition 2.21], we know that $\overline{BP}_{(c)}(M \otimes M^\vee)$ is a union of finitely presented BP_*BP modules, which by the result of Landweber [12, Lemma 3.3] are extensions of finitely many modules of the type $\overline{BP}/I(m)$. Since some power of v_r annihilates our module, all these m are $> r$. But, for such m, $Tor_i^{\overline{BP}}(\overline{BP}/I(m), K(r)) = 0$, for all i, as can be seen from the Koszul resolution of the $\overline{BP}$-module $\overline{BP}/I(m)$. Since Tors commute with filtered colimits, we obtain that $Tor_i^{\overline{BP}}(\overline{BP}_{(c)}(M \otimes M^\vee), K(r)) = 0$, for all i. This implies that (for the codimension of support filtration F_l on $\overline{BP}$) the map

$$F_{c+1}\overline{BP}(M \otimes M^\vee) \otimes_{\overline{BP}} K(r) \hookrightarrow F_c\overline{BP}(M \otimes M^\vee) \otimes_{\overline{BP}} K(r)$$

is injective (actually, an isomorphism).

(2) The space of $\overline{BP}$-generators of $\overline{BP}_{(c)}(M \otimes M^{\vee})$ can be identified with $\mathrm{CH}^c(M \otimes M^{\vee})/p$. From [24, Proof of Theorem 4.1], we know that for any subspace of generators, the relations in the respective $\overline{BP}$-submodule are generated by elements of positive co-dimension. Let y be such a generator that $v_r \notin \sqrt{Ann(y)}$ (we know that such y exists). Let N be the minimal quotient-module of $\overline{BP}_{(c)}(M \otimes M^{\vee})$, obtained by moding out a $\overline{BP}$-submodule generated by some subspace of generators with the property that the image of y there is not annihilated by any power of v_r. I claim that N is finitely generated $\overline{BP}$-module. Note that N still has the property that the submodule generated by any subspace of generators of it has relations generated in positive co-dimension. Hence, if the submodule $\overline{BP} \cdot x$ generated by some generator x does not contain elements of the type $y \cdot v_r^k$ of positive co-dimension, it does not contain such elements at all (of any co-dimension). But since there are only finitely many elements of $\overline{BP}$ of bounded dimension, the space of generators of N has a subgroup C of finite index, so that for every $x \in C$, the submodule $\overline{BP} \cdot x$ does not contain elements of the form $y \cdot v_r^k$ of positive co-dimension. If C is nonzero, this contradicts the minimality of N (as we may mod-out the $\overline{BP}$-submodule, generated by x). Hence, N has only finitely many generators. Since all the relations are in positive co-dimension, it is finitely presented (there are only finitely many such relations).

Since N is a finitely presented BP_*BP-module, by the results of Landweber [12, Lemma 3.3], N is an extension of finitely many BP_*BP-modules of the type $\overline{BP}/I(m)$. Since no power of v_r annihilates the image of y and so, N itself, we see that among these pieces there should be, at least, one with $m \leqslant r$. Now, from the Koszul resolution of a $\overline{BP}$-module $\overline{BP}/I(m)$, we see that $\left(\overline{BP}/I(m)\right) \otimes_{\overline{BP}} K(r) = K(r)$, for $r \geqslant m$. Looking at the "most outer" piece with $m \leqslant r$ (and recalling that, for $l > r$, $Tor_i^{\overline{BP}}(\overline{BP}/I(l), K(r)) = 0$, for all i), we obtain that $N \otimes_{\overline{BP}} K(r) \neq 0$. Since N is a quotient-module of $\overline{BP}_{(c)}(M \otimes M^{\vee})$, we get that $\overline{BP}_{(c)}(M \otimes M^{\vee}) \otimes_{\overline{BP}} K(r)$ is nonzero too. And so is $F_c\overline{BP}(M \otimes M^{\vee}) \otimes_{\overline{BP}} K(r)$.

Since we know that there exists c, such that $v_r \notin \sqrt{Ann(\overline{BP}_{(c)}(M \otimes M^{\vee}))}$, combining cases (1) and (2), we obtain that $\overline{BP}(M \otimes M^{\vee}) \otimes_{\overline{BP}} K(r) = K(r)(M \otimes M^{\vee})$ is nonzero. Hence, $M^{K(r)} \neq 0$. ■

Thus, our notion of the *p-level* agrees with that of the *type* in topology; see [19, Definition 1.5.3].

Remark 2.14. Since $K(1)$ is just a re-orientation (as in Quillen [18] or Panin and Smirnov [17]) of the p-localized K-theory K_0, we see that (p-localized) Chow motives M of p-level

$\leqslant 1$ are distinguished by the property that their K-motives are nontrivial. For motives of p-level > 1, their K-motive is zero. So, these are "K-phantom motives". △

Moreover, the Morava K-theory of M disappers together with the "derived versions" of it.

Proposition 2.15. Let M be a (p-localized) Chow motive of p-level n. Then the following conditions are equivalent:

(1) $r < n$;
(2) $Tor_i^{BP}(BP(M \otimes M^{\vee}), K(r)) = 0$, for all i.

Proof. By the very definition, (1) implies that some power of v_r annihilates $BP(M \otimes M^{\vee})$. But by the results of Sechin [21, Proposition 2.21] and Landweber [12, Lemma 3.3], this module is a union of finitely presented BP-modules, each of which is an extension of finitely many modules of the type $BP/I(m)$. All these ms should be $> r$ due to mentioned annihilation. But as we discussed above, for $m > r$, $Tor_i^{BP}(BP/I(m), K(r)) = 0$, for all i. So, the same is true about $BP(M \otimes M^{\vee})$ that implies (2).

Conversely, since $Tor_0^{BP}(BP(M \otimes M^{\vee}), K(r)) = K(r)(M \otimes M^{\vee})$, the $i = 0$ case of (2) implies (1) by Proposition 2.13. ■

The following result shows that the p-level is stable under $\otimes$-powers.

Lemma 2.16. Let A^* be an oriented cohomology theory and $M \in Ob(Chow^A(k))$. Then $M \neq 0 \Rightarrow M^{\otimes n} \otimes (M^*)^{\otimes m} \neq 0$, $\forall n, m$, where $M^* = \underline{\mathrm{Hom}}(M, T)$.

Proof. It is sufficient to prove it for $n = m = 1$. This is a standard argument for tensor rigid categories: $M \neq 0 \Rightarrow id_M \neq 0 \Rightarrow T \stackrel{\Delta}{\to} M \otimes M^*$ is nonzero $\Rightarrow M \otimes M^* \neq 0$. ■

Applying this to Morava-motives of M, we obtain the first statement of the following.

Proposition 2.17. The p-level of $M^{\otimes n}$ coincides with that of M. More precisely, we have the following. The motives $M^{\otimes n} \otimes (M^{\vee})^{\otimes m}$, for all $(n, m) \neq (0, 0)$, have the same J and $\overline{J}$.

Proof. It is sufficient to compare $(1, 0)$ and $(1, 1)$. In the notations of the proof of Lemma 2.16, $z \cdot id_M = 0 \Leftrightarrow z \cdot \Delta = 0 \Leftarrow z \cdot id_{M \otimes M^*} = 0 \Leftarrow z \cdot id_M = 0$ in both Ω^*

and $Gr\Omega^*$, where $M = (X, \rho)$ and $M^\vee = M^*(d)[2d]$ for $d = \dim(X)$ is given by the dual projector. ■

The following result allows to obtain a bound on the possible size of a torsion motive in terms of the p-level of it.

Proposition 2.18. Let Y be a smooth variety, $z \in \mathrm{CH}^r_{\mathbb{Z}_{(p)}}(Y)$ and $\alpha \equiv v_m^k (mod\, I(m)) \in BP$ be such that $z \cdot \alpha = 0 \in GrBP(Y)$. Then,

(1) either $z = 0$, or $r \geqslant \frac{p^{m+1}-1}{p-1}$;
(2) if $r \leqslant \frac{(2p-1)(p^m-1)}{p-1}$, then there exists an α as above with $k = 1$.

Proof. The proof is based on *symmetric operations* of [25]. The total symmetric operation $\Phi : \Omega^*_{\mathbb{Z}_{(p)}} \to \Omega^*_{\mathbb{Z}_{(p)}}[t^{-1}]$ is the nonpositive part of the total Quillen's type Steenrod operation on Ω^* divided by "formal" $[p]$; see [25, Theorem 7.1]. This operation can be extended to the graded algebraic cobordism $Gr\Omega^*$ and to the BP^*-theory (as well as to the graded version of it); see [24]. By [25, Proposition 7.14], the action of Φ on $GrBP^*$ is described as follows: for $z \in \mathrm{CH}^r(X)$, with $r > 0$, and $u \in BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, we have

$$\Phi(z \cdot u) = z \cdot t^{r(p-1)} \cdot \mathbf{i}^r \cdot \Phi(u)_{\leqslant -r(p-1)}, \tag{1}$$

where $\mathbf{i}$ is some integer, invertible in $\mathbb{Z}_{(p)}$. Here, $\Phi_{\leqslant a}$ is the part of Φ of t-degree $\leqslant a$.

The idea is to "divide" the relation $\alpha \cdot z = 0$ by v_m using symmetric operations (cf. [25, Corollary 7.11]). This is possible as long as codimension of z is not very large. Namely, by [22, Proposition 7.16], we know about the action of Φ on BP that if $\alpha \equiv v_n^l (mod\, I(n))$, for $l > 0$, then $\Phi_{-l(p-1)(p^n-1)-(p^n-1)}(\alpha) \equiv -v_n^{l-1} (mod\, I(n))$. If $r = codim(z) \leqslant \frac{p(p^m-1)}{p-1}$, then $-l(p-1)(p^m-1) - (p^m-1) \leqslant -r(p-1)$, and so (1) applies. This shows by decreasing induction on l, that for every $k \geqslant l \geqslant 0$, there is $\alpha_l \equiv v_m^l (mod\, I(m))$ such that $\alpha_l \cdot z = 0 \in GrBP^*(Y)$. More precisely, we apply $\Phi_{r(p-1)-l(p-1)(p^m-1)-(p^m-1)}$ to $\alpha_l \cdot z = 0$ and get $\alpha_{l-1} = -\mathbf{i}^{-r} \cdot \Phi_{-l(p-1)(p^m-1)-(p^m-1)}(\alpha_l)$ of the needed form. Finally, we obtain $\alpha_0 \equiv 1 (mod\, I(m))$ such that $\alpha_0 \cdot z = 0 \in GrBP^*$. Hence, either $z = 0$, or $r > \frac{p(p^m-1)}{p-1} = \frac{p^{m+1}-1}{p-1} - 1$. This proves (1).

If $r \leqslant \frac{(2p-1)(p^m-1)}{p-1}$, then $-l(p-1)(p^m-1) - (p^m-1) \leqslant -r(p-1)$ as long as $l \geqslant 2$, and so, there is $\alpha_1 \equiv v_m (mod\, I(m))$ such that $\alpha_1 \cdot z = 0 \in GrBP^*(Y)$, which gives (2). ■

Theorem 2.19. Let $M = (X, \rho) \in Ob(Chow(k, \mathbb{Z}_{(p)}))$ be a motive of p-level n. Then $\dim(X) \geqslant \frac{p^n-1}{p-1}$.

Proof. This follows from Proposition 2.18 (1) applied to $Y = X^{\times 2}$, $r = \dim(X)$, $m = n-1$, and $z = \rho$, $\alpha = v_{n-1}^k \in \overline{J}(M)$. ■

More generally, we obtain the following.

Proposition 2.20. Let $M = (X, \rho)$ be a torsion motive of p-level n. Then $\mathrm{CH}^i_{\mathbb{Z}_{(p)}}(M_F) = 0$, for $i < \frac{p^n-1}{p-1}$ and any field extension F/k.

Proof. Observe that every element $\alpha \in BP$ annihilating ρ will annihilate $\mathrm{CH}^*_{\mathbb{Z}_{(p)}}(M_F)$ in $GrBP$. It remains to apply Proposition 2.18 (1). ■

Proposition 2.21. Let $M = (X, \rho) \in Ob(Chow(k; \mathbb{Z}_{(p)}))$ be a torsion motive of p-level n with $\dim(X) \leqslant \frac{(2p-1)(p^{n-1}-1)}{p-1}$. Then $\overline{J}(M) = I(n) = \sqrt{J(M)}$.

Proof. By Proposition 2.18 (2) applied to $z = \rho$, we know that there exists some $\alpha \equiv v_{n-1} (mod\, I(n-1)) \in \overline{J}(M)$. Then it follows from Observation 2.2 that $I(n) \subset \overline{J}(M)$. Indeed, if $\alpha \equiv v_{n-1}(mod\, I(n-1))$, then $\alpha \equiv v_{n-1}(mod\, I^2)$, where $I = I(\infty)$ is the ideal generated by all v_is. Applying appropriate Landweber–Novikov operations to α, we get elements $\alpha_i \equiv v_i(mod\, I^2)$ in $\overline{J}(M)$, for any $0 \leqslant i \leqslant n-1$; see, for example, [24, Proposition 3.1]. Then an increasing induction on i shows that $v_i \in \overline{J}(M)$, for any $0 \leqslant i \leqslant n-1$. Finally, in light of Proposition 2.9, our embedding is an equality. ■

In addition, we can observe that torsion motives are not present in the motives of curves.

Proposition 2.22. If $M = (X, \rho)$ is a torsion motive, then $\dim(X) > 1$.

Proof. The fact that torsion motives cannot be direct summands of 0-dimensional varieties is obvious, since the Chow-endomorphism rings of the latter are torsion-free.

Suppose X is a curve. To prove that $\rho = 0$, it is sufficient to show that it acts trivially on Chow groups of X over any field extension E/k. The action on $\mathrm{CH}^0(X) = \mathbb{Z}$ is clearly zero. By the same reason, the action on $\mathrm{CH}_0(X)$ passes through 0-cycles of degree 0 and so defines an action on $J(E)$—the E-rational points of the Jacobian of X. It is given by an algebraic map $J \to J$ that should be the projection to the zero point, as it is torsion. Hence, $\rho = 0$. ■

Torsion motives do exist.

Example 2.23. In [3], [8], and [1] examples of the so-called "quasiphantom" subcategories of $D^b(coh(S))$ were constructed for some surfaces (over the field $\mathbb{C}$). Namely, for

Godeaux, Beauville, and Burniat surfaces. It was shown by Gorchiskiy and Orlov in [10, Proposition 2.2] that Chow motives of respective surfaces contain direct summands M such that $\mathrm{CH}^*(M) = \mathrm{CH}^1(M) = \mathrm{Pic}(S)_{tors}$. The latter group is $\mathbb{Z}/5\mathbb{Z}$, $(\mathbb{Z}/5\mathbb{Z})^2$ and $(\mathbb{Z}/2\mathbb{Z})^6$, for Godeaux, Beauville, respectively, Burniat surface. Moreover, it follows from [10, Proposition 2.3] that any $n \in \mathbb{N}$ that annihilates $\mathrm{Pic}(S)_{tors}$, annihilates M itself. Thus, the motives M are torsion motives. These were used by Gorchinsky and Orlov to construct the 1st known example of a "phantom category"; see [10].

Since our motives M are killed by 5, respectively, 2, their ideals $J(M)$ involve single primes only. Moreover, since their K-motives (=$K(1)$-motives) are nontrivial (as $K_0(M) = \mathrm{Pic}(S)_{tors} \neq 0$), it follows from Proposition 2.13 that p-levels of M are 1. The same can be seen from Theorem 2.19, as motives of surfaces cannot contain torsion motives of level > 1. Thus, $\sqrt{J(M)}$ is $I(5,1)$, $I(5,1)$, $I(2,1)$ for the Godeaux, Beauville, respectively Burniat torsion motive.

3 Motivic Cohomology of Torsion Motives

More subtle results on torsion motives can be obtained with the help of their motivic cohomology.

3.1 Pre-morphisms of theories

The results of this subsection hold over any field (of any characteristic).

Definition 3.1. Let A^* and B^* be oriented cohomology theories in the sense of Levine–Morel; [13, Definition 1.1.2] (no (LOC) axiom). A "pre-morphism" of theories $G : A^* \to B^*$ is an additive morphism of functors on $\mathbf{Sm}_k$ that, in addition, respects maps of multiplication by the 1st Chern classes of line bundles.

