Determination of the Spherical Couple-Stress in Polar Linear Isotropic Elasticity

K.P. Soldatos[1]

Abstract
It is well known that the conventional couple-stress theory leaves the spherical part of the couple-stress indeterminate. This indeterminacy problem is recently resolved for fibrous composites subjected to either small or large deformations and containing a single family of fibres resistant in bending (Soldatos in Math. Mech. Solids, 2021, https://doi.org/10.1177/10812865211061595). However, the problem remains still unsolved in simpler cases where the implied preference material direction is not related to fibre bending resistance, and even in the simplest possible case where the polar material of interest is linearly elastic and isotropic. This communication aims (i) to show that a relevant virtual spin concept employed (Soldatos in Math. Mech. Solids, 2021, https://doi.org/10.1177/10812865211061595) is further applicable in the latter case of polar linear isotropic elasticity, (ii) to demonstrate the process in which that concept thus leads to determination of the spherical part of the couple-stress, (iii) to exemplify this process by providing a couple of simple illustrative examples, (iv) to specify and discuss the reason that the outlined method meets a hurdle in cases of linear anisotropic elasticity that is due to one or more preferential material directions, and, hence, (v) to further discuss the manner in which that newly identified difficulty is currently confronted, and may thus be handled successfully.

Keywords Couple-stress theory · Elasticity · Generalised Cosserat theory · Isotropic elasticity · Polar elasticity · Spherical part of the couple-stress

Mathematics Subject Classification 74A10 · 74A35 · 74A40 · 74A60 · 74B99 · 74E10

1 Introduction

The conventional couple-stress theory as its origin in the 1909 publication of the Cosserat brothers [1] (see also [2]). It took, however, a little more than five decades to become openly understood that the theory leaves the spherical part of the couple-stress indeterminate [3, 4] and may, therefore, be somehow incomplete. By mentioning that the relevant analysis presented in part I of Reference [4] (pp. 17-29) had been developed between 1949 and 1950,

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Koiter implied that he essentially knew of that drawback of the theory more than a decade before [3] and [4] were published. Koiter may therefore have been the first who recorded that drawback, at least privately, and he has certainly been the first who claimed [4] that only five of the six boundary conditions required for the formulation of a relevant, well-posed boundary value problem are independent.

On its own, the latter result may now give rise to the feeling that the remaining, sixth boundary condition may somehow be related to action of the indeterminate spherical part of the couple-stress (termed, for brevity, spherical couple-stress in what follows). Nevertheless, Koiter [4] pointed also out, perhaps not unreasonably at the time, that "the reason for this indeterminacy is that an arbitrary distribution of the first invariant of the couple-stress tensor" (that is, the spherical couple-stress) "and the associated skew-symmetric force-stress tensor" (that is, the antisymmetric part of the stress tensor) "do not work in any velocity distribution over the body". He thus concluded [4] that, "without any physical loss of generality, we may henceforward put the invariant" (namely, the spherical couple-stress) "equal to zero."

It will be seen in what follows that the latter claim is an eventually incomplete conclusion that is reached, not unreasonably, due to apparent lack of complete relevant information. Still though, the truth of the matter is that the mentioned, now well-known indeterminacy drawback of the conventional couple-stress theory remains still unresolved, despite that it has since become an issue of considerable controversy and debate (e.g., [5–12] and references therein). Moreover, that drawback has re-emerged, more recently, in the development of a relevant hyperelasticity theory of fibre-reinforced solids with embedded fibres that resist bending [13]. Like their conventional couple-stress theory counterpart [1–4], both the linear and the non-linear versions of the polar elasticity theory presented in [13] leave the spherical couple-stress indeterminate as well.

It is recalled in this context, that Koiter [4] paid exclusive attention to a version of the conventional couple-stress theory that is adequate for studying the behaviour of linearly elastic isotropic solids subjected to infinitesimally small deformations; Mindlin and Tiersten [3] had earlier done the same, though not exclusively, to a considerable extent. In contrast, the theoretical framework developed in [13] was principally interested to model large elastic deformations of polar fibre-reinforced and, therefore, anisotropic material behaviour. However, the linearised version of that anisotropic material framework [13] is applicable also within the small deformation regime only, provided that the implied anisotropic features are due to the presence of fibres resistant in bending (see also [14]).

It though further happened that, unlike its linearly elastic isotropic counterpart [1–4], the linear version of the theory developed in [13] revealed that the energy stored in a fibrous elastic solid with fibre-bending stiffness contains a couple-stress related term that does not contribute into the couple-stress constitutive equation. A feeling thus emerges that the appearance of that extra energy term relates to elastic energy contribution that is stored in the material through action of the indeterminate spherical couple-stress. That feeling has recently led to a refined formulation of [13] that, finally, does enable determination of the spherical couple-stress [15].

The present study is driven by the thought that a similar, "hidden" strain energy term may also be present in the case of the isotropic polar linear elasticity that References [3] and [4] were principally interested on. If present, such a term must then enter the strain energy function in a manner that is not directly detectable through the conventional equilibrium considerations. As happens in the case of polar fibre-reinforced materials [15], such an energy term must instead represent work produced through interaction of the spherical couple-stress with the gradient of some auxiliary, virtual spin vector that differs from the standard, displacement generated spin vector employed in the conventional theory. The principal purpose

of the present communication thus is (i) to show that the implied, new virtual spin concept [15] is indeed applicable to problems of polar linear isotropic elasticity, (ii) to demonstrate the process that this concept leads to identification of the hidden or missing strain energy term and, hence, to the determination of the spherical couple-stress, and (iii) to exemplify the proposed theoretical development by successfully applying it on the relatively simple boundary value problems considered previously in [4].

In this context, Sect. 2 outlines the basic theoretical background required for the presentation of the implied theoretical development. This background includes the standard kinematic and equilibrium considerations met in conventional couple-stress theory, it, further, introduces an auxiliary spin vector field that replaces its standard, displacement generated counterpart [1–4] and, thus, leads to the development of a relevant refined theoretical formulation. Section 3 then appropriately refines the energy principles and relevant considerations employed in the conventional theory and arrives to a necessary additional condition that, as happens in [15], may lead to the determination of the spherical couple-stress.

Section 4 next proceeds to develop the method that enables the outlined theoretical refinement to determine the spherical couple-stress in the case of the constitutional setting of current interest, namely the material setting of linear isotropic elasticity. Section 5 then revisits the fundamental problems of polar linear isotropy considered in [4], namely the torsion of a circular cylindrical bar and the pure bending problem of a rectangular plate and, in each case, demonstrates the manner that the spherical couple-stress is determined. Section 6 summarises the main conclusions drawn from the presented analysis and comments on the nature of a newly identified operational difficulty met in the case of material anisotropy. Moreover, that concluding Section discusses the way that this difficulty is currently dealt with, and thus highlights directions that would enable more successful handling of the same.

2 Theoretical Foundation

Most of the principal equations and associated concepts of the refined, linear couple-stress theory of interest are essentially available in [16, 17] and, for self-consistency of the present article, are quoted in this section without detailed attention to their derivation. These naturally begin with the standard decomposition,

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \tag{2.1}$$

of the components of the non-symmetric stress tensor, $\boldsymbol{\sigma}$, into symmetric and antisymmetric parts, where indices refer to a suitable three-dimensional Cartesian co-ordinate framework, Ox_i, and thus take the values 1, 2 and 3.

By neglecting body forces and body couples for simplicity, the standard equations of equilibrium are written as follows:

$$\sigma_{ij,i} = 0, \qquad \varepsilon_{kji}\sigma_{ji} + m_{\ell k,\ell} = 0, \tag{2.2}$$

where $\boldsymbol{m}$ and $\boldsymbol{\varepsilon}$ denote the couple-stress tensor and the alternating tensor, respectively. Nevertheless, with use of (2.1) and appropriate exploitation of the alternating tensor properties, (2.2b) converts into the following:

$$\sigma_{[ij]} = \frac{1}{2}\varepsilon_{kji}m_{\ell k,\ell}, \tag{2.3}$$

and may thus be perceived as a constitutive equation for the antisymmetric part of the stress.

