SOLVABLE CROSSED PRODUCT ALGEBRAS REVISITED

C. BROWN AND S. PUMPLÜN

Abstract. For any central simple algebra over a field F which contains a maximal subfield M with non-trivial automorphism group $G = \mathrm{Aut}_F(M)$, G is solvable if and only if the algebra contains a finite chain of subalgebras which are generalized cyclic algebras over their centers (field extensions of F) satisfying certain conditions. These subalgebras are related to a normal subseries of G. A crossed product algebra F is hence solvable if and only if it can be constructed out of such a finite chain of subalgebras. This result was stated for division crossed product algebras by Petit, and overlaps with a similar result by Albert which, however, was not explicitly stated in these terms. In particular, every solvable crossed product division algebra is a generalized cyclic algebra over F.

Introduction

Let F be a field. A central simple algebra A over F of degree n is a *crossed product algebra* if it contains a maximal subfield M (i.e. with $[M : F] = n$) that is Galois. To be more precise, A is also called a *G-crossed product algebra*, if $G = \mathrm{Gal}(M/F)$ is the Galois group of M/F. Crossed product algebras play an important role in the theory of central simple algebras: every element in the Brauer group of F is similar to a crossed product algebra, moreover, their multiplicative structure can be described by a group action. It is well known that any central simple algebra of degree 2, 3, 4, 6 or 12 is a crossed product algebra. Moreover, any central simple algebra over a local or global field is a crossed product algebra (in that case the algebras even contain a maximal subfield that is cyclic).

Skew polynomial rings have been successfully used in the past to construct central simple algebras. These appear for instance as quotient algebras $D[t;\sigma]/(f)$ when factoring out a two-sided ideal generated by a twisted polynomial $f \in D[t;\sigma]$ with D a finite-dimensional central division algebra over F in [2] or [7, Sections 1.5, 1.8, 1.9]. Following Jacobson [7, p. 19], when $\sigma|_F$ has finite order m and $f(t) = t^m - d \in D[t;\sigma]$, $d \in (\mathrm{Fix}(\sigma) \cap F)^\times$, is a right invariant polynomial, such a quotient algebra is also called a *generalized cyclic algebra*, and denoted (D, σ, d). In characteristic zero, generalized cyclic division algebras can be considered to be the noncommutative

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2010 *Mathematics Subject Classification.* Primary: 16S35; Secondary: 16K20.

Key words and phrases. Skew polynomial ring, skew polynomial, solvable crossed product algebra, generalized cyclic algebra, cyclic subalgebra, crossed product subalgebra, admissible group.

analogue of simple algebraic field extensions. To our knowledge, generalized cyclic division algebras appear for the first time in a paper by Amitsur [2], where they are examples of noncommutative cyclic extensions. They are examples of crossed products of central simple algebras which were introduced by Teichmüller [19] in 1940.

In this paper, we will revisit a result on the structure of crossed product algebras with solvable Galois group due to both Albert [1, p. 182-187] and Petit [13, Section 7].

To be more precise, we write up the proof for Albert's result following the approach given by Petit, using generalized cyclic algebras (none of Petit's results are proved). In the process, we generalize some results to central simple algebras which need neither be crossed products nor division algebras. In order to do so, we extend the classical definition of a generalized cyclic algebra (D, σ, d) as we do not assume that D needs to be a division algebra.

As a special case we obtain that a G-crossed product algebra is solvable if and only if it can be constructed as a finite chain of subalgebras over F which are generalized cyclic algebras over their centers, which are field extensions of F. The generalized cyclic algebras appearing in this chain correspond to the normal subgroups in a chain of normal subgroups of the solvable group G. We highlight how the structure of the solvable group (i.e., its chain of normal subgroups G_i) is connected to the structure of the algebra, and how each subalgebra is related to a normal subgroup G_i in the chain and the order of the factor groups G_{i+1}/G_i.

The paper is structured as follows. After the basic terminology in Section 1 we look at the existence of crossed product algebras and in particular, of cyclic algebras, inside central simple algebras in Section 2. As a byproduct, we show that even if a central division algebra A over F is a noncrossed product, if it contains a maximal field extension M with a non-trivial $\sigma \in G = \mathrm{Aut}_F(M)$ of order h, then it contains a cyclic division algebra of degree h, and a crossed product algebra $(M, G, \mathfrak{a})$ of degree $|G|$ as well, both of them not necessarily with center F, however (Theorem 4).

The first results on the structure of central simple algebras which contain a maximal subfield with non-trivial solvable group $G = \mathrm{Aut}_F(M)$ are stated in Section 3 (Theorems 7 and 13). These algebras have certain chains of generalized cyclic algebras (with centers larger than F) as subalgebras.

As a consequence, we can show in Section 4 that all solvable crossed product algebras can be constructed as chains of such generalized cyclic algebras and that if a central simple algebra contains a maximal subfield with $G = \mathrm{Aut}_F(M)$ that this G is solvable exactly if there is such a chain (Theorems 14 and Corollary 16). In particular, every solvable G-crossed product division algebra is a generalized cyclic algebra (Corollary 18). Some straightforward applications to admissible groups are given in Section 5. In Section 6 we generalize a result on crossed product algebras with Galois group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Albert [1, p. 186], cf. also [7, Theorem 2.9.55], to crossed product algebras with G any abelian group, and give a recipe how to construct central division algebras containing a given Galois field extension with abelian Galois group from a chain of generalized cyclic algebras, complementing the construction of such algebras via generic algebras by Amitsur and Saltman described in [7, 4.6].

Most of the results presented here are part of the first author's PhD thesis [4] written under the supervision of the second author.

1. Preliminaries

1.1. **Twisted polynomial rings and (nonassociative) algebras.** In the following, we recall some results from [7] and [13] for the convenience of the reader.

Let S be a unital (associative, not necessarily commutative) ring and σ an injective ring endomorphism of S. The *twisted polynomial ring* $R = S[t;\sigma]$ is the set of twisted polynomials $a_0 + a_1 t + \cdots + a_n t^n$ with $a_i \in S$, where addition is defined term-wise and multiplication by $ta = \sigma(a)t$ for all $a \in S$. For $f = a_0 + a_1 t + \cdots + a_n t^n$ with $a_n \neq 0$ define $\deg(f) = n$ and $\deg(0) = -\infty$. An element $f \in R$ is *irreducible* in R if it is not a unit and it has no proper factors, i.e if there do not exist $g, h \in R$ with $\deg(g), \deg(h) < \deg(f)$ such that $f = gh$. An element $f \in R$ is called *right invariant* (or *two-sided*) if Rf is a two-sided ideal in R.

We now briefly explain how classical quotient algebras R/Rf, f right invariant, fit into the nonassociative setting of Petit's paper [13]:

In the following, we always assume that $f(t) \in S[t;\sigma]$ is monic of degree $m > 1$. Then for all $g, f \in R$, $g \neq 0$, there exist unique $r, q \in R$ such that $\deg(r) < \deg(f)$ and $g = qf + r$, e.g. see [15].

In [13] and [15], it is shown that the additive group $S_f = \{g \in R \,|\, \deg(g) < m\}$ of twisted polynomials of degree less that m is a nonassociative unital ring together with the multiplication given by

$$g \circ h = gh \bmod_r f,$$

where $\bmod_r f$ denotes the remainder $\bmod_r f$ of right division by f. This algebra is also denoted by R/Rf.

Note that since the remainders are uniquely determined, the elements in the set S_f also canonically represent the elements of the left $S[t;\sigma]$-module $S[t;\sigma]/S[t;\sigma]f$.

$S_0 = \{a \in S \,|\, ah = ha \text{ for all } h \in S_f\}$ is a commutative subring of S, and S_f is a unital algebra over S_0. If S is a division ring, the structure of S_f is extensively investigated in [13], else see [15]. For instance, if S is a division ring and the S_0-algebra S_f is finite-dimensional, then S_f is a division algebra if and only if $f(t)$

is irreducible [13, (9)]. In the following, we will only be interested in the case that S_f is a unital associative algebra, which happens if and only if f is a *right invariant* polynomial in R, i.e. generates a two-sided ideal Rf in R [13]. In that case, $S_f = R/Rf$ is the well known quotient algebra obtained by factoring out the two-sided ideal in R generated by f.

We will moreover only need the case that S is a finite-dimensional algebra over a field F with center F and only consider automorphisms σ of S such that $\sigma|_F$ has finite order m. Then, by the Theorem of Skolem-Noether, σ^m is an inner automorphism $I_u(y) = uyu^{-1}$ of S [7, Sec. 1.4].

1.2. **Generalized cyclic algebras and generalized cyclic extensions.** Let S be a finite-dimensional simple algebra of degree n over its center $F = C(S)$, and $\sigma \in \mathrm{Aut}(S)$ such that $\sigma|_F$ has finite order m and fixed field $F_0 = \mathrm{Fix}(\sigma)$.

Generalizing Jacobson's definition [7, p. 19], which assumes that S is a division algebra, we define a *generalized cyclic algebra* as an associative algebra of the type $S_f = S[t;\sigma]/S[t;\sigma]f(t)$ which is constructed using a right invariant twisted polynomial

$$f(t) = t^m - d \in S[t;\sigma],$$

with $d \in \mathrm{Fix}(\sigma)^\times$ non-zero.

