Online Appendix for Equilibrium Analysis in Behavioral One-Sector Growth Models

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Proof of Lemma 3. To shorten expressions we set $A = [\underline{a}, \overline{a}]$ and $z = (l, \epsilon) \in Z = [l_{\min}, l_{\max}] \times E$ where E is as given in Assumption 1. Let S denote the lattice of equivalence classes of measurable functions from $A \times Z$ into A with the pointwise order, $s \succeq \tilde{s} \Leftrightarrow s(a,z) \geq \tilde{s}(a,z)$ for *a.e.* (a,z). Define the parametrized fixed point correspondence $F(\rho, s) = \{\tilde{s} \in S : L(\tilde{s}(a,z), (a,z), s, \rho) = 0$ for *a.e.* $(a,z)\}$. Then s^* solves (11) if and only if $s^* \in F(\rho, s^*)$.

By assumption, $L(a', (a,z), s, \rho)$ is continuous in a', and $L(\underline{a}, (a,z), s, \rho) \geq 0$ and $L(\overline{a}, (a,z), s, \rho) < 0$. The Mean-value Theorem therefore implies the existence of a solution $\tilde{s}(a,z)$ such that $L(\tilde{s}(a,z), (a,z), s, \rho) = 0$, for all $(a,z) \in A \times Z$. By continuity in (a', a, z), $\tilde{s}$ must be measurable in (a,z) and it is also clear that for fixed (a,z), the set of solutions is compact. In particular, there will therefore exist (measurable) least and greatest solutions, $\tilde{s}^{\min}(a,z) \leq \tilde{s}(a,z) \leq \tilde{s}^{\max}(a,z)$ where $\tilde{s}$ is any solution (specifically $\tilde{s}^{\min}$ and $\tilde{s}^{\max}$ are the pointwise minimum and maximum, respectively). Let $f^{\min}(\rho, s), f^{\max}(\rho, s) \in F(\rho, s)$ denote these least and greatest solutions, now viewed as elements in S (hence functions of ρ and s). By monotonicity in future savings, continuity of L in a' and the boundary conditions, it follows from an application of the standard "curve shifting" argument for continuous functions discussed in Appendix B that $f^{\min}(\rho, s)$ and $f^{\max}(\rho, s)$ are increasing in s with the order $\succeq$. Because S is chain-complete and has least and greatest elements, by the Knaster-Tarski fixed point theorem (Granas and Dugundji (2003), Theorem 1.1) $f^{\min}(\rho, \cdot)$ has a least fixed point s_* and $f^{\max}(\rho, \cdot)$ has a greatest fixed point s^*. Let $s \in F(\rho, s)$ be any fixed point of F. Then $f^{\max}(\rho, s) \succeq s$. Since $f^{\max}(\rho, \cdot)$ is increasing, it maps $\{t \succeq s\}$ into $\{t \succeq s\}$. Because $\{t \succeq s\}$ is chain-complete and has least element s, there exists $\tilde{s} = f^{\max}(\rho, \tilde{s})$ (again by the Knaster-Tarski fixed point theorem) where, necessarily, $s^* \succeq \tilde{s} \succeq s$. This shows that the greatest fixed point of $f^{\max}(\rho, \cdot)$ is the greatest fixed point of $F(\rho, \cdot)$. Similarly, s_* is the least fixed point of $F(\rho, \cdot)$. We conclude, as in the lemma, that (11) has least and greatest solutions.

Now let s^* denote the largest fixed point given ρ^* and consider $\rho^{**} > \rho^*$. As previously, denote the largest element in $F(\rho^{**}, s^*)$ by $f^{\max}(\rho^{**}, s^*)$. Again because $L(\cdot, (a,z), s^*, \rho^*)$ is continuous and begins above and ends below 0, and because $0 = L(s^*(a,z), (a,z), s^*, \rho^*)$ and by assumption $L(s^*(a,z), (a,z), s^*, \rho^*) \leq L(s^*(a,z), (a,z), s^*, \rho^{**})$ whenever $(a,z) \in \lambda_{s^*}$, it must hold that $f^{\max}(\rho^{**}, s^*) \succeq s^*$. By monotonicity in future savings $(0 <)$ $L(s^*(a,z), (a,z), s^*, \rho^{**}) \leq L(s^*(a,z), (a,z), t, \rho^{**}) \leq L(s^*(a,z), (a,z), t', \rho^{**})$ whenever $t' \succeq t \succeq s^*$. Therefore $f^{\max}(\rho^{**}, t') \succeq f^{\max}(\rho^{**}, t)$ whenever $t' \succeq t$ $(\succeq s^*)$. We conclude that $f^{\max}(\rho^{**}, \cdot)$ is a monotone self-map on $\{t \succeq s^*\}$. As above, there must then exist s^{**} with $s^{**} = f^{\max}(\rho^{**}, s^{**})$. Necessarily, $s^{**} \in F(\rho^{**}, s^{**})$ and $s^{**} \succeq s^*$. Because the greatest solution to the Euler equations given ρ^{**} must be greater than s^{**}, it must be greater than s^*. This establishes both the first and the last itemized claims of the lemma (in particular, if the Euler equations have a unique solution, then this is s^* given ρ^* and s^{**} given ρ^{**}). One proves analogously that the least solution to the Euler equations must decrease if $\rho^{**} < \rho^*$. In the case where L is continuous and the change in ρ is infinitesimal, the graph of $F(\rho, \cdot)$ is compact and the claim follows similarly to in the proof of Theorem 2. ■