Provided that the couple-stresses are continuous and at least twice differentiable functions, further use of (2.1) enables the reduced pair of equilibrium equations (2.2a) and (2.3) to contract into the single equation

$$\sigma_{(ij),i} + \frac{1}{2}\varepsilon_{kji}\bar{m}_{\ell k,\ell i} = 0, \tag{2.4}$$

where the appearance of the deviatoric couple-stress tensor

$$\bar{m}_{\ell k} = m_{\ell k} - \frac{1}{3}m_{rr}\delta_{\ell k}, \tag{2.5}$$

reveals that standard equilibrium considerations are not influenced by the spherical couple-stress, m_{rr}; the appearing Kronecker's delta represents the components of the unit matrix.

In the usual manner, the components of the traction and the couple-traction vectors acting on any internal or bounding surface of the material are respectively given as follows:

$$T_i^{(n)} = \sigma_{ji}n_j, \qquad L_i^{(n)} = m_{ji}n_j, \tag{2.6}$$

where n denotes the outward unit normal of that surface. It is recalled in passing that Koiter [4] have shown that, in dealing with the bounding surface of a polar material, only two of the three boundary conditions (2.6b) can be set independently of the deformation.

The total energy stored within an arbitrary volume, V, of such a polar elastic material is

$$E = \frac{1}{2}\int_S \left(T_i^{(n)}u_i + L_i^{(n)}\Phi_i\right)dS \geq 0, \tag{2.7}$$

where S denotes the surface that surrounds V, dS stands for the corresponding surface element, and the equality sign holds only in the absence of deformation. Moreover, $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{\Phi}$ represents some unspecified, auxiliary spin-type vector. Although unspecified, that vector is considered dependent on the deformation field and relates to a corresponding, antisymmetric, auxiliary rotation tensor, φ, through the standard relationships

$$\Phi_i = \frac{1}{2}\varepsilon_{ijk}\varphi_{kj}, \quad \varphi_{kj} = \varepsilon_{ijk}\Phi_i. \tag{2.8}$$

The conventional couple-stress theory [1–4] may thus be obtained as a special case of this consideration, by setting

$$\varphi_{kj} \equiv \omega_{kj} = \frac{1}{2}\left(u_{k,j} - u_{j,k}\right), \qquad \Phi_i \equiv \Omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj}, \tag{2.9}$$

where the implied antisymmetric part, $\boldsymbol{\omega}$, of the displacement gradient represents the standard, actual rotation field of the deformation. In the usual manner, the symmetric part of the displacement gradient, namely

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.10}$$

represents the corresponding infinitesimal strain field.

It can readily be verified that

$$\Omega_{i,i} = 0, \qquad (2.11)$$

and, for reasons explained in what follows, this identity makes the choice (2.9) a special case of secondary interest. In this context, it is generally considered that

$$\Phi_i \neq \alpha Q_{ij}\Omega_j \quad \implies \quad \Phi_{i,i} \neq \alpha Q_{ij}\Omega_{j,i} = \alpha \Omega_{i,i} = 0, \qquad (2.12)$$

where α is a non-zero real constant, and $\boldsymbol{Q}$ is a constant orthogonal matrix that represents an arbitrary rotation/reflection of the co-ordinate system.

It is reemphasised that $\boldsymbol{\Phi}$ or, equivalently, $\boldsymbol{\varphi}$ is considered related with the displacement field and its gradient, but the implied relevance is unknown and, therefore, unspecified. In this context, $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$ are essentially regarded as virtual spin and rotation fields. In contrast, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are actual relevant fields that may be identified uniquely, through the solution of the outlined governing equations as soon as the constitutive equations of the material are specified.

3 Elastic Energy Considerations

3.1 Conventional Couple-Stress Theory

It is now recalled that, in the conventional couple-stress theory, the energy stored internally in a polar, linearly elastic material attains the form (e.g., [3, 4])

$$W\left(e_{ij}, \Omega_{i,j}\right) = W^e\left(e_{ij}\right) + W^\Omega\left(\Omega_{i,j}\right), \qquad (3.1)$$

where the positive semidefinite quantities

$$W^e\left(e_{ij}\right) = \frac{1}{2}\sigma_{(ji)}e_{ij} \geq 0,$$
$$W^\Omega\left(\Omega_{i,j}\right) = \frac{1}{2}m_{\ell i}\Omega_{i,\ell} = \frac{1}{2}\bar{m}_{\ell i}\Omega_{i,\ell} \geq 0, \qquad (3.2)$$

are quadratic in their arguments and represent, respectively, the strain energy function met in non-polar linear elasticity and its spin-gradient counterpart that is here due to the assumed polar material response.

It is emphasised that, in the derivation of (3.2b), use is also made of (2.5). The spherical couple-stress, m_{rr}, thus fails to record its contribution into the polar part, W^Ω, of the total strain energy function, due to its subsequent reciprocation with the divergence (2.11) of the actual, displacement generated spin vector, $\Omega_{i,i}$. As a result, m_{rr} emerges as an indeterminate quantity, because it fails to record a contribution not only in the equilibrium equations (2.4), but also in the strain energy function (3.1); and subsequently in the constitutive equations,

$$\sigma_{(ij)} = \frac{\partial W^e}{\partial e_{ij}}, \qquad \bar{m}_{ji} = \frac{\partial W^\Omega}{\partial \Omega_{i,j}}, \qquad (3.3)$$

employed in the conventional theory.

It may be claimed at this point that a third energy term that potentially involves mixed products of displacement and spin gradients may appear in the right-hand side of (3.1). Such an energy term had indeed initially been included in, and subsequently dropped from either version of conventional polar linear elasticity presented in [3] and [4]. Details that justify rejection of that energy term are provided in both [3, 4], and the provided reasoning is here considered still valid.

Under these considerations, a revisit now becomes necessary of the concept of the displacement-gradient energy function introduced in [16]. Namely,

$$U\left(e_{ij}, \omega_{ij}\right) = \frac{1}{2}\sigma_{ji}u_{i,j} = \frac{1}{2}\left(\sigma_{(ji)} + \sigma_{[ji]}\right)\left(e_{ij} + \omega_{ij}\right) = \frac{1}{2}\left(\sigma_{(ij)}e_{ij} + \sigma_{[ji]}\omega_{ij}\right)$$
$$= W^e\left(e_{ij}\right) + W^\omega\left(\omega_{ij}\right), \tag{3.4}$$

where the appearing rotation energy that is apparently stored in the material is

$$W^\omega\left(\omega_{ij}\right) = \frac{1}{2}\sigma_{[ji]}\omega_{ij} = \frac{1}{4}\varepsilon_{kij}m_{\ell k,\ell}\omega_{ij} = -\frac{1}{2}\Omega_k m_{\ell k,\ell} = \frac{1}{2}\Omega_{k,\ell}m_{\ell k} - \frac{1}{2}\left(\Omega_k m_{\ell k}\right)_{,\ell}. \tag{3.5}$$

It is initially observed that, like its strain energy counterpart (3.1), the displacement-gradient energy (3.4) splits, naturally, into two quadratic terms. The first term still coincides with the strain energy function met in non-polar linear elasticity while, within the afore-mentioned reasoning detailed in [3, 4], the rotation generated part (3.5) is regarded relevant to purely polar material response.

It is further observed that the displacement gradient energy (3.4) and the strain energy (3.1) naturally coincide in the case of non-polar linear elasticity, where $W^\Omega = W^\omega = 0$. However, these expressions now differ to each other, due to the different positive values attained by the appearing spin- and rotation-generated energies. Indeed, a comparison of (3.2b) and (3.5) yields the energy difference

$$W^m = W^\Omega\left(\Omega_{i,j}\right) - W^\omega\left(\omega_{ij}\right) = \frac{1}{2}\left(m_{\ell k}\Omega_k\right)_{,\ell}. \tag{3.6}$$

Moreover, (3.5) suggests that W^ω is an amount of stored energy that takes into consideration the action of the full couple-stress tensor, $\boldsymbol{m}$, while (3.2b) indicates that W^Ω accounts only for the deviatoric part, $\bar{\boldsymbol{m}}$, of the same. The latter observation makes it understood that the difference (3.6) of those polar-response parts of the stored energy relates to spherical couple-stress action.