Proposition 3.2. Let k be any field. A "pre-morphism" of theories is $\mathbb{L}$-linear.

Proof. Consider the map

$$X \times \left(\mathbb{P}^\infty \times \mathbb{P}^\infty \stackrel{Segre}{\to} \mathbb{P}^\infty\right).$$

Let $x^{A,B}, y^{A,B}, t^{A,B}$ be the 1st Chern classes of the line bundles $O(1)$ lifted from various components, in A^*, respectively, B^*-theory. Then $Segre^*(t^C) = F_C(x^C, y^C)$, where F_C is the formal group law of the oriented theory C. Let $\alpha \in A^*(X)$. Then, using the fact that G

respects multiplication by the 1st Chern classes, we get

$$\sum_{i,j} G(\alpha \cdot a^A_{i,j}) \cdot (x^B)^i (y^B)^j = \sum_{i,j} G(\alpha \cdot a^A_{i,j} (x^A)^i (y^A)^j) = G(\alpha \cdot F_A(x^A, y^A)) =$$

$$G(\alpha \cdot Segre^*(t^A)) = (id_X \times Segre)^* G(\alpha \cdot t^A) = (id_X \times Segre)^* (G(\alpha) \cdot t^B) =$$

$$G(\alpha) \cdot Segre^*(t^B) = G(\alpha) \cdot F_B(x^B, y^B) = \sum_{i,j} G(\alpha) \cdot a^B_{i,j} \cdot (x^B)^i (y^B)^j.$$

Comparing coefficients at $(x^B)^i (y^B)^j$, we obtain $G(\alpha \cdot a^A_{i,j}) = G(\alpha) \cdot a^B_{i,j}$. Since the coefficients of the universal formal group law generate the Lazard ring $\mathbb{L}$ additively, we get that G is $\mathbb{L}$-linear. ■

Proposition 3.3. Let k be an arbitrary field and $G : A^* \to B^*$ be an additive operation, where A^* and B^* are oriented cohomology theory in the sense of Levine–Morel and B^*, in addition, satisfies the weak form of (LOC) axiom (exactness in the middle) as in Panin [16, Definition 1.1.7]. Then the following conditions are equivalent.

(1) G respects push-forwards.

(2) G commutes with maps of multiplication by the 1st Chern classes of line bundles.

Proof. (1) → (2). Let L be a line bundle on X. We will denote the total space of it by the same symbol L. The zero section provides a regular closed embedding $f : X \to L$, s.t. $f_* f^*$ coincides with the multiplication by the 1st Chern class of L. Since G commutes with f^* and f_*, it commutes with the multiplication by this Chern class.

(2) → (1). Since any projective morphism can be decomposed into a composition of a regular embedding and a trivial projective fibration: $X \to \mathbb{P}^n \times Y \to Y$, it is sufficient to check the statement for these two types of maps.

1) Let $f : X \to Y$ be a regular emebedding with the normal bundle N_f. Let μ^B_i, $i \in \overline{n} = \{1, \dots, n\}$ be the B^*-*roots* of N_f. Then, by the general Riemann–Roch theorem—[27, Theorem 5.19], for any $\alpha \in A^*(X)$,

$$G(f_*(\alpha)) = f_* \operatorname*{Res}_{t=0} \frac{G(\prod_{i \in \overline{n}} x^A_i \cdot \alpha)(x^B_i = t +_B \mu^B_i|_{i \in \overline{n}}) \cdot \omega^B_t}{t \cdot \prod_{i \in \overline{n}} (t +_B \mu^B_i)},$$

where $x^{A,B}_i$ are the 1st Chern classes of the line bundle $O(1)$ lifted from the i-th component in $A^*(X \times (\mathbb{P}^\infty)^{\times n})$, respectively, $B^*(X \times (\mathbb{P}^\infty)^{\times n})$, and we make a substitution instead of x^B_i-variables. Since G commutes with the multiplication by the 1st Chern classes, this can be rewritten as

$$f_* \operatorname*{Res}_{t=0} \frac{G(\alpha) \cdot \prod_{i \in \overline{n}} (t +_B \mu_i^B) \cdot \omega_t^B}{t \cdot \prod_{i \in \overline{n}} (t +_B \mu_i^B)} = f_*(G(\alpha)).$$

2) Let $\pi : Y \times \mathbb{P}^n \to Y$ be the projection and $\alpha \in A^*(Y \times \mathbb{P}^n)$. By the projective bundle theorem, it can be uniquely written as $\alpha = \sum_{i=0}^n \pi^*(a_i) \cdot (\xi^A)^i$, where $\xi^A = c_1^A(O(1))$ and $a_i \in A^*(Y)$. Then, by the projection formula, $\pi_*(\alpha) = \sum_{i=0}^n a_i \cdot [\mathbb{P}^{n-i}]^A$. So, by Proposition 3.2 and (2),

$$G(\pi_*(\alpha)) = \sum_{i=0}^n G(a_i \cdot [\mathbb{P}^{n-i}]^A) = \sum_{i=0}^n G(a_i) \cdot [\mathbb{P}^{n-i}]^B = \pi_*(\sum_{i=0}^n \pi^* G(a_i) \cdot (\xi^B)^i) = \pi_*(G(\alpha)).$$

■

Remark 3.4. Proposition 3.3 shows that, if the target theory B^* has (a weak form of) localization, then "pre-morphisms" of theories can be defined as additive transformations respecting both pull-backs and push-forwards (for projective morphisms). This agrees with and provides a simplification of the [27, Definition 2.10]. So the only difference with genuine "morphisms" of theories is that "pre-morphisms" do not have to respect the multiplicative structure. △

3.2 Milnor's operations

In this subsection, our ground field is any field of characteristic zero.

Let

$$Q_r : \mathrm{H}_{\mathcal{M}}^{a,b}(X, \mathbb{F}_p) \to \mathrm{H}_{\mathcal{M}}^{a+2p^r-1, b+p^r-1}(X, \mathbb{F}_p)$$

be the motivic analogues of Milnor's operations introduced by Voevodsky [31]. These operations are differentials: $Q_r \circ Q_r = 0$ and anti-commute with each other: $Q_r \circ Q_l = -Q_l \circ Q_r$, for $r \neq l$. For $p = 2$ and $I \subset \overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$, let us denote as Q_I the composition $\circ_{i \in I} Q_i$. Then the behavior of Milnor's operations with respect to the multiplicative structure is given by: $Q_r(x \cdot y) = \mu(\Delta(Q_r)(x \otimes y))$, where

$$\Delta(Q_r) = \begin{cases} Q_r \otimes 1 + 1 \otimes Q_r, & \text{for } p > 2; \\ \sum_{2^I + 2^J = 2^r} Q_I \otimes Q_J \cdot \{-1\}^{|I|+|J|-1}, & \text{for } p = 2, \end{cases} \quad (2)$$

where $2^I = \sum_{i \in I} 2^i$.

Proposition 3.5. Milnor's operations are Chow group linear.

Proof. Milnor's operations act trivially on Chow groups of smooth varieties, as their bi-degree is $(p^r-1)[2p^r-1]$ and so the potential target resides in bi-degrees, where smooth varietieties have no motivic cohomology (above the slope $=2$ line). Then the coproduct formula (2) shows that $Q_r(u\cdot\alpha) = u\cdot Q_r(\alpha)$, for any Chow group element u. ■

As 1st Chern classes of line bundles reside in Chow groups, we immediately get the following.

Corollary 3.6. Milnor's operations are pre-morphisms of theories.

Taking into account that motivic cohomology satisfy the conditions of Proposition 3.3, combining further Proposition 3.5 with Proposition 3.3, we obtain the following result (cf. [33, Lemma 9.1]).

Corollary 3.7. Milnor's operations commute with (push-forwards for) correspondences and so define a morphism of functors on the category of Chow motives.

Proof. Let $\varphi : X \rightsquigarrow Y$ be a correspondence (between smooth projective varieties). Then φ_* is the composition of a pull-back, followed by a multiplication by a certain Chow group element, followed by a push-forward. By Propositions 3.5 and 3.3, Milnor's operations commute with all these ingredients. ■

Remark 3.8. Of course, nothing of this sort is true for "non-pure" motives, as Milnor's operations are not linear with respect to (non-pure) motivic cohomology elements. △

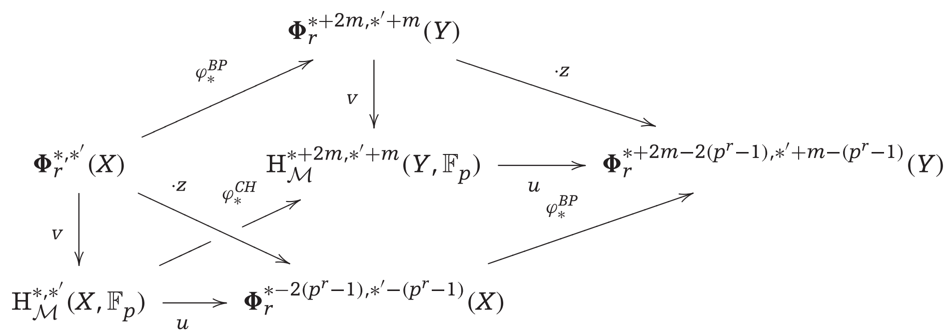
Following Voevodsky, let us introduce the spectrum $\mathbf{\Phi}_r$ in $\mathcal{SH}_{\mathbb{A}^1}(k)$ from the distinguished triangle

$$T^{p^r-1}\wedge \mathbf{H}_{\mathbb{F}_p} \xrightarrow{u} \mathbf{\Phi}_r \xrightarrow{v} \mathbf{H}_{\mathbb{F}_p} \xrightarrow{Q_r} S^1_s \wedge T^{p^r-1}\wedge \mathbf{H}_{\mathbb{F}_p},$$

where $\mathbf{H}_{\mathbb{F}_p}$ is the *Eilenberg–MacLane spectrum*. It follows from [30, Lemma 3.23] that $\mathbf{\Phi}_r$ has a structure of an MGL-module and so an orientation compatible with this distinguished triangle. In other words, that the respective cohomology theory has a structure of push-forwards compatible with push-forwards for motivic cohomology with $\mathbb{F}_p$-coefficients. Thus, we also have the action of (algebraic cobordism of Levine–Morel) Ω^*-correspondences on our distinguished triangle.

Proposition 3.9. Let X and Y be smooth projective varieties and $\varphi^{BP} : X \rightsquigarrow Y$ be a BP^*-correspondence. Let $z \in BP$ be such ν_r-element that $z \cdot \varphi^{BP} = 0 \in GrBP$. Then the Chow trace φ^{CH} acts trivially on the r-th Margolis cohomology. That is, if $Q_r(x) = 0$, for some $x \in \mathrm{H}_{\mathcal{M}}(X)$, then $\varphi^{CH}_*(x) = Q_r(y)$, for some $y \in \mathrm{H}_{\mathcal{M}}(Y)$.

Proof. Combining the action of Ω^*-correspondences with [30, Proposition 3.24], we get a diagram



commuting up to an invertible element of $\mathbb{F}_p$ (here, m is the degree of φ). Since $\mathbf{\Phi}_r^{2*,*}(V) = BP^*(V) \otimes_{BP} \mathbb{Z}/p[v_r]/(v_r^2)$, we see that, for $x \in BP^*(V)$, if $z \cdot x = 0 \in GrBP(V)$, then $z \cdot x = 0 \in \mathbf{\Phi}_r(V)$. Hence, the composition $u \circ \varphi^{CH}_* \circ v$ is zero, which means exactly that φ^{CH}_* acts as zero on $\mathrm{Ker}(Q_r)/\,\mathrm{Im}(Q_r)$. ■

Corollary 3.10. If M is a torsion motive such that $\overline{J}(M_{\mathbb{Z}_{(p)}}) \supset I(n)$, then the r-th Margolis cohomology of M are trivial, for all $0 \leqslant r < n$.

Proof. Apply Proposition 3.9 to $\varphi = \rho_M$. ■

Remark 3.11. More generally, one can show that if $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains some $\alpha \equiv v_{n-1}^k (mod\, I(n-1))$, then the spectral sequence $\mathrm{H}_{\mathcal{M}}^{*,*'}(M, \mathbb{Z}/p) \Rightarrow \mathrm{K}(r)(M)$ (with the target being zero by Proposition 2.13) should degenerate after k differentials, for all $0 \leqslant r < n$. △

3.3 Consequences for torsion motives

In this subsection, our ground field is any field of characteristic zero. It appears that motivic cohomology of torsion motives are absent in some regions depending on the level.

Proposition 3.12. Let $M = (X, \rho)$ be a torsion motive such that $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains $\alpha \equiv v_{n-1}(mod\, I(n-1))$. Then $\mathrm{H}^{a,b}_{\mathcal{M}}(M_F, \mathbb{Z}/p) = 0$, for $a < n$, as well as for $b < n$, over any F/k.

Proof. By the Beilinson–Lichtenbaum "conjecture", it is sufficient to prove the statement for $b < n$. Induction on b. For $b < 0$ we know it. Suppose, $u \in \mathrm{H}^{a,b}_{\mathcal{M}}(M, \mathbb{Z}/p)$ is a nonzero element. Again, by the Beilinson–Lichtenbaum and the inductive assumption, we can assume that $a \geqslant b$ and, in the case $a = b$, u is not in the image of Q_0. Now, by induction on $0 \leqslant r < n$, we show that $Q_r \circ \ldots \circ Q_0(u)$ is nonzero. Suppose not. Then, by Corollary 3.10, $Q_{r-1} \circ \ldots \circ Q_0(u) = Q_r(v)$, for some v. But the "round" degree of v is equal to $b + (p^0 - 1) + \ldots + (p^{r-1} - 1) - (p^r - 1) = b + \frac{p^r - 1}{p-1} - r - (p^r - 1) < b$. Thus, $v = 0$ by the inductive assumption—a contradiction. Hence, $Q_{n-1} \circ \ldots \circ Q_0(u) \neq 0$. But this element resides in some group $\mathrm{H}^{c,d}_{\mathcal{M}}(M, \mathbb{Z}/p)$ with $c > 2d$—a contradiction. Therefore, $u = 0$. ■

Combining it with the Beilinson–Lichtenbaum "conjecture", we obtain the following.

Corollary 3.13. Let $M = (X, \rho)$ be a torsion motive such that $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains $\alpha \equiv v_{n-1}(mod\, I(n-1))$. Then $\mathrm{H}^a_{et}(M_F, \mathbb{Z}/p) = 0$, for any $a < n$ and any F/k.

Remark 3.14. In view of the Remark 3.11, similar arguments imply that the result still holds if $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains some $\alpha \equiv v^k_{n-1}(mod\, I(n-1))$, for $k < p$. △

In the case of an algebraically closed field, we obtain more.

Proposition 3.15. Let $M = (X, \rho)$ be a torsion motive such that $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains $\alpha \equiv v_{n-1}(mod\, I(n-1))$. Then $\mathrm{H}^a_{et}(M_{\overline{k}}, \mathbb{Z}/p) = 0$, for any $a < n$ and for $a > 2d - n$, where $d = \dim(X)$. In other words, the singular cohomology (with $\mathbb{Z}/p$-coefficients) H^a of the topological realization of M are absent in this range.

Proof. Let $M^\vee = \underline{\mathrm{Hom}}(M, \mathbb{Z}(d)[2d])$ be the dual direct summand. Then $\overline{J}(M^\vee) = \overline{J}(M)$ and, by Corollary 3.13, $\mathrm{H}^a_{et}(M_{\overline{k}}, \mathbb{Z}/p) = 0$, for $a < n$, and the same is true for $M^\vee$. Since $\mathrm{H}^a_{et}(M_{\overline{k}}, \mathbb{Z}/p) = \left(\mathrm{H}^{2d-a}_{et}(M^\vee_{\overline{k}}, \mathbb{Z}/p)\right)^*$, we get that $\mathrm{H}^a_{et}(M_{\overline{k}}, \mathbb{Z}/p) = 0$, for $a > 2d - n$. ■

The above proposition shows that a torsion motive of a high p-level cannot reside just "under the surface" but should be "sufficiently deep" inside the motive of the respective variety.