We write $S_f = (S, \sigma, d)$ for this algebra. (S, σ, d) is a central simple algebra over $F_0 = \mathrm{Fix}(\sigma)$ of degree mn and the centralizer of S in (S, σ, d) is F ([7, p. 20] if S is division, else [20]).

Note that this definition canonically generalizes the one of a cyclic algebra $(F/F_0, \sigma, d)$, where $f(t) = t^m - d \in F[t;\sigma]$, and F/F_0 is a cyclic Galois extension of degree m with Galois group $G = <\sigma>$. This is the algebra $S_f = F[t;\sigma]/F[t;\sigma](t^m - d)$, cf. [7, p. 19] or [13, p. 13-13]. This case appears when $S = F$ above.

Generalized cyclic algebras are a special case of generalized crossed products, i.e. crossed products of simple algebras cf. for instance [6, p. 35], [10], [20]. We will mostly need crossed products involving Galois fields:

1.3. **Crossed product algebras.** Let F be a field and A be a (finite-dimensional) central simple algebra over F of degree n. A is called a *G-crossed product algebra* or *crossed product algebra* if it contains a maximal field extension K/F which is Galois with Galois group $G = \mathrm{Gal}(K/F)$.

Equivalently, we can define a (G-)crossed product algebra $(M, G, \mathfrak{a})$ over F via factor sets starting with a finite Galois field extension as follows: Take a finite Galois field extension M/F of degree n with Galois group G. Suppose $\{a_{\sigma,\tau} \mid \sigma, \tau \in G\}$ is a set of elements of $M^\times$ such that

$$a_{\sigma,\tau} a_{\sigma\tau,\rho} = a_{\sigma,\tau\rho} \sigma(a_{\tau,\rho}), \tag{1}$$

for all $\sigma, \tau, \rho \in G$. Then a map $\mathfrak{a} : G \times G \to M^\times$, $(\sigma, \tau) \mapsto a_{\sigma,\tau}$, is called a *factor set* or *2-cocycle* of G.

An associative multiplication is defined on the F-vector space $\bigoplus_{\sigma\in G} Mx_\sigma$ by

$$x_\sigma m = \sigma(m)x_\sigma, \tag{2}$$

$$x_\sigma x_\tau = a_{\sigma,\tau}x_{\sigma\tau}, \tag{3}$$

for all $m \in M$, $\sigma, \tau \in G$. This way $\bigoplus_{\sigma\in G} Mx_\sigma$ becomes an associative central simple F-algebra that contains a maximal subfield isomorphic to M. This algebra is denoted by $(M, G, \mathfrak{a})$ and is a G-crossed product algebra over F. If G is solvable then A is also called a *solvable G-crossed product*.

In the following, we will only consider unital algebras A over a field F which are finite-dimensional without explicitly saying so. We denote the set of invertible elements of A by $A^\times$.

2. Cyclic and crossed product subalgebras of central simple algebras

In this section, let M/F be a field extension of degree n, and $G = \mathrm{Aut}_F(M)$ the group of automorphisms of M which fix the elements of F. Let A be a central simple algebra of degree n over F and suppose that M is contained in A, i.e. is a maximal subfield of A. For a subset B in A, let $\mathrm{Cent}_A(B)$ denote the centralizer of B in A.

Some of the results in this section are stated for central division algebras A over F for instance in [13], and none of them are proved there. The following generalizes [13, (27)] to central simple algebras with a maximal subfield M as above. The result was before only stated for division algebras and also not in terms of crossed product algebras:

Theorem 1. *(i) A contains a subalgebra $M(G)$ which is a crossed product algebra $(M, G, \mathfrak{a})$ of degree $|G|$ over* $\mathrm{Fix}(G)$ *with maximal subfield M.*
(ii) $A = M(G)$ if and only if M is a Galois field extension of F. In that case, A is a G-crossed product algebra over F.
(iii) For any subgroup H of G, there is a subalgebra $M(H)$ of both $M(G)$ and A which is a H-crossed product algebra of degree $|H|$ over $\mathrm{Fix}(H)$ *with maximal subfield M.*

Proof. (i) Define $M(G) = \mathrm{Cent}_A(\mathrm{Fix}(G))$, then $M(G)$ is a central simple algebra over $\mathrm{Fix}(G)$ by the Centralizer Theorem for central simple algebras [5, Theorem III.5.1]. Since M is a maximal subfield of $M(G)$ and $M/\mathrm{Fix}(G)$ is a Galois field extension with Galois group G, $M(G)$ is a G-crossed product algebra.
(ii) Since $[M : F] = n$, A has dimension n^2 over F. Moreover, $M(G)$ has a basis $\{x_\sigma |\ \sigma \in G\}$ as a vector space over M. If M is not a Galois extension of F, then

$|G| < n$ and thus $\{x_\sigma \mid \sigma \in G\}$ cannot be a set of generators for A as a vector space over M. Conversely, if M/F is a Galois extension, then the elements in $|G| = n$ and since $\{x_\sigma \mid \sigma \in G\}$ are linearly independent over M, counting dimensions yields $M(G) = A$. The rest of the assertion is trivial.
(iii) For any subgroup H of G, let $M(H) = \mathrm{Cent}_A(\mathrm{Fix}(H))$. The proof now follows exactly as in (i). □

More precisely, a closer look at the above proof reveals:

Lemma 2. *(i) For any subgroup H of G, $M(H)$ is a H-crossed product algebra over its center with*

$$M(H) = (M, H, \mathfrak{a}_H),$$

where $\mathfrak{a}_H$ denotes the factor set $\mathfrak{a}$ of the crossed product algebra $M(G) = (M, G, \mathfrak{a})$, restricted to the elements in H.
(ii) For any subgroup H of G, $M(H)$ is the centralizer of $\mathrm{Fix}(H)$ in A.
(iii) If H is a cyclic subgroup of G of order $h > 1$ generated by $\sigma \in G$, then there exists $c \in \mathrm{Fix}(\sigma)^\times$ such that

$$M(H) \cong M[t;\sigma]/M[t;\sigma](t^h - c)$$

is a cyclic algebra of degree h over $\mathrm{Fix}(\sigma)$ and an F-subalgebra of A.

Proof. (i) and (ii) are trivial.
(iii) $M(H)$ is a H-crossed product algebra $(M, H, \mathfrak{a}_H)$ over $\mathrm{Fix}(\sigma)$ of degree h by Theorem 1 and H is a cyclic group, therefore $M(H)$ is a cyclic algebra over $\mathrm{Fix}(\sigma)$ of degree h, i.e. there exists $c \in \mathrm{Fix}(\sigma)^\times$ such that $M(H) \cong M[t;\sigma]/M[t;\sigma](t^h - c)$, e.g. see [7, p. 19]. □

The following generalises [13, (26)] to any central simple algebra A with a maximal subfield M, it is well-known for Galois extensions M/F:

Lemma 3. *(i) For any $\sigma \in G$ there exists an invertible $x_\sigma \in A$ such that the inner automorphism*

$$I_{x_\sigma} : A \to A,\ y \mapsto x_\sigma y x_\sigma^{-1}$$

restricted to M is σ.
(ii) Given any $\sigma \in G$, we have $\{x \in A^\times \mid I_x|_M = \sigma\} = M^\times x_\sigma$.
(iii) The set of cosets $\{M^\times x_\sigma \mid \sigma \in G\}$ together with the multiplication given by

$$M^\times x_\sigma M^\times x_\tau = M^\times x_{\sigma\tau} \tag{4}$$

is a group isomorphic to G, where σ and $M^\times x_\sigma$ correspond under this isomorphism.

Proof. A contains the G-crossed product algebra $M(G)$ by Theorem 1, hence (i) and (iii) follow from (2) and (3).
(ii) We have

$$I_{(mx_\sigma)}(y) = (mx_\sigma)y(mx_\sigma)^{-1} = (mx_\sigma)y(x_\sigma^{-1}m^{-1}) = m\sigma(y)m^{-1} = \sigma(y),$$

for all $m, y \in M^\times$, and thus $M^\times x_\sigma \subset \{x \in A^\times \mid I_x|_M = \sigma\}$.

Suppose $u \in \{x \in A^\times \mid I_x|_M = \sigma\}$. As u and x_σ are invertible, we can write $u = vx_\sigma$ for some $v \in A^\times$. We still have to prove that $v \in M^\times$. We have

$$\sigma(y) = I_u(y) = (vx_\sigma)y(vx_\sigma)^{-1} = vx_\sigma y x_\sigma^{-1}v^{-1} = v\sigma(y)v^{-1},$$

for all $y \in M$, and so $\sigma(y)v = v\sigma(y)$ for all $y \in M$, that is $mv = vm$ for all $m \in M$ since σ is bijective. Therefore v is contained in the centralizer of M in A, which is equal to M because M is a maximal subfield of A. □

We conclude that even if a central division algebra A over F is a noncrossed product, if A contains a maximal field extension M with a non-trivial $\sigma \in G = \mathrm{Aut}_F(M)$ of order h, then it contains a cyclic division algebra of degree h (though generally not with center F):

Theorem 4. *Let A be a central division algebra over F with maximal subfield M and non-trivial $\sigma \in G = \mathrm{Aut}_F(M)$ of order h. Then A contains the cyclic division algebra*

$$(M/\mathrm{Fix}(\sigma), \sigma, c) = M[t;\sigma]/M[t;\sigma](t^h - c)$$

of degree h over $\mathrm{Fix}(\sigma)$ as an F-subalgebra.