Proof of Lemma 4 in Appendix A. From the proof of Lemma 2 we know that $\mathcal{M}^\theta(k) = \{K : K \in F_k(K,\theta)\}$ where $F_k(K,\theta) = \int \mathcal{A}_k^{\theta,i}(K)\, di$. $F_{k^*}(\cdot,\theta^*)$ and $F_{k^*}(\cdot,\theta^{**})$ are upper hemicontinuous, convex valued, and necessarily begin above and end below the diagonal since they are decreasing correspondences (see the proof of Lemma 2). For clarify, we first consider the case where households' assets distributions are uniquely determined in steady state (in principle, different assets distributions might support the same steady state).

Let k^* be the greatest equilibrium. If the population's mean asset holdings increase at k^*, then $F_{k^*}(k^*,\theta^{**}) \geq k^*$ and there therefore exists $K \in F_{k^*}(K,\theta^{**})$ with $K \geq k^*$. Since $K \in \mathcal{M}^{\theta^{**}}(k^*) \Leftrightarrow K \in F_{k^*}(K,\theta^{**})$, it follows that the market correspondence shifts up at k^*. This argument also applies if k^* is the least equilibrium since F is decreasing in K.

To see that an increase in mean asset holdings is also necessary for the market correspondence to shift up, use that if there does not exist $\hat{k} \in F_{k^*}(k^*,\theta^{**})$ with $\hat{k} \geq k^*$, then because $F_k(K,\theta^{**})$ is convex valued with least and greatest selections that are decreasing in K, there is not a $K \in F_{k^*}(K,\theta^{**})$ with $K \geq k^*$, and so the population's mean asset holdings does not increase as θ^* changes to θ^{**}.

If households' steady-state asset distributions are not uniquely determined from k, one one considers instead the greatest mean asset holdings: $A_+^{\theta,i}(k) = \sup\{\mathbb{E}[\hat{a}^i] : \hat{a}^i \in S^{\theta,i}_{w(k),R(k)}(\hat{a}^i)\}$, and define the greatest aggregate asset holdings across the agents (given θ and the steady state k): $A_+^\theta(k) = \int A_+^{\theta,i}(k)\, di$. Then the change in environment from θ^* to θ^{**} shifts the market correspondence up at k^* if and only if $k^* \leq A_+^{\theta^{**}}(k^*)$ (see the proof of Proposition 4). Trivially the left-hand side of this inequality, k^*, is the aggregate asset holding across the households at the steady state k^*. So, the necessary and sufficient condition is that the greatest aggregate asset holding after the change in environment is above the aggregate asset holdings before the change. ■

Proof of Proposition 2. We can ignore the parameter ϵ^i from the general model and take $z^i = l^i$ (as in the previous proof we also suppress the transfers $T^i = 0$, and from now on household indices are omitted). Fixing throughout steady-state prices w and R determined by the initial capital-labor ratio k^*, a behavioral household with assets a and labor endowment l will save

$s^{\text{Beh.}}(a;l) = a'$ where a' solves

$$-u'((1+R)a + wl - a') + \max\{\delta(1+R)\int u'((1+R)a' + W - s^{\text{Beh.}}(a';l'))\mu_W^m(dW|w), u'((1+R)a + wl - \bar{a})\} = 0 \quad \text{(A1)}$$

The rational households' time-stationary saving function $s^{\text{Neocl.}}(a;l)$ is determined similarly except that μ_W^m is replaced with the true distribution μ_W defined in the text prior to the proposition. We consider only the "optimistic" case where $\mu_W^m(\cdot|w)$ first-order stochastically dominates $\mu_W(\cdot|w)$ (the "pessimistic" case is proved analogously). Because the left-hand side of (A1) must decline if $\mu_W(\cdot|w)$ is replaced with $\mu_W^m(\cdot|w)$ (u' is decreasing), it immediately follows from Lemma 3 that $s^{\text{Beh.}}(a;l) \leq s^{\text{Neocl.}}(a;l)$, that is, behavioral households who overestimate their future labor incomes will save less than rational households (all else equal). Observe that whether behavioral households over- or underestimate future labor incomes at other wage levels than w is not relevant to this argument. The rest of the proof proceeds as in Proposition 1 and can be omitted. ■