For a number $x \in \mathbb{R}$, denote as $\lceil x \rceil$ the smallest integer $\geqslant x$.

Proposition 3.16. Let $M = (X, \rho) \in Ob(Chow(k; \mathbb{Z}_{(p)}))$ be a torsion motive of p-level n with $\dim(X) < \lceil\frac{n}{2}\rceil + \frac{p^n-1}{p-1}$. Then $M_{et}|_{\overline{k}} = 0$.

Proof. For $n = 1$, we know it from Proposition 2.22. So, assume that $n > 1$. Since $\lceil\frac{n}{2}\rceil + \frac{p^n-1}{p-1} - 1 \leqslant \frac{(2p-1)(p^{n-1}-1)}{p-1}$, for $n > 1$, from Proposition 2.21, we obtain that $I(n) \subset \overline{J}(M)$. Then Proposition 3.15 implies that $\mathrm{H}^a_{et}(M_{\overline{k}}, \mathbb{Z}/p)$ can be nonzero only for $n \leqslant a \leqslant 2d - n$, where $d = \dim(X)$.

It follows from Corollary 3.10 and the Beilinson–Lichtenbaum "conjecture" that Q_r is an exact differential on $\mathrm{H}^*_{et}(M_{\overline{k}}, \mathbb{Z}/p)$, for any $0 \leqslant r < n$. This means that $\mathrm{H}^*_{et}(M_{\overline{k}}, \mathbb{Z}/p)$ is a free module over $\Lambda_{\mathbb{Z}/p}(Q_0, \dots, Q_{n-1})$. But the "height" of this algebra is the "square" degree of $Q_{n-1} \circ \dots \circ Q_0$, which is $(2p^0 - 1) + \dots + (2p^{n-1} - 1) = 2\frac{p^n-1}{p-1} - n > 2d - 2n$. Hence, $\mathrm{H}^*_{et}(M_{\overline{k}}, \mathbb{Z}/p) = 0$. Since the category of etale motives with $\mathbb{Z}/p$-coefficients over $\overline{k}$ is the derived category of $\mathbb{F}_p$-vector spaces, we obtain the following: $M_{et}|_{\overline{k}}(mod\, p) = 0$, and so, $M_{et}|_{\overline{k}} = 0$, as M is killed by some power of p. ■

Corollary 3.10 permits to structurize the motivic cohomology of M (provided $\overline{J}(M_{\mathbb{Z}_{(p)}})$ contains a v_{n-1} element). Let us denote as Λ the ring $\Lambda_{\mathbb{F}_p}(Q_0, \dots, Q_{n-1})$.

Proposition 3.17. Suppose $I(n) \subset \overline{J}(M_{\mathbb{Z}_{(p)}})$. Then $\mathrm{H}^{*,*'}_{\mathcal{M}}(M, \mathbb{Z}/p)$ has a filtration $F^{(a,b)}$ by $\Lambda[\tau]$-modules, where (a, b) runs through all pairs $2b \geqslant a \geqslant b \geqslant 0$ ordered lexicographically, $F^{(a,b)}/F^{<(a,b)}$ is a direct sum of modules of the type $\Lambda[\tau] \cdot u_\alpha$ and $\Lambda[\tau]/(\tau^{l_\beta}) \cdot v_\beta$, where $\deg(u_\alpha) = \deg(v_\beta) = (b)[a]$, $0 \leqslant l_\beta \leqslant a - b - 1$, and the quotient module $G = \mathrm{H}^{*,*'}_{\mathcal{M}}(M, \mathbb{Z}/p)/F^{(a,b)}$ satisfies $G^{c,d} = 0$, for $c < a$ and for $c = a$, $d \leqslant b$.

Proof. We can take $F^{<(0.0)} = 0$. Suppose, we already constructed $F^{<(a,b)}$. By inductive assumption, we have an exact sequence:

$$0 \to F^{<(a,b)} \to \mathrm{H}^{*,*'}_{\mathcal{M}}(M, \mathbb{Z}/p) \stackrel{g}{\to} G \to 0,$$

where $G^{c,d} = 0$, for $c < a$ and for $c = a$, $d < b$. Then Q_r, $0 \leqslant r \leqslant n - 1$ act as exact differentials on G as well. By the standard arguments, using the fact that G is trivial below the level a, we obtain that $Q_{n-1} \circ \dots \circ Q_0$ is injective on $G^{a,*}$. We have a filtration $G^{a,b}_l = \mathrm{Ker}(\cdot\tau^l)$ on $G^{a,b}$. Let $G^{a,b}_\infty = \cup_l G^{a,b}_l$. By the Beilinson–Lichtenbaum "conjecture", $G^{a,b}_\infty = G^{a,b}_{a-b-1}$. Choose an $\mathbb{F}_p$-vector subspace H_l of $G^{a,b}_l$ that $G^{a,b}_l = G^{a,b}_{l-1} \oplus H_l$ and a subspace H_∞ of $G^{a,b}$ such that $G^{a,b} = G^{a,b}_\infty \oplus H_\infty$. Then,

$$H = \left(\bigoplus_{0 \leqslant l \leqslant a-b-1} \Lambda[\tau]/(\tau^l) \otimes_{\mathbb{F}_p} H_l\right) \oplus \left(\Lambda[\tau] \otimes_{\mathbb{F}_p} H_\infty\right)$$

is a Λ-submodule of G such that $H^{a,b} = G^{a,b}$. It remains to take $F^{(a,b)} = g^{-1}(H)$. Induction step is proven. ■

This permits to improve a bit the bound of Theorem 2.19, at least, for algebraically closed fields.

Proposition 3.18. Let $M = (X, \rho) \in Ob(Chow(k; \mathbb{Z}_{(p)}))$ be a torsion motive of p-level n with $\dim(X) = \frac{p^n-1}{p-1} + i$, where $i < \lceil \frac{n}{2} \rceil$ and $i \leqslant 1$. Then $M_{\overline{k}} = 0$.

Proof. By Proposition 2.21, $\overline{J}(M) = I(n)$. Hence, Proposition 3.17 applies and $\mathrm{H}^{*,*'}_{\mathcal{M}}(M_F, \mathbb{Z}/p)$ is an extension of modules of the type $\Lambda[\tau] \cdot u_\alpha$ and $\Lambda[\tau]/(\tau^{l_\beta}) \cdot v_\beta$, where $\deg(v_\beta) = (b)[a]$ and $l_\beta \leqslant a - b - 1$. Since $\dim(X) \leqslant \frac{p^n-1}{p-1} + 1$ and Q_r has the "diagonal degree" p^r, the generators v_β (as well as u_α) of our modules reside on the diagonals with numbers $\leqslant 1$. But multiplication by τ is injective on such diagonals by the Beilinson–Lichtenbaum "conjecture". Hence, the modules of the 2nd kind (τ-torsion ones) are absent, and so multiplication by τ is injective on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M_F, \mathbb{Z}/p)$, for any extension F/k. Consider now some finitely generated extension $F/\overline{k}$. From Proposition 3.16, $M_{et}|_{\overline{k}} = 0$, which implies that $M_{et}|_F = 0 \Rightarrow \mathrm{H}^*_{et}(M_F, \mathbb{Z}/p) = 0$. Since multiplication by τ is injective, by the Beilinson–Lichtenbaum "conjecture", we obtain that $\mathrm{H}^{*,*'}_{\mathcal{M}}(M_F, \mathbb{Z}/p) = 0$ for all finitely generated extensions $F/\overline{k}$. As M is killed by some power of p, the same is true about integral motivic cohomology. In particular, all Chow groups of M are zero, for any such F. Then $M_{\overline{k}} = 0$. ■

Corollary 3.19. Let $k = \overline{k}$ and $M = (X, \rho)$ be a torsion motive of p-level $n \leqslant 4$. Then,

$$\dim(X) \geqslant \lceil \frac{n}{2} \rceil + \frac{p^n - 1}{p - 1}.$$

In the light of expected properties of Chow motives (in particular, *Rost nilpotence conjecture*), it is natural to propose the following.

Conjecture 3.20. Let $M = (X, \rho)$ be a torsion motive of p-level n. Then $\dim(X) \geqslant \lceil \frac{n}{2} \rceil + \frac{p^n-1}{p-1}$.

4 Torsion Motives of Surfaces

4.1 Godeaux torsion motive

The aim of this subsection is to have a closer look at the torsion direct summand in the motive of the Godeaux surface. Our ground field will be the field of complex numbers $\mathbb{C}$.

Let X be a *Godeaux surface*, that is, the quotient of the Fermat quintic $x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$ in $\mathbb{P}^3$ by the $\mathbb{Z}/5$-action given by $\sigma(x_1, x_2, x_3, x_4) = (\xi x_1, \xi^2 x_2, \xi^3 x_3, \xi^4 x_4)$, for a generator σ of $\mathbb{Z}/5$, where ξ is a primitive root of 1 of degree 5.

It is a smooth projective surface whose singular cohomology $\mathrm{H}^*(X, \mathbb{Z}) = \mathrm{H}^*_{sing}(X(\mathbb{C}), \mathbb{Z})$ are given by $\mathrm{H}^i(X, \mathbb{Z}) = \mathbb{Z}$, for $i = 0, 4$; $\mathrm{H}^1(X, \mathbb{Z}) = 0$, $\mathrm{H}^3(X, \mathbb{Z}) = \mathbb{Z}/5$; $\mathrm{H}^2(X, \mathbb{Z}) = \mathbb{Z}^9 \oplus \mathbb{Z}/5$; [3, Section 2].

It was shown by Gorchinskiy and Orlov [10, Proposition 2.2] that the torsion part of the cohomology corresponds to a direct summand of $M(X)$. It is not difficult to see that the respective projector can be made self-dual.

Proposition 4.1. There exists a self dual with respect to $\underline{\mathrm{Hom}}(-, \mathbb{Z}(2)[4])$ direct summand M of $M(X)$ such that $\mathrm{H}^*(M, \mathbb{Z}) = \mathrm{H}^*(X, \mathbb{Z})_{tors}$.

Proof. By [3, Corollary 4.15], there is a $\mathbb{Z}$-basis of the free part of the Picard group $N(X) = \mathbb{Z}^9$ that the respective intersection matrix A is an invertible (integral valued symmetric) matrix. Let e_i, $i = 1, \dots, 9$ be such a basis, and let B be some symmetric integral valued 9×9 matrix. We can assign to it the cycle $\Phi_B = \sum_{i,j} b_{i,j} \cdot e_i \times e_j \in \mathrm{CH}^2(X \times X)$. If $f = \sum_k \lambda_k e_k \in N(X)$, then $(\Phi_B)_*(f) = B \cdot A \cdot f$. Thus, if we take $B = A^{-1}$, then $(\Phi_B)_*$ will act identically on $N(X)$. By the same reason, $\rho_1 = \Phi_{A^{-1}}$ is a projector. Since A^{-1} is a symmetric matrix, this projector is symmetric. It splits $(\mathbb{Z}(1)[2])^{\oplus 9}$ from the motive of X. In addition, we have a projector $\rho_2 = [X \times pt] + [pt \times X]$ orthogonal to ρ_1, and so, the symmetric projector $\rho = \rho_1 + \rho_2$ splits the free part from the cohomology of X. That is, $M(X) = \rho M(X) \oplus (1 - \rho)M(X)$, where $\rho M(X) = \mathbb{Z} \oplus (\mathbb{Z}(1)[2])^{\oplus 9} \oplus \mathbb{Z}(2)[4]$ and $\mathrm{H}^*((1 - \rho)M(X), \mathbb{Z}) = \mathrm{H}^*(X, \mathbb{Z})_{tors}$. Let us denote $(1 - \rho)M(X)$ as M. Since $(1 - \rho)$ is symmetric, we have a natural identification $\underline{\mathrm{Hom}}(M, \mathbb{Z}(2)[4]) = M$. ∎

Because $\mathrm{H}^i(M, \mathbb{Z}) = \mathbb{Z}/5$, for $i = 2, 3$ and zero for all other i, we obtain that $\mathrm{H}^1(M, \mathbb{Z}/5) = \mathbf{u} \cdot \mathbb{Z}/5$, $\mathrm{H}^2(M, \mathbb{Z}/5) = Q_0(\mathbf{u}) \cdot \mathbb{Z}/5 \oplus \mathbf{v} \cdot \mathbb{Z}/5$, $\mathrm{H}^3(M, \mathbb{Z}/5) = Q_0(\mathbf{v}) \cdot \mathbb{Z}/5$, where Q_0 is the Bockstein, with all other cohomology groups trivial.

By the Artin comparison theorem [2, Exp.XI, Theorem 4.4], $\mathrm{H}^*_{et}(X, \mathbb{Z}/5) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Z}/5)$. Hence, $\mathrm{H}^*_{et}(M, \mathbb{Z}/5) = \mathrm{H}^*(M, \mathbb{Z}/5)$.

Proposition 4.2. Suppose N is a Chow motive over $\mathbb{C}$ and $k/\mathbb{C}$ is some field extension and l is prime. Then,

$$\mathrm{H}^*_{et}(N_k, \mathbb{Z}/l) = \mathrm{H}^*(N(\mathbb{C}), \mathbb{Z}/l) \otimes_{\mathbb{Z}/l} \mathrm{H}^*_{et}(k, \mathbb{Z}/l).$$

Proof. Since $\mathrm{H}^*(N(\mathbb{C}),\mathbb{Z}/l)=\mathrm{H}^*_{et}(N_{\mathbb{C}},\mathbb{Z}/l)$, we have a natural map

$$\mathrm{H}^*(N(\mathbb{C}),\mathbb{Z}/l)\otimes_{\mathbb{Z}/l}\mathrm{H}^*_{et}(k,\mathbb{Z}/l)\to\mathrm{H}^*_{et}(N_k,\mathbb{Z}/l),$$

which we claim to be an isomorphism. Our motive N is a direct summand in $M(Y)$ for some smooth projective $\mathbb{C}$-variety Y. It is sufficient to prove the statement for $M(Y)$.