This generalizes [13, (28)] to central simple algebras with a maximal subfield M such that $G = \mathrm{Aut}_F(M)$ is not trivial.

Remark 5. The question when a central division algebra A over F has a cyclic subalgebra of prime degree was recently raised in [12, Question 1]. If F is a Henselian field such that $\overline{F}$ is a global field, and A is an central division algebra over F such that $\mathrm{char}(\overline{F})$ does not divide $\deg(A)$, then A contains a cyclic division algebra of prime degree [12, Theorem 3].

By Theorem 4, any central division algebra A over F containing a maximal subfield M with some $\sigma \in G = \mathrm{Aut}_F(M)$ of prime order p contains a cyclic division algebra of prime degree p.

A central division algebra of prime degree over F is cyclic if and only if it has a cyclic subalgebra of prime degree (not necessarily with center F) [12, p. 2]. Theorem 4 yields the following observations:

Corollary 6. *Let A be a central division algebra over F.*
(i) If A has prime degree then either A is a cyclic algebra or each of its maximal subfields M has trivial automorphism group $\mathrm{Aut}_F(M)$.
(ii) Suppose A contains a maximal subfield M such that $G = \mathrm{Aut}_F(M)$ is non-trivial. Then A contains the G-crossed product division algebra $M(G) = (M, G, \mathfrak{a})$ of degree $|G|$ over $\mathrm{Fix}(G)$ as a subalgebra.

Proof. (i) If $G = \mathrm{Aut}_F(M)$ is non-trivial, then A contains the cyclic algebra $M(G)$ of degree $|G| \leq p$ over $F_0 = \mathrm{Fix}(G)$ as subalgebra. Since M/F_0 is a maximal subfield of $M(G)$, it also has degree $|G|$. Looking at the possible degrees of the intermediate field extensions of M/F we have $[M : F_0] = 1$ or $[M : F_0] = p$, so $|G| = 1$ or $|G| = p$. If $|G| = p$ then A is a cyclic algebra. Hence if A is not cyclic then each of its maximal subfields M must have trivial automorphism group $\mathrm{Aut}_F(M)$.
(ii) is trivial. □

3. Central simple algebras containing maximal subfields with a solvable F-automorphism group

Let G be finite a solvable group, i.e. there exists a chain of subgroups

$$\{1\} = G_0 \leq G_1 \leq \ldots \leq G_k = G, \tag{5}$$

such that G_j is normal in G_{j+1} and G_{j+1}/G_j is cyclic of prime order q_j for all $j \in \{0, \ldots, k-1\}$, that is

$$G_{j+1}/G_j = \{G_j, G_j\sigma_{j+1}, \ldots\}, \tag{6}$$

for some $\sigma_{j+1} \in G_{j+1}$. Lemma 3, Theorem 1 and Corollary 2 yield the following generalization of [13, (29)], which only claims the result for central division algebras over F:

Theorem 7. *Let M/F be a field extension of degree n with non-trivial solvable $G = \mathrm{Aut}_F(M)$, and A a central simple algebra of degree n over F with maximal subfield M. Then there exists a chain of subalgebras*

$$M = A_0 \subset A_1 \subset \ldots \subset A_k = M(G) \subset A, \tag{7}$$

of A which are G_i-crossed product algebras over $Z_i = \mathrm{Fix}(G_i)$ and where

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i) \tag{8}$$

for all $i \in \{0, \ldots, k-1\}$, such that
(i) q_i is the prime order of the factor group G_{i+1}/G_i in the chain of normal subgroups (5),
(ii) τ_i is an F-automorphism of A_i of inner order q_i which restricts to the automorphism $\sigma_{i+1} \in G_{i+1}$ that generates G_{i+1}/G_i, and
(iii) $c_i \in \mathrm{Fix}(\tau_i)$ is invertible.

Note that the inclusion $M(G) \subset A$ in (7) is an equality if and only if M/F is a Galois extension by Theorem 1, i.e. if and only if A is a G-crossed product algebra.

Proof. Define $A_i = M(G_i)$ for all $i \in \{1, \dots, k\}$. A_i is a G_i-crossed product algebra over $\mathrm{Fix}(G_i)$ by Theorem 1.

$G_1/G_0 \cong G_1$ is a cyclic subgroup of G of order q_0 generated by some $\sigma_1 \in G$. Let $\tau_0 = \sigma_1$, then there exists $c_0 \in \mathrm{Fix}(\tau_0)$ such that $A_1 = M(G_1)$ is F-isomorphic to

$$M[t_0; \tau_0]/M[t_0; \tau_0](t_0^{q_0} - c_0)$$

by Corollary 2, which is a cyclic algebra of prime degree q_0 over $\mathrm{Fix}(\tau_0)$.

Now $G_1 \triangleleft G_2$ and G_2/G_1 is cyclic of prime order q_1 with

$$G_2/G_1 = \{G_1, G_1\sigma_2, \dots, G_1\sigma_2^{q_1-1}\} \tag{9}$$

for some $\sigma_2 \in G_2$. Hence we can write $G_2 = \{h\sigma_2^i \mid h \in G_1, 0 \leq i \leq q_1 - 1\}$ and thus the crossed product algebra $A_2 = M(G_2)$ has a basis

$$\{x_{h\sigma_2^j} \mid h \in G_1,\ 0 \leq j \leq q_1 - 1\},$$

as an M-vector space. Recall

$$M^\times x_{h\sigma_2^j} = M^\times x_h x_{\sigma_2^j} = M^\times x_h x_{\sigma_2}^j$$

for all $h \in G_1$ by Lemma 3, and $\{1, x_{\sigma_2}, \dots, x_{\sigma_2}^{q_1-1}\}$ is a basis for A_2 as a left A_1-module, i.e.

$$A_2 = A_1 + A_1 x_{\sigma_2} + \ldots + A_1 x_{\sigma_2}^{q_1-1}. \tag{10}$$

We have $G_2G_1 = G_1G_2$ as G_1 is normal in G_2 and so for every $h \in G_1$, we get $\sigma_2 h = h'\sigma_2$ for some $h' \in G_1$. Choose the basis $\{x_h \mid h \in G_1\}$ of A_1 as a vector space over M. By (3) we obtain

$$x_{\sigma_2} x_h = a_{\sigma_2,h} x_{\sigma_2 h} = a_{\sigma_2,h} x_{h'\sigma_2} = a_{\sigma_2,h}(a_{h',\sigma_2})^{-1} x_{h'} x_{\sigma_2}. \tag{11}$$

Recall $x_{\sigma_2} \in A^\times$ by Lemma 3. The inner automorphism

$$\tau_1 : A \to A,\ z \mapsto x_{\sigma_2} z x_{\sigma_2}^{-1}$$

restricts to σ_2 on M. Moreover,

$$\begin{aligned} \tau_1(x_h) &= x_{\sigma_2} x_h x_{\sigma_2}^{-1} = a_{\sigma_2,h}(a_{h',\sigma_2})^{-1} x_{h'} x_{\sigma_2} x_{\sigma_2}^{-1} \\ &= a_{\sigma_2,h}(a_{h',\sigma_2})^{-1} x_{h'} \in A_1, \end{aligned} \tag{12}$$

for all $h \in G_1$, i.e. $\tau_1|_{A_1}(y) \in A_1$ for all $y \in A_1$ and so $\tau_1|_{A_1}$ is an F-automorphism of A_1. Moreover,

$$x_{\sigma_2} x_h = \tau_1|_{A_1}(x_h) x_{\sigma_2},$$

for all $h \in G_1$ by (11), (12), and

$$x_{\sigma_2} m = \sigma_2(m) x_{\sigma_2} = \tau_1|_{A_1}(m) x_{\sigma_2},$$

for all $m \in M$. We conclude that

$$x_{\sigma_2} y = \tau_1|_{A_1}(y) x_{\sigma_2} \tag{13}$$

for all $y \in A_1$. Define $c_1 = x_{\sigma_2}^{q_1}$, then $\sigma_2^{q_1} \in G_1$ by (6) which implies $c_1 \in A_1$. Furthermore c_1 is invertible since x_{σ_2} is invertible. Also,

$$\tau_1|_{A_1}(c_1) = x_{\sigma_2} x_{\sigma_2}^{q_1} x_{\sigma_2}^{-1} = c_1$$

which means $c_1 \in \mathrm{Fix}(\tau_1|_{A_1})^\times$. Since

$$x_{\sigma_2^{-q_1}} x_{\sigma_2^{q_1}} = a_{\sigma_2^{-q_1}, \sigma_2^{q_1}} x_{\mathrm{id}} \in M^\times,$$

it follows that $c_1^{-1} = x_{\sigma_2^{q_1}}^{-1} \in M^\times x_{\sigma_2^{-q_1}} \in A_1$ as $\sigma_2^{-q_1} \in G_1$. Hence $\tau_1|_{A_1}$ has inner order q_1, as indeed

$$(\tau_1|_{A_1})^{q_1} : A_1 \to A_1, z \mapsto c_1 z c_1^{-1},$$

is an inner automorphism.