Proof of Proposition 3. Let $R = (1-\tau)\hat{R} - 1$ where $\hat{R} = f'(k^*)$ is the rental rate in the initial steady state k^* (*i.e.*, before the change in the capital income tax τ). w is the corresponding wage rate as given by (1). A rational households with assets a and labor endowment l will in the initial steady state save $s^{\text{Neocl.}}(a;l,w,R) = a'$ where a' solves

$$-u'((1+R)a + wl - a') + \max\{\delta(1+R)\int u'((1+R)a' + W - s^{\text{Neocl.}}(a';l',w,R))\mu_W(dW|w), u'((1+R)a + wl - \bar{a})\} = 0\,, \quad \text{(A2)}$$

and $\mu_W(A|w_{t'}) = \mu\{l \in [l_{\min}, l_{\max}] : w_{t'}l \in A\}$ is the true distribution of labor income (here A is any Borel subset of $\mathbb{R}_+$).[43] By Theorem 1 in Light (2020), $s^{\text{Neocl.}}(a;l,w,R)$ is increasing in R whenever $\chi \geq 1$. Let $\tilde{\chi}$ denote the smallest value for which the rational households will increase their aggregate savings given w, $\hat{R}$ and the specific tax reduction $\tau^{**} < \tau^*$. Then if $\chi \geq \tilde{\chi}$, because $R = (1-\tau)\hat{R}$, it immediately follows that the rational households' direct effect is positive (given the initial steady state prices, in particular, given w and $\hat{R}$). To determine the behavioral households' saving function we consider again (A2) except with $\mu_W^m(\cdot|w,\tau)$ replacing μ_W. If the behavioral households are "pessimistic", it is immediate that the left-hand side of (A2) will be increasing faster as a function of a' than it will if households are rational (this is because $\mu_W^m(\cdot|w,\tau)$ is first-order stochastically decreasing in τ and u' is decreasing). In particular, by

[43] Because the labor endowment shocks are *i.i.d.*, we have $s^{\text{Neocl.}}(a';l',w,R) = \hat{s}(a';W,w,R)$ for some function $\hat{s}$. Because we condition on the prices w and R, it is therefore not necessary to take the distribution of l' separately into account in the expected (marginal) utility term of the Euler equation.

Lemma 3, the behavioral households must (also) increase savings if the capital income tax is reduced. The first statement of the proposition now follows from Theorem 1 (the steady-state is unique; see Light (2020) and especially Section 3.3. concerning ex-ante heterogeneity as in our model).

Next consider the "optimistic" case. Since nothing changes for the rational households, their direct effect remains positive assuming $\chi > \tilde{\chi}$. Because all functional forms involved are continuous and because $\mu_W^m(\cdot|w,\tau)$ is first-order stochastically increasing in τ and u' strictly decreasing, when χ is sufficiently close to $\tilde{\chi}$, the behavioral households' savings must decrease as τ^* is reduced to τ^{**}. So the behavioral households' direct effect will be negative. The average direct effect across all households equals the α-weighted average of the rational and the behavioral households. But since $\tilde{\chi}$ is the break-even value where the rational households' direct effect is zero, and the behavioral households' direct effect becomes negative as $\chi \downarrow \tilde{\chi}$, and is bounded away from 0, it follows that if χ is close enough to $\tilde{\chi}$ then the average direct effect must be negative. The statement of the proposition now follows from Theorem 1 as in the previous case. ■

Proof of Proposition 4. The notation is the same as in the proof of Proposition 3. Because the intertemporal elasticities of substitution (IES) are greater than 1, that proof is identical when the behavioral subset is pessimistic, establishing that the decline in the capital income tax from τ^* to τ^{**} increases the steady-state capital-labor ratio. We are again assuming full depreciation to simplify expressions. Then, the (after-tax) market rate of interest increases from $R^b = (1 - \tau^*)\hat{R}^* - 1$ to $R^a = (1 - \tau^{**})\hat{R}^{**} - 1 > R^b$, and the equilibrium wage rate increases from $w^b = w((1+R^b)/(1-\tau^*))$ to $w^a = w((1+R^a))/(1-\tau^{**})) > w^b$ where $w((1+R)/(1-\tau)) = f((f')^{-1}((1+R)/(1-\tau)) - f'((f')^{-1}((1+R)/(1-\tau)))(f')^{-1}((1+R)/(1-\tau))$ (this follows from (1)-(2) in view of the fact that labor is not taxed).