We have the Hochschild–Serre spectral sequence:

$$E_2=\mathrm{H}^*_{et}(k,\mathrm{H}^*_{et}(Y_{\overline{k}},\mathbb{Z}/l))\Rightarrow\mathrm{H}^*_{et}(Y_k,\mathbb{Z}/l).$$

The E_2-term of this sequence is a free $\mathrm{H}^*_{et}(k,\mathbb{Z}/l)$-module with the basis $\mathrm{H}^0_{et}(k,\mathrm{H}^*_{et}(Y_{\overline{k}},\mathbb{Z}/l))=\mathrm{H}^*_{et}(Y_{\overline{k}},\mathbb{Z}/l)=\mathrm{H}^*(Y(\mathbb{C}),\mathbb{Z}/l)$, since the natural map $\mathrm{H}^*_{et}(Y_{\mathbb{C}},\mathbb{Z}/l)\to\mathrm{H}^*_{et}(Y_{\overline{k}},\mathbb{Z}/l)$ is an isomorphism by the smooth base change theorem [15, Chapter VI, Corollary 4.3], and so the $Gal(\overline{k}/k)$-action on the latter module is trivial. Differentials in this sequence are $\mathrm{H}^*_{et}(k,\mathbb{Z}/l)$-linear, while the basis elements survive in the E_∞-term by the same smooth base change result, since these come from $\mathrm{H}^*_{et}(Y_{\mathbb{C}},\mathbb{Z}/l)$. Hence, the sequence degenerates from the E_2-term and so our original map is an isomorphism. ■

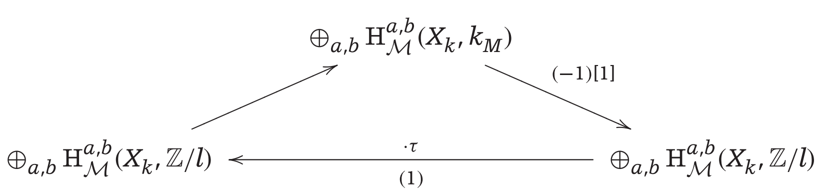
We have the Brown–Gersten–Quillen-type spectral sequence

$$E_1^{p,q}(coniveau)=\oplus_{x\in X^{(p)}}\mathrm{H}^q_{et}(k(x),\mathbb{Z}/l)\Rightarrow\mathrm{H}^{q+2p}_{et}(X_k,\mathbb{Z}/l),$$

inducing the coniveau filtration on $\mathrm{H}^*_{et}(X_k,\mathbb{Z}/l)$. Staring with the E_2-page this spectral sequence can be identified with the τ-spectral sequence (see, e.g., [23, Section 2]).

$$E_1^{a,b}(tau)=\mathrm{H}^{a,b}_{\mathcal{M}}(X_k,k_M)\Rightarrow\mathrm{H}^a_{et}(X_k,\mathbb{Z}/l)$$

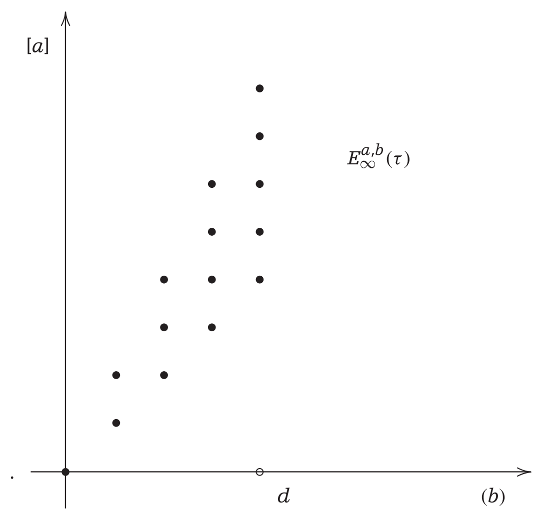
given by the exact pair



coming from the distinguished triangle $\mathbb{Z}/l(-1)\stackrel{\tau}{\to}\mathbb{Z}/l\to k_M\to\mathbb{Z}/l(-1)[1]$ (where τ corresponds to some choice of the primitive l-th root of 1 in $\mathbb{C}\subset k$). Note that k_M belongs to the heart of the homotopic t-structure [7] and is represented there by the Milnor's K-theory mod p Rost cycle module $k_M^* = K_M^*/p$ (so the name). We have $E_2^{p,q}(coniveau) = E_1^{2p+q,p+q}(tau)$, by the results of Deligne and Parajape; see [23, Theorem 2.4]. On the level of associated filtrations on etale cohomology, this is reflected by the fact that elements of $\mathrm{H}^n_{et}(X_k, \mathbb{Z}/l) = \mathrm{H}^{n,n}_{\mathcal{M}}(X_k, \mathbb{Z}/l)$ supported in codimension $\geqslant c$ are exactly those that are divisible by τ^c in $\mathrm{H}^{*,*'}_{\mathcal{M}}$.

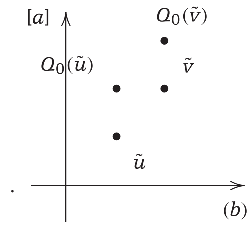
The differential d_r of the τ-sequence acts as follows: $d_r : E_r^{a,b} \to E_r^{a+1,b-r}$ and, for a d-dimensional variety, $E_1^{a,b}$ is nonzero only for $a-2b \leqslant 0 \leqslant a-b \leqslant d$. In particular, the sequence degenerates at E_d. The diagonal part $E_1^{a,a}$ is nothing else, but the *unramified cohomology* $\mathrm{H}^a_{nr}(k(X)/k, \mathbb{Z}/l)$ of the generic point of X.

If the ground field $k = \overline{k}$ is algebraically closed, then $\mathrm{H}^q_{et}(\overline{k}(x), \mathbb{Z}/l) = 0$, for points of co-dimension $p > d - q$. Hence, the $E(tau)$ spectral sequence is concentrated; in this case, in the region $a - 2b \leqslant 0 \leqslant a - b$; $b \leqslant d$. The E_∞ page of it provides us with some sort of "Hodge half-diamond" describing the codimension of support of elements in $\mathrm{H}^*(X(\mathbb{C}), \mathbb{Z}/l)$.



Note that the τ-spectral sequence makes sense for arbitrary motives. In particular, for Chow motives. If a Chow motive M is a direct summand of a motive of a surface, then this sequence degenerates in the E_2-term and if, moreover, $k = \overline{k}$ is algebraically closed, then it degenerates already in the E_1-term. Since singular cohomology of M are finite-dimensional vector spaces, this implies that the same holds about k_M-motivic cohomology, in this case.

Returning to our Godeaux motive M, we conclude (taking into account the action of Bockstein) that the Hodge half-diamond for $M_{\mathbb{C}}$ looks as



where each $\bullet$ corresponds to a one-dimensional $\mathbb{Z}/5$-space spanned by the respective element and elements $\tilde{u}$ and $\tilde{v}$ project to $\mathbf{u}$ and $\mathbf{v} \in \mathrm{H}^*(M(\mathbb{C}), \mathbb{Z}/5)$. Taking into account that Tate-motives contribute only to the slope$=2$ part $a = 2b$ of the half-diamond, we see that the diagonal part $a = b$ of the E_1-term for the Godeaux surface X itself is $1 \cdot \mathbb{Z}/5 \oplus \tilde{u} \cdot \mathbb{Z}/5 \oplus \tilde{v} \cdot \mathbb{Z}/5$. This is exactly the unramified cohomology $\mathrm{H}^*_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/5)$ of the generic point of X over $\mathbb{C}$. So, $\tilde{u}$ is represented by some unramified element of $K^M_1(\mathbb{C}(X))/5$ and $\tilde{v}$ by some unramified element of $K^M_2(\mathbb{C}(X))/5$. Since k_M-motivic cohomology fits into the exact sequence:

$$0 \to \mathrm{Coker}(\mathrm{H}^{a,b-1}_{\mathcal{M}} \stackrel{\tau}{\to} \mathrm{H}^{a,b}_{\mathcal{M}}) \to \mathrm{H}^{a,b}_{\mathcal{M}}(-, k_M) \to \mathrm{Ker}(\mathrm{H}^{a+1,b-1}_{\mathcal{M}} \stackrel{\tau}{\to} \mathrm{H}^{a+1,b}_{\mathcal{M}}) \to 0,$$

where $\mathrm{H}^{*,*'}_{\mathcal{M}} = \mathrm{H}^{*,*'}_{\mathcal{M}}(-, \mathbb{Z}/5)$, and $\mathrm{H}^{2,0}_{\mathcal{M}}(X) = \mathrm{H}^{3,1}_{\mathcal{M}}(X) = 0$, we see that $\tilde{u}$ and $\tilde{v}$ lift to elements $u \in \mathrm{H}^{1,1}_{\mathcal{M}}(M_{\mathbb{C}}, \mathbb{Z}/5)$ and $v \in \mathrm{H}^{2,2}_{\mathcal{M}}(M_{\mathbb{C}}, \mathbb{Z}/5)$ that are not divisible by τ and project to $\mathbf{u}$ and $\mathbf{v}$ in $\mathrm{H}^*(M(\mathbb{C}), \mathbb{Z}/5)$.

From the description of the Hodge half-diamond for M and the fact that Tate-motives do not have $\mathrm{H}^{3,*}_{\mathcal{M}}(-, \mathbb{Z}/5)$ over $\mathbb{C}$, we see that the product $u \cdot v$ in $\mathrm{H}^{*,*'}_{\mathcal{M}}(X_{\mathbb{C}}, \mathbb{Z}/5)$ is equal to $m \cdot \tau Q_0(v)$, for some $m \in \mathbb{Z}/5$. In particular, $Q_0(u \cdot v) = 0$. Since $M(\mathbb{C})$ is self-dual with respect to $\underline{\mathrm{Hom}}(-, \mathbb{Z}/5[4])$ in $D^b(\mathbb{Z}/5)$, we can assume w.l.o.g. that $\mathbf{u} \cdot Q_0(\mathbf{v}) = \mathfrak{e} \in \mathrm{H}^4(X, \mathbb{Z}/5)$ the dual of the class of a point. In other words, the nondegenerate pairing on $\mathrm{H}^*(M, \mathbb{Z}/5)$ satisfies $\langle \mathbf{u}, Q_0(\mathbf{v}) \rangle = 1$. Then, since $Q_0(\mathbf{u} \cdot \mathbf{v}) = 0$, we have $\langle Q_0(\mathbf{u}), \mathbf{v} \rangle = 1$. Because multiplication by τ is injective on $\mathrm{H}^{4,*}_{\mathcal{M}}(X_{\mathbb{C}}, \mathbb{Z}/5)$ (note that $M_{\mathbb{C}}$ has no motivic cohomology on this row), we obtain that $u \cdot Q_0(v) = \varepsilon \cdot \tau$ and $Q_0(u) \cdot v = \varepsilon \cdot \tau$ for the class of a $\mathbb{C}$-point $\varepsilon \in \mathrm{H}^{4,2}_{\mathcal{M}}(X_{\mathbb{C}}, \mathbb{Z}/5)$.

It follows from the proof of [10, Proposition 2.3] that the identity morphism of M is killed by multiplication by 5. That is, M is a torsion motive of exponent 5. This is a consequence of the Beilinson–Lichtenbaum conjecture and the computation of singular cohomology of M.

Let now $k/\mathbb{C}$ be some field extension. Denote as $\mathrm{H}^{\{i\}}_{\mathcal{M}}$ the i-th diagonal $\mathrm{H}^{i+*,*}_{\mathcal{M}}$ in motivic cohomology. Since integral motivic cohomology of M is killed by 5, Bockstein Q_0 acts without cohomology on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M_k,\mathbb{Z}/5)$. By the Beilinson–Lichtenbaum conjecture and Proposition 4.2, we have

$$\mathrm{H}^{\{0\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5) = u\cdot \mathrm{H}^*_{et} \oplus v\cdot \mathrm{H}^*_{et} \oplus \tau Q_0(u)\cdot \mathrm{H}^*_{et} \oplus \tau Q_0(v)\cdot \mathrm{H}^*_{et},$$

where $\mathrm{H}^*_{et} = \mathrm{H}^*_{et}(k,\mathbb{Z}/5)$. Multiplication by τ is injective on $\mathrm{H}^{\{1\}}_{\mathcal{M}}$ and so

$$\mathrm{H}^{\{1\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5) = \tau^{-1}\cdot (u,v)\cdot A \oplus\ Q_0(u)\cdot \mathrm{H}^*_{et} \oplus Q_0(v)\cdot \mathrm{H}^*_{et},$$

where $A = \mathrm{Ker}(\mathrm{H}^*_{et}\oplus \mathrm{H}^*_{et} \xrightarrow{(\tilde{u},\tilde{v})} \mathrm{H}^*_{et}(k(X)))$ (recall that $\mathrm{H}^{n,n}_{\mathcal{M}}(M_k,k_M)$ is a direct summand in $\mathrm{H}^n_{nr}(k(X)/k,\mathbb{Z}/5)$, so a linear combination of u and v is divisible by τ if and only if the respective combination of $\tilde{u}$ and $\tilde{v}$ vanishes in unramified cohomology). The image of $Q_0 : \mathrm{H}^{\{0\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5) \to \mathrm{H}^{\{1\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5)$ is $Q_0(u)\cdot \mathrm{H}^*_{et} \oplus Q_0(v)\cdot \mathrm{H}^*_{et}$, and as Q_0 acts as an exact differential, it is injective on $\tau^{-1}\cdot(u,v)\cdot A$. On the other hand, the map $Q_0 : \mathrm{H}^{\{1\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5) \to \mathrm{H}^{\{2\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5)$ is surjective, since Q_0 is trivial on the 2nd diagonal (X is 2-dimensional). Hence,

$$\mathrm{H}^{\{2\}}_{\mathcal{M}}(M_k,\mathbb{Z}/5) = \tau^{-1}\cdot Q_0(u,v)\cdot A.$$

In particular, we see that multiplication by τ is injective on the 2nd, and so all the diagonals. Hence, all the differentials d_r are zero in the τ-spectral sequence of M, which implies that our spectral sequence degenerates at the 1st page.

Considered for all finitely generated field extensions $k/\mathbb{C}$, A forms a cycle module of Rost [20], which is a submodule of $\mathrm{H}^*_{et}\oplus \mathrm{H}^*_{et}$ (with appropriate degree shift). Note that this module has only a zero section over $\mathbb{C}$, since there are no relations among $\tilde{u},\tilde{v} \in \mathrm{H}^*_{nr}(\mathbb{C}(X)/\mathbb{C},\mathbb{Z}/5)$ as $\mathbb{C}$ has cohomological dimension zero. But below, we will see that over appropriate field extensions such relations will appear.

We can express the group of zero cycles on M.

Corollary 4.3. The group $\mathrm{CH}_0(M_k) = \mathrm{CH}^2(M_k)$ can be identified with

$$\mathrm{Ker}(\mathrm{H}^2_{et}(k)\oplus \mathrm{H}^1_{et}(k) \xrightarrow{(\tilde{u},\tilde{v})} \mathrm{H}^3_{et}(k(X))).$$

To get hold of our cycle module A, we will first establish certain orthogonality relations among its sections.

Proposition 4.4. Let $k/\mathbb{C}$ be some field extension and $(\lambda_a, \mu_a), (\lambda_b, \mu_b)$ be two sections of A over k. Then,

$$\mu_b \lambda_a + (-1)^{\deg(\lambda_a)\deg(\lambda_b)} \mu_a \lambda_b = 0 \in \mathrm{H}^*_{et}(k). \tag{3}$$

Proof. We know that $u \cdot Q_0(v) = Q_0(u) \cdot v = \varepsilon \cdot \tau \in \mathrm{H}^{4,3}_{\mathcal{M}}(X_k, \mathbb{Z}/5)$, where ε is the class of a $\mathbb{C}$-point. Let $\alpha = (u \cdot \lambda_a + v \cdot \mu_a)$ and $\beta = (Q_0(u) \cdot \lambda_b + Q_0(v) \cdot \mu_b)$

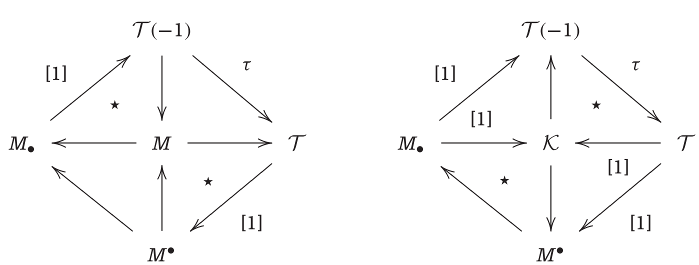
Since $\tau^{-1} \cdot \alpha \in \tau^{-1} \cdot (u, v) \cdot A$ belongs to the 1st diagonal, while $\tau^{-1} \cdot \beta$ belongs to the 2nd diagonal in motivic cohomology of M, their product in $\mathrm{H}^{*,*'}_{\mathcal{M}}(X, \mathbb{Z}/5)$ must be zero. Then so is the product of α and β. But the push-forward $\pi_*(\alpha \cdot \beta)$ of the latter product to the point is equal to $\tau \cdot ((-1)^{\deg(\lambda_a)} \lambda_a \mu_b + \mu_a \lambda_b)$. Since multiplication by τ is injective in $\mathrm{H}^{*,*'}_{\mathcal{M}}(k, \mathbb{Z}/5)$, and elements anticommute with respect to the square degree (and $\deg(\lambda) = \deg(\mu) + 1$), the triviality of this element is equivalent to the equation we need to prove. ■

We have a map $M \xrightarrow{(u,v)} \mathbb{Z}/5(1)[1] \oplus \mathbb{Z}/5(2)[2]$. The dual of it under $\underline{\mathrm{Hom}}(-, \mathbb{Z}(2)[4])$ will be $\mathbb{Z}/5[1] \oplus \mathbb{Z}/5(1)[2] \xrightarrow{(v^\vee, u^\vee)} M$ (note that $\underline{\mathrm{Hom}}(\mathbb{Z}/5, \mathbb{Z}) = \mathbb{Z}/5[-1]$). Let $\pi : X \to \mathrm{Spec}(\mathbb{C})$ be the projection. Since $\pi_*(u \cdot Q_0(v)) = \pi_*(v \cdot Q_0(u)) = \tau$ and $\pi_*(v \cdot Q_0(v)) = \pi_*(u \cdot Q_0(u)) = 0$ in $\mathrm{H}^{*,*'}_{\mathcal{M}}(\mathbb{C}, \mathbb{Z}/5)$, it follows that $u \circ v^\vee = v \circ u^\vee = \cdot\tau$, while $u \circ u^\vee = v \circ v^\vee = 0$ and we obtain a self-dual composition

$$\mathbb{Z}/5[1] \oplus \mathbb{Z}/5(1)[2] \xrightarrow[(v^\vee, u^\vee)]{} M \xrightarrow[(u,v)]{} \mathbb{Z}/5(1)[1] \oplus \mathbb{Z}/5(2)[2] \,. \quad (\text{composite: } \cdot\tau)$$

Since M is killed by multiplication by 5, by the Beilinson–Lichtenbaum "conjecture", the map $\mathrm{H}^{4,2}_{\mathcal{M}}(M \otimes M, \mathbb{Z}) \to \mathrm{H}^4_{et}(M \otimes M, \mathbb{Z}/5)$ is injective. Since (u, v) induces an isomorphism on etale cohomology, our composition shows that $M_{et} = (\mathbb{Z}/5)_{et}[1] \oplus (\mathbb{Z}/5)_{et}[2]$.