It is finally observed that integration of (3.6) over an arbitrary volume, V, of the material, followed by application of the divergence theorem and a subsequent use of (2.6b), yields

$$\int_V W^m dV = \frac{1}{2}\int_V \left(m_{\ell k}\Omega_k\right)_{,\ell}dV = \frac{1}{2}\int_S m_{\ell k}\Omega_k n_\ell dS = \frac{1}{2}\int_S L_k^{(n)}\Omega_k dS. \tag{3.7}$$

This result reveals that the contribution of W^m into the total elastic energy stored in an arbitrary volume of the material equals one half of the total work done through the interaction of the set of couple-tractions, $L_i^{(n)}$, acting on the bounding surface of V with the actual spin vector, Ω_i.

3.2 Refined Couple-Stress Theory

Validity of (2.12) enables the proposed, refined couple-stress theory to employ a generalised polar part of the strain energy function,

$$W^{\Phi}\left(\Phi_{i,j}\right) = \frac{1}{2} m_{\ell i} \Phi_{i,\ell} = \frac{1}{2}\left(\frac{1}{3} m_{ii} \Phi_{\ell,\ell} + \bar{m}_{\ell i} \Phi_{i,\ell}\right), \tag{3.8}$$

that can explicitly record the stored energy contribution of the spherical couple-stress, m_{ii}.

The relevant constitutive equation,

$$m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}}, \tag{3.9}$$

then implies that the expression

$$m_{rr} = \frac{\partial W^{\Phi}}{\partial \Phi_{r,r}} \tag{3.10}$$

now acquires mathematical sense and, by thus enabling the spherical couple-stress to be treated as a potentially measurable quantity, paves a route to its determination.

On the other hand, introduction of (2.6) into (2.7), followed by application of the divergence theorem and a relevant process detailed in [16, 17], reveals that the total elastic energy stored in the material attains the form

$$E = \int_{V} \left(W^{e} + W^{\Phi}\right) dV = \int_{V} W dV, \tag{3.11}$$

where the polar part (3.8) of the appearing strain energy necessarily satisfies the condition

$$W^{\Phi} = \frac{1}{2}\left[m_{\ell i,\ell}\left(\Phi_{i} - \Omega_{i}\right) + m_{\ell i}\Phi_{i,\ell}\right] = \frac{1}{2}\left\{[m_{\ell i}\left(\Phi_{i} - \Omega_{i}\right)]_{,\ell} + m_{\ell i}\Omega_{i,\ell}\right\} \geq 0. \tag{3.12}$$

A comparison of the right-hand sides of (3.12) with (3.2b) thus yields

$$W^{\Phi} - W^{\Omega} = \frac{1}{2}\left[m_{\ell i}\left(\Phi_{i} - \Omega_{i}\right)\right]_{,\ell}, \tag{3.13}$$

which holds for any suitable auxiliary spin vector $\boldsymbol{\Phi}$. Evidently, by ignoring (2.12), one may select $\boldsymbol{\Phi}$ to coincide with $\boldsymbol{\Omega}$ in the sense implied by (2.9b), and, hence, enable (3.13) to trivially yield

$$W^{\Phi} = W^{\Omega}. \tag{3.14}$$

It is already mentioned though, that (2.9b) represents a case of secondary interest, in which the conventional couple-stress theory essentially forces the strain energy expressions (3.2) and (3.8) to become identical. Nevertheless, alternative vector values of $\boldsymbol{\Phi}$ that (i) are consistent with (2.9b) and (ii) may lead to forms of W^{Φ} that still satisfy (3.14) may become available through satisfaction of the condition

$$(m_{\ell i}\Phi_{i})_{,\ell} = (m_{\ell i}\Omega_{i})_{,\ell} = 2W^{m}, \tag{3.15}$$

where (3.6) is also accounted for.

As a matter of fact, (3.15) is essentially an extra equation that emerges at this point as an enhanced form of (3.6). It is noted with profound interest that (3.15) is practically identical with the corresponding condition noted as Eq. (23) in [15], namely with the condition that made compatible and therefore complementary the non-linear hyperelasticity analyses detailed in [13] and [14].

Hence, as is also pointed out in [13], the middle side of (3.15) represents energy per unit volume due to interaction of the couple-stress tensor with the actual, displacement generated spin vector, while the left-hand side is an equal amount of energy that is due to interaction of the couple-stress with an auxiliary spin vector. Integration of (3.15) over an arbitrary volume, V, of the material, followed by application of the divergence theorem and a subsequent use of (2.6), thus yields

$$\int_S L_i^{(n)} \Phi_i dS = \int_S L_i^{(n)} \Omega_i dS = 2 \int_V W^m dV, \qquad (3.16)$$

which is naturally in agreement with (3.7).

It is now recalled that the actual spin vector, $\boldsymbol{\Omega}$, as well as the deviatoric part of couple-stress, $\bar{\boldsymbol{m}}$, can become known from the solution of the conventional theory equations. Hence, in the light of the outlined analysis and relevant observations detailed in [15], a combination of (3.15) with (2.5) yields

$$\Omega_\ell m_{rr,\ell} = 6W^m - 3\left(\bar{m}_{\ell i}\Omega_i\right)_{,\ell}. \qquad (3.17)$$

This is then regarded as a partial differential equation (PDE), which may be solved for the spherical couple-stress as soon as the energy term W^m is specified. Potential determination of W^m will thus become a main issue of priority in the subsequent Section.

It is also pointed out that the remaining, unused part of the equation (3.15) is insufficient for unique determination of all three components of the spin vector Φ_i. Hence, since there exists an infinite number of such auxiliary vectors, $\boldsymbol{\Phi}$ remains unspecified and thus is essentially recognised as a virtual spin vector.

It is further emphasised that, validity of (3.15) and, therefore, (3.14) implies that the alteration of the principal spin vector employed in the refined couple-stress theory, from $\boldsymbol{\Omega}$ to $\boldsymbol{\Phi}$, leaves unaltered the value of the spin-gradient part of the elastic energy stored in the material. Hence, (3.1) may be re-rewritten in the following augmented form:

$$W\left(e_{ij}, \Omega_{i,j}\right) = W^e\left(e_{ij}\right) + W^\Omega\left(\Omega_{i,j}\right) = W^e\left(e_{ij}\right) + W^\Phi\left(\Phi_{i,j}\right). \qquad (3.18)$$

As is detailed in the subsequent Section, explicit expressions of W^Φ and W^Ω may be sought and, thus, be associated with the outlined analysis only after the material symmetries of the linearly elastic polar solid of interest are specified. Nevertheless, (3.14) or, equivalently, (3.18) will still hold for the implied vector values of $\boldsymbol{\Phi}$, but expressions (3.8) and (3.2b) of the corresponding polar parts of the strain energy function will remain different. It follows that (3.8) will still contain the additional piece of information that (3.10) associates with the value of the spherical couple-stress.

A more comprehensive view of (3.12) suggests that, while the extra energy contribution W^m enters the expression of the virtual strain energy function through its association with the virtual spin field, $\boldsymbol{\Phi}$, that contribution remains ultimately unrecorded in the value of W^Φ because it is essentially counteracted by an equal amount of energy that is taken off through equivalent association with the actual spin field, $\boldsymbol{\Omega}$. The effect of this observation passes naturally undetectable in the special case of the conventional theory, where validity of (2.9b) thus leads to the controversial impression [4] that the strain energy function is completely uninfluenced by the action of the spherical couple-stress.

4 Constitution – Material Symmetry Considerations

Since the first term, W^e, in the right-hand side of (3.18) is identical with its non-polar material counterpart, no further direct attention will be paid to either that term or to the symmetric part of the stress. It will instead be assumed that W^e naturally conforms with whichever material symmetries are assumed in the material of interest and, hence, that the symmetric part of the stress also does so through subsequent use of (3.3a).

Since (3.9) represents the couple-stress part of the refined constitutive equations, it is initially observed that, unlike its small strain counterpart employed in (3.3a), the appearing spin gradient $\Phi_{i,j}$ is a nonsymmetric tensorial quantity. The search for corresponding forms of W^Φ that remain invariant under rigid body rotation thus requires the split of $\Phi_{i,j}$ into symmetric and antisymmetric parts

$$\Phi_{(i,j)} = \frac{1}{2}\left(\Phi_{i,j} + \Phi_{j,i}\right), \qquad \Phi_{[i,j]} = \frac{1}{2}\left(\Phi_{i,j} - \Phi_{j,i}\right), \tag{4.1}$$

respectively.