To prove the first claim in the proposition, note, first, that because any household $i \in [0,1]$ has $\chi_i > 1$ and is either rational, or behavioral and "pessimistic", its savings function $s^i(a;l,w,R)$ is increasing in R and decreasing in w (see Light (2020) and Acikgoz (2018) and the proof of Proposition 3 for the relationship between the rational and behavioral cases). Second, let us set $B > 0$ sufficiently small and then select any (measurable) subset of households $J \subseteq [0,1]$ with $\mu(J) \leq B$. Then the equilibrium interest and wage rate increases ($R^a - R^b > 0$ and $w^a - w^b > 0$) will be arbitrarily close to those of a hypothetical (or limit) economy consisting only of the households in $[0,1]\backslash J$ subject to the same before and after capital income tax rates (see the proof of Proposition 3 for the continuity arguments involved in this claim).

Now given $B > 0$ sufficiently small, we take a (measurable) subset of the households $J \subset$

$[0,1]$ with $\mu(J) \leq B$ and keep their preferences and biases fixed. Because the increase in the market interest rate in the limit economy with households $[0,1]\backslash J$ is small relative to the increase in the wage rate, the aggregate stationary equilibrium asset holdings of all households in J must decline when the capital income tax is reduced.

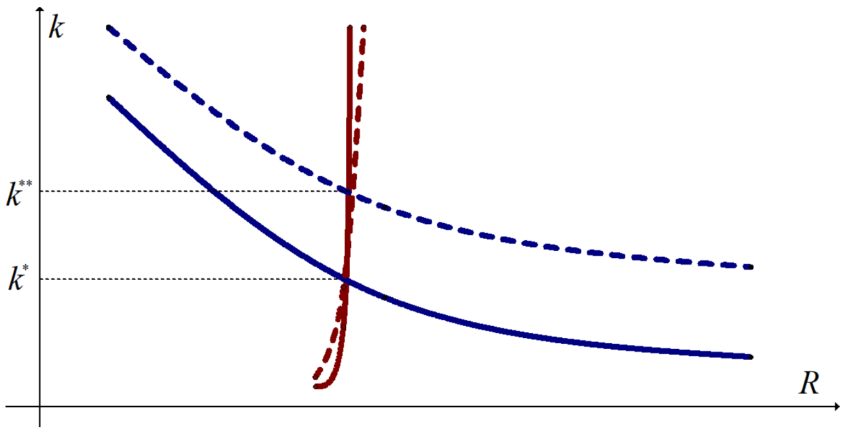

Figure 6: Blue curves show the demand for capital from the production side, $k = (f')^{-1}((1+R)/(1-\tau))$, before (solid) and after (dashed) the reduction in the capital income tax rate from τ^* to τ^{**}, $\tau^* > \tau^{**}$. The red curves show the supply of savings from the household side, $k = s(R, w((1+R)/(1-\tau)))$, where $w((1+R)/(1-\tau)) = f((f')^{-1}((1+R)/(1-\tau)) - f'((f')^{-1}((1+R)/(1-\tau)))(f')^{-1}((1+R)/(1-\tau))$. The figure depicts an economy where the large fraction of households $[0,1]\backslash J$ has very elastic savings supply.

To see that we can ensure the previous (relative) price changes in the limit economy, note from the Euler equation (A2) in the proof of Proposition 3 that we can choose $\mu_W^{m,i}$ ("the biases") and χ_i, $i \in [0,1]\backslash J$, such that these households exhibit arbitrarily high stationary savings supply elasticity (for example, we could impose that they are all rational and have IESs sufficiently close to 1). Therefore, provided that f' is decreasing (f strictly concave), the interest rate increase can be made arbitrarily small relative to the change in the wage rate in the limit economy.[44]

The main idea of this argument is summarized in the "Aiyagari diagram" in Figure 6 which illustrates the market outcome given $B > 0$ and some $J \subseteq [0,1]$ with $\mu(B) \leq B$ or, alternately, it illustrates the outcome in the limit economy (when B is small, these are approximately the same). This finishes the proof of of the first claim in the proposition.

The second claim is proved analogously by choosing the biases and intertemporal elasticities of substitutions for households in $[0,1]\backslash J$ that ensure that the increase in the wage rate is now small relative to the increase in the after-tax interest rate, and we omit the details. ■

[44] In particular, the increase in the wage rate is fully pinned down by the change in the capital-labor ratio, and can be lower-bounded by choosing f suitably given the preferences and biases of the households in $[0,1]\backslash J$.