Consider the algebra

$$B_2 = A_1[t_1; \tau_1|_{A_1}]/A_1[t_1; \tau_1|_{A_1}](t_1^{q_1} - c_1)$$

with center

$$C(B_2) \supset \{b \in A_1 \mid bh = hb \text{ for all } h \in B_2\} = C(A_1) \cap \mathrm{Fix}(\tau_1) \supset F.$$

By (10) and (13), the F-linear map

$$\phi : A_2 \to B_2,\ y x_{\sigma_2}^i \mapsto y t_1^i \qquad (y \in A_1),$$

is an isomorphism. In addition, by a straightforward calculation we have

$$\phi\big((y x_{\sigma_2}^i)(z x_{\sigma_2}^j)\big) = \phi(y x_{\sigma_2}^i)\phi(z x_{\sigma_2}^j)$$

for all $y, z \in A_1$, $i, j \in \{0, \dots, q_1 - 1\}$, so ϕ is also multiplicative, thus an F-algebra isomorphism. Continuing in this manner for $G_2 \triangleleft G_3$ etc. yields the assertion. □

Theorem 1 immediately yields that the algebras A_i are the centralizers of $\mathrm{Fix}(G_i)$ in A:

Corollary 8. *Let M/F be a field extension of degree n with non-trivial solvable $G = \mathrm{Aut}_F(M)$ with normal series* (5)*, and A a central simple algebra of degree n over F with maximal subfield M. Then $A_i = \mathrm{Cent}_A(\mathrm{Fix}(G_i))$ for all $i \in \{0, \dots, k-1\}$, where $A_i = M(G_i)$ are as in Theorem 7.*

Corollary 9. *Let A be a central division algebra over F containing a maximal subfield M with non-trivial solvable $G = \mathrm{Aut}_F(M)$. Then:*
(i) A contains the cyclic division algebra $(M/\mathrm{Fix}(\sigma_1), \sigma_1, c_0)$ of prime degree q_0 over $\mathrm{Fix}(\sigma_1)$ as a subalgebra.
(ii) There is a non-central element $t_0 \in A$ such that $t_0^{q_0} \in \mathrm{Fix}(\sigma_1)^\times$ and $t_0^m \notin \mathrm{Fix}(\sigma_1)$

for all $1 \leq m < q_0$.
Here, q_0 *is the order of the cyclic subgroup* G_1 *of the normal subseries (5) of* G.

Additionally, we obtain the following straightforward observations:

Corollary 10. *Let* A_i *be as in (7) of Theorem 7,* $i \in \{0, \dots, k\}$.
(i) A_i *is a generalized cyclic algebra over* $Z_i = \mathrm{Fix}(G_i)$ *of degree*

$$\deg(A_{i-1})q_{i-1} = \prod_{l=0}^{i-1} q_l$$

and

$$M = Z_0 \supset \ldots \supset Z_{k-1} \supset Z_k \supset F. \tag{14}$$

(ii) M/Z_i *is a Galois extension and* M *is a maximal subfield of* A_i.
(iii) Z_{i-1}/Z_i *has prime degree* q_{i-1} *for all* $i \in \{1, \dots, k\}$.

Proof. A_i is a generalized cyclic algebra as defined in 1.2. Since $G_{i-1} \leq G_i$ we have

$$Z_i = \mathrm{Fix}(G_i) \subset \mathrm{Fix}(G_{i-1}) = Z_{i-1},$$

for all $i \in \{1, \dots, k\}$. We know that A_i has $\deg(A_i) = \deg(A_{i-1})q_{i-1}$ over its center. By induction we obtain (i).
(ii) is trivial by Theorem 7.
(iii) We have $n = [M : F] = [M : Z_i][Z_i : F] = |G_i|[Z_i : F]$ for all i, therefore

$$[Z_{i-1} : Z_i] = \frac{[Z_{i-1} : F]}{[Z_i : F]} = \frac{|G_i|}{|G_{i-1}|} = q_{i-1}$$

for all $i \in \{1, \dots, k\}$ as required. □

Hence even if a central division algebra A over F is a noncrossed product, if A contains a maximal field extension M with non-trivial solvable $G = \mathrm{Aut}_F(M)$ then it contains a chain of generalized cyclic division algebras:

Corollary 11. *Let* M/F *be a field extension of degree* n *with non-trivial solvable* $G = \mathrm{Aut}_F(M)$*, and* A *a central division algebra over* F *with maximal subfield* M*. Then* A *contains a chain of generalized cyclic division algebras* A_i *over intermediate fields* $Z_i = \mathrm{Fix}(G_i)$ *of* M/F *as in (7). Here,* q_i *is the order of the cyclic factor group* G_{i+1}/G_i *of the normal subseries (5) of* G.

If A is a division algebra in the above setup then G solvable implies that $A^\times$ contains an solvable subgroup:

Lemma 12. *Suppose* A *is a central division algebra over* F*. If* A *contains a maximal subfield* M/F *with solvable* $G = \mathrm{Aut}_F(M)$ *then* $A^\times$ *contains an solvable subgroup. If* M/F *is Galois, i.e.* A *a* G*-crossed product algebra, then this solvable subgroup is irreducible.*

Proof. As noted in [8, Lemma 1] (where M/F is Galois, but the argument is the same), $N = \bigcup_{\sigma\in G} M^\times x_\sigma \subset A^\times$ and it is easy to see that $M^\times$ is a normal subgroup of N, and that N is the normalizer of $M^\times$ in $A^\times$. Therefore $N/M^\times \cong G$ as in Lemma 3 (iii) and if G is solvable as assumed in later sections, we see that in fact N is a solvable subgroup of $A^\times$. Since $N = \bigcup M^\times x_\sigma$, if M/F is Galois, i.e. A a G-crossed product algebra, then N is irreducible, i.e. the F-algebra generated by elements of N, $F[N]$, is A by [8, Lemma 1]. □

We note that it is known that given a finite-dimensional central division algebra D over F, $\mathrm{Mat}_m(D)$ is a crossed product algebra over a maximal subfield if and only if there is an irreducible subgroup G' of $\mathrm{Mat}_m(D)$ and a normal abelian subgroup G_0 of G', such that the centralizer $C_{G'}(G_0)$ of G_0 in G is G_0, and such that the F-subalgebra $F[G_0]$ of $\mathrm{Mat}_m(D)$ generated by elements of G_0 over F does not contain zero divisors [9, Theorem 1].

Our next result generalizes [14, (9)] and characterizes all the algebras with a maximal subfield M/F that have a solvable automorphism group $G = \mathrm{Aut}_F(M)$ via generalized cyclic algebras:

Theorem 13. *Let M/F be a field extension of degree n with non-trivial $G = \mathrm{Aut}_F(M)$, and A be a central simple algebra over F with maximal subfield M. Then G is solvable if there exists a chain of subalgebras*

$$M = A_0 \subset A_1 \subset \ldots \subset A_k \subset A \tag{15}$$

of A which all have maximal subfield M, where A_k is a G-crossed product algebra over $\mathrm{Fix}(G)$*, and where*

$$A_{i+1} \cong A_i[t_i;\tau_i]/A_i[t_i;\tau_i](t_i^{q_i} - c_i), \tag{16}$$

for all $i \in \{0,\ldots,k-1\}$, with

(i) *q_i a prime,*
(ii) *τ_i an F-automorphism of A_i of inner order q_i which restricts to an automorphism $\sigma_{i+1} \in G$, and*
(iii) *$c_i \in \mathrm{Fix}(\tau_i)^\times$.*

Proof. Suppose there exists a chain of algebras A_i, $i \in \{0,\ldots,k\}$ satisfying the above assumptions. Put $G_k = G$. Since each A_i has center $Z_i = Z_{i-1} \cap \mathrm{Fix}(\tau_{i-1})$, so that by induction

$$Z_i = \mathrm{Fix}(\tau_0) \cap \mathrm{Fix}(\tau_1) \cap \cdots \cap \mathrm{Fix}(\tau_{i-1}) \supset F,$$

M/Z_i is a Galois extension contained in A_i. Put $G_i = \mathrm{Gal}(M/Z_i)$, then each A_i is a G_i-crossed product algebra. In particular, G_i is a subgroup of G_{i+1}.

We use induction to prove that each G_i, thus G, is a solvable group.

For $i = 1$,

$$A_1 \cong M[t_0; \sigma_1]/M[t_0; \sigma_1](t_0^{q_0} - c_0)$$

is a cyclic algebra of degree q_0 over $\text{Fix}(\sigma_1)$. $G_1 =< \sigma_1 >$ is a cyclic group of prime order q_0 and therefore solvable.