W^Φ is then required to be an isotropic invariant function of these quantities and, therefore, must be a function of a corresponding complete set of relevant invariants that represent the material symmetry group of the polar elastic solid of interest. In the context of linear elasticity though, W^Φ is also required to be quadratic in its arguments. Hence, relevant invariants of order higher than two need not be listed in that set.

4.1 Polar Isotropic Materials

The class of linearly elastic polar isotropic materials received exclusive attention in Reference [4], as well as substantial attention in Reference [3], and, thus, has naturally dominated research interest in subsequent relevant investigations. Within this material class, the required list of first- and second-order invariants of the tensor quantities defined in (4.1) is as follows (e.g., [18]):

$$I_1 = \Phi_{(i,i)} \equiv \Phi_{i,i}, \qquad I_2 = \Phi_{(i,j)}\Phi_{(j,i)}, \qquad I_3 = \Phi_{[i,j]}\Phi_{[j,i]}. \tag{4.2}$$

The most general, quadratic form of the relevant polar part of the strain energy function thus is

$$W^\Phi\left(\Phi_{m,n}\right) = \frac{1}{2}\left(\eta_0 I_1^2 + \eta_1 I_2 + \eta_2 I_3\right) = \frac{1}{2}\left[\eta_0\left(\Phi_{m,m}\right)^2 + \eta_1\Phi_{(m,n)}\Phi_{(n,m)} + \eta_2\Phi_{[m,n]}\Phi_{[n,m]}\right], \tag{4.3}$$

where η_0, η_1 and η_2 are appropriate material moduli having dimensions of force. The inequality noted in (3.12) is then satisfied regardless of the deformation only for positive values of these moduli.

With use of (4.3), (3.9) yields the couple-stress constitutive equation as follows:

$$m_{ji} = \frac{\partial W^\Phi}{\partial \Phi_{i,j}} = \eta_0\Phi_{m,m}\delta_{ij} + 2\eta_1\Phi_{(i,j)} + 2\eta_2\Phi_{[j,i]}$$
$$= \eta_0\Phi_{m,m}\delta_{ij} + (\eta_1 + \eta_2)\Phi_{j,i} + (\eta_1 - \eta_2)\Phi_{i,j}. \tag{4.4}$$

Hence, by contracting the appearing free indices, one obtains

$$m_{rr} = \frac{\partial W^\Phi}{\partial \Phi_{r,r}} = (3\eta_0 + 2\eta_1)\Phi_{r,r} \neq 0, \tag{4.5}$$

where (2.12) is also taken into consideration.

In the special case of conventional polar elasticity, where (2.12) is dismissed and (2.9) thus holds, (4.3) reduces to

$$W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}\left(\eta_1\Omega_{(i,j)}\Omega_{(j,i)} + \eta_2\Omega_{[i,j]}\Omega_{[j,i]}\right), \qquad (4.6)$$

where the definition of the appearing symmetric and antisymmetric parts of the actual spin vector, $\boldsymbol{\Omega}$, are analogous to their (4.1) counterparts. It is noted in this regard that the pair of elastic moduli η_1 and η_2, is equivalent to either of its counterpart pair employed in [3] and [4].

These considerations make it understood that $\eta_0\left(\Phi_{m,m}\right)^2/2$ emerges in (4.3) as an extra energy term or, more precisely, as an energy term that necessarily escapes the attention of and, therefore, becomes unaccountable by the conventional couple-stress theory. This is because validity of the identity (2.11) prevents (4.6) from explicitly recording the existence of such an energy contribution in this special case.

The additional modulus appearing in (4.3), namely η_0, is here associated with energy contribution that is clearly due to interaction of the virtual spin divergence with the spherical couple-stress. In this context, (4.4) and (4.5) dispute Koiter's early conclusion [4] (see also Introduction), which suggests that the spherical couple-stress does not produce work and, without loss of generality, may thus be put equal to zero.

It is worth further noting that upon dismissing (2.12) and, hence, directing attention to the special case of the conventional couple-stress theory, (4.4) reduces to

$$m_{ji} = \frac{\partial W^{\Omega}}{\partial \Omega_{i,j}} = 2\eta_1\Omega_{(i,j)} + 2\eta_2\Omega_{[j,i]} = (\eta_1 + \eta_2)\,\Omega_{j,i} + (\eta_1 - \eta_2)\,\Omega_{i,j}. \qquad (4.7)$$

In a hypothetical situation that (2.11) was invalid, and a contraction of the appearing free indices thus was allowed, this expression would lead to

$$m_{rr} = 0, \qquad (4.8)$$

which, to a certain extent, would justify Koiter's early, essentially incomplete relevant conclusion. Most interestingly, a combination of (4.8) with (2.5) would then enable (4.7) to reduce to

$$\bar{m}_{ji} = \frac{\partial W^{\Omega}}{\partial \Omega_{i,j}} = 2\eta_1\Omega_{(i,j)} + 2\eta_2\Omega_{[j,i]} = (\eta_1 + \eta_2)\,\Omega_{j,i} + (\eta_1 - \eta_2)\,\Omega_{i,j}, \qquad (4.9)$$

which is clearly in agreement with the constitutive equation (3.3b) employed in the conventional theory. In this manner, as soon as the condition (2.12) would be removed, the conventional couple-stress theory might emerge, along with Koiter's conclusion, as a special case of the presented, more general theoretical formulation.

However, validity of the identity (2.11) suggests that the free indices contraction that might have led to led from (4.4) to (4.8) is mathematically incompatible with the middle term of (4.7). In this regard, the conventional couple-stress theory or, equivalently, (2.9) represents an exceptional, essentially singular case. In that case, (3.10) becomes invalid and, for the well-known reasons adopted in the conventional theory [3, 4], (3.9) is considered directly reducible to (3.3b).

Under these considerations, it is revealing to further note that a combination of (4.4) with (2.5) yields the deviatoric part of the couple-stress as follows:

$$\bar{m}_{ji} = 2\eta_1 \left(\Phi_{(i,j)} - \frac{1}{3}\Phi_{r,r}\delta_{ij} \right) + 2\eta_2 \Phi_{[j,i]} = -\frac{2\eta_1}{3}\Phi_{m,m}\delta_{ij} + (\eta_1 + \eta_2)\,\Phi_{j,i} + (\eta_1 - \eta_2)\,\Phi_{i,j}. \tag{4.10}$$

While the divergence, $\Phi_{m,m}$, of the virtual spin vector does mark its contribution into this expression, the additional elastic modulus that is relevant to the energy contribution of the spherical couple-stress, namely η_0, does not do so. Expression (4.10) then naturally reduces to its conventional theory counterpart (4.9) as soon as the actual spin vector (2.9b) is allowed to replace its appearing virtual spin counterpart.

While the conventional rotation/spin choice (2.9) thus represents an exceptional, essentially singular case, all the information that the conventional couple-stress theory makes available through the solution of a relevant, well-posed boundary value problem is still valid. As is already mentioned, this information includes the ultimate forms of the displacement vector, the small strain tensor, and the actual spin vector, $\boldsymbol{\Omega}$, as well as the relevant stress and the deviatoric couple-stress fields.

Hence, by further requiring from the unaccounted energy part, $\eta_0 \left(\Phi_{m,m} \right)^2 /2$, to balance the unrecorded energy contribution (3.15), the present analysis enables the additional equation (3.17) to obtain the more specific form

$$\Omega_\ell m_{rr,\ell} = 3\eta_0 \left(\Phi_{m,m} \right)^2 - 3\left(\bar{m}_{\ell i}\Omega_i \right)_{,\ell}, \tag{4.11}$$

which evidently holds outside the conventional equilibrium framework. A combination with (4.5) then eliminates the appearing, unknown, virtual spin divergence and leads to the following, first-order, non-linear PDE:

$$\Omega_\ell m_{rr,\ell} = \eta m_{rr}^2 - 3\left(\bar{m}_{\ell i}\Omega_i \right)_{,\ell}, \quad \eta = \frac{3\eta_0}{\left(3\eta_0 + 2\eta_1 \right)^2} > 0, \tag{4.12}$$

for the unknown spherical couple-stress, m_{rr}.