The above composition can be extended to a self-dual octahedron:



where $\mathcal{T} = \mathbb{Z}/5(1)[1] \oplus \mathbb{Z}/5(2)[2]$ and $\mathcal{K} = k_M(1) \oplus k_M(2)[1]$. All the objects here are compact. The computation of motivic cohomology of M above gives that $\mathrm{H}^{\{i\}}_{\mathcal{M}}(M^\bullet, \mathbb{Z}) = 0$, for $i \neq 2$, while $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M^{\bullet},\mathbb{Z})$ can be identified with $\delta(\tau^{-1}\cdot(u,v)\cdot A)$, where $\delta : \mathrm{H}^{*,*'}_{\mathcal{M}}(-,\mathbb{Z}/5)\to\mathrm{H}^{*+1,*'}_{\mathcal{M}}(-,\mathbb{Z})$ is the connecting homomorphism. Under the pull-back with respect to $\mathcal{K}\to M^{\bullet}$, it is identified with the submodule $(\overline{u},\overline{v})\cdot A$ of $\overline{u}\cdot\mathrm{H}^*_{et}\oplus\overline{v}\cdot\mathrm{H}^*_{et} = \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{K},\mathbb{Z})$, where $(\overline{u},\overline{v})$ is δ of the canonical projection $\mathcal{K}\to\mathcal{T}(-1)$. This implies that motivic cohomology of $M_{\bullet}$ is concentrated on the 3rd diagonal and $\mathrm{H}^{\{3\}}_{\mathcal{M}}(M_{\bullet},\mathbb{Z}) = (\overline{u},\overline{v})\cdot\left(\mathrm{H}^*_{et}\oplus\mathrm{H}^*_{et}/A\right)$ [1]. The dual calculations with $\mathrm{H}^{\mathcal{M}}_{\{i\}}$ - diagonals $\mathrm{H}^{\mathcal{M}}_{i+*,*}$ in motivic homology show that we have an exact sequence:

$$0\to\mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet},\mathbb{Z})\to\left(\overline{v}^{\vee}\cdot\mathrm{H}^*_{et}\oplus\overline{u}^{\vee}\cdot\mathrm{H}^*_{et}\right)\to\mathrm{H}^{\mathcal{M}}_{\{-1\}}(M^{\bullet},\mathbb{Z})\to 0,$$

where $\mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet},\mathbb{Z}) = (\overline{v}^{\vee},\overline{u}^{\vee})\cdot A$, $\mathrm{H}^{\mathcal{M}}_{\{-1\}}(M^{\bullet},\mathbb{Z}) = (\overline{v}^{\vee},\overline{u}^{\vee})\cdot\left(\mathrm{H}^*_{et}\oplus\mathrm{H}^*_{et}/A\right)$ and all other diagonals are zero. Since motivic homology of $M_{\bullet}$ are concentrated on the zeroth diagonal, this object belongs to the heart of the homotopy t-structure on $\mathrm{DM}(k)$; see [7]. This heart can be identified with the abelian category of Rost cycle modules. The cycle module of Rost corresponding to $M_{\bullet}$ is exactly A. Under this identification, A is a graded submodule of $\mathrm{H}^*_{et}\langle -1\rangle\oplus\mathrm{H}^*_{et}\langle -2\rangle$, where $\langle m\rangle$ indicates the shift in degrees. We will use this convention below.

We have an adjoint pair

$$\mathrm{DM}(k;\mathbb{Z})\ \underset{\nu^*}{\overset{\nu_*}{\rightleftarrows}}\ \mathrm{DM}(k;\mathbb{Z}/5),$$

where ν^* respects the $\otimes$ and ν_* satisfies the projection formula. Since the cycle module A is 5-torsion, it comes from $\mathrm{DM}(k;\mathbb{Z}/5)$ together with the map $M_{\bullet}\to\mathcal{K}[1]$. By obvious reasons, the same is true about the map $\mathcal{K}\to\mathcal{T}(-1)$. Hence, the motive M itself is $\nu_*(\mathbf{M})$, for some motive $\mathbf{M}\in\mathrm{DM}(k;\mathbb{Z}/5)$.

Let $\Delta_M\in\mathrm{H}^{4,2}_{\mathcal{M}}(M\otimes M,\mathbb{Z})$ be the composition $M\otimes M\to M(X\times X)\stackrel{\Delta_X}{\to}\mathbb{Z}(2)[4]$. The etale version of it modulo 5 is detected in the topological realization and so is equal to

$$(\overline{\Delta}_M)_{et} = -u_{et}\times Q_0(v_{et}) + v_{et}\times Q_0(u_{et}) + Q_0(v_{et})\times u_{et} + Q_0(u_{et})\times v_{et}.$$

Since M is killed by 5, the modulo 5 version $\overline{\Delta}_M$ is Q_0 of some element from $\mathrm{H}^{3,2}_{\mathcal{M}}(M\otimes M,\mathbb{Z}/5)$. But the map from Nisnevich to etale topology is injective on $\mathrm{H}^{\{1\}}_{\mathcal{M}}(M\otimes M,\mathbb{Z}/5)$. Hence, $\overline{\Delta}_M = Q_0(\tau^{-1}(u\times v + v\times u))$ and so, $\Delta_M = \delta(\tau^{-1}(u\times v+v\times u))$. Since Δ_M is nothing else but the projector $(1-\rho)$ in Proposition 4.1, and the remaining summands are Tate, we readily obtain the following result.

Proposition 4.5. The class of the diagonal of the Godeaux surface can be expressed as

$$\Delta_X = 1 \times \varepsilon + \varepsilon \times 1 + \sum_i \alpha_i \times \beta_i + \delta(\tau^{-1}(u \times v + v \times u)),$$

where $\alpha_i, \beta_i \in \mathrm{CH}^1(X)$, $u \in \mathrm{H}^{1,1}_{\mathcal{M}}(X, \mathbb{Z}/5)$, $v \in \mathrm{H}^{2,2}_{\mathcal{M}}(X, \mathbb{Z}/5)$ are elements introduced above and ε is the class of a $\mathbb{C}$-point.

Restricting to the generic point of X (on one of the components of $X \times X$), and taking into account that $Q_0(u)$ and $Q_0(v)$ disappear, when restricted from X to $\mathrm{Spec}(\mathbb{C}(X))$, we obtain that the class η of the generic point of X in $\mathrm{CH}^2(X_{\mathbb{C}(X)})/5$ is equal to

$$\eta = \overline{\Delta}_X|_{\mathbb{C}(X)} = \varepsilon + \tau^{-1}(Q_0(v) \cdot \tilde{u} + Q_0(u) \cdot \tilde{v}).$$

Hence, $\tau^{-1}(Q_0(v) \cdot \tilde{u} + Q_0(u) \cdot \tilde{v})$ is the difference $\eta - \varepsilon$ between the classes of the generic and complex points. This zero cycle of degree zero on X represents a nonzero element in $\mathrm{CH}^2(M_{\mathbb{C}(X)})$. Thus, $(\tilde{v}, \tilde{u}) \in A(\mathbb{C}(X))$ is a nontrivial section of our cycle module A (indeed, we know that the elements $\tilde{u}, \tilde{v} \in \mathrm{H}^*_{et}(\mathbb{C}(X))$ are nonzero). It appears that this section generates A as a Rost cycle module.

Proposition 4.6. The Rost cycle module A is generated by the section $(\tilde{v}, \tilde{u}) \in A(\mathbb{C}(X))$.

Proof. From the above, we know that the cycle module A can be identified with the second diagonal in motivic cohomology of M, which is $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M, \mathbb{Z}/5) = \mathrm{Ker}(\mathrm{H}^{\{2\}}_{\mathcal{M}}(X, \mathbb{Z}/5) \stackrel{\pi_*}{\to} \mathrm{H}^*_{et})$, where π is the projection to the point. By [32, Lemma 4.11] and [20], for a d-dimensional variety X, $\mathrm{H}^{\{d\}}_{\mathcal{M}}(X, \mathbb{Z})$ coincides with the Rost cycle module of "Chow groups with coefficients" in Milnor's K-theory $\mathrm{H}^d(X, \underline{K}^M_*)$. And the projection $\mathrm{H}^{\{d\}}_{\mathcal{M}}(X, \mathbb{Z}) \to \mathrm{H}^{\{d\}}_{\mathcal{M}}(X, \mathbb{Z}/p)$ is surjective. That means that, in our case, $\mathrm{H}^{\{2\}}_{\mathcal{M}}(X_k, \mathbb{Z}/5)$ is additively generated by elements of the form $\mathrm{Tr}_{E/k}([q] \cdot \alpha)$, where E/k is some finite extension, $q \in X(E)$ and $\alpha \in k^M_*(E) = K^M_*(E)/5$. Then the $\mathrm{Ker}(\mathrm{H}^{\{2\}}_{\mathcal{M}}(X, \mathbb{Z}/5) \stackrel{\pi_*}{\to} \mathrm{H}^*_{et})$ is additively generated by the elements of the form $\mathrm{Tr}_{E/k}(([q] - \varepsilon) \cdot \alpha)$, where ε is the class of a $\mathbb{C}$-point. But $[q] - \varepsilon$ is a specialization of $\eta - \varepsilon$. Hence, $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M, \mathbb{Z}/5)$ as a Rost cycle module is generated by $\eta - \varepsilon = \tau^{-1}(Q_0(u) \cdot \tilde{v} + Q_0(v) \cdot \tilde{u})$. In other words, the cycle module A is generated by $(\tilde{v}, \tilde{u}) \in A(\mathbb{C}(X))$. ■

In particular, $\mathrm{CH}_0(M_{\mathbb{C}(X)}) = \mathbb{Z}/5$ spanned by $(\tilde{v}, \tilde{u})$. We also obtain the following.

Proposition 4.7. A is a "Lagrangian" submodule of $\mathrm{H}^*_{et}\langle -1 \rangle \oplus \mathrm{H}^*_{et}\langle -2 \rangle$, that is, a maximal submodule satisfying the orthogonality conditions (3).

Proof. We know that $(\tilde{v}, \tilde{u}) \in A(\mathbb{C}(X))$. On the other hand, for any field $k/\mathbb{C}$, $(\lambda, \mu) \in A(k)$ if and only if $\tilde{u} \cdot \lambda + \tilde{v} \cdot \mu = 0 \in \mathrm{H}^*_{et}(k(X))$. Hence, $A(k) = (\mathrm{H}^*_{et}(k)\langle -1 \rangle \oplus \mathrm{H}^*_{et}(k)\langle -2 \rangle) \cap A(k(X))^{\perp}$ and so our submodule cannot be enlarged without breaking the orthogonality conditions. ■

Remark 4.8. From Propositions 4.6 and 4.7, we see that A is the unique submodule of $\mathrm{H}^*_{et}\langle -1 \rangle \oplus \mathrm{H}^*_{et}\langle -2 \rangle$ containing $(\tilde{v}, \tilde{u})$ and satisfying the orthogonality conditions.

4.2 p-Torsion motives of surfaces

In this subsection, unless specified otherwise, our ground field k will be an algebraically closed field of characteristic zero. Let us start with some general facts about torsion direct summands of surfaces.

Proposition 4.9. Let M and N be torsion direct summands in the motives of surfaces. Then the group $\mathrm{Hom}(M, N)$ is finite.

Proof. We will prove a stronger result: it is sufficient to assume that one of M or N is torsion. If M and N are direct summands in the motives of surfaces X and Y, respectively, and n kills M, or N, then $\mathrm{Hom}(M, N) \hookrightarrow \mathrm{CH}^2(X \times Y)_{n-tors}$. But

$$\mathrm{H}^{4,2}_{\mathcal{M}}(X \times Y, \mathbb{Z})_{n-tors} \overset{\delta}{\twoheadleftarrow} \mathrm{H}^{3,2}_{\mathcal{M}}(X \times Y, \mathbb{Z}/n) \hookrightarrow H^3_{et}(X \times Y, \mathbb{Z}/n),$$

where the last inclusion follows from the Beilinson–Lichtenbaum "conjecture". The rightmost group is finite, since k is algebraically closed. ■

This readily implies the following.

Theorem 4.10. Let k be a field of characteristic zero. Then torsion direct summands in the motives of surfaces over k satisfy Krull–Schmidt principle.

Proof. By the result of Gille [9], the restriction to the algebraic closure functor is conservative on the direct summands of the motives of surfaces. Then Proposition 4.9 implies that every torsion direct summand in a motive of a surface decomposes into a direct sum of finitely many simple ones. Now, suppose that a simple torsion motive M is a direct summand in $N \oplus L$. Then there are maps $N \underset{\varphi_1}{\overset{\psi_1}{\rightleftarrows}} M \underset{\varphi_2}{\overset{\psi_2}{\leftrightarrows}} L$ such that $\alpha = \psi_1 \circ \varphi_1$ and $\beta = \psi_2 \circ \varphi_2$ satisfy $\alpha + \beta = id_M$. Denote as $\overline{\alpha}$, etc. the restrictions to $\overline{k}$. Since $\mathrm{End}(\overline{M})$ is finite, some powers $\overline{\alpha}^n$ and $\overline{\beta}^m$ are idempotents. If these are both zero, then $id_{\overline{M}} = (\overline{\alpha} + \overline{\beta})^{n+m} = 0$ (note that α and β commute), which gives a contradiction. Hence, some power of $\overline{\alpha}$ or $\overline{\beta}$ is equal to a nonzero idempotent in $\mathrm{End}(\overline{M})$. Since $\mathrm{Ker}(\mathrm{End}(M) \to \mathrm{End}(\overline{M}))$ consists of nilpotents by the mentioned result of Gille, and M is indecomposable, by [29, Lemma 2.4], there exists some integral polynomial ϕ without constant term, whose value on α or β is equal to id_M. Then M is a direct summand of one of N, or L. This shows that the decomposition into irreducibles is unique up to reordering. ■

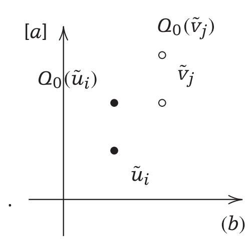
Corollary 4.11. In the situation of Theorem 4.10, any indecomposable object is a direct summand of the motive of a connected surface.

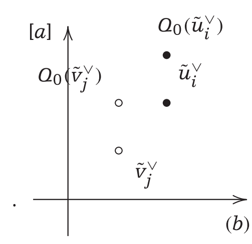
Proposition 4.12. In the situation of Theorem 4.10, the maximal torsion direct summand of the motive of a surface X is a (poly-)birational invariant of X.

Proof. It follows from the proof of Theorem 4.10 that the maximal torsion direct summand of the motive of a surface is well defined. If two surfaces X and Y have the same set of generic points, then one can get from X to Y by a series of blow ups and downs in smooth 0-dimensional centers. So, it is sufficient to consider a single blow-up $Y \to X$. But then $M(Y) = M(X) \oplus M(P)(1)[2]$, where P is the center. Since $M(P)(1)[2]$ does not contain torsion direct summands, any torsion direct summand of $M(Y)$ should be a direct summand of $M(X)$, by the mentioned arguments of Theorem 4.10. The converse is obvious. ■

We will try to classify the p-torsion motives appearing as direct summands in the motives of surfaces.