We assume as induction hypothesis that if there exists a chain $M = A_0 \subset \ldots \subset A_j$ of algebras such that (16) holds for all $i \in \{0, \ldots, j-1\}$, $j \geq 1$, then G_j is solvable.

For the induction step we take a chain of algebras $M = A_0 \subset \ldots \subset A_j \subset A_{j+1}$,

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

where τ_i is an automorphism of A_i of inner order q_i which induces an automorphism $\sigma_{i+1} \in G$, $c_i \in \text{Fix}(\tau_i)$ is invertible and q_i is prime, for all $i \in \{0, \ldots, j\}$. By the induction hypothesis, G_j is a solvable group.

We show that G_{j+1} is solvable: t_j is an invertible element of

$$A_{j+1} \cong A_j[t_j; \tau_j]/A_j[t_j; \tau_j](t_j^{q_j} - c_j),$$

with inverse $c_j^{-1} t_j^{q_j - 1}$.

A_j is a G_j-crossed product algebra over Z_j with maximal subfield M. The F-automorphism τ_j on A_j satisfies $t_j l = \tau_j(l) t_j$ for all $l \in A_j$ which implies the inner automorphism

$$I_{t_j} : A \to A, \ d \mapsto t_j d t_j^{-1}$$

restricts to τ_j on A_j and to σ_{j+1} on M.

For any $\sigma \in G$ there exists an invertible $x_\sigma \in A$ such that the inner automorphism

$$I_{x_\sigma} : A \to A, \ y \mapsto x_\sigma y x_\sigma^{-1}$$

restricted to M is σ. Hence we have $x_{\sigma_{j+1}} = t_j$ with $x_{\sigma_{j+1}}$ as defined in Lemma 3. We know that $\{1, t_j, \ldots, {t_j}^{q_j - 1}\}$ is a basis for A_{j+1} as a left A_j-module. By (3) we have $x_{\sigma_{j+1}^2} = a_1 {t_j}^2$, $x_{\sigma_{j+1}^3} = a_2 {t_j}^3, \ldots$ for suitable $a_i \in M^\times$, so that w.l.o.g. $\{1, x_{\sigma_{j+1}}, \ldots, x_{\sigma_{j+1}^{q_j - 1}}\}$ is a basis for A_{j+1} as a left A_j-module.

Since A_j is a G_j-crossed product algebra, it has an M-basis $\{x_\rho \mid \rho \in G_j\}$, and hence A_{j+1} has basis

$$\{x_\rho x_{\sigma_{j+1}^i} \mid \rho \in G_j, \ 0 \leq i \leq q_j - 1\}$$

as M-vector space. Additionally, $x_\rho x_{\sigma_{j+1}^i} \in M^\times x_{\rho \sigma_{j+1}^i}$ by Lemma 3 (iii) and thus A_{j+1} has the M-basis

$$\{x_{\rho \sigma_{j+1}^i} \mid \rho \in G_j, \ 0 \leq i \leq q_j - 1\}. \tag{17}$$

A_{j+1} is a G_{j+1}-crossed product algebra and thus also has the M-basis $\{x_\sigma \mid \sigma \in G_{j+1}\}$. We use these two basis to show that $G_{j+1} = G_j < \sigma_{j+1} >$: Write

$$x_{\rho \sigma_{j+1}^i} = \sum_{\sigma \in G_{j+1}} m_\sigma x_\sigma$$

for some $m_\sigma \in M$, not all zero. Then

$$x_{\rho\sigma^i_{j+1}} m = \sum_{\sigma \in G_{j+1}} m_\sigma x_\sigma m = \sum_{\sigma \in G_{j+1}} m_\sigma \sigma(m) x_\sigma,$$

and

$$x_{\rho\sigma^i_{j+1}} m = \rho\sigma^i_{j+1}(m) x_{\rho\sigma^i_{j+1}} = \rho\sigma^i_{j+1}(m) \sum_{\sigma \in G_{j+1}} m_\sigma x_\sigma,$$

for all $m \in M$. Let $\sigma \in G_{j+1}$ be such that $m_\sigma \neq 0$, then in particular $m_\sigma \sigma(m) x_\sigma = \rho\sigma^i_{j+1}(m) m_\sigma x_\sigma$ for all $m \in M$, that is $\sigma = \rho\sigma^i_{j+1}$. This means that $\{\rho\sigma^i_{j+1} \mid \rho \in G_j,\ 0 \leq i \leq q_j - 1\} \subset G_{j+1}$. Both sets have the same size so must be equal and we conclude $G_{j+1} = G_j < \sigma_{j+1} >$.

Finally we prove G_j is a normal subgroup of G_{j+1}: the inner automorphism $I_{x_{\sigma_{j+1}}}$ restricts to the F-automorphism τ_j of A_j. This implies

$$x_{\sigma_{j+1}} x_\rho x^{-1}_{\sigma_{j+1}} \in A_j,$$

for all $\rho \in G_j$. Furthermore,

$$x_{\sigma_{j+1}\rho\sigma^{-1}_{j+1}} \in M^\times x_{\sigma_{j+1}} x_\rho x_{\sigma^{-1}_{j+1}} = M^\times x_{\sigma_{j+1}} x_\rho x^{-1}_{\sigma_{j+1}} \subset A_j,$$

for all $\rho \in G_j$ by Lemma 3. Hence $\sigma_{j+1}\rho\sigma^{-1}_{j+1} \in G_j$ because A_j is a G_j-crossed product algebra. Similarly, we see $\sigma^r_{j+1}\rho\sigma^{-r}_{j+1} \in G_j$ for all $r \in \mathbb{N}$. Let $g \in G_{j+1}$ be arbitrary and write $g = h\sigma^r_{j+1}$ for some $h \in G_j$, $r \in \{0, \dots, q_j - 1\}$ which we can do because $G_{j+1} = G_j < \sigma_{j+1} >$. Then

$$g\rho g^{-1} = (h\sigma^r_{j+1})\rho(h\sigma^r_{j+1})^{-1} = h(\sigma^r_{j+1}\rho\sigma^{-r}_{j+1})h^{-1} \in G_j,$$

for all $\rho \in G_j$ so G_j is indeed normal.

It is well known that a group G is solvable if and only if given a normal subgroup H of G, both H and G/H are solvable. It is clear now that G_{j+1}/G_j is cyclic and hence solvable, which implies G_{j+1} is solvable as required. □

4. Solvable crossed product algebras

We keep the assumptions from the previous section, but from now on we focus on the case that M/F is a Galois extension, i.e. now A is a G-crossed product algebra. We obtain the next result as a special case of Theorem 7 and Corollaries 8 and 10:

Theorem 14. *Let M/F be a Galois field extension of degree n with non-trivial solvable $G = \mathrm{Aut}_F(M)$, and A a central simple algebra of degree n over F with maximal subfield M. Then A is a G-crossed product algebra and there exists a chain of subalgebras*

$$M = A_0 \subset A_1 \subset \ldots \subset A_k = M(G) = A,$$

of A which are generalized cyclic algebras of degree $\prod_{l=0}^{i-1} q_l$ over $Z_i = \mathrm{Fix}(G_i)$ of the type

$$A_{i+1} \cong A_i[t_i;\tau_i]/A_i[t_i;\tau_i](t_i^{q_i} - c_i)$$

for all $i \in \{0,\dots,k-1\}$, satisfying (i), (ii), (iii) in Theorem 7. Additionally the following holds for all $i \in \{1,\dots,k\}$:
(iv) Z_{i-1}/Z_i has prime degree q_{i-1} and A_i is the centralizer of $\mathrm{Fix}(G_i)$ in A.

In general, it is not always easy to decide if a given crossed product algebra is a division algebra or not.

Theorem 15. *In the setup of Theorem 14, the solvable crossed product algebra A is a division algebra if and only if*

$$b\tau_i(b)\cdots\tau_i^{q_i-1}(b) \neq c_i \tag{18}$$

for all $b \in A_i$ and $i \in \{0,\dots,k-1\}$.

Proof. If A is a division algebra then so are all the subalgebras A_i, $i \in \{0,\dots,k-1\}$. In particular, this means that $t_i^{q_i} - c_i \in A_i[t_i;\tau_i]$ is an irreducible twisted polynomial for all $i \in \{0,\dots,k-1\}$, i.e.

$$b\tau_i(b)\cdots\tau_i^{q_i-1}(b) \neq c_i$$

for all $b \in A_i$ [7, 1.3.16].