The spherical couple-stress can then be determined as a solutions of this PDE, which may generally be pursued analytically or computationally, subject to some appropriate set of boundary conditions. It is recalled in this context, that, as is shown by Koiter [4], only five of the six couple-traction boundary conditions (2.6) are formally independent in the case of the conventional couple-stress theory. Those independent conditions are necessarily used earlier, for the determination of the deviatoric couple-stress tensor and the actual spin vector appearing in (4.12). Hence, a sixth, unused boundary condition will still remain available to be associated with the solution of the first-order PDE (4.12).

After m_{rr} thus is determined, (4.5) may be reemployed to provide the value of the appearing virtual spin divergence, $\Phi_{m,m}$. Moreover, some alternative form of the couple-stress constitutive equation might be felt desirable or even sought using (4.4) and (4.10). Still though, not enough information appears to be available for unique determination of all three components of the unknown spin vector Φ_i, which thus remains an auxiliary, virtual spin vector that needs not be determined.

Solution of the PDE (4.12) thus generally depends on the boundary value problem of interest and the characteristics of its conventional couple-stress theory solution, including the relevant forms attained by the appearing actual spin vector, $\boldsymbol{\Omega}$, and the deviatoric couple-stress tensor (4.9). Moreover, the non-linear character of this PDE implies that its anticipated solution may be non-unique, while the superposition principle of particular solutions is not

applicable. Under these considerations, a search for a general or any particular solution of (4.12) needs not be pursued at this point. However, a couple of simple relevant applications are presented later, in Sect. 5 below.

Finally, a remarkable similarity is noted between (4.12) and its counterpart that is noted as Eq. (4.16) in [15]. The only difference between these two PDEs is confined into the form of the first term appearing in their right-hand sides. In the hyperelasticity analysis detailed in [15], that energy representing term attains a general form that depends on the influence that a fibre stretching type of invariant, termed there as I_{20}, exerts on the strain energy density. In (4.12) though, the present, linear elasticity formulation naturally requires from such an energy generated term to be quadratic on the unknown spherical couple-stress, m_{rr}.

4.2 Polar Material Anisotropy Due to Presence of Preferential Material Directions

Consider next the class of polar transverse isotropic solids, and denote the unit vector that defines the single relevant direction of material preference with $\boldsymbol{a}$. For this material class, the complete list of invariants required for an adequately general description of W^{Φ} is formed by complementing (4.2) with the following:

$$I_4 = a_i \Phi_{(i,j)} a_j = a_i \Phi_{i,j} a_j, \qquad I_5 = a_i \Phi_{(i,j)} \Phi_{(j,k)} a_k,$$
$$I_6 = a_i \Phi_{[i,j]} \Phi_{[j,k]} a_k, \qquad I_7 = a_i \Phi_{(i,j)} \Phi_{[j,k]} a_k. \tag{4.13}$$

The quadratic form (4.3) of the polar part of the strain energy function thus is augmented accordingly, and takes the form

$$W^{\Phi}\left(\Phi_{i,j}\right) = \frac{1}{2}\left[\eta_0 I_1\left(I_1 + 2\hat{\eta}_0 I_4\right) + \eta_1 I_2 + \eta_2 I_3 + \eta_3 I_4^2 + \eta_4 I_5 + \eta_5 I_6 + 2\eta_6 I_7\right], \tag{4.14}$$

where η_3 to η_6 and the product $\eta_0\hat{\eta}_0$ represent five additional material moduli having dimensions of force. It is noted in this regard that the nondimensional parameter $\hat{\eta}_0$ is essentially associated with strain energy contribution that may be due to interaction of the spherical couple-stress with the invariant I_4.

It is further noted with interest that I_4 represents a stretching type of deformation along the direction of transverse isotropy. In this context, this invariant is completely analogous with its fibre-stretching type of invariant termed as I_{20} in the hyperelasticity analysis detailed in [15].

It is then fitting to also note that, in the case of conventional polar elasticity, (4.14) reduces into the following:

$$W^{\Omega}\left(\Omega_{i,j}\right) = \frac{1}{2}\left[\eta_1 \Omega_{(i,j)}\Omega_{(j,i)} + \eta_2 \Omega_{[i,j]}\Omega_{[j,i]} + \eta_3\left(a_i \Omega_{(i,j)} a_j\right)^2\right.$$
$$\left. + \eta_4 \Omega_{(i,j)}\Omega_{(j,k)} a_k + \eta_5 a_i \Omega_{[i,j]}\Omega_{[j,k]} a_k + 2\eta_6 a_i \Omega_{(i,j)}\Omega_{[j,k]} a_k\right], \tag{4.15}$$

which, like its isotropic material counterpart (4.6), naturally fails to explicitly record any kind of strain energy contributions that may be due to spherical couple-stress action.

In contrast, (4.14) states explicitly that $\eta_0 I_1\left(I_1 + 2\hat{\eta}_0 I_4\right)/2$ does represent such an amount of energy contribution. By thus requiring from the latter, essentially unaccounted energy part to balance the extra energy contribution captured in (3.15), one finds that (4.11) is now replaced by the following:

$$\Omega_\ell m_{rr,\ell} = 3\eta_0 \Phi_{r,r}\left(\delta_{mn} + 2\hat{\eta}_0 a_m a_n\right)\Phi_{(m,n)} - 3\left(\bar{m}_{\ell i}\Omega_i\right)_{,\ell}. \tag{4.16}$$

On the other hand, the transverse isotropic counterpart of the constitutive equation (4.4) is found to be

$$m_{ji} = \frac{\partial W^{\Phi}}{\partial \Phi_{i,j}} = \eta_0 \left(\delta_{ij} + \hat{\eta}_0 a_i a_j \right) \Phi_{m,m} + 2\eta_1 \Phi_{(i,j)} + 2\eta_2 \Phi_{[j,i]} + 2\eta_3 \Phi_{(m,n)} a_m a_n a_i a_j$$
$$+ 2\eta_4 \Phi_{(j,k)} a_i a_k + 2\eta_5 \Phi_{[j,k]} a_i a_k + 2\eta_6 \left(a_i \Phi_{[j,k]} + \Phi_{(k,i)} a_j \right) a_k, \qquad (4.17)$$

and a contraction of the appearing free indices yields now the relationship

$$m_{rr} = \frac{\partial W^{\Phi}}{\partial \Phi_{r,r}} = \left[(3 + \hat{\eta}_0) \eta_0 + 2\eta_1 \right] \Phi_{r,r} + 2 \left(\eta_3 + \eta_4 + \eta_6 \right) a_m a_n \Phi_{(m,n)}. \qquad (4.18)$$

However, the pair of equations (4.16) and (4.18) is algebraically complicated and, unlike their isotropic material counterparts (4.11) and (4.5), does not seem to make feasible a conversion of (4.16) into a single PDE for the unknown spherical couple stress. It follows that, in this case, m_{rr} cannot be determined in the manner already detailed for the case of polar material isotropy.

The analysis becomes even more cumbersome in cases involving advance anisotropy that is due to presence of two or more preferential material directions. As is demonstrated in Appendix A, this is because, in such cases, a considerable number of additional deformation invariants enter and complement their counterparts listed in (4.2) and (4.13). It then becomes understood that determination of the spherical couple-stress in the case of polar material anisotropy requires accessibility to additional information regarding the nature of the virtual spin vector, $\boldsymbol{\Phi}$, or its relationship with the observed elastic deformation.

In this context, References [15] and [17] are referred to as relevant examples in which such a kind of additional information has indeed been made available, though only in problems that polar transverse isotropy is due to the presence of a single family of fibres resistant in bending. For that class of polar fibre-reinforced materials, Reference [17] has accordingly specified $\boldsymbol{\Phi}$ as a spin vector that is relevant to the fibre deformation. Alternatively, Reference [15] employs the concept of fibre bending resistance in a manner that enables the extra energy term, W^m, to be expressed in terms of the second gradient of the displacement vector.

These observations make it evident that further relevant developments are certainly feasible and will continue to appear in the case of polar material anisotropy. The additional illustrating examples presented in the next Section thus refer only to the subclass of polar material isotropy, which has also been the principal subject of interest in [3, 4] and in subsequent, more recent relevant publications.