Suppose M is a torsion direct summand in the motive of some (smooth projective) surface X (possibly disconnected), such that $p \cdot id_M = 0$. From Proposition 3.15, we know that $\mathrm{H}^i_{et}(M, \mathbb{Z}/p) = 0$, for $i = 0$ and $i = 4$ and since $p \cdot id_M = 0$, the (mod p) Bockstein acts as an exact differential on $\mathrm{H}^*_{et}(M, \mathbb{Z}/p)$. And the same is true about $M^\vee = \underline{\mathrm{Hom}}(M, \mathbb{Z}(2)[4])$. This implies that the Hodge half-diamonds for M and $M^\vee$ look as





where each • represents an s-dimensional vector space over $\mathbb{F}_p$ spanned by classes: $\tilde{u}_i$ for deg = (1)[1], $Q_0(\tilde{u}_i)$ for deg = (1)[2], $\tilde{u}_i^\vee$ for deg = (2)[2], and $Q_0(\tilde{u}_i^\vee)$ for deg = (2)[3], and each ∘ represents a t-dimensional vector space over $\mathbb{F}_p$ spanned by classes: $\tilde{v}_j^\vee$ for deg = (1)[1], $Q_0(\tilde{v}_j^\vee)$ for deg = (1)[2], $\tilde{v}_j$ for deg = (2)[2], and $Q_0(\tilde{v}_j)$ for deg = (2)[3]. The pairing

$$\mathrm{H}^a_{et}(M,\mathbb{Z}/p)\times \mathrm{H}^c_{et}(M^\vee,\mathbb{Z}/p)\to \mathrm{H}^{a+c-4}_{et}(k,\mathbb{Z}/p)$$

is given by

$$\langle \tilde{u}_i, Q_0(\tilde{u}_i^\vee)\rangle = \langle Q_0(\tilde{u}_i),\tilde{u}_i^\vee\rangle = 1,\ \ \text{for } 1\leqslant i\leqslant s$$
$$\langle \tilde{v}_j, Q_0(\tilde{v}_j^\vee)\rangle = -\langle Q_0(\tilde{v}_j),\tilde{v}_j^\vee\rangle = 1,\ \ \text{for } 1\leqslant j\leqslant t,$$

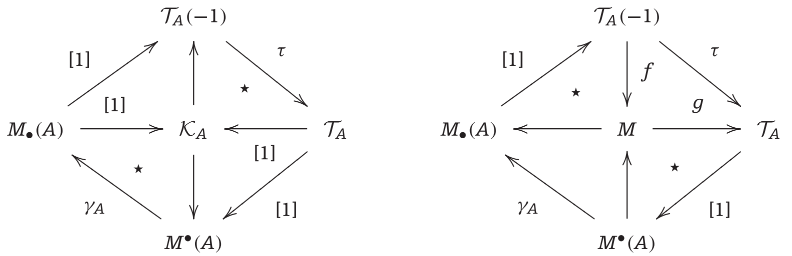
with all other combinations being zero. By the Beilinson–Lichtenbaum “conjecture”, $\tilde{u}_i$ and $\tilde{v}_j$ can be lifted to elements $u_i\in \mathrm{H}^{1,1}_{\mathcal{M}}(M,\mathbb{Z}/p)$ and $v_j\in \mathrm{H}^{2,2}_{\mathcal{M}}(M,\mathbb{Z}/p)$, while $\tilde{v}_j^\vee$ and $\tilde{u}_i^\vee$ can be lifted to elements $v_j^\vee\in \mathrm{H}^{1,1}_{\mathcal{M}}(M^\vee,\mathbb{Z}/p)$ and $u_i^\vee\in \mathrm{H}^{2,2}_{\mathcal{M}}(M^\vee,\mathbb{Z}/p)$, such that $\langle u_i, Q_0(u_i^\vee)\rangle = \langle Q_0(u_i),u_i^\vee\rangle = \langle v_j, Q_0(v_j^\vee\rangle = -\langle Q_0(v_j),v_j^\vee\rangle = \tau$ (zero otherwise), for the pairing

$$\mathrm{H}^{a,b}_{\mathcal{M}}(M,\mathbb{Z}/p)\times \mathrm{H}^{c,d}_{\mathcal{M}}(M^\vee,\mathbb{Z}/p)\to \mathrm{H}^{a+c-4,b+d-2}_{\mathcal{M}}(k,\mathbb{Z}/p).$$

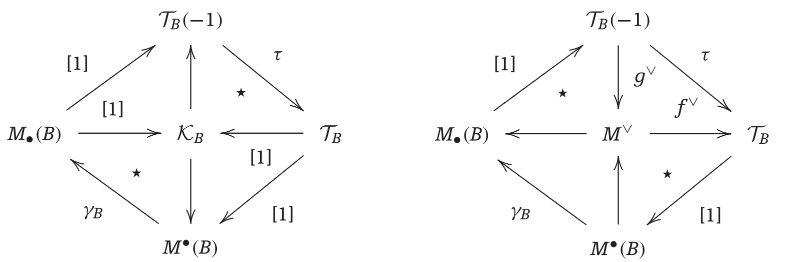
In other words, if we denote $\mathcal{T}_A=\mathbb{Z}/p(1)[1]^{\oplus s}\oplus\mathbb{Z}/p(2)[2]^{\oplus t}$ and $\mathcal{T}_B=\mathbb{Z}/p(1)[1]^{\oplus t}\oplus\mathbb{Z}/p(2)[2]^{\oplus s}$, then the compositions

$$\mathcal{T}_A(-1)\xrightarrow{(u_i^\vee,v_j^\vee)} M\xrightarrow{(u_i,v_j)}\mathcal{T}_A \qquad\qquad \mathcal{T}_B(-1)\xrightarrow{(v_j,u_i)} M^\vee\xrightarrow{(v_j^\vee,u_i^\vee)}\mathcal{T}_B \tag{4}$$

coincide with the multiplication by τ. We can complete these to dual octahedra:

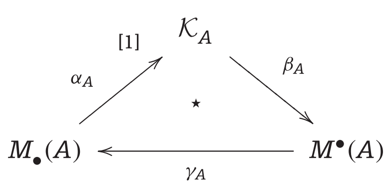


and

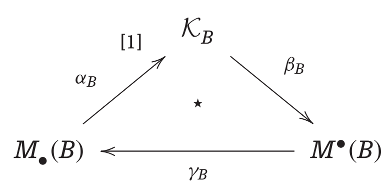


, (5)

where $\mathcal{K}_A = k_M(1)^{\oplus s} \oplus k_M(2)[1]^{\oplus t}$, $\mathcal{K}_B = k_M(1)^{\oplus t} \oplus k_M(2)[1]^{\oplus s}$ and $k_M = \mathrm{Cone}(\mathbb{Z}/p(-1) \xrightarrow{\cdot\tau} \mathbb{Z}/p)$. Note that k_M belongs to the heart of the homotopic t-structure [7] and represents there the Rost cycle module $k_*^M = K_*^M/p$. In particular, we get dual to each other distinguished triangles in $\mathrm{DM}_{gm}(k)$:



and



. (6)

We will use the standard notations $\mathrm{H}_{\mathcal{M}}^{i,j}(N,\mathbb{Z}) := \mathrm{Hom}_{DM(k)}(N,\mathbb{Z}(j)[i])$ for *motivic cohomology*, respectively, $\mathrm{H}^{\mathcal{M}}_{i,j}(N,\mathbb{Z}) := \mathrm{Hom}_{DM(k)}(\mathbb{Z}(j)[i],N)$ for *motivic homology*.

In view of (4), $M_{et} = (\mathbb{Z}/p)_{et}[1]^{\oplus s} \oplus (\mathbb{Z}/p)_{et}[2]^{\oplus t}$ and $f_{et} = id_{M_{et}} = g_{et}$. By the Beilinson–Lichtenbaum "conjecture", g^* is an isomorphism on motivic diagonals $\mathrm{H}_{\mathcal{M}}^{\{i\}} = \mathrm{H}_{\mathcal{M}}^{i+*,*}$ with $i \leqslant 1$, and so $\mathrm{H}_{\mathcal{M}}^{\{i\}}(M^\bullet(A),\mathbb{Z}) = 0$, for $i \neq 2$, while $\mathrm{H}_{\mathcal{M}}^{\{2\}}(M^\bullet(A),\mathbb{Z}) = \mathrm{H}_{\mathcal{M}}^{\{2\}}(M,\mathbb{Z})$, which can be identified with $\mathrm{H}^{\mathcal{M}}_{\{0\}}(M^\vee,\mathbb{Z})$. By the same reason, f^* is injective on diagonals $\mathrm{H}_{\mathcal{M}}^{\{\leqslant 2\}}$, and so on all of them (as $\dim(X) = 2$) and $\mathrm{H}_{\mathcal{M}}^{\{i\}}(M_\bullet(A),\mathbb{Z}) = 0$, for $i \neq 3$, while $\mathrm{H}_{\mathcal{M}}^{\{3\}}(M_\bullet(A),\mathbb{Z})$ can be identified with $H_B/\mathrm{H}^{\mathcal{M}}_{\{0\}}(M^\vee,\mathbb{Z})$, where $H_B = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus t} \oplus \mathrm{H}^*_{et}\langle -2\rangle^{\oplus s}$ and $\mathrm{H}^*_{et} = \mathrm{H}^*_{et}(k,\mathbb{Z}/p)$. In particular, $\gamma_A^* = 0$.

The same considerations apply to the 2nd triangle, $M^{\bullet}(B)$ and $M_{\bullet}(B)$. In particular, $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M^{\bullet}(B),\mathbb{Z})$ can be identified with $\mathrm{H}^{\mathcal{M}}_{\{0\}}(M,\mathbb{Z})$ and $\mathrm{H}^{\{3\}}_{\mathcal{M}}(M_{\bullet}(B),\mathbb{Z})$—with $H_A/\mathrm{H}^{\mathcal{M}}_{\{0\}}(M,\mathbb{Z})$, where $H_A = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus s} \oplus \mathrm{H}^*_{et}\langle -2\rangle^{\oplus t}$. By duality, $(\gamma_{A,B})_* = 0$ as well. This shows that motivic homology of $M_{\bullet}(A)$ and $M_{\bullet}(B)$ is concentrated on the 0-th diagonal and so these objects belong to the heart of the homotopic t-structure. There, it is represented by some Rost cycle submodules $A \subset H_A$ and $B \subset H_B$. Since $A = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet}(A),\mathbb{Z}) = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M,\mathbb{Z})$ and $B = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet}(B),\mathbb{Z}) = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M^{\vee},\mathbb{Z})$, we obtain that $\mathrm{H}^{\mathcal{M}}_{i,i}(M_{\bullet}(A),\mathbb{Z}) = 0$ and $\mathrm{H}^{\mathcal{M}}_{i,i}(M_{\bullet}(B),\mathbb{Z}) = 0$, for $i > 0$, that is, A and B are concentrated in nonnegative degrees (recall that the generators of the modules H_A and H_B are in degrees -1 and -2).

Conversely, let $\mathcal{K}_A = k_M(1)^{\oplus s} \oplus k_M(2)[1]^{\oplus t}$, $\mathcal{K}_B = k_M(1)^{\oplus t} \oplus k_M(2)[1]^{\oplus s}$, and suppose we have a pair of dual with respect to $\underline{\mathrm{Hom}}(-,\mathbb{Z}(2)[4])$ distinguished triangles (6) of geometric motives, where $\gamma^*_{A,B}$ is zero on $\mathrm{H}^{*,*'}_{\mathcal{M}}$ ($\Leftrightarrow$ $(\gamma_{A,B})_* = 0$) and $\mathrm{H}^{\mathcal{M}}_{i,i}(M_{\bullet}(A),\mathbb{Z}) = 0$, $\mathrm{H}^{\mathcal{M}}_{i,i}(M_{\bullet}(B),\mathbb{Z}) = 0$, for $i > 0$.

The (integral) motivic homology of $\mathcal{K}_A$ is a free module over H^*_{et} of rank $(s+t)$ with generators $\overline{v}^{\vee}_j$ and $\overline{u}^{\vee}_i$ in degrees (1) and (2)[1] which via $\underline{\mathrm{Hom}}(\mathcal{K}_A,\mathbb{Z}(2)[4]) = \mathcal{K}_B[1]$ can be identified with the motivic cohomology of $\mathcal{K}_B$, which is a similar free module of rank $(s+t)$ with generators $\overline{u}_i$ and $\overline{v}_j$ in degree [2] and (1)[3], where we identify $U_i := \overline{u}_i = \overline{u}^{\vee}_i$, $V_j := \overline{v}_j = \overline{v}^{\vee}_j$. Similarly, motivic homology of $\mathcal{K}_B$ can be identified with the motivic cohomology of $\mathcal{K}_A$. Since $\gamma^*_{A,B} = 0$, for $M_{\bullet} = M_{\bullet}(A,B)$ and $M^{\bullet} = M^{\bullet}(A,B)$, we obtain

$$
\begin{aligned}
&\mathrm{H}^{\{l\}}_{\mathcal{M}}(M^{\bullet},\mathbb{Z}) = 0, \ \text{for} \ l \neq 2, \quad \mathrm{H}^{\{l\}}_{\mathcal{M}}(M_{\bullet},\mathbb{Z}) = 0, \ \text{for} \ l \neq 3;\\
&\mathrm{H}^{\mathcal{M}}_{\{l\}}(M^{\bullet},\mathbb{Z}) = 0, \ \text{for} \ l \neq -1, \quad \mathrm{H}^{\mathcal{M}}_{\{l\}}(M_{\bullet},\mathbb{Z}) = 0, \ \text{for} \ l \neq 0,
\end{aligned}
$$

and via the above identification, $\mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet}(A),\mathbb{Z}) = \mathrm{H}^{\{2\}}_{\mathcal{M}}(M^{\bullet}(B),\mathbb{Z})$, $\mathrm{H}^{\{3\}}_{\mathcal{M}}(M_{\bullet}(A),\mathbb{Z}) = \mathrm{H}^{\mathcal{M}}_{\{-1\}}(M^{\bullet}(B),\mathbb{Z})$ (and similar with A and B swapped) that fit into the exact sequences

$$
\begin{aligned}
&0 \to \mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet}(A),\mathbb{Z}) \xrightarrow{(\alpha_A)_*} H_A \xrightarrow{\alpha^*_B} \mathrm{H}^{\{3\}}_{\mathcal{M}}(M_{\bullet}(B),\mathbb{Z}) \to 0;\\
&0 \to \mathrm{H}^{\mathcal{M}}_{\{0\}}(M_{\bullet}(B),\mathbb{Z}) \xrightarrow{(\alpha_B)_*} H_B \xrightarrow{\alpha^*_A} \mathrm{H}^{\{3\}}_{\mathcal{M}}(M_{\bullet}(A),\mathbb{Z}) \to 0.
\end{aligned}
\tag{7}
$$

Thus, $M_{\bullet}(A)$ and $M_{\bullet}(B)$ belong to the heart of the homotopic t-structure and correspond to Rost cycle submodules $A \subset H_A$ and $B \subset H_B$. Moreover, the dual of these should be given by the respective quotient cycle modules (up to appropriate shift).

Combining our distinguished triangles with the following ones

$$\mathcal{T}_{A,B}(-1) \xrightarrow{\cdot\tau} \mathcal{T}_{A,B} \longrightarrow \mathcal{K}_{A,B}[1] \longrightarrow \mathcal{T}_{A,B}(-1)[1],$$

where $\mathcal{T}_A = \mathbb{Z}/p(1)[1]^{\oplus s} \oplus \mathbb{Z}/p(2)[2]^{\oplus t}$ and $\mathcal{T}_B = \mathbb{Z}/p(1)[1]^{\oplus t} \oplus \mathbb{Z}/p(2)[2]^{\oplus s}$, we get a pair of dual w.r.to $\underline{\mathrm{Hom}}(-,\mathbb{Z}(2)[4])$ octahedra (5) consisting of compact objects. In particular, motive M appears.

By the very construction, M comes from $\mathrm{DM}(k;\mathbb{Z}/p)$ (since it is so for the morphism $M_\bullet(A) \to \mathcal{T}_A(-1)$). In particular, it is p-torsion. Since $\mathrm{H}^{\{i\}}_{\mathcal{M}}(\mathcal{T}_A,\mathbb{Z}) = 0$, for $i > 1$ and $\mathrm{H}^{\{i\}}_{\mathcal{M}}(M^\bullet(A),\mathbb{Z}) = 0$, for $i \neq 2$, we obtain that $\mathrm{H}^{\{i\}}_{\mathcal{M}}(M,\mathbb{Z}) = 0$, for $i > 2$ and so the map $M \longrightarrow M_\bullet(A)$ is zero on cohomology. By the same reason, the map $\mathcal{T} \xrightarrow{[1]} M^\bullet$ is also trivial on cohomology. We have the pairing $\langle -,- \rangle$

$$\mathrm{H}^{a,b}_{\mathcal{M}}(M,\mathbb{Z}/p) \otimes \mathrm{H}^{c,d}_{\mathcal{M}}(M^\vee,\mathbb{Z}/p) \longrightarrow \mathrm{H}^{a+c-4,b+d-2}_{\mathcal{M}}(k,\mathbb{Z}/p),$$

which is $\mathrm{H}^{*,*'}_{\mathcal{M}}(k,\mathbb{Z}/p)$-linear with respect to the left action on the left factor and the right action on the right one. It is also compatible with the pairing

$$\mathrm{H}^{a,b}_{\mathcal{M}}(\mathcal{T}_A(-1),\mathbb{Z}/p) \otimes \mathrm{H}^{c,d}_{\mathcal{M}}(\mathcal{T}_B,\mathbb{Z}/p) \longrightarrow \mathrm{H}^{a+c-4,b+d-2}_{\mathcal{M}}(k,\mathbb{Z}/p)$$

in the sense that $\langle f^*(x), y\rangle = \langle x, (f^\vee)^*(y)\rangle$, for $x \in \mathrm{H}^{*,*'}_{\mathcal{M}}(M,\mathbb{Z}/p)$, $y \in \mathrm{H}^{*,*'}_{\mathcal{M}}(\mathcal{T}_B,\mathbb{Z}/p)$. The map $f^\vee$ is given by $M^\vee \xrightarrow{(v_j^\vee,u_i^\vee)} \mathcal{T}_B$ and so we get

$$\langle u_i, Q_0(u_i^\vee)\rangle = \langle Q_0(u_i), u_i^\vee\rangle = \langle v_j, Q_0(v_j^\vee)\rangle = -\langle Q_0(v_j), v_j^\vee\rangle = \tau, \tag{8}$$

with all other combinations equal to zero.

Because $\mathrm{H}^{\mathcal{M}}_{i,i}(M_\bullet(A),\mathbb{Z}) = 0$, for $i > 0$, we obtain that $\mathrm{H}^{\mathcal{M}}_{i,j}(M,\mathbb{Z}) = 0$, for $i < j$, for $i < 2j$ and for $i > 2$. And the same is true about $M^\vee$. By duality, $\mathrm{H}^{i,j}_{\mathcal{M}}(M,\mathbb{Z}) = 0$ for $i-j > 2$, for $i > 2j$ and for $i < 0$. This holds over any extension K/k.

It follows from Proposition 5.1 that our motive M is a Chow motive. Then Proposition 5.4 implies that M is a direct summand in the motive $M(X)$ of a smooth projective (possibly reducible) surface X.

Thus, we have proven the following result.

Theorem 4.13. The assignment $M \mapsto A = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M,\mathbb{Z})$, $B = \mathrm{H}^{\mathcal{M}}_{\{0\}}(M^\vee,\mathbb{Z})$ defines a 1-to-1 correspondence between isomorphism classes of

(1) p-torsion direct summands in the motives of smooth projective surfaces (possibly, disconnected); and

(2) pairs of Rost submodules $A \xrightarrow{(\alpha_A)_*} H_A$, $B \xrightarrow{(\alpha_B)_*} H_B$, where $H_A = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus s} \oplus \mathrm{H}^*_{et}\langle -2\rangle^{\oplus t}$, $H_B = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus t} \oplus \mathrm{H}^*_{et}\langle -2\rangle^{\oplus s}$, for some s and t, such that

   (a) the respective objects $M_\bullet(A)$ and $M_\bullet(B)$ of the heart of the homotopic t-structure are compact;

   (b) $A^j = B^j = 0$, for $j < 0$;

   (c) the respective triangles (6) are dual with respect to $\underline{\mathrm{Hom}}(-, \mathbb{Z}(2)[4])$, namely, $\gamma_B = \gamma_A^\vee$ (here $\mathcal{K}_A = k_M(1)^{\oplus s} \oplus k_M(2)[1]^{\oplus t}$ and $\mathcal{K}_B = k_M(1)^{\oplus t} \oplus k_M(2)[1]^{\oplus s}$).

Remark 4.14. Of course, in the above theorem, the Rost submodule B is determined by A, and vice versa. So, the result can be formulated in terms of A only. But then the conditions will be less natural. The motive M will be self-dual exactly when $s = t$ and $A = B$. △

Furthermore, such a pair (A, B) of Rost submodules enjoys various additional properties.

Proposition 4.15. For any field extension F/k and any sections (μ^i_a, λ^j_a) and (μ^j_b, λ^i_b) of $A(F)$ and $B(F)$, respectively, the following orthogonality relation holds:

$$\sum_j (-1)^{\deg(\lambda_a)\deg(\lambda_b)} \mu^j_b \lambda^j_a + \sum_i \mu^i_a \lambda^i_b = 0 \in H^*_{et}(F, \mathbb{Z}/p). \tag{9}$$

Proof. Because M comes from $\mathrm{DM}(k; \mathbb{Z}/p)$, the Bockstein Q_0 acts as an exact differential on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M^\vee, \mathbb{Z}/p)$. Since $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M_F, \mathbb{Z}) \to \mathrm{H}^{\{2\}}_{\mathcal{M}}(M^\bullet(A)_F, \mathbb{Z})$ and $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M^\vee_F, \mathbb{Z}) \to \mathrm{H}^{\{2\}}_{\mathcal{M}}(M^\bullet(B)_F, \mathbb{Z})$ are isomorphisms, $(\mu^i_a, \lambda^j_a) \in A(F)$ corresponds to $\tilde{a} = \tau^{-1}(\sum_j v^\vee_j \cdot \lambda^j_a + \sum_i u^\vee_i \cdot \mu^i_a) \in \mathrm{H}^{\{1\}}_{\mathcal{M}}(M^\vee_F, \mathbb{Z}/p)$ and $(\mu^j_b, \lambda^i_b) \in B(F)$ corresponds to $Q_0(\tilde{b}) = \tau^{-1}(\sum_i Q_0(u_i) \cdot \lambda^i_b + \sum_j Q_0(v_j) \cdot \mu^j_b) \in \mathrm{H}^{\{2\}}_{\mathcal{M}}(M_F, \mathbb{Z}/p)$. Since $\langle Q_0(\tilde{b}), \tilde{a}\rangle \in \mathrm{H}^{\{1\}}_{\mathcal{M}}(F, \mathbb{Z}/p) = 0$, the pairing $\langle \tau Q_0(\tilde{b}), \tau\tilde{a}\rangle$ is also zero. Using (8), we obtain the orthogonality relation. ■

Let $M = (X, \rho)$ and $X = \coprod_r X_r$ be the decomposition of X into a disjoint union of its connected components. Since the E_1-term of the *tau*-spectral sequence coincides with the E_2-term of the *coniveau*-spectral sequence, we have an embedding

$$\frac{\mathrm{H}^{\{0\}}_{\mathcal{M}}(M_F, \mathbb{Z}/p)}{\tau \cdot \mathrm{H}^{\{1\}}_{\mathcal{M}}(M_F, \mathbb{Z}/p)} \subset \oplus_r \mathrm{H}^*_{nr}(F(X_r)/F, \mathbb{Z}/p) \tag{10}$$

of $\mathrm{H}^*_{et}(F)$-modules, for any F/k.

As was discussed above, $M_{et} = (\mathbb{Z}/p)_{et}[1]^{\oplus s} \oplus (\mathbb{Z}/p)_{et}[2]^{\oplus t}$ and for the modulo p version $\overline{M}_{et}$ of it, we have

$$(\overline{\Delta}_M)_{et} = \sum_i \left[-(u_i)_{et} \times Q_0((u_i^\vee)_{et}) + Q_0((u_i)_{et}) \times (u_i^\vee)_{et} \right] + \sum_j \left[(v_j)_{et} \times Q_0((v_j^\vee)_{et}) + Q_0((v_j)_{et}) \times (v_j^\vee)_{et} \right]$$

and so

$$(\Delta_M)_{et} = \delta \left(\sum_i (u_i)_{et} \times (u_i^\vee)_{et} + \sum_j (v_j)_{et} \times (v_j^\vee)_{et} \right).$$

Since $Q_0(u_i^\vee)$ and $Q_0(v_j^\vee)$ are contained in $\mathrm{H}^{\{1\}}_{\mathcal{M}}(X, \mathbb{Z}/p)$, these have a positive co-dimension of support and so

$$(\overline{\Delta}_M|_{k(X_r)})_{et} = \sum_i Q_0((u_i)_{et}) \cdot (u_i^\vee)_{k(X_r)} + \sum_j Q_0((v_j)_{et}) \cdot (v_j^\vee)_{k(X_r)}.$$

Since the map $M \to M_\bullet(A)$ is zero on cohomology, the multiplication by τ is injective on $\mathrm{H}^{*,*'}_{\mathcal{M}}(M_E, \mathbb{Z}/p)$, for any field extension E/k (this implies, in particular, that the *tau*-spectral sequence for M degenerates on the E_1-page). From the injectivity of the map $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M, \mathbb{Z}/p) \xrightarrow{\cdot\tau} \mathrm{H}^{\{1\}}_{\mathcal{M}}(M, \mathbb{Z}/p)$, we obtain that

$$\Delta_M|_{k(X_r)} = \delta \left(\tau^{-1} \left(\sum_i u_i \cdot (u_i^\vee)_{k(X_r)} + \sum_j v_j \cdot (v_j^\vee)_{k(X_r)} \right) \right). \tag{11}$$

Since $\Delta_M|_{k(X_r)} \in \mathrm{H}^{\{2\}}_{\mathcal{M}}(M_{k(X_r)}, \mathbb{Z})$, we obtain that $((v_j^\vee)_{k(X_r)}, (u_i^\vee)_{k(X_r)}) \in B(k(X_r))$. In the same way, $((u_i)_{k(X_r)}, (v_j)_{k(X_r)}) \in A(k(X_r))$. Moreover, in the light of the embedding (10), we obtain the following.

Proposition 4.16. For any extension K/k,

$$(\mu_a^i, \lambda_a^j) \in A(K) \Leftrightarrow \sum_j (v_j^\vee)_{K(X_r)} \cdot \lambda_a^j + \sum_i (u_i^\vee)_{K(X_r)} \cdot \mu_a^i = 0 \in \mathrm{H}^*_{nr}(K(X_r)/K, \mathbb{Z}/p), \forall r.$$

In other words, $A(K)$ consists exactly of those elements of $H_A(K)$ that are orthogonal to $((v_j^\vee)_{K(X_r)}, (u_i^\vee)_{K(X_r)})$ in $H_A(K(X_r))$, for all r, in terms of the relations (9). Similarly, $B(K)$

consists exactly of those elements of $H_B(K)$ that are orthogonal to $((u_i)_{k(X_r)},(v_j)_{k(X_r)})$ in $H_B(K(X_r))$, for all r.

Corollary 4.17. $A = B^{\perp}$ and $B = A^{\perp}$ as Rost cycle modules, in terms of the relation (9).

By [32, Lemma 4.11] and [20], $\mathrm{H}^{\{2\}}_{\mathcal{M}}(X,\mathbb{Z})$ coincides with the Rost cycle module of "Chow groups with coefficients" in Milnor's K-theory $\mathrm{H}^2(X,\underline{K}^M_*)$. Thus, $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M_K,\mathbb{Z})$ is additively generated by elements of the form $\mathrm{Tr}_{E/K}(\rho_M([q]\cdot\alpha))$, where E/K is some finite extension, $q\in X(E)$, $\alpha\in K^M_*(E)$, and ρ_M is the projector defining M. Since q is a specialization of one of the generic points $\mathrm{Spec}(E(X_r))$ of X_E, we get from (11) that

$$\rho_M([q]\cdot\alpha) = \rho_M([q])\cdot\alpha = \delta(\tau^{-1}\Big(\sum_i u_i\cdot(u_i^{\vee})_{E(X_r)} + \sum_j v_j\cdot(v_j^{\vee})_{E(X_r)}\Big))(q)\cdot\alpha.$$

Since $\mathrm{H}^{\{2\}}_{\mathcal{M}}(M,\mathbb{Z})$ can be identified with the Rost cycle module B, we obtain the following.

Proposition 4.18. The Rost cycle module B is generated by elements $((u_i^{\vee})_{k(X_r)},(v_j^{\vee})_{k(X_r)})\in B(k(X_r))$, numbered by the connected components of X. Similarly, the Rost cycle module A is generated by elements $((u_i)_{k(X_r)},(v_j)_{k(X_r)})\in A(k(X_r))$, numbered by the same components.

Remark 4.19. In the light of Corollary 4.11, for an irreducible p-torsion motive M, the respective Rost cycle modules A and B are principal (generated by a single generator). △

For a smooth projective curve C/k having m connected components, $M(C) = \mathbb{Z}^{\oplus m}\oplus\overline{M}(C)\oplus\mathbb{Z}(1)[2]^{\oplus m}$, where $\overline{M}(C)$ is the *reduced motive* of C (recall that our field k is algebraically closed and so every variety has a k-rational point).

Using arguments completely parallel to the proof of Theorem 4.13, one gets the following.

Theorem 4.20. The assignment $N\mapsto C = \mathrm{H}^{\mathcal{M}}_{\{0\}}(N,\mathbb{Z}/p)$, $D = \mathrm{H}^{\mathcal{M}}_{\{0\}}(N^{\vee},\mathbb{Z}/p)$ defines a 1-to-1 correspondence between isomorphism classes of

(1) direct summands in the reduced motives of smooth projective (possibly, disconnected) curves in $Chow(k,\mathbb{Z}/p)$; and
(2) pairs of Rost submodules $C\xrightarrow{(\alpha_C)_*}H_C$, $D\xrightarrow{(\alpha_D)_*}H_D$, where $H_C = H_D = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus r}$, for some r, such that
  (a) the respective objects $N_{\bullet}(C)$ and $N_{\bullet}(D)$ of the heart of the homotopic t-structure (on $\mathrm{DM}(k;\mathbb{Z}/p)$) are compact;

(*b*) $C^j = D^j = 0$, for $j < 0$;

(*c*) the respective triangles (with $\mathcal{K}_C = \mathcal{K}_D = k_M(1)^{\oplus r}$)

$$\begin{array}{ccccc} & & \mathcal{K}_C & & \\ & \overset{[1]}{\nearrow}{\scriptstyle \alpha_C} & \star & \searrow{\scriptstyle \beta_C} & \\ N_{\bullet}(C) & & \xleftarrow[\gamma_C]{} & & N^{\bullet}(C) \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \mathcal{K}_D & & \\ & \overset{[1]}{\nearrow}{\scriptstyle \alpha_D} & \star & \searrow{\scriptstyle \beta_D} & \\ N_{\bullet}(D) & & \xleftarrow[\gamma_D]{} & & N^{\bullet}(D) \end{array} .$$

are dual w.r.t. to $\underline{\mathrm{Hom}}_{\mathrm{DM}(k;\mathbb{Z}/p)}(-, \mathbb{Z}/p(1)[2])$, namely $\gamma_D = \gamma_C^{\vee}$.

This result permits to draw some conclusions about p-torsion motives of surfaces.

Proposition 4.21. In the notations of Theorem 4.13, for nonzero M, both s and t are non-zero.

Proof. Suppose, $s \cdot t = 0$. Changing M by $M^{\vee}$, if necessary, we can assume that $t = 0$. Then, $A \subset H_A = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus s}$, $B \subset H_B = \mathrm{H}^*_{et}\langle -2\rangle^{\oplus s}$. Consider $C = A$, $D = B\langle 1\rangle$. Then $C \subset H_C = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus s}$, $D \subset H_D = \mathrm{H}^*_{et}\langle -1\rangle^{\oplus s}$ and this pair of Rost submodules satisfies all the conditions of Theorem 4.20(2) (note that for the adjunction

$$\mathrm{DM}(k;\mathbb{Z}) \underset{\nu^*}{\overset{\nu_*}{\rightleftarrows}} \mathrm{DM}(k;\mathbb{Z}/p),$$

we have: $\underline{\mathrm{Hom}}_{\mathrm{DM}(k;\mathbb{Z})}(\nu_*(U), \mathbb{Z}(2)[4]) = \nu_*\underline{\mathrm{Hom}}_{\mathrm{DM}(k;\mathbb{Z}/p)}(U, \mathbb{Z}/p(2)[3])$). Thus, it defines a direct summand N in the reduced motive of some smooth projective curve, in particular, a Chow motive with $\mathbb{Z}/p$-coefficients. But since $B^j = 0$, for $j < 0$, it follows that $D^j = 0$, for $j \leqslant 0$. Hence, the Chow motive $N^{\vee}$ has no Chow groups CH_* over any field extension F/k. By the standard arguments [14], this implies that $N^{\vee} = 0 = N$, and so $s = 0$ and $M = 0$. ■

Thus, in the Hodge half-diamond of M all four potential positions are filled. In particular, we get the following.

Corollary 4.22. For any nonzero p-torsion direct summand M in the motive of a surface (over any field of characteristic zero), $Pic(M_{\overline{k}}) \neq 0$.

Proof. By the result of Gille [9], the passage to the algebraic closure is conservative for direct summands of motives of surfaces. The case of an algebraically closed field follows from Proposition 4.21. ■

In the end, let me say few words about endomorphisms of p-torsion motives of surfaces.

The octahedra (5) appear to be functorial.

Proposition 4.23. Let M and N be p-torsion motives of surfaces. Then any map $\varphi : M \to N$ extends uniquely to the map of octahedra (5).

Proof. Since $M_\bullet(A)$ is the 0-th slice of M and $M^\bullet(A)$ is the dual to the 0-th slice of $M^\vee$, the assignment $M \mapsto M_\bullet(A), M^\bullet(A)$ is functorial. Hence, φ extends to a map of upper halves of octahedra (the ones, containing M (respectively, N) and $M^\vee$ (respectively, $N^\vee$)). Since $M_{et} = (\mathcal{T}_A)_{et}$ and $\mathrm{Hom}(\mathcal{T}_A(M), \mathcal{T}_A(N)) = \mathrm{Hom}(\mathcal{T}_A(M)_{et}, \mathcal{T}_A(N)_{et})$, the extensions to $\mathcal{T}_A$ and $\mathcal{T}_A(-1)$ are unique. Since $\mathrm{Hom}(\mathcal{T}_A(M)(-1)[1], N_\bullet(A)) = 0$, the extension to $M_\bullet(A)$ is unique. By the dual argument, the same is true about extension to $M^\bullet(A)$ (as well as to $M_\bullet(B)$ and $M^\bullet(B)$). Because $\mathrm{Hom}(\mathcal{T}_A(M)(-1), \mathcal{K}_A(N)) = 0$, we obtain the unique extension to the $N - E$ triangles in the lower halves of octahedra. Finally, because $\mathrm{Hom}(\mathcal{T}_A(M)(-1), N^\bullet(A)) = 0$, from the fact that our extension commutes with $\mathcal{K}_A(L) \stackrel{[1]}{\leftarrow} \mathcal{T}_A(L) \stackrel{[1]}{\to} L^\bullet(A)$ (for $L = M, N$), it follows that it commutes with β_A and, by duality, with α_A. ■

In particular, we obtain a natural homomorphism

$$\mathrm{End}(M) \to \mathrm{End}(\mathcal{K}_A(M)) = \mathrm{Mat}(s,t) := \mathrm{Mat}_{s\times s}(\mathbb{F}_p) \times \mathrm{Mat}_{t\times t}(\mathbb{F}_p).$$

Denote the image of it as $R(A)$.

Proposition 4.24.

(1) $R(A) = \mathrm{End}(M)/\mathit{Nilp}$, where Nilp is finite nilpotent ideal.

(2) $\psi \in \mathrm{Mat}(s,t)$ belongs to $R(A)$ if and only if it maps A to itself, if and only if it maps $A^0 = \mathrm{CH}_0(M)$ (considered over all generic points of X) to itself.

Proof. The kernel of the surjection $\mathrm{End}(M) \twoheadrightarrow R(A)$ consists of those φ, which are trivial on $M_\bullet$, $M^\bullet$, and $\mathcal{K}$ (A and B). Then φ is nilpotent on $\mathcal{T}$ and $\mathcal{T}(-1)$ and so on M. By Proposition 4.9, the kernel is finite.

Clearly, any element of $R(A)$ maps A to itself. Conversely, suppose ψ maps A to itself. Then it lifts uniquely to an endomorphism of $M_\bullet(A)[-1] \to \mathcal{K}_A \to M^\bullet(A) \to M_\bullet(A)$ and $\varphi^\vee$—to an endomorphism of a similar B-triangle. Any $\psi \in \mathrm{End}(\mathcal{K}_A)$ lifts to an endomorphism of $\mathcal{K}_A \to \mathcal{T}_A(-1) \stackrel{\cdot\tau}{\to} \mathcal{T}_A \to \mathcal{K}_A[1]$. Since $\mathrm{Hom}(M_\bullet(A), M) = 0$, we have: $\mathrm{End}(M_\bullet(A) \stackrel{[1]}{\to} \mathcal{T}_A(-1)) = \mathrm{End}(M_\bullet(A) \stackrel{[1]}{\to} \mathcal{T}_A(-1) \to M \to M_\bullet(A))$ and, by duality, $\mathrm{End}(\mathcal{T}_A \stackrel{[1]}{\to} M^\bullet(A)) = \mathrm{End}(\mathcal{T}_A \stackrel{[1]}{\to} M^\bullet(A) \to M \to \mathcal{T}_A)$. Finally, because $\mathrm{Hom}(M_\bullet(A), \mathcal{T}_A) = 0$, if our extension commutes with f and $(\cdot\tau)$, then it commutes with g.

It remains to note that by Proposition 4.18, A is generated by A^0 as a Rost cycle module. So, ψ respects A if and only if it respects A^0 (sufficient to know for generic points of X). ■

Remark 4.25. Observe that we obtain a canonical splitting $R(A) \to \mathrm{End}(M)$. Also, we get a 1-to-1 correspondence between idempotents of $\mathrm{End}(M)$ and that of $R(A)$.

Example 4.26. Let M be the Godeaux torsion motive. Then $s = t = 1$ and $\mathrm{CH}_0(M_{\mathbb{C}(X)}) = \mathbb{Z}/5$ spanned by $(\widetilde{v}, \widetilde{u})$. Thus, $R(A) = \mathbb{F}_5$ embedded diagonally into $\mathrm{Mat}(1,1) = \mathbb{F}_5 \times \mathbb{F}_5$. In particular, M is indecomposable (which is also clear from Corollary 4.22).

It is easy to see that the ideal *Nilp*, in this case, is one-dimensional, spanned by the composition $M \stackrel{u}{\to} \mathbb{Z}/5(1)[1] \stackrel{Q_0}{\to} \mathbb{Z}/5(1)[2] \stackrel{u^\vee}{\to} M$. Recall that $M = \nu_*(\mathbf{M})$, for some motive $\mathbf{M} \in \mathrm{DM}(k; \mathbb{Z}/5)$. The same calculations give $\mathrm{End}_{\mathrm{DM}(k;\mathbb{Z}/5)}(\mathbf{M}) = \mathbb{F}_5$.

5 Appendix: Chow Motives vis Geometric Motives

The purpose of this appendix is to show that looking at motivic homology and cohomology of a geometric motive M one can check, if the object is pure and, moreover, one can determine the dimension of the respective smooth projective variety (M is a direct summand of) as well as the Tate-shift involved. The main tool that permits to see it is the weight structure of Bondarko [4] on $\mathrm{DM}_{gm}(k)$, and these results can be alternatively deduced from those of [5, 6]. Below, k is any field of characteristic zero.

Recall the standard notations for *motivic homology* $\mathrm{H}^{\mathcal{M}}_{i,j}(M, \mathbb{Z}) := \mathrm{Hom}_{DM(k)}(\mathbb{Z}(j)[i], M)$ and *motivic cohomology* $\mathrm{H}^{i,j}_{\mathcal{M}}(M, \mathbb{Z}) := \mathrm{Hom}_{DM(k)}(M, \mathbb{Z}(j)[i])$ of the motive M.

Proposition 5.1. Suppose $M \in \mathrm{DM}_{gm}(k)$. Then the following conditions are equivalent:

(1) $M \in Chow(k)$;

(2) $\mathrm{H}^{\mathcal{M}}_{i,j}(M_K, \mathbb{Z}) = 0$, for $i < 2j$, while $\mathrm{H}^{i,j}_{\mathcal{M}}(M_K, \mathbb{Z}) = 0$, for $i > 2j$, for any finitely generated field extension K/k.

Proof. The implication (1) $\Rightarrow$ (2) is clear from the standard properties of motivic cohomology; see [32]. (2) $\Rightarrow$ (1): as was shown by Bondarko [4], on $\mathrm{DM}_{gm}(k)$, one has a (unique) bounded nondegenerate weight structure whose heart is the category of Chow-motives $Chow(k)$. Let $\mathcal{D}^{\leqslant m}$ and $\mathcal{D}^{\geqslant m}$ be the respective subcategories.

Let us show that $M \in \mathcal{D}^{\leqslant 0}$. Indeed, since the weight structure is bounded, $M \in \mathcal{D}^{\leqslant m}$, for some m. If $m \leqslant 0$, we are done. Suppose $m > 0$. Then there exists an exact triangle

$$\omega_{\leqslant m-1}M \to M \to \omega_{\geqslant m}M \to (\omega_{\leqslant m-1}M)[1]$$

Since $M \in \mathcal{D}^{\leqslant m}$, here $\omega_{\geqslant m}M = U[m]$, for some $U \in Chow(k)$. Then $\mathrm{Hom}(M, U[m]) = \mathrm{Hom}(M \otimes U^{\vee}, \mathbb{Z}[m])$ that is a direct summand in $\mathrm{Hom}(M \otimes M(X), \mathbb{Z}(i)[2i+m])$, for some smooth projective variety X and some $i \in \mathbb{Z}$. Since, for any point $y \in X$ of co-dimension c, the group $\mathrm{Hom}(M_{k(y)}, \mathbb{Z}(i-c)[2(i-c)+m])$ is zero by our condition, it follows from the Brown–Gersten–Quillen-type arguments that $\mathrm{Hom}(M \otimes M(X), \mathbb{Z}(i)[2i+m]) = 0$. Thus, the morphism $M \to \omega_{\geqslant m}M$ is zero, and so M is a direct summand of $\omega_{\leqslant m-1}M$. Hence, $M \in \mathcal{D}^{\leqslant m-1}$. By induction, we obtain that $M \in \mathcal{D}^{\leqslant 0}$.

Applying the same arguments to the dual $M^{\vee}$ of M and using triviality of motivic homology of M below the slope $= 2$ line, we obtain that $M \in \mathcal{D}^{\geqslant 0}$. Thus, M belongs to the heart $\mathcal{D}^{\geqslant 0} \cap \mathcal{D}^{\leqslant 0} = Chow(k)$. ■

One can also deduce the above result from [5, Theorem 3.3.3].

Proposition 5.2. (cf. [6, Theorem 3.2.1(1)]) Suppose $N \in Chow(k)$ be such a Chow motive that $\mathrm{H}^{\mathcal{M}}_{2i,i}(N_K, \mathbb{Z}) = 0$, for $i < r$ and any finitely generated extension K/k. Then there exists a smooth projective (possibly reducible) variety X over k such that N is a direct summand of $M(X)(r)[2r]$.

Proof. Since N, up to Tate-shift, is a direct summand in the motive of some smooth projective variety, which we may assume equi-dimensional (by multiplying the components of it by projective spaces of appropriate dimension), the statement follows from the following result. ■

Lemma 5.3. If N is a direct summand of $M(Y)$, for some smooth projective equi-dimensional variety Y and $\mathrm{CH}_0(N_K) = 0$, for any finitely generated extension K/k, then N is a direct summand of $M(P)(1)[2]$, for some smooth projective equi-dimensional variety P/k with $\dim(P) = \dim(Y) - 1$.

Proof. Let $Y = \coprod_i Y_i$ be the decomposition of Y into connected components and $E_i = k(Y_i)$ be their function fields. N is given by some projector $\varphi \in \mathrm{End}_{Corr}(Y)$, represented by some cycle $\Phi \in \mathrm{CH}^*(Y \times Y)$. Since $\mathrm{CH}_0(N) = 0$, this projector acts as zero on zero-cycles over any field extension. In particular, it sends the generic point of Y_i over E_i to zero. This means that the restriction of $\Phi_{k(Y_i)} \in \mathrm{CH}^*(Y_{k(Y_i)})$ is zero. Thus, Φ is rationally equivalent to some cycle supported on $Z \times Y$, where $Z = \coprod_i Z_i$ and $Z_i \subset Y_i$ is a reduced closed proper subscheme. By the results of Hironaka [11], we can resolve the singularities of Z_i by blowing up smooth centers in the singular locus of Z_i. Let R_i be the disjoint union of all such centers for Z_i, and $\widetilde{Z}_i \to Z_i$ be the resulting smooth model. We have the natural proper map $P_i := \widetilde{Z}_i \coprod R_i \to Z_i$. It follows from [26, Proposition 7.7] that the respective push-forward $\mathrm{CH}_*(P_i \times Y) \to \mathrm{CH}_*(Z_i \times Y)$ is surjective. By multiplying the components of P_i by projective spaces of appropriate dimension, we may assume that P is equi-dimensional of dimension one less than Y. In light of the above surjectivity, the map $\varphi : M(Y) \to M(Y)$ decomposes as $M(Y) \xrightarrow{\alpha} M(P)(1)[2] \xrightarrow{\beta} M(Y)$. Since $\varphi = \beta \circ \alpha$ is a projector, so is $\psi = \alpha \circ \beta \circ \alpha \circ \beta$, and the Chow-motive $N = (Y, \varphi)$ is isomorphic to a direct summand of $M(P)(1)[2]$. ■

Proposition 5.4. Suppose $N \in Chow(k)$ be such a Chow motives that $\mathrm{H}^{\mathcal{M}}_{2i,i}(N_K, \mathbb{Z}) = 0$, for $i < 0$, and $\mathrm{H}^{2i,i}_{\mathcal{M}}(N_K, \mathbb{Z})$, for $i > r$, over any finitely generated field extension K/k. Then N is a direct summand in the motive $M(X)$ of some smooth projective (possibly reducible) variety X of dimension r.

Proof. From Proposition 5.2, N is a direct summand in the motive $M(Y)$ of some smooth projective (possibly disconnected) equi-dimensional variety Y/k. Let $d = \dim(Y)$. If $d \leqslant r$, we are done (can always multiply Y by an appropriate projective space, if needed). Assume $d > r$. Consider $N^\vee$—a direct summand given in $M(Y)$ by the dual projector. Then $\mathrm{H}^{\mathcal{M}}_{2i,i}(N^\vee_K, \mathbb{Z}) = \mathrm{H}^{2(d-i),d-i}_{\mathcal{M}}(N_K, \mathbb{Z})$ because $N^\vee = \underline{\mathrm{Hom}}(N, \mathbb{Z}(d)[2d])$. In particular, $\mathrm{CH}_0(N^\vee) = 0$, for any field extension K/k and so, by Lemma 5.3, $N^\vee$ is a direct summand of $M(P)(1)[2]$, where P is a smooth projective variety of dimension $d-1$. Since $M(P) = \underline{\mathrm{Hom}}(M(P)(1)[2], \mathbb{Z}(d)[2d])$, we get that N is a direct summand of $M(P)$. Applying this argument inductively, we obtain that N is a direct summand of the motive $M(X)$ of a smooth projective variety X of dimension r. ■

This result can be also deduced from [6, Theorem 3.2.1(1), Corollary 2.2.4(2), Proposition 5.2.1].

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