Conversely suppose (18) holds for all $b \in A_i$ and $i \in \{0,\dots,k-1\}$. We prove by induction that then A_i is a division algebra for all $i \in \{0,\dots,k\}$, thus in particular so is $A = A_k$: $A_0 = M$ is a field. Assume as induction hypothesis that A_j is a division algebra for some $j \in \{0,\dots,k-1\}$. By the proof of Theorem 7, $\tau_j^{q_j}$ is the inner automorphism $I_{c_j}(z) = c_j z c_j^{-1}$ on A_j. Therefore

$$A_{j+1} \cong A_j[t_j;\tau_j]/A_j[t_j;\tau_j](t_j^{q_j} - c_j)$$

is a division algebra since $t_j^{q_j} - c_j \in A_j[t_j;\tau_j]$ is irreducible by [7, 1.3.16], because by assumption

$$b\tau_j(b)\cdots\tau_j^{q_j-1}(b) \neq c_j,$$

for all $b \in A_j$. Thus A_i is a division algebra for all $i \in \{0,\dots,k\}$ by induction. □

The next result follows from Theorem 13. It generalizes [14, (9)] and characterizes solvable crossed product algebras via generalized cyclic algebras:

Corollary 16. *Let A be a crossed product algebra of degree n over F with maximal subfield M such that M/F is a Galois field extension. Then $G = \mathrm{Gal}(M/F)$ is solvable if there exists a chain of subalgebras $M = A_0 \subset A_1 \subset \ldots \subset A_k = A$ of A which all have maximal subfield M, and are generalized cyclic algebras*

$$A_{i+1} \cong A_i[t_i;\tau_i]/A_i[t_i;\tau_i](t_i^{q_i} - c_i),$$

over their centers for all $i \in \{0, \dots, k-1\}$, where q_i is a prime, τ_i is an F-automorphism of A_i of inner order q_i which restricts to an automorphism $\sigma_{i+1} \in G$, and $c_i \in \mathrm{Fix}(\tau_i)^\times$.

Remark 17. Let M/F be a finite Galois field extension with non-trivial solvable Galois group G and A a solvable crossed product algebra over F with maximal subfield M.

A close inspection of Albert's proof [1, p. 182-187] shows that he constructs the same chain of algebras

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

inside a solvable crossed product A as we obtain in Theorem 14, but they are not explicitly identified as generalized cyclic algebras. We also obtain a converse of Albert's statement (Corollary 16).

Theorem 14 also tells us something about the existence of n-central elements in a solvable crossed product algebra A, as t_{k-1} is a q_{k-1}-central element in A. Recall that for a central simple algebra A over F whose degree is a multiple of n, $u \in A \setminus F$ is called an *n-central element* if $u^n \in F^\times$ and $u^m \notin F$ for all $1 \leq m < n$. The n-central elements play an important role in the structure of central simple algebras.

Corollary 18. *Let A be a solvable G-crossed product division algebra over F. Then*

$$A \cong D[t; \tau]/D[t; \tau](t^q - c) = (D, \tau, c)$$

is a generalized cyclic algebra, where D is either a central simple algebra over its center and τ a suitable automorphism of D of finite inner order which is a prime q, or D is a cyclic Galois field extension of F of prime degree q with Galois group $G =< \tau >$. A contains a q-central element.

Proof. The first assertion follows directly from Theorem 14. In particular, then $t \in A$ is a non-central element such that $t^q \in F^\times$ and $t^m \notin F$ for all $1 \leq m < q$. □

5. Some simple consequences for admissible groups

A finite group G is called *admissible* over a field F, if there exists a G-crossed product division algebra over F [16].

Suppose G is a finite solvable group, so that we have a chain of normal subgroups $\{1\} = G_0 \leq \ldots \leq G_k = G$, where $G_j \triangleleft G_{j+1}$ and G_{j+1}/G_j is cyclic of prime order q_j for all $j \in \{0, \dots, k-1\}$ as in (5) and (6).

Suppose G is admissible over F. Then Theorem 14 shows that the subgroups G_i of G appearing in the chain of normal subgroups of G are admissible over suitable intermediate fields of M/F:

Theorem 19. *Suppose G is admissible over a field F. Then each G_i in the above chain is admissible over the intermediate field $Z_i = \mathrm{Fix}(G_i)$ of M/F and*

$$[Z_i : F] = \prod_{j=i}^{k-1} q_j,$$

$i \in \{1, \dots, k\}$. *In particular, G_{k-1} is admissible over $Z_{k-1} = \mathrm{Fix}(G_{k-1})$ which has prime degree q_{k-1} over F.*

Proof. As G is F-admissible there exists a G-crossed product division algebra A over F and a chain of generalized cyclic division algebras $M = A_0 \subset \ldots \subset A_k = A$ over F, such that

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i) \tag{19}$$

for all $i \in \{0, \dots, k-1\}$, where τ_i is an automorphism of A_i of inner order q_i which restricts to an automorphism $\sigma_{i+1} \in G$ and $c_i \in \mathrm{Fix}(\tau_i)$ is invertible (Theorem 14). A_i is a G_i-crossed product division algebra over Z_i with maximal subfield M and M/Z_i is a Galois field extension with $\mathrm{Gal}(M/Z_i) = G_i$, i.e. G_i is Z_i-admissible. □

Example 20. Let $G = \mathbf{S_4}$, then G is $\mathbb{Q}$-admissible [16, Theorem 7.1], so there exists a central simple division algebra D over $\mathbb{Q}$ with maximal subfield M, such that M/F is a Galois field extension and $\mathrm{Gal}(M/F) = G$ is a finite solvable group. Let

$$\{\mathrm{id}\} \lhd <(12)(34)> \lhd \mathbf{K} \lhd \mathbf{A_4} \lhd \mathbf{S_4}$$

be its subnormal series, where $\mathbf{K}$ is the Klein four-group and $\mathbf{A_4}$ is the alternating group, and

$$\mathbf{S_4}/\mathbf{A_4} \cong \mathbb{Z}/2\mathbb{Z}, \mathbf{A_4}/\mathbf{K} \cong \mathbb{Z}/3\mathbb{Z},\ \mathbf{K}/<(12)(34)> \cong \mathbb{Z}/2\mathbb{Z},\ <(12)(34)>/\{\mathrm{id}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

By Corollary 16, there exists a corresponding chain of division algebras

$$M = A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4 = D$$

over $\mathbb{Q}$, such that

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

is a generalized cyclic division algebra over its center for all $i \in \{0, 1, 2, 3\}$, where τ_i is an automorphism of A_i, whose restriction to M is $\sigma_{i+1} \in G$, $c_i \in \mathrm{Fix}(\tau_i)$, and τ_i has inner order $2, 2, 3, 2$ for $i = 0, 1, 2, 3$ respectively. Moreover, we have $q_0 = q_1 = q_3 = 2$ and $q_2 = 3$, and A_i has degree $\prod_{l=0}^{i-1} q_l$ over its center Z_i for all $i \in \{1, 2, 3, 4\}$ by Theorem 14. In addition, by Theorem 19 we conclude:

(i) $\mathbf{A_4}$ is admissible over the quadratic field extension $Z_3 = \mathrm{Fix}(\mathbf{A_4}) \subset M$ of $\mathbb{Q}$.
(ii) $\mathbf{K}$ is admissible over the field extension $Z_2 = \mathrm{Fix}(\mathbf{K}) \subset M$ of $\mathbb{Q}$ of degree 6.
(iii) $<(12)(34)>$ is admissible over the field extension $\mathrm{Fix}(<(12)(34)>) \subset M$ of $\mathbb{Q}$ of degree 12 which is Z_1.

Schacher proved that for every finite group G, there exists an algebraic number field F such that G is admissible over F [16, Theorem 9.1]. Combining this with Theorem 14 we obtain:

Corollary 21. *Let G be a finite solvable group. Then there exists an algebraic number field F and a G-crossed product division algebra A over F. Furthermore, there exists a chain of division algebras $M = A_0 \subset A_1 \subset \ldots \subset A_k = A$ over F, such that*

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

is a generalized cyclic algebra over its center Z_i for all $i \in \{0, \ldots, k-1\}$, satisfying the properties listed in Theorems 7 and 14.
In particular, each G_i is admissible over Z_i.

Proof. Such a field F and division algebra D exist by [16, Theorem 9.1]. The assertion follows by Corollary 16. □

In [18, Theorem 1], Sonn proved that a finite solvable group is admissible over $\mathbb{Q}$ if and only if all its Sylow subgroups are metacyclic, i.e. if every Sylow subgroup H of G has a cyclic normal subgroup N, such that H/N is also cyclic. Combining this with Theorem 14 we conclude:

Corollary 22. *Let G be a finite solvable group such that all its Sylow subgroups are metacyclic. Then there exists a G-crossed product division algebra A over $\mathbb{Q}$, and a chain of division algebras $M = A_0 \subset A_1 \subset \ldots \subset A_k = A$ over $\mathbb{Q}$, such that*

$$A_{i+1} \cong A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

is a generalized cyclic algebra over its center Z_i for all $i \in \{0, \ldots, k-1\}$ satisfying the properties listed in Theorems 7 and 14.
In particular, each G_i is admissible over the field extension Z_i of $\mathbb{Q}$.

6. How to construct crossed product division algebras containing a given abelian Galois field extension as a maximal subfield

For S a unital ring and τ an injective endomorphism of S, the twisted polynomial $f(t) = t^q - c \in S[t; \tau]$ is right invariant, if

$$\tau^q(z)c = c\tau^i(z) \text{ and } \tau(c) = c$$

for all $z \in S$, $0 \leq i < q$.