5 Illustrating Examples

This Section revisits the pair of relatively simple isotropic elasticity problems employed in Koiter's polar elasticity analysis [4] and demonstrates the manner that the presented refined theoretical formulation enables determination of the spherical couple-stress. These are the classical problems of the torsion of a circular cylindrical bar and the pure bending of a rectangular plate, which are of fundamental importance in non-polar linear isotropic elasticity (e.g., [19, 20]).

Since the material is considered isotropic, the symmetric stress constitutive equation (3.3a) attains the standard form of the nonpolar material Hooke's law,

$$\sigma_{(ij)} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}, \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}, \qquad (5.1)$$

where λ and μ is the pair of the Lamé elastic moduli, while E and ν represent the equivalent such pair of the Young's modulus and the Poisson's ratio, respectively.

5.1 Torsion of a Cylindrical Bar with Circular Cross-Section

By considering that the axis of the circular cylindrical bar coincides with the x_1-axis of the co-ordinate system, description of the solution of the non-polar version of this problem may begin with the following displacement field,

$$u_1 = 0, \qquad u_2 = -\tau x_1 x_3, \qquad u_3 = \tau x_1 x_2, \tag{5.2}$$

where τ is a torsion parameter and, as is seen in (5.4) below, τx_1 represents the cross-sectional angle of rotation at a distance x_1 from the origin.

It follows that the only nonzero strain components are

$$e_{12} = -\frac{1}{2}\tau x_3, \qquad e_{13} = \frac{1}{2}\tau x_2, \tag{5.3}$$

the components of the corresponding spin vector are

$$\Omega_1 = \tau x_1, \qquad \Omega_2 = -\frac{1}{2}\tau x_2, \qquad \Omega_3 = -\frac{1}{2}\tau x_3, \tag{5.4}$$

and, hence, only three spin gradients are nonzero, namely

$$\Omega_{1,1} = \tau, \qquad \Omega_{2,2} = \Omega_{3,3} = -\frac{\tau}{2}. \tag{5.5}$$

The constitutive equations (5.1) and (4.9) then reveal that

$$\sigma_{(12)} = -\mu\tau x_3, \qquad \sigma_{(13)} = \mu\tau x_2, \tag{5.6}$$

and

$$\bar{m}_{11} = 2\tau\eta_1, \qquad \bar{m}_{22} = \bar{m}_{33} = -\tau\eta_1, \tag{5.7}$$

are the only nonzero stress and deviatoric couple-stress components, respectively. Since these are all constant, the equilibrium equations (2.4) are satisfied identically.

The last remaining unknown is the spherical couple-stress, m_{rr}, which must be determined by solving (4.12) independently of the equilibrium equations. Use of the outlined results thus enables (4.12) to attain the form of the following PDE for the unknown spherical couple-stress:

$$2\tau x_1 m_{rr,1} - \tau x_2 m_{rr,2} - \tau x_3 m_{rr,3} = 2\eta m_{rr}^2 - 18\eta_1\tau^2. \tag{5.8}$$

The simplest solution of (5.8) is evidently the constant solution

$$m_{rr} = \pm 3\tau\sqrt{\frac{\eta_1}{\eta}} = \pm 3\tau\left(3\eta_0 + 2\eta_1\right)\sqrt{\frac{\eta_1}{3\eta_0}}, \tag{5.9}$$

which, when combined with (2.5) and (5.7), yields

$$m_{11} = \tau\left(\pm\sqrt{\frac{\eta_1}{\eta}} + 2\eta_1\right), \qquad m_{22} = m_{33} = \tau\left(\pm\sqrt{\frac{\eta_1}{\eta}} - \eta_1\right), \tag{5.10}$$
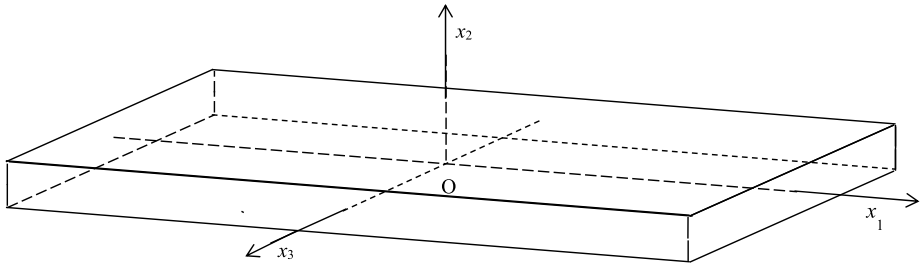
Fig. 1 Schematic representation of a rectangular plate in a suitably selected Cartesian co-ordinate system, Ox_i ($-L_1 \leq x_1 \leq L_1$, $-h/2 \leq x_2 \leq h/2$, $-L_3 \leq x_3 \leq L_3$)

throughout the body of the circular cylindrical bar.

Such a constant couple-stress field can be maintained by the bar only if the normal couple-traction (5.10a) is applied externally on the outer cross-sections and, similarly, the normal couple-stress field (5.10b) is also applied in the radial direction, throughout the bar later surface. Implementation of these boundary conditions thus enable the outlined derivations to extent validity of the well-known non-polar elasticity solution of the bar torsion problem into the polar elasticity regime. Nevertheless, non-constant solutions of the PDE (5.8) may also exist but are not pursued further.

5.2 Pure Bending of a Rectangular Plate

This polar linear elasticity problem is identical to that referred in Ref. [4] as "pure bending of a bar with rectangular cross-section". The problem and its solution are here presented in slightly different, though equivalent forms that resemble their counterparts detailed and solved in [17] for a corresponding transverse isotropic plate containing fibres resistant in bending

Accordingly, Fig. 1 places the isotropic rectangular plate of interest into to the Cartesian co-ordinate system Ox_i, where the plate dimensions are such that $|x_1| \leq L_1$, $|x_2| \leq h/2$ and $|x_3| \leq L_3$. At the edges $x_1 = \pm L_1$, the plate is subjected to the externally applied normal stress distribution

$$\sigma_{11} = \hat{\sigma}_1 x_2, \tag{5.11}$$

where $\hat{\sigma}_1$ is a known positive constant (see Fig. 2).

No other traction is applied externally on any of the six boundary planes. In the non-polar elasticity case, the plate then bends under the action of a pair of terminal couples with magnitude

$$M_3 = \int_{-h/2}^{h/2} \sigma_{11} \big|_{x_1=\pm L_1} x_2 dx_2 = \frac{h^3}{12}\hat{\sigma}_1, \tag{5.12}$$

per unit plate width.

By considering that (5.11) and

$$\sigma_{22} = \sigma_{33} = 0, \qquad \sigma_{12} \equiv \sigma_{(12)} = 0, \qquad \sigma_{13} \equiv \sigma_{(13)} = 0,$$
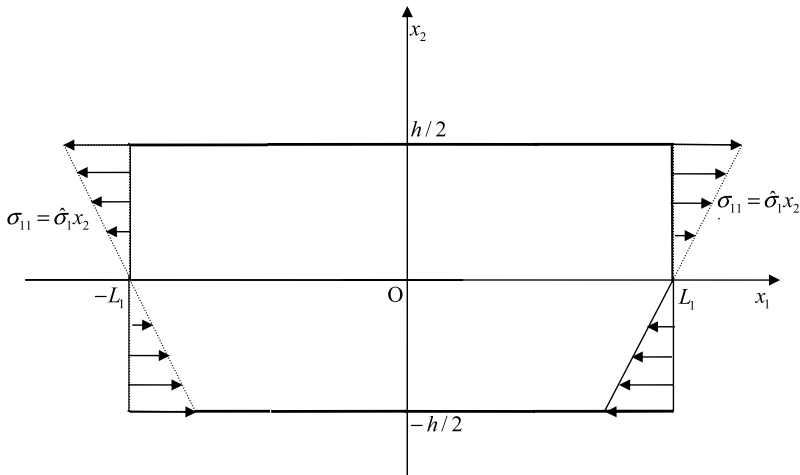$$\sigma_{23} \equiv \sigma_{(23)} = 0, \qquad \sigma_{[ij]} \equiv 0, \tag{5.13}$$

Fig. 2 Schematic representation of a plate cross-section that is normal to the x_3-axis, featuring the boundary traction distributions (5.11) that create pure bending in the case of non-polar linearly elastic material behaviour

represent the stress distribution throughout the body of the plate, the implied all-around traction boundary conditions, and the equilibrium equations (2.2a), are satisfied exactly. Hence, this stress distribution and the corresponding displacement field, namely

$$u_1 = \frac{\hat{\sigma}_1}{E} x_1 x_2, \qquad u_2 = \frac{\hat{\sigma}_1}{2E} \left[\nu \left(x_3^2 - x_2^2 \right) - x_1^2 \right], \qquad u_3 = -\frac{\nu \hat{\sigma}_1}{E} x_3 x_2, \qquad (5.14)$$

form the unique, exact solution of the non-polar version of this boundary value problem ($\boldsymbol{m} \equiv \boldsymbol{0}$).