Proof of Proposition 5. Under the conditions imposed on page 28, the value function W is unique and strictly concave, and time-stationary savings is characterized by a Generalized Euler Equation (GEE): $s(a;l) = a'$ where a' solves

$$-u'((1+R)a+wl-a') - \lambda v'((1+R)a+wl-a') + \max\{\delta(1+R)\int u'((1+R)a'+wl'-s(a';l'))+ \lambda v'((1+R)a'+wl'-s(a';l')) - \lambda v'((1+R)a'+wl'-\underline{a})\mu(dl'),$$

$$u'((1+R)a+wl-\underline{a}) + \lambda v'((1+R)a+wl-\underline{a})\} = 0\,.$$

It is easily verified that the conditions of Lemma 3 are satisfied — in particular, the left-hand side of the GEE is increasing in "future savings" $s(a';l')$ because $u+\lambda v$ is concave. The statements in the proposition now follow straight-forwardly from Lemma 3. It is useful to rewrite the GEE as follows:

$$\max\{\delta(1+R)\int u'((1+R)a'+wl'-s(a';l')) + \lambda v'((1+R)a'+wl'-s(a';l')) -\lambda v'((1+R)a'+wl'-\underline{a})\mu(dl') - u'((1+R)a+wl-a') - \lambda v'((1+R)a+wl-a')$$

$$, u'((1+R)a+wl-\underline{a})+\lambda v'((1+R)a+wl-\underline{a})-u'((1+R)a+wl-a')-\lambda v'((1+R)a+wl-a')\} = 0\,.$$

For any (a,l), if the second term is (weakly) larger than the first term, then $a' = s(a;l) = \underline{a}$. When $a' = s(a;l)$, an increase in λ will therefore weakly increase the left-hand side if (13) holds, and if (13) holds with the inequality reversed, the left-hand side must decline.[45] The condition in Lemma 3 therefore holds if (13) holds for all $(a,l) \in \mu^*$ where μ^* is the initial steady-state distribution. Hence the households' direct responses to an increase in λ are positive. The proposition's statement now follows from our main Theorems 1-3.

If (13) holds with the inequality reversed, the direct responses are instead negative and the (least and greatest) steady-state capital labor ratio(s) must instead decrease.

The proof is similar in the case where we reduce $\underline{a}$ and v is convex, because in this case both terms on the left-hand side of the GEE must decline (recall that, by assumption, $u+\lambda v$ is concave). If we reduce $\underline{a}$ and v is concave, then the first term will increase and the second term will decrease. So households who are not borrowing constrained will raise savings while households whose marginal utility of savings is strictly negative at the borrowing limit may fully exhaust the (loosened) borrowing constraint. The statement in the proposition follows. Note here that — unlike in the model with no temptation — when there is significant temptation, borrowing constraints will frequently not bind for any household (corresponding to no

[45] Note that when (13) holds, an increase in λ may cause some households to come off the borrowing constraint, and when (13) holds with the inequality reversed, additional households may be further tempted and become borrowing constrained when the temptation intensity increases.

one falling prey to temptation). In particular, savings will often not take the form of a "renewal process" (*e.g.*, see Deaton (1991)) in the model with temptation, something which is already clear from the deterministic version of the model considered by Gul and Pesendorfer (2004) (in this case, consumption stationarity implies $\delta(1+R) > 1$; see Gul and Pesendorfer (2004), p.137). ■

Proof of Proposition 6. Denote market prices given the initial tax τ^* by w and R, so that $1+R = (1-\tau^*)\hat{R}$ under full depreciation. Time-stationary savings is given by the GEE: $s(a;l) = a'$ where a' solves

$$-u'((1-\tau)\hat{R}a+wl-a')-v'((1-\tau)\hat{R}a+wl-a')+\max\{\delta(1-\tau)\hat{R}\int u'((1-\tau)\hat{R}a'+wl'-s(a';l'))+ v'((1-\tau)\hat{R}a'+wl'-s(a';l'))-v'((1-\tau)\hat{R}a'+wl'-\underline{a})\mu(dl'), u'((1-\tau)\hat{R}a+wl-\underline{a})+v'((1-\tau)\hat{R}a+wl-\underline{a})\} = 0\ .$$

The condition in the proposition is found by simply differentiating the left-hand side with respect to τ and evaluate at τ^*, which ensures the conclusion in either direction via Lemma 3 and our main theorems. The details can be omitted. Note that given the above GEE, the most precise approach (if and only if, given explicit functional forms) would solve numerically and explicitly verify Definition 5 for the specific tax change. This type of "blended" numerical-qualitative analysis is discussed more in Section 4.5 in the paper. ■

Proof of Proposition 7. As in the text, we fix an initial capital-labor ratio k^* and associated steady-state prices w and R. s is a TSSF if for almost every a and l, $s(a,l) = x$ where x solves the GEE $-u'((1+R)a+wl-x)-v'((1+R)a+wl-x)+\max\{\delta(1+R)\int u'((1+R)x+wl'-s(x,l'))+ v'((1+R)x+wl'-s(x,l'))-v'((1+R)x+wl'-\underline{a})\mu(dl'), u'((1+R)a+wl-\underline{a})+v'((1+R)a+wl-\underline{a})\} = 0$. The assumptions of Lemma 3 clearly hold and — holding the future savings function $s(x,l')$ fixed and considering only the initial invariant distribution — the left-hand side of this expression increases when μ is replaced with μ^* where μ^* is a mean-preserving spread of μ if and only if (16) holds (note that in (16) we have set $y = (1+R)a+wl$ and inserted the consumption function). So a mean-preserving spread to the perceived labor endowments shifts the GEE up in the sense of Lemma 3. The conclusions of the proposition now follow from our main results as in the previous proofs (the details can be omitted). ■

Proof of Proposition 8. Let ϵ_t^i denote the discount factor at date t. At each date t, both the naïve ($\beta < 1$) and the rational households believe that $\hat{P}(\epsilon_\tau^i = \delta) = 1$ for $\tau > t$. By Definition 1, both groups correctly anticipate their future selves' saving function conditional on these beliefs. For the rational households, the belief is correct (the objective distribution is degenerate *i.i.d.* with unit mass at δ). For the naïve households, it is incorrect (the objective distribution has unit mass at $(\beta\delta, \beta\delta^2, \ldots)$). In both cases, denoting the current discount factor by $\epsilon_t \in \{\beta\delta, \delta\}$, the Euler equation is $-u'((1+R)a+wl-a')+\max\{\epsilon_t(1+R)\int u'((1+R)a'+wl'-s(a';l',\delta))\mu(dl'), u'((1+R)a+wl-\bar{a})\} = 0$, where $(a,l,\epsilon_t) \in [\underline{a}, \overline{a}] \times [l_{\min}, l_{\max}] \times \{\beta\delta, \delta\}$, and $a' = s(a;l,\epsilon_t)$ is the solution given ϵ_t where, as usual, we have fixed w and R at their initial steady-state values (and suppressed them). For the rational households, $\epsilon_t = \delta$, so this is the benchmark Euler equation and it immediately follows by Lemma 3 that their saving function $s(a';l',\delta)$ is increasing in δ ($= \epsilon_t$) because the Euler equation's left-hand side is increasing in ϵ_t. For the naïve households, the Euler equation (also) takes $s(\cdot;l',\delta)$ as given but now $\epsilon_t = \beta\delta$. It follows, again from Lemma 3, that the resulting saving function is increasing in β.[46] In particular, rational households must save more than naïve households for all a and l, $\beta < 1$ and $\delta < 1$. The naïve households' saving function is also increasing in δ: the Euler equation's left-hand side increases both because $\beta\delta$ ($= \epsilon_t$) increases when δ increases and because when δ increases, $s(a';l',\delta)$ increases and the Euler equation satisfies "Monotonicity in Future Savings". The rest of the proposition is similar to the last part of the proof of Proposition 1 and can therefore be omitted. ■

The Sophisticated Quasi-hyperbolic Model with Vanishing Endowment Uncertainty (Section 4.4)

All households have quasi-hyperbolic CRRA preferences with rate of risk aversion $\gamma > 0$ and date t labor endowment $l_t = \tilde{l}_t + \hat{l}$ where $\hat{l} \in \mathbb{R}_{++}$ and $l_t \sim \mu(\cdot)$. $\mu(\cdot)$ is *i.i.d.* with mean zero and compact support. Whenever we say that l_t is almost degenerate below, what we mean is that μ and the degenerate measure with unit mass on 0 is sufficiently small in the Levy-Prokhorov metric. The borrowing constraint is zero (it can be ignored in steady state as will become clear). Capital is taxed at the rate τ, and tax receipts are rebated back to households lump-sum. There is full depreciation for notational simplicity, hence given the pre-tax rental rate of capital $\hat{R}_t$, the after-tax interest rate is $R_t = (1-\tau)\hat{R}_t - 1$.

The generalized Euler equation is

$$(C_\tau(y_t))^{-\gamma} = (1-\tau)\hat{R}\delta \int \left(1 + (\beta - 1)C'_\tau(y_{t+1})\right)(C_\tau(y_{t+1}))^{-\gamma}\mu(d\tilde{l}_{t+1}), \tag{A3}$$

where $y_t = (1-\tau)\hat{R}a_t + w \cdot (\hat{l} + l_t) + T_t$, $y_{t+1} = (1-\tau)\hat{R}[y_t - C_\tau(y_t)] + w \cdot (\hat{l} + \tilde{l}_{t+1}) + T_{t+1}$, a_t is total wealth inclusive of expected discounted future labor income, and C_τ the consumption function. Note that in terms of the paper's general notation, $(s(a_t;\hat{l}+l_t, w, R, T^i) =)\ s(a_t;\hat{l}+l_t) =$

[46] Note that in this application of Lemma 3, the future saving function is fixed, hence the GEE trivially exhibits monotonicity in future savings.

$(1-\tau)\hat{R}a_t+w\cdot(\hat{l}+l_t)+T_t-C_\tau(\theta\hat{R}a_t+w\cdot(\hat{l}+l_t)+T_t)$ where $R=\theta\hat{R}-1$. The consumption-saving model is Harris and Laibson (2001).

We first roll the deterministic component of labor income into assets using $\hat{R}$ as discount rate: The original budget equation is

$$a_{t+1}=\hat{R}a_t+w\cdot(\hat{l}+l_t)+\tau\hat{R}\cdot(k_t-a_t)-c_t$$

Inserting $\tilde{a}_t=a_t+(w\hat{l})/(\hat{R}-1)$, we get

$$c_t+\tilde{a}_{t+1}=\hat{R}\tilde{a}_t+w\cdot l_t+\tau\hat{R}\cdot(\tilde{k}_t-\tilde{a}_t)\ ,$$

where $\tilde{k}_t=k_t+(w\hat{l})/(\hat{R}-1)$ (so that $\tilde{k}_t=\int\tilde{a}_t\,di$). It clearly makes no difference to us whether we work in the original units or in these "tilde" units because we get to keep all of the variables in $(w\hat{l})/(\hat{R}-1)$ fixed in our all-else equal analysis. Further, since we one only inserts c_t and c_{t+1} into the Euler condition, this looks exactly the same as above except now $y_t=\hat{R}\tilde{a}_t+w\cdot l_t+\tau\hat{R}\cdot(\tilde{k}_t-\tilde{a}_t)$ and $y_{t+1}=\hat{R}[y_t-C_\tau(y_t)]+w\cdot\tilde{l}_{t+1}+\tau\hat{R}\cdot(\tilde{k}_{t+1}-[y_t-C_\tau(y_t)])$. The "trick" is then to rewrite the GEE as follows:

$$\left(\frac{C_\tau(\hat{R}\cdot[y_t-C_\tau(y_t)])}{C_\tau(y_t)}\right)^\gamma=(1-\tau)\hat{R}\delta\int\left(1+(\beta-1)C'_\tau(y_{t+1})\right)\left(\frac{C_\tau(\hat{R}\cdot[y_t-C_\tau(y_t)])}{C_\tau(y_{t+1})}\right)^\gamma\mu(d\tilde{l}_{t+1})$$

To clarify, all that has happened at this point is that the GEE has been divided through with $(C_\tau(\hat{R}\cdot[y_t-C_\tau(y_t)]))^\gamma$. Consider the term in the large parenthesis on the right-hand side which, defining $\hat{y}_t\equiv y_t-C_\tau(y_t)$, we can write

$$\left(\frac{C_\tau(\hat{R}\cdot[y_t-C_\tau(y_t)])}{C_\tau(y_{t+1})}\right)^\gamma=\left(\frac{C_\tau(\hat{R}\cdot\hat{y}_t)}{C_\tau(\hat{R}\hat{y}_t+w\cdot\tilde{l}^i_{t+1}+\tau\hat{R}\cdot(\tilde{k}_{t+1}-\hat{y}_t))}\right)^\gamma\ ,$$

Probabilistically, this is a random variable whose distribution converges to a degenerate distribution with unit mass at 1 if the random variable $\tilde{l}^i_t$ converges to a random variable with a degenerate distribution. Hence the previous term will be approximately equal to 1 . To see this, note that since C_τ is continuous, as $\tilde{l}^i_{t+1}$ becomes degenerate, $\hat{R}\cdot\hat{y}_t\approx\hat{R}\hat{y}_t+w\cdot\tilde{l}^i_{t+1}+\tau\hat{R}\cdot(\tilde{k}_{t+1}-\hat{y}_t)$ (in probability) where it is also used that households' are homogenous if $\tilde{l}^i_{t+1}$ is degenerate and that we are considering a steady state (these imply that next period assets $\hat{y}_t$ will be close to the mean assets $\tilde{k}_t=\tilde{k}_{t+1}$).

Now we can just test-and-verify (in fact, this is familiar from the CRRA case with no uncertainty), that the limit GEE is solved by a linear consumption function, $C_\tau(y)=C'_\tau(y)z$ where the MPC ($\partial C_\tau(y)/\partial y=C'_\tau(y)$) is constant given the tax. We then get

$$(\hat{R}\cdot(1-C'_\tau(y))^\gamma=(1-\tau)\hat{R}\delta\left(1+(\beta-1)C'_\tau(y)\right)\Omega(y),\tag{A4}$$

where $\Omega(y)\equiv\left(\frac{C_\tau(\hat{R}\cdot[z-C_\tau(y)])}{C_\tau(y)}\right)^\gamma\approx1$ and where it has been used that under the linear functional form

$$\frac{C_\tau(\hat{R}\cdot[y_t - C_\tau(y_t)])}{C_\tau(y_t)} = \hat{R}\cdot(1 - C'_\tau(y_t))\ .$$

Note that because $\Omega(y)$ is only approximately equal to 1, the solution ("the slope" $C'_\tau(y_t)$) depends on y_t. But as $\tilde{l}_t$ approaches a degenerate distribution, the interval of solutions will narrow towards a single point. That is why we can analyze the comparative statics with a single application of the Implicit Function Theorem (IFT) and be sure that the resulting (strict) inequality condition must hold for all households if it holds for one of the households.

In the application of the IFT, setting $\Omega = [\int(\frac{y+w\tilde{l}_{t+1}}{y})^{-\gamma}\hat{\mu}(d\tilde{l}_{t+1})]$, we use that the savings function is linear and keep y fixed and treat the MPS, $\bar{s} = 1 - C'_\tau(y)$, as the unknown variable. We thus get,

$$\gamma\hat{R}^\gamma\bar{s}^{\gamma-1}\,d\bar{s} = -\Omega[\hat{R}\delta(1+(\beta-1)[1-\bar{s}])]\,d\tau - \Omega[(1-\tau)\hat{R}\delta(\beta-1)]\,d\bar{s}\ .$$

Because (A4) holds, we can divide through with this equation and rearrange to see that $d\bar{s}/d\tau < 0 \Leftrightarrow$ (20).

Proof of Theorem 4 and the Corollary in Appendix B. We first prove the theorem: Consider the greatest fixed point $k_L^{\theta^*}$ given some $\theta^* \in \Theta$. To simplify notation, we take $\Theta \subseteq \mathbb{R}$ (but the argument is true in general). Since $m_L^\theta(k_L^{\theta^*}) \geq m_L^{\theta^*}(k_L^{\theta^*}) = k_L^{\theta^*}$ for $\theta^* + \epsilon > \theta > \theta^*$, $m_L^\theta(\cdot)$ begins above the 45° line and ends below it on the interval $[k_L^{\theta^*}, \sup K]$. Since $\mathcal{M}$ has convex values, $\mathcal{M}^\theta(\cdot)$ therefore has a fixed point on this interval, and so $k_L^\theta \geq k_L^{\theta^*}$. This argument clearly extends to any $\theta > \theta^*$ (not necessarily in a neighborhood) since we may reach any such θ in a finite number of steps (Θ is compact so any open cover contains a finite subcover). The more difficult case is when $\theta^* - \epsilon < \theta < \theta^*$. Assume for a contradiction that $k_L^\theta > k_L^{\theta^*}$. Consider θ_n, where $\theta < \theta_n < \theta^*$. Since $\theta_n > \theta$, it follows from the first part of the proof that $k_L^{\theta^n} \geq k_L^\theta > k_L^{\theta^*}$. Note that these inequalities hold for any $\theta_n \in (\theta, \theta^*)$. Since K is compact, we may consider a sequence $n = 0, 1, 2, \ldots$ with $\theta_n \uparrow \theta^*$ and such that $\lim_{n\to\infty} k^*(\theta_n)$ exists. $k_L^{\theta_n} \in \mathcal{M}^{\theta_n}(k_L^{\theta_n})$ for all n and $\mathcal{M}$ has a closed graph, hence $\lim_{n\to\infty} k_L^{\theta_n} \in \mathcal{M}^{\theta^*}(\lim_{n\to\infty} k_L^{\theta_n})$. But since $\lim_{n\to\infty}\lim_{n\to\infty} k_L^{\theta_n} \geq k_L^\theta > k_L^{\theta^*}$, this contradicts that $k_L^{\theta^*}$ is the greatest fixed point. The parallel statement for the least fixed point $k_S^{\theta^*}$ is shown by a dual argument (in this case the situation where $\theta^* - \epsilon < \theta < \theta^*$ is simple while the limit sequence argument must be used for the case where $\theta^* + \epsilon > \theta > \theta^*$).

The following corollary is immediate by combining Theorem 4 with Lemma 4, as we did in the proof of Theorem 1:

Corollary 1 ***(Main Comparative Statics Result, Topological Case)*** *Let the assumptions of Theorem 1 hold and assume in addition that Θ is a compact subset of an ordered topological space and that the market correspondence $\mathcal{M}^{\theta}(k)$ is upper hemi-continuous in (θ, k). Then the greatest and least steady states k_S^{θ} and k_L^{θ} are increasing in θ if for all $\theta^* \in \Theta$ and all $\theta_a < \theta_b$ in a neighborhood of θ^*, the change in the environment from θ_a to θ_b raises mean savings at $k_L^{\theta^*}$ as well as at $k_S^{\theta^*}$.*

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