Let M/F be a Galois field extension of degree n with abelian Galois group $G = \mathrm{Gal}(M/F)$. We now show how to canonically construct crossed product division algebras of degree n over F containing M as a subfield. This generalizes a result by Albert in which $n = 4$ and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ [1, p. 186], cf. also [7, Theorem 2.9.55]: For $n = 4$ every central division algebra containing a quartic abelian extension M

with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be obtained this way [1, p. 186], that means as a generalized cyclic algebra (D, τ, c) with D a quaternion algebra over its center.

Another way to construct such a crossed product algebra is via generic algebras, using a process going back to Amitsur and Saltman [3], described also in [7, 4.6].

As G is a finite abelian group, we have a chain of subgroups $\{1\} = G_0 \leq \ldots \leq G_k = G$, such that $G_j \triangleleft G_{j+1}$ and G_{j+1}/G_j is cyclic of prime order $q_j > 1$ for all $j \in \{0, \ldots, k-1\}$. We use this chain to construct the algebras we want:

$G_1 =< \sigma_1 >$ is cyclic of prime order $q_0 > 1$ for some $\sigma_1 \in G$. Let $\tau_0 = \sigma_1$. Choose any $c_0 \in F^\times$ that satisfies

$$z\tau_0(z) \cdots \tau_0^{q_0-1}(z) \neq c_0$$

for all $z \in M$ and define

$$f(t_0) = t_0^{q_0} - c_0 \in M[t_0; \tau_0].$$

Since τ_0 has order q_0, we have $\tau_0^{q_0}(z)c_0 = zc_0 = c_0 z$ for all $z \in M$, so that $f(t_0)$ is a right invariant twisted polynomial and hence

$$A_1 = M[t_0; \tau_0]/M[t_0; \tau_0](t_0^{q_0} - c_0)$$

is an associative algebra which is cyclic of degree q_0 over $\mathrm{Fix}(\tau_0)$. Moreover, $f(t_0)$ is irreducible by [7, 2.6.20 (i)] and therefore A_1 is a division algebra.

Now G_2/G_1 is cyclic of prime order q_1, say

$$G_2/G_1 = \{\sigma_2^i G_1 \mid i \in \mathbb{Z}\}$$

for some $\sigma_2 \in G_2$ where $\sigma_2^{q_1} \in G_1$. As $\sigma_2^{q_1} \in G_1$ we have $\sigma_2^{q_1} = \sigma_1^\mu$ for some $\mu \in \{0, \ldots, q_0 - 1\}$. Define $c_1 = l_1 t_0^\mu$ for some $l_1 \in F^\times$ and define the map

$$\tau_1 : A_1 \to A_1,\ \sum_{i=0}^{q_0-1} m_i t_0^i \mapsto \sum_{i=0}^{q_0-1} \sigma_2(m_i) t_0^i,$$

which is an automorphism of A_1 by a straightforward calculation.

Denote the multiplication in A_1 by $\circ$. Then $\tau_1(c_1) = \sigma_2(l_1)t_0^\mu = l_1 t_0^\mu = c_1$. We have

$$\tau_1^{q_1}\Big(\sum_{i=0}^{q_0-1} m_i t_0^i\Big) \circ c_1 = \sum_{i=0}^{q_0-1} \sigma_2^{q_1}(m_i) t_0^i \circ l_1 t_0^\mu = \sum_{i=0}^{q_0-1} l_1 \sigma_1^\mu(m_i) t_0^i \circ t_0^\mu$$

and

$$c_1 \circ \sum_{i=0}^{q_0-1} m_i t_0^i = l_1 t_0^\mu \circ \sum_{i=0}^{q_0-1} m_i t_0^i = \sum_{i=0}^{q_0-1} l_1 \sigma_1^\mu(m_i) t_0^\mu \circ t_0^i$$

for all $m_i \in M$. Hence $\tau_1^{q_1}(z) \circ c_1 = c_1 \circ z$ for all $z \in A_1$ and $\tau_1(c_1) = c_1$, thus $f(t_1) = t_1^{q_1} - c_1 \in A_1[t_1;\tau_1]$ is a right invariant twisted polynomial and

$$A_2 = A_1[t_1;\tau_1]/A_1[t_1;\tau_1](t_1^{q_1} - c_1)$$

is a finite-dimensional associative algebra over $\mathrm{Fix}(\tau_1) \cap C(A_1) = \mathrm{Fix}(\tau_1) \cap \mathrm{Fix}(\tau_0) \supset F$ [15].

Again, G_3/G_2 is cyclic of prime order q_2, say

$$G_3/G_2 = \{\sigma_3^i G_2 \mid i \in \mathbb{Z}\}$$

for some $\sigma_3 \in G$ with $\sigma_3^{q_2} \in G_2$. Write $\sigma_3^{q_2} = \sigma_2^{\lambda_1}\sigma_1^{\lambda_0}$ for some $\lambda_1 \in \{0,\dots,q_1-1\}$ and $\lambda_0 \in \{0,\dots,q_0-1\}$. The map

$$H_{\sigma_3} : A_1 \to A_1,\ \sum_{i=0}^{q_0-1} m_i t_0^i \mapsto \sum_{i=0}^{q_0-1} \sigma_3(m_i) t_0^i,$$

is an automorphism of A_1 by a straightforward calculation. Define

$$\tau_2 : A_2 \to A_2,\ \sum_{i=0}^{q_1-1} x_i t_1^i \mapsto \sum_{i=0}^{q_1-1} H_{\sigma_3}(x_i) t_1^i \ (x_i \in A_1).$$

Then a straightforward calculation using that H_{σ_3} commutes with τ_1 and $H_{\sigma_3}(c_1) = c_1$ shows that τ_2 is an automorphism of A_2. Define

$$c_2 = l_2 t_0^{\lambda_0} t_1^{\lambda_1}$$

for some $l_2 \in F^\times$. Denote the multiplication in A_i by $\circ_{A_i}$ and let $x_i = \sum_{j=0}^{q_0-1} y_{ij} t_0^j \in A_1$, $y_{ij} \in M$, $i \in \{0,\dots,q_1-1\}$. Then

$$\tau_2(c_2) = \tau_2(l_2 t_0^{\lambda_0} t_1^{\lambda_1}) = H_{\sigma_3}(l_2 t_0^{\lambda_0}) t_1^{\lambda_1} = l_2 t_0^{\lambda_0} t_1^{\lambda_1} = c_2.$$

Furthermore we have

$$\begin{aligned}
\tau_2^{q_2}\Big(\sum_{i=0}^{q_1-1} x_i t_1^i\Big) \circ_{A_2} c_2 &= \sum_{i=0}^{q_1-1} H_{\sigma_3}^{q_2}(x_i) t_1^i \circ_{A_2} l_2 t_0^{\lambda_0} t_1^{\lambda_1} \\
&= \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_0-1} \sigma_3^{q_2}(y_{ij}) t_0^j t_1^i \circ_{A_2} l_2 t_0^{\lambda_0} t_1^{\lambda_1} \\
&= \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_0-1} \sigma_2^{\lambda_1}(\sigma_1^{\lambda_0}(y_{ij})) t_0^j t_1^i \circ_{A_2} l_2 t_0^{\lambda_0} t_1^{\lambda_1} \\
&= \sum_{i=0}^{q_1-1} \Big(\sum_{j=0}^{q_0-1} \sigma_2^{\lambda_1}(\sigma_1^{\lambda_0}(y_{ij})) t_0^j \circ_{A_1} \tau_1^i(l_2 t_0^{\lambda_0})\Big) t_1^i \circ_{A_2} t_1^{\lambda_1} \\
&= \sum_{i=0}^{q_1-1} \Big(\sum_{j=0}^{q_0-1} l_2 \sigma_2^{\lambda_1}(\sigma_1^{\lambda_0}(y_{ij})) t_0^j \circ_{A_1} t_0^{\lambda_0}\Big) t_1^i \circ_{A_2} t_1^{\lambda_1},
\end{aligned}$$

and

$$\begin{aligned} c_2 \circ_{A_2} \sum_{i=0}^{q_1-1} x_i t_1^i &= l_2 t_0^{\lambda_0} t_1^{\lambda_1} \circ_{A_2} \sum_{i=0}^{q_1-1} x_i t_1^i = \sum_{i=0}^{q_1-1} \big(l_2 t_0^{\lambda_0} \circ_{A_1} \tau_1^{\lambda_1}(x_i)\big) t_1^{\lambda_1} \circ_{A_2} t_1^i \\ &= \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_0-1} \big(l_2 t_0^{\lambda_0} \circ_{A_1} \sigma_2^{\lambda_1}(y_{ij}) t_0^j\big) t_1^{\lambda_1} \circ_{A_2} t_1^i \\ &= \sum_{i=0}^{q_1-1} \Big(\sum_{j=0}^{q_0-1} l_2 \sigma_1^{\lambda_0}(\sigma_2^{\lambda_1}(y_{ij})) t_0^{\lambda_0} \circ_{A_1} t_0^j \Big) t_1^{\lambda_1} \circ_{A_2} t_1^i. \end{aligned}$$