The quadratic form of the displacement field (5.14) gives rise to the linear spin field

$$\Omega_1 = -\frac{\nu \hat{\sigma}_1}{E} x_3, \qquad \Omega_2 = 0, \qquad \Omega_3 = -\frac{\hat{\sigma}_1}{E} x_1, \qquad (5.15)$$

and since

$$\Omega_{1,3} = \Omega_{3,1} = -\frac{\nu \hat{\sigma}_1}{E} \qquad (5.16)$$

are the only non-zero spin gradients, the constitutive equation (4.9) reveals that, in the corresponding polar elasticity problem, there exist only two nonzero deviatoric couple-stresses,

$$\bar{m}_{13} = -\frac{\hat{\sigma}_1}{E} \left[\eta_1 \left(1 + \nu \right) + \eta_2 \left(\nu - 1 \right) \right] = -\hat{m}_3,$$

$$\bar{m}_{31} = -\frac{\hat{\sigma}_1}{E} \left[\eta_1 \left(1 + \nu \right) + \eta_2 \left(1 - \nu \right) \right] = -\hat{m}_1, \qquad (5.17)$$

and these are also constant. It follows that the equilibrium equations (2.4) are still satisfied identically in the non-polar elasticity case.

Equations (5.11), (5.13), (5.14) and (5.17) thus represent the solution of the described pure bending boundary value problem within the framework of the conventional couple-stress theory, provided that the problem description is augmented to further require from
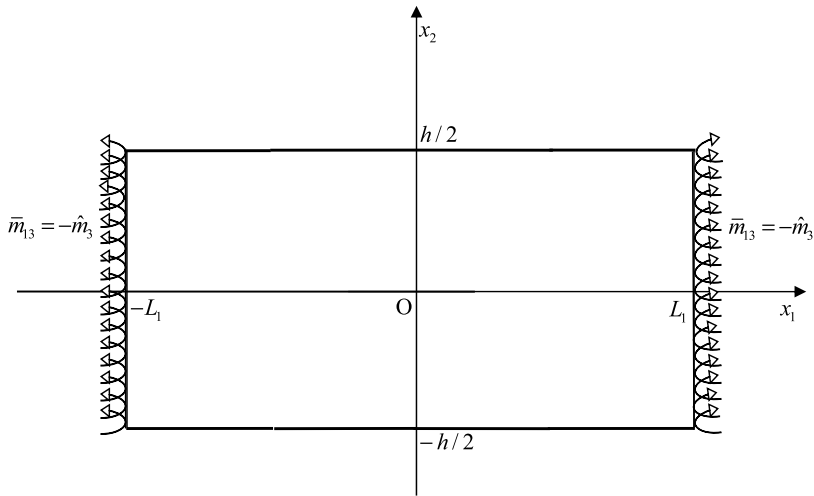
Fig. 3 Schematic representation of the constant couple-traction distributions (5.17a) that must accompany the boundary traction depicted in Fig. 2, for a corresponding polar linearly elastic isotropic plate to maintain the pure bending deformation (5.14). The corresponding effect of the boundary couple-traction (5.17b), which must also accompany (5.11) and (5.17a), is similarly depicted in this Figure, by interchanging everywhere the appearing suffices 1 and 3

the boundary normal traction (5.11) to be accompanied by the constant in-plane boundary couple-tractions (5.17a) and (5.17b). These must be applied externally on the boundaries $x_1 = \pm L_1$ and $x_3 = \pm L_3$, respectively and Fig. 3 depicts their effect on these boundaries in the case that the value of $\hat{m}_3$ or $\hat{m}_1$ is positive.

The spherical couple-stress is then determined by inserting the outlined results into the PDE (4.12), which thus reduces to

$$\nu x_3 m_{rr,1} + x_1 m_{rr,3} + \frac{\eta E}{\hat{\sigma}_1} m_{rr}^2 = 6 \frac{\eta_1 \hat{\sigma}_1}{E} (1 + \nu).$$ (5.18)

If the Poisson ratio attains its exceptional, rather uninteresting, limiting negative value $\nu = -1$, this PDE admits the trivial solution $m_{rr} = 0$. Nevertheless, the non-zero, and more interesting constant solution

$$m_{rr} = \pm \frac{\hat{\sigma}_1}{E} \sqrt{6 (1 + \nu) \frac{\eta_1}{\eta}}$$ (5.19)

is consistent with any other value of the Poisson ratio $(-1 < \nu \le 1/2)$.

A combination of (5.19) with (2.5) then reveals that the normal couple-stresses attain one of the alternative constant values

$$m_{11} = m_{22} = m_{33} = \pm \frac{\hat{\sigma}_1}{3E} \sqrt{6 (1 + \nu) \frac{\eta_1}{\eta}},$$ (5.20)

throughout the body of the plate. The physical meaning of this result is, in fact, that the pure bending deformation (5.14) can be maintained by the polar isotropic plate if, along with the boundary tractions (5.11) and their associated couple-traction counterparts (5.17), either of

the alternative sets (5.20) of normal couple-tractions is externally applied on the appropriate plate boundaries.

Non-constant, spatially variable solution of the PDE (5.18) may also be possible, and one of them is determined in Appendix B with use of the method of characteristics. This, as well as other potential solutions that may be pursued and found analytically or computationally, can be handled in a manner analogous to that described in this Section for the constant solutions (5.9) and (5.19).

6 Further Discussion and Conclusions

When combined with the polar, linearly elastic isotropic analysis detailed in Sect. 4.1, the refined couple-stress theory formulation presented earlier in Sects. 2 and 3 forms a useful and proper extension of its conventional counterpart, which had essentially commenced in [3] and continued in [4]. This is because by reinforcing, rather than dismissing the validity of the conventional couple-stress theory [1–4], the implied theoretical combination succeeds to determine the spherical part of the couple-stress and, thus, resolves the relevant, long-standing indeterminacy problem.

Still though, the search for this refined couple-stress theory formulation, which is based on a virtual than on the actual rotation/spin vector of the deformation, might have not come up without an earlier crucial observation. Namely, the fact that, as is also detailed in the Introduction, a relevant, conventional couple-stress theory formulation that refers to anisotropic composites with embedded fibres resistant in bending [13, 14] revealed the unexpected existence of an extra energy term. This is a couple-stress generated energy term that, like the spherical couple-stress, affects neither the constitutive equations nor the equilibrium of the polar material of interest.

That observation, followed by further relevant studies (e.g., [16] and references therein) led gradually to the search for a refined/generalised couple-stress theory formulation that is analogous to its counterpart detailed in Sects. 2 and 3. That new development embraced the contribution of the implied extra energy term, and thus succeeded initially to resolve the spherical couple-stress indeterminacy issue for anisotropic fibrous composites with fibres resistant in bending [15, 17]. Through the thus gained experience, the present analysis has now succeeded to do the same for the apparently simpler material configuration of polar linearly elastic isotropy.

This rather strange, at least chronologically development is attributed to the physical complexity and subsequent difficulty of the problem in hand. Retrospectively, for instance, it now seems easier for one to accept that energy contributions due to a fibre rotation/spin vector may be more influential in modelling polar material behaviour than relevant contributions due to the general, displacement generated spin vector. It may then feel more natural for one to search for an alternative couple-stress theory formulation starting from the material configuration of a transverse isotropic fibrous composite, rather than from a simpler isotropic material configuration that does not seem to possess an alternative to the standard, deformation generated spin/rotation field.

It though happens that, regardless of whether the polar material of interest is a fibrous composite or just a simple isotropic such, the implied alternative rotation/spin vector is and remains virtual, and, as such, it does not need to be determined. Whether its potential determination is necessary, or at all needed, is currently perceived as an issue of secondary importance that might be faced in the future. It is though necessary for the present communication to end the presented analysis by (i) summarising the progress made so far in

the search for a mechanism that leads to the spherical couple-stress determination, and (ii) placing that progress in a chronologically proper, or more reasonable order.

Accordingly, as is mentioned at the beginning of this Section, the linearly elastic isotropic analysis detailed in Sect. 4.1 is regarded as a direct extension of its conversional counterpart, presented about six decades ago [3, 4], towards the successful determination of the spherical couple-stress. A key point of the observed success is the fact that, through inversion of equation (4.5), the simple material configuration of polar isotropy enables elimination of the appearing virtual spin divergence and, thus, conversion of the extra energy balance equation (4.11) into a PDE, namely (4.12), for the unknown spherical couple-stress. Examples of the manner that this equation can then be solved, when connected with the extra boundary condition mentioned by Koiter [4], are further detailed in Sect. 5.

However, this approach meets an operative difficulty and apparently fails in cases of polar material anisotropy that is due to the presence of one of more material directions of preference. Regardless of whether such a preferential material direction relates to fibre presence, or signifies something different, the observed failure is due to algebraic hurdles which, due to the complicated form of equation (4.18), do not anymore enable direct reduction of equation (4.16) into a suitable PDE for the unknown spherical couple-stress.

The implied equation reduction may though still become possible if sufficient additional information becomes available, in some essentially technical or intuitive manner, regarding the unspecified nature of the involved virtual spin field. For instance, this route is already followed in [17] where, for the special case of unidirectional fibre reinforced materials, the spherical couple-stress is determined by considering that the virtual spin field coincides with the rotation/spin field of the fibre direction.

Moreover, for the same class of unidirectional fibre-reinforced materials, Reference [13] employed a slightly different consideration, according to which the observed material anisotropy is due to the presence of fibres that resist bending (see also [13, 14]), and thus obtained a slightly different result. In that case, the employed concept of fibre bending resistance (i) enabled determination of the spherical couple-stress by directly incorporating the influence of the deformed fibre curvature into the strain energy function of the material, and thus (ii) led to calculations that avoid excessive use of the gradient of either the actual or the virtual spin vector.

Evidently, lack of fibre-reinforcement makes the theoretical analysis presented in [13, 14] unsuitable for the principal subject of the present investigation, which has direct application only within the bounds of polar material isotropy. Nevertheless, by examining the nature and, thus, underpinning a solution of the long-standing problem of the spherical couple-stress indeterminacy, this communication provides, along with recent relevant developments [15–17], new insights of the still developing subject of polar material elasticity.

Appendix A: Polar Material Anisotropy Due to Two Distinct Directions of Material Preference

As is mentioned in Sect. 4.2, the presented analysis becomes more cumbersome in cases involving advanced anisotropy that is due to presence of two or more preferred material directions. Consider, for instance, a material class of polar anisotropic solids characterised by the unit vectors, $a^{(1)}$ and $a^{(2)}$, of two different preference material directions. By assuming that these directions are not mutually orthogonal, so that

$$a_i^{(1)} a_i^{(2)} = \cos\theta \neq 0, \tag{A.1}$$

one essentially considers symmetry features of monoclinic material response, which is sufficiently general in studying structural material behaviour.

For that material class, the complete list of at most quadratic deformation invariants required for adequate description of W^Φ begins with the invariants listed in (4.2) and is complemented by the following:

$$\mathrm{I}_4 = a_i^{(1)} \Phi_{(i,j)} a_j^{(1)}, \qquad \mathrm{I}_5 = a_i^{(1)} \Phi_{(i,j)} \Phi_{(j,k)} a_k^{(1)}, \qquad \mathrm{I}_6 = a_i^{(1)} \Phi_{[i,j]} \Phi_{[j,k]} a_k^{(1)},$$

$$\mathrm{I}_7 = a_i^{(1)} \Phi_{(i,j)} \Phi_{[j,k]} a_k^{(1)}, \qquad \mathrm{I}_8 = a_i^{(2)} \Phi_{(i,j)} a_j^{(2)}, \qquad \mathrm{I}_9 = a_i^{(2)} \Phi_{(i,j)} \Phi_{(j,k)} a_k^{(2)},$$

$$\mathrm{I}_{10} = a_i^{(2)} \Phi_{[i,j]} \Phi_{[j,k]} a_k^{(2)}, \qquad \mathrm{I}_{11} = a_i^{(2)} \Phi_{(i,j)} \Phi_{[j,k]} a_k^{(2)}, \qquad \mathrm{I}_{12} = a_i^{(1)} \Phi_{(i,j)} a_j^{(2)},$$

$$\mathrm{I}_{13} = a_i^{(1)} \Phi_{(i,j)} \Phi_{(j,k)} a_k^{(2)}, \qquad \mathrm{I}_{14} = a_i^{(1)} \Phi_{[i,j]} \Phi_{[j,k]} a_k^{(2)}, \qquad \mathrm{I}_{15} = a_i^{(1)} \Phi_{(i,j)} \Phi_{[j,k]} a_k^{(2)},$$

$$\mathrm{I}_{16} = a_i^{(1)} \left(\Phi_{(i,j)} \Phi_{[j,k]} + \Phi_{[i,j]} \Phi_{(j,k)} \right) a_k^{(2)}, \qquad \mathrm{I}_{17} = a_i^{(1)} a_i^{(2)} = \cos\theta. \tag{A.2}$$

Such an augmented list of invariants produces a similarly augmented form of W^Φ, that is more cumbersome than (4.14) even in the special case of polar material orthotropy ($\cos\theta = 0$). The corresponding forms of both the couple-stress constitutive equation (4.17) and the spherical couple-stress (4.18) thus become also more cumbersome. Unless some further, additional information is provided regarding either the nature of the spin vector, $\boldsymbol{\Phi}$, or its relationship with the observed deformation, this result aggravates the algebraic hurdles described already in Sect. 4.2.

Appendix B: A Spatially Variable Solution of the Partial Differential equation (5.18)

Solution of the PDE (5.18) with the method of characteristic lines, requires initially the search for plane curves whose tangent satisfies the equation

$$\frac{dx_3}{dx_1} = \frac{x_1}{v x_3} \tag{B.1}$$

on the $x_1 x_3$-plane. Integration of this separable differential equation yields

$$x_3^2 - \frac{x_1^2}{v} = c^2 (x_2), \tag{B.2}$$

where c is either an arbitrary constant or, at most, an arbitrary function of x_2. It follows that the characteristic lines sought are either hyperbolas ($0 < v \le 1/2$) or ellipses ($-1 < v < 0$).

With use of (B.1), the PDE (5.18) is next transformed into the following first-order ordinary differential equation

$$\frac{dm_{rr}}{m_{rr}^2 - b(1+v)/a} = -\frac{a}{v x_3} dx_1, \tag{B.3}$$

where the constants

$$a = \eta E / \hat{\sigma}_1, \, b = 6 \eta_1 \hat{\sigma}_1 / E, \tag{B.4}$$

depend on the material moduli and on loading features, and thus are assumed known.

Direct integration of (B.3) then yields

$$\frac{1}{2a} \ln \left| \frac{m_{rr} + b\,(1+\nu)\,/a}{m_{rr} - b\,(1+\nu)\,/a} \right| = \frac{ax_1}{\nu x_3} + \hat{c}, \tag{B.5}$$

or, after some necessary algebraic simplification,

$$m_{rr} = \frac{b\,(1+\nu)}{a} \left[\frac{\bar{c} \exp \left(2\frac{a^2 x_1}{\nu x_3} \right) + 1}{\bar{c} \exp \left(2\frac{a^2 x_1}{\nu x_3} \right) - 1} \right], \tag{B.6}$$

where either of $\hat{c}$ and $\bar{c}$ represents the involved arbitrary constant of integration

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Declarations

Competing interests The authors declare no competing interests.

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