Hence $\tau_2^{q_2}(z) \circ_{A_2} c_2 = c_2 \circ_{A_2} z$ for all $z \in A_2$ and $\tau_2(c_2) = c_2$, therefore

$$f(t_2) = t_2^{q_2} - c_2 \in A_2[t_2; \tau_2]$$

is a right invariant twisted polynomial and thus

$$A_3 = A_2[t_2; \tau_2]/A_2[t_2; \tau_2](t_2^{q_2} - c_2)$$

is a finite-dimensional associative algebra over

$$\mathrm{Fix}(\tau_2) \cap C(A_2) = \mathrm{Fix}(\tau_0) \cap \mathrm{Fix}(\tau_1) \cap \mathrm{Fix}(\tau_2) \supset F$$

[15]. Continuing in this manner we obtain a chain $M = A_0 \subset \ldots \subset A_k$ of finite-dimensional associative algebras

$$A_{i+1} = A_i[t_i; \tau_i]/A_i[t_i; \tau_i](t_i^{q_i} - c_i)$$

over

$$\mathrm{Fix}(\tau_i) \cap C(A_i) = \mathrm{Fix}(\tau_0) \cap \mathrm{Fix}(\tau_1) \cap \cdots \cap \mathrm{Fix}(\tau_i) \supset F,$$

for all $i \in \{0, \ldots, k-1\}$, where $\tau_0 = \sigma_1$ and τ_i restricts to σ_{i+1} on M for all $i \in \{0, \ldots, k-1\}$. Moreover,

$$[A_i : M] = [A_i : A_{i-1}] \cdots [A_1 : M] = \prod_{l=0}^{i-1} q_l$$

hence $[A_k : F] = \big(\prod_{l=0}^{k-1} q_l\big) n = n^2$, and A_k contains M as a subfield.

Lemma 23. *For all $i \in \{0, \ldots, k-1\}$, $\tau_i : A_i \to A_i$ has inner order q_i.*

Proof. The automorphism $\tau_0 = \sigma_1 : M \to M$ has inner order q_0.

Fix $i \in \{1, \ldots, k-1\}$. A_i is finite-dimensional over F, so it is also finite-dimensional over its center $C(A_i) \supset F$. Recall that $\tau_i^{q_i}(z) c_i = c_i z$ for all $z \in A_i$, in particular $\tau_i^{q_i}|_{C(A_i)} = id$. As q_i is prime this means either $\tau_i|_{C(A_i)} = id$ or $\tau_i|_{C(A_i)}$ has order $q_i > 1$.

Assume that $\tau_i|_{C(A_i)} = id$, then τ_i is an inner automorphism of A_i by the Theorem of Skolem-Noether, say $\tau_i(z) = uzu^{-1}$ for some invertible $u \in A_i$, for all $z \in A_i$. In particular $\tau_i(m) = \sigma_{i+1}(m) = umu^{-1}$ for all $m \in M$. Write

$$u = \sum_{j=0}^{q_{i-1}-1} u_j t_{i-1}^j$$

for some $u_j \in A_{i-1}$, thus

$$\begin{aligned}\sigma_{i+1}(m)u = \sigma_{i+1}(m) \sum_{j=0}^{q_{i-1}-1} u_j t_{i-1}^j &= \sum_{j=0}^{q_{i-1}-1} u_j t_{i-1}^j m \\ &= \sum_{j=0}^{q_{i-1}-1} u_j \tau_{i-1}^j(m) t_{i-1}^j = \sum_{j=0}^{q_{i-1}-1} u_j \sigma_i^j(m) t_{i-1}^j.\end{aligned}$$

for all $m \in M$. Choose η_i with $u_{\eta_i} \neq 0$ then

$$\sigma_{i+1}(m) u_{\eta_i} = u_{\eta_i} \sigma_i^{\eta_i}(m), \tag{20}$$

for all $m \in M$.

If $i = 1$ we are done. If $i \geq 2$ then we can also write $u_{\eta_i} = \sum_{l=0}^{q_{i-2}-1} w_l t_{i-2}^l$ for some $w_l \in A_{i-2}$, therefore (20) yields

$$\begin{aligned}\sigma_{i+1}(m) \sum_{l=0}^{q_{i-2}-1} w_l t_{i-2}^l = \sum_{l=0}^{q_{i-2}-1} w_l t_{i-2}^l \sigma_i^{\eta_i}(m) &= \sum_{l=0}^{q_{i-2}-1} w_l \tau_{i-2}^l(\sigma_i^{\eta_i}(m)) t_{i-2}^l \\ &= \sum_{l=0}^{q_{i-2}-1} w_l \sigma_{i-1}^l(\sigma_i^{\eta_i}(m)) t_{i-2}^l,\end{aligned}$$

for all $m \in M$. Choose η_{i-1} with $w_{\eta_{i-1}} \neq 0$, then

$$\sigma_{i+1}(m) w_{\eta_{i-1}} = w_{\eta_{i-1}} \sigma_{i-1}^{\eta_{i-1}}(\sigma_i^{\eta_i}(m)),$$

for all $m \in M$.

Continuing in this manner we see that there exists $s \in M^\times$ such that

$$\sigma_{i+1}(m)s = s\sigma_1^{\eta_1}(\sigma_2^{\eta_2}(\cdots(\sigma_i^{\eta_i}(m)\cdots),$$

for all $m \in M$, hence

$$\sigma_{i+1}(m) = \sigma_1^{\eta_1}(\sigma_2^{\eta_2}(\cdots(\sigma_i^{\eta_i}(m)\cdots),$$

for all $m \in M$ where $\eta_j \in \{0, \dots, q_{j-1} - 1\}$ for all $j \in \{1, \dots, i\}$. But $\sigma_{i+1} \notin G_i$ and thus $\sigma_{i+1} \neq \sigma_1^{\eta_1} \circ \sigma_2^{\eta_2} \circ \cdots \circ \sigma_i^{\eta_i}$, a contradiction.

It follows that $\tau_i|_{C(A_i)}$ has order $q_i > 1$. By the Skolem-Noether Theorem the kernel of the restriction map $\mathrm{Aut}(A_i) \to \mathrm{Aut}(C(A_i))$ is the group of inner automorphisms of A_i, and so τ_i has inner order q_i. □

Let us furthermore assume that each c_i above, $i \in \{0, \dots, k-1\}$, is successively chosen such that

$$z\tau_i(z) \cdots \tau_i^{q_i - 1}(z) \neq c_i \tag{21}$$

for all $z \in A_i$, then using that τ_i has inner order q_i,

$$f(t_i) = t_i^{q_i} - c_i \in A_i[t_i; \tau_i]$$

is an irreducible twisted polynomial by Lemma 23 and thus A_{i+1} is a division algebra [7, 1.3.16].

Proposition 24. $C(A_k) = F$.

Proof. $F \subset C(A_k)$ by construction. Let now $z = z_0 + z_1 t_{k-1} + \ldots + z_{q_{k-1}-1} t_{k-1}^{q_{k-1}-1} \in C(A_k)$ where $z_i \in A_{k-1}$. Then z commutes with all $l \in A_{k-1}$, hence $lz_i = z_i \tau_{k-1}^i(l)$ for all $i \in \{0, \dots, q_{k-1} - 1\}$. This implies $z_0 \in C(A_{k-1})$ and $z_i = 0$ for all $i \in \{1, \dots, q_{k-1} - 1\}$, otherwise z_i is invertible and τ_{k-1}^i is inner, a contradiction by Lemma 23. Thus $z = z_0 \in C(A_{k-1})$. A similar argument shows $z \in C(A_{k-1})$ and continuing in this manner we conclude $z \in M = C(A_0)$.

Suppose for contradiction $z \notin F$, then $\rho(z) \neq z$ for some $\rho \in G$. Since the σ_{i+1} were chosen so that they generate the cyclic factor groups G_{i+1}/G_i, we can write $\rho = \sigma_1^{i_0} \circ \sigma_2^{i_1} \circ \cdots \circ \sigma_k^{i_{k-1}}$ for some $i_s \in \{0, \dots, q_s - 1\}$. We have

$$\begin{aligned} t_0^{i_0} t_1^{i_1} \cdots t_{k-1}^{i_{k-1}} z &= \sigma_1^{i_0}(\sigma_2^{i_1}(\cdots(\sigma_k^{i_{k-1}}(z) \cdots) t_0^{i_0} t_1^{i_1} \cdots t_{k-1}^{i_{k-1}} \\ &= \rho(z) t_0^{i_0} t_1^{i_1} \cdots t_{k-1}^{i_{k-1}} \neq z t_0^{i_0} t_1^{i_1} \cdots t_{k-1}^{i_{k-1}}, \end{aligned}$$

contradicting the assumption that $z \in C(A_k)$. Therefore $C(A_k) \subset F$. □

This yields a recipe for constructing a G-crossed product division algebra $A = A_k$ over F with maximal subfield M provided it is possible to find suitable c_i's satisfying (21). By Corollary 16, every abelian crossed product division algebra that is solvable can be obtained this way, starting with a suitable M/F.

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Email address: christian__jb@hotmail.co.uk; susanne.pumpluen@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom