SUPPLEMENTAL MATERIAL

Many-body radiative decay in strongly interacting Rydberg ensembles

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I. DERIVATION OF THE MASTER EQUATION WITH COLLECTIVE JUMP OPERATORS

In this section, we derive equation (5) from the main paper. We start with the *Redfield equation*, where the Born-Markov approximations have already been assumed [1, 2]. We label the trace over the bath degrees of freedom with $\mathrm{tr}_E\{\cdot\}$ and the Hamiltonian in the interaction picture with $H''(t)$. The time evolution of the system at time t is exclusively dependent on the current state $\rho_{\mathrm{int}}(t) =: \rho_{\mathrm{int}}$ in the rotating frame and reads:

$$\dot{\rho}_{\mathrm{int}} = -\int_0^t \mathrm{d}\tau\, \mathrm{tr}_E\left\{[H''(t),[H''(\tau),\rho_{\mathrm{int}}\otimes\rho_E(0)]]\right\}\,. \tag{1}$$

To obtain a Markovian master equation, the explicit dependence on the absolute time t must be eliminated. We therefore substitute $\tau = t - t'$ into (1) and assume that the bath correlation time τ_E is sufficiently small compared to τ ($\tau_E \ll \tau$). The integrand thus decays sufficiently fast, and the upper integration limit can be considered to be infinity to a good approximation. We now obtain the Markovian quantum master equation:

$$\dot{\rho}_{\mathrm{int}} = -\int_0^\infty \mathrm{d}\tau\, \mathrm{tr}_E\left\{[H''(t),[H''(t-\tau),\rho_{\mathrm{int}}\otimes\rho_E(0)]]\right\}\,. \tag{2}$$

It should be noted that the above equation does not resolve the dynamics on time scales of the correlation time τ_{E}. In this sense, the equation describes a coarse-grained time evolution. In the later calculation, we have to perform a rotating wave approximation (also called *secular approximation*) which means that we have to neglect fast oscillating terms in the order of magnitude of the correlation time τ_{E} by averaging. Only in this way, we end up with a Markovian master equation that is a generator of a quantum dynamical semigroup [1].

The Hamiltonian of the whole system is obtained by summing up Eq. (1), (2) and (3) from the main paper:

$$H = \underbrace{\omega_a \sum_k n_k + \frac{V}{2}\sum_{\langle km\rangle} n_k n_m}_{H_{\mathrm{atom}}} + \underbrace{\sum_{\mathbf{q}s}\omega_q a^\dagger_{\mathbf{q}s}a_{\mathbf{q}s}}_{H_{\mathrm{rad}}} + \underbrace{\sum_{k,\mathbf{q},s}\left(g_{\mathbf{q}s}a^\dagger_{\mathbf{q}s}e^{i\mathbf{q}\cdot\mathbf{r}_k} + \mathrm{h.c.}\right)\left(\sigma_k^+ + \sigma_k^-\right)}_{H_{\mathrm{int}}}\,. \tag{3}$$

We transform H into the interaction picture via a unitary transformation:

$$H''(t) = U H_{\mathrm{int}} U^\dagger \tag{4}$$
$$U = e^{it(H_{\mathrm{atom}}+H_{\mathrm{rad}})}\,. \tag{5}$$

We then use the Baker-Campbell-Hausdorff formula [3] to simplify H'' and subsequently obtain:

$$H''(t) = \sum_{k,\mathbf{q},s}\left(g_{\mathbf{q}s}a^\dagger_{\mathbf{q}s}\mathrm{e}^{-i(\mathbf{q}\cdot\mathbf{r}_k-\omega_q t)} + \mathrm{h.c.}\right)\cdot\underbrace{\left(\sigma_k^+\mathrm{e}^{it(\omega_a+V\sum_{m\in\mathcal{I}_k}n_m)} + \mathrm{h.c.}\right)}_{\mathcal{X}_k(t)}\,. \tag{6}$$

We label all atoms in the neighborhood of atom k with $m \in \mathcal{I}_k$.

Afterwards we introduce the projectors motivated and defined in figure (1) of the main paper. This allows the exponential term in $\mathcal{X}_k(t)$ to be represented in the basis of the neighbouring atoms. For $\mathcal{X}_k(t)$ one obtains:

$$\mathcal{X}_k(t) = \sigma_k^+\mathrm{e}^{it\omega_a}\sum_{\xi=0}^{2d}P_k^\xi\mathrm{e}^{it\xi V} + \mathrm{h.c.}\,. \tag{7}$$

We substitute $H''(t)$ from (6) into the Markovian quantum master equation (2). First, we simplify the occurring double commutator expression in (2). For the bath, we assume the vacuum state, i.e., a temperature $T = 0$. Thus, according to [1], we have for the expectation values of the bath operators:

$$\langle a_{\mathbf{q}} \rangle = \langle a_{\mathbf{q}}^\dagger \rangle = \langle a_{\mathbf{q}} a_{\mathbf{q}'} \rangle = \left\langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}'}^\dagger \right\rangle = \langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}'} \rangle = 0 \qquad \text{and} \qquad \left\langle a_{\mathbf{q}} a_{\mathbf{q}'}^\dagger \right\rangle = \delta_{\mathbf{q},\mathbf{q}'} \,. \tag{8}$$

After applying the trace over the degrees of freedom of the bath, the double commutator reads:

$$\begin{aligned} &\mathrm{tr}_E \left\{ \sum_{k,k'} \sum_{\mathbf{q},\mathbf{q}'} \sum_{s,s'} \left[H''_{\mathbf{q}k}(t), \left[H''_{\mathbf{q}'k'}(t-\tau), \rho_{\mathrm{int}} \otimes \rho_E(0) \right] \right] \right\} \\ &= \sum_{k,k'} \sum_{\mathbf{q}s} |g_{\mathbf{q}s}|^2 \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}_{kk'}} \left[\mathcal{X}_k(t), \mathcal{X}_{k'}(t-\tau)\rho_{\mathrm{int}} \right] \mathrm{e}^{-i\omega_q \tau} + \mathrm{h.c.}\,. \end{aligned} \tag{9}$$

Note that we now sum up pairwise over all atoms with index k and k'. If we add up all the intermediate results, the Markov master equation looks like this:

$$\dot{\rho}_{\mathrm{int}} = -\sum_{k,k'} \sum_{\mathbf{q}s} |g_{\mathbf{q}s}|^2 \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}_{kk'}} \int_0^\infty \mathrm{d}\tau \left[\mathcal{X}_k(t), \mathcal{X}_{k'}(t-\tau)\rho_{\mathrm{int}} \right] \mathrm{e}^{-i\omega_q \tau} + \mathrm{h.c.}\,,$$

where we have already interchanged integration and summation.

Next, we consider the integration of the time difference $\mathrm{d}\tau$. To do this, we insert the expression for $\mathcal{X}_k(t)$ from (7) and multiply the entire expression out. Summand by summand integrals of similar form are to be solved. Using the formula from [1]:

$$\int_0^\infty d\tau \mathrm{e}^{-i\tau(\omega_q \pm \omega_a)} = \pi\delta(\omega_q \pm \omega_a) - i\mathcal{P}\left(\frac{1}{\omega_q \pm \omega_a}\right) \tag{10}$$

and application of the rotating wave approximation for all fast oscillating terms with factors $\mathcal{O}(\mathrm{e}^{itV})$ or $\mathcal{O}(\mathrm{e}^{2itV})$, the equation can be further simplified.

Now the Markovian master equation is in a known form for several interacting atoms. The summation over the bath modes $\mathbf{q}$ with polarisation s is transformed into an integration analogously to what is done in Ref. [4]. After solving this one obtains a Markovian master equation:

$$\dot{\rho}_{\mathrm{int}} = \sum_{\xi=0}^{2d} \left(-i \sum_{k\neq k'} \tilde{V}^\xi_{kk'} \left[\sigma_k^+ P_k^\xi P_{k'}^\xi \sigma_{k'}, \rho_{\mathrm{int}} \right] + \sum_{k,k'} \tilde{\Gamma}^\xi_{kk'} \left[\sigma_{k'} P_{k'}^\xi \rho_{\mathrm{int}} P_k^\xi \sigma_k^+ - \frac{1}{2} \left\{ \sigma_k^+ P_k^\xi P_{k'}^\xi \sigma_{k'}, \rho_{\mathrm{int}} \right\} \right] \right) . \tag{11}$$

Where

$$\tilde{V}^\xi_{kk'} = -\frac{3\gamma^\xi}{4} \left[\alpha_{kk'} \frac{\cos b_{kk'}}{b_{kk'}} - \beta_{kk'} \left(\frac{\sin b_{kk'}}{b_{kk'}^2} + \frac{\cos b_{kk'}}{b_{kk'}^3} \right) \right] , \tag{12}$$

$$\tilde{\Gamma}^\xi_{kk'} = \frac{3\gamma^\xi}{2} \left[\alpha_{kk'} \frac{\sin b_{kk'}}{b_{kk'}} + \beta_{kk'} \left(\frac{\cos b_{kk'}}{b_{kk'}^2} - \frac{\sin b_{kk'}}{b_{kk'}^3} \right) \right] , \tag{13}$$

utilizing $b_{kk'} = \omega_a |\mathbf{r}_{kk'}|/c = 2\pi |\mathbf{r}_{kk'}|/\lambda_a$, $\alpha_{kk'} = 1 - (\mathbf{d}\cdot\mathbf{r}_{kk'})^2$, $\beta_{kk'} = 1 - 3(\mathbf{d}\cdot\mathbf{r}_{kk'})^2$ and $\gamma^\xi = \left|\mathbf{d}^2\right| (\omega_a + \xi V)^3 / 3\pi\epsilon_0 c^3$.

We use now the assumptions of our model where $\omega_a \ll V$ thus $\gamma^\xi \approx \gamma$. Furthermore, we use the fact that we are in a regime where the distance $|\mathbf{r}_{kk'}|$ between the Rydberg atoms is large with respect to the wavelength of the atomic transition λ_a. Therefore the off-diagonal elements of $\tilde{V}_{kk'}$ and $\tilde{\Gamma}_{kk'}$ vanish. Moreover $\tilde{\Gamma}_{kk'} \approx \gamma \,\forall k = k'$. Using the fact $\sum_\xi P_k^\xi = 1$ to simplify the anti-commutator we obtain:

$$\dot{\rho}_{\mathrm{int}} = \gamma \sum_k \left[\sum_{\xi=0}^{2d} \sigma_k^- P_k^\xi \, \rho_{\mathrm{int}} \, P_k^\xi \sigma_k^+ - \frac{1}{2} \{ n_k, \rho_{\mathrm{int}} \} \right] . \tag{14}$$

Measurements and simulations are always carried out in the laboratory frame. For this reason, we transform back into the same laboratory frame. It is valid using the chain rule and with U from (5):

$$\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}\left(U^\dagger \rho_{\mathrm{int}} U \right) = -i[H_{\mathrm{atom}}, \rho] + U^\dagger \dot{\rho}_{\mathrm{int}} U \,. \tag{15}$$

Let us now consider further the dissipative term. We apply Eq. (14) and obtain:

$$U^\dagger \dot{\rho}_{\rm int} U = \gamma \sum_k \left[\sum_{\xi=0}^{2d} \underbrace{U^\dagger \sigma_k^- P_k^\xi \, \rho_{\rm int} \, P_k^\xi \sigma_k^+ U}_{=: \, \mathcal{J}_1} - \frac{1}{2} \underbrace{U^\dagger \{n_k, \rho_{\rm int}\} U}_{=: \, \mathcal{J}_2} \right] . \tag{16}$$

It can be easily shown by recalculation that:

$$U^\dagger \sigma_k^- P_k^\xi U = e^{i\varphi} \sigma_k^- P_k^\xi \tag{17}$$

holds, where $\varphi \in \mathbb{R}$. Taking advantage of this identity, we transform the term $\mathcal{J}_1$:

$$\begin{aligned}
\mathcal{J}_1 &= U^\dagger \, \sigma_k^- P_k^\xi \qquad \rho_{\rm int} \qquad P_k^\xi \sigma_k^+ U & (18)\\
&= U^\dagger \, \sigma_k^- P_k^\xi U U^\dagger \rho_{\rm int} U U^\dagger P_k^\xi \sigma_k^+ U & (19)\\
&= e^{i\varphi} \sigma_k^- P_k^\xi \qquad \rho \qquad P_k^\xi \sigma_k^+ e^{-i\varphi} & (20)\\
&= \qquad \sigma_k^- P_k^\xi \qquad \rho \qquad P_k^\xi \sigma_k^+ \, . & (21)
\end{aligned}$$

We find that the jump operators do not change due to the transformation into the laboratory frame. For the transformation of $\mathcal{J}_2$, it follows analogously:

$$\mathcal{J}_2 = U^\dagger \{n_k, \rho_{\rm int}\} U = \{n_k, \rho\} \, . \tag{22}$$

It follows:

$$\dot{\rho} = -i[H_{\rm atom}, \rho] + \gamma \sum_k \left[\sum_{\xi=0}^{2d} \sigma_k^- P_k^\xi \, \rho \, P_k^\xi \sigma_k^+ - \frac{1}{2} \{n_k, \rho\} \right] , \tag{23}$$

which is the master equation with collective jump operators in the laboratory frame Eq. (5) from the main paper.

II. DECOHERENCE DYNAMICS

We analyse the time evolution of the coherence $X(t)$ defined in Eq. (6) in the main text. The time evolution can be calculated using the adjoint Lindblad operator $\mathcal{L}^\dagger$:

$$\frac{\partial}{\partial t} P_k^\xi \sigma_k^- = \mathcal{L}^\dagger [P_k^\xi \sigma_k^-] = \underbrace{i \Big[H_{\rm atom}, P_k^\xi \sigma_k^- \Big]}_{\text{coherent}} + \underbrace{\mathcal{D}^\dagger [P_k^\xi \sigma_k^-]}_{\text{dissipative}} . \tag{24}$$

For evaluating the coherent part we use

$$i \Big[H_{\rm atom}, P_k^\xi \sigma_k^- \Big] = i \frac{V}{2} \sum_{l, m \in \mathcal{I}_l} \Big[n_l n_m, P_k^\xi \sigma_k^- \Big] + i \omega_a \sum_l \Big[n_l, P_k^\xi \sigma_k^- \Big], \tag{25}$$

where we have labeled all atoms in the neighborhood of atom l with $m \in \mathcal{I}_l$. By using the canonical commutator relations for Pauli matrices and the relations

$$\sum_{m \in \mathcal{I}_l} n_m = \sum_{\eta=0}^{2d} \eta P_l^\eta \tag{26}$$

$$\Big[n_k, P_l^\xi \Big] = 0 \ \forall k, l \tag{27}$$

one obtains

$$i \Big[H_{\rm atom}, P_k^\xi \sigma_k^- \Big] = -i(\omega_a + \xi V) P_k^\xi \sigma_k^- . \tag{28}$$

The dissipative part yields different results depending on whether single-atom or many-body decay is considered. For the collective many-body decay one finds:

$$\mathcal{D}_{\mathrm{c}}^{\dagger}[P_k^{\xi}\sigma_k^-] = \gamma \sum_m \left[\sum_{\eta=0}^{2d} P_m^{\eta}\sigma_m^+ P_k^{\xi}\sigma_k^- P_m^{\eta}\sigma_m^- - \frac{1}{2}\Big\{n_m, P_k^{\xi}\sigma_k^-\Big\} \right] = -\gamma \left(\frac{1}{2}P_k^{\xi}\sigma_k^- + \sum_{m\in\mathcal{I}_k} n_m P_k^{\xi}\sigma_k^- \right). \tag{29}$$

After applying Eq. 26 this yields

$$\mathcal{D}_{\mathrm{c}}^{\dagger}[P_k^{\xi}\sigma_k^-] = -\gamma\left(\frac{1}{2}+\xi\right) P_k^{\xi}\sigma_k^- . \tag{30}$$

On the other hand, in the single-atom dissipation case, one obtains

$$\mathcal{D}_{\mathrm{s}}^{\dagger}[P_k^{\xi}\sigma_k^-] = \gamma \sum_m \left[\sigma_m^+ P_k^{\xi}\sigma_k^-\sigma_m^- - \frac{1}{2}\Big\{n_m, P_k^{\xi}\sigma_k^-\Big\} \right] \tag{31}$$

$$= -\gamma \left(\frac{1}{2}P_k^{\xi}\sigma_k^- + \xi P_k^{\xi}\sigma_k^- - \sum_{m\in\mathcal{I}_k} \sigma_m^+ P_k^{\xi}\sigma_m^-\sigma_k^- \right). \tag{32}$$

Again, applying Eq. 26 we find that

$$\mathcal{D}_{\mathrm{s}}^{\dagger}[P_k^{\xi}\sigma_k^-] = -\gamma\left(\frac{1}{2}+\xi\right) P_k^{\xi}\sigma_k^- + \gamma(\xi+1)P_k^{\xi+1}\sigma_k^- . \tag{33}$$

Taking the expectation value of Eq. (24), summing over k and dividing by the number of particles N then yields for collective many-body decay:

$$\begin{aligned}\frac{1}{N}\sum_k \left\langle \frac{\partial}{\partial t} P_k^{\xi}\sigma_k^- \right\rangle = \frac{\partial}{\partial t}\frac{1}{N}\sum_k \left\langle P_k^{\xi}\sigma_k^- \right\rangle = \frac{\partial}{\partial t}X_{\xi}^c &= -i(\omega_a+\xi V)\frac{1}{N}\sum_k \left\langle P_k^{\xi}\sigma_k^- \right\rangle - \gamma\left(\frac{1}{2}+\xi\right)\frac{1}{N}\sum_k \left\langle P_k^{\xi}\sigma_k^- \right\rangle \\ &= -i(\omega_a+\xi V)X_{\xi}^c - \gamma\left(\frac{1}{2}+\xi\right)X_{\xi}^c. \end{aligned}\tag{34}$$

With single-body decay one obtains on the other hand

$$\frac{\partial}{\partial t}X_{\xi}^s = -i(\omega_a+\xi V)X_{\xi}^s - \gamma\left(\frac{1}{2}+\xi\right)X_{\xi}^s + \gamma(\xi+1)X_{\xi+1}^s. \tag{35}$$

These equations can be readily integrated. For the product state

$$|\Psi_0\rangle = \left(\frac{1}{2}\right)^{\frac{N}{2}} \bigotimes_k \left[|\downarrow\rangle_k + |\uparrow\rangle_k\right], \tag{36}$$

on obtains the initial values

$$X_{\xi}^{\mathrm{s/c}}(0) = \frac{1}{N}\sum_k \langle\Psi_0|P_k^{\xi}\sigma_k^-|\Psi_0\rangle = 2^{-2d-1}\binom{2d}{\xi}. \tag{37}$$

Using these, one finds (for $d=1,2,3$)

$$|X^c(t)| = |\sum_{\xi} X_{\xi}^c(t)| = \frac{1}{2^{d+1}}e^{-\frac{(2d+1)\gamma t}{2}}\left[\cos(Vt)+\cosh(\gamma t)\right]^d \tag{38}$$

and

$$|X^s(t)| = |\sum_{\xi} X_{\xi}^s(t)| = \frac{1}{2^{d+1}}e^{-\frac{(2d+1)\gamma t}{2}}\left[\frac{(\gamma^2+V^2)\cosh(\gamma t) + V\left(V\cos(Vt)+2\gamma\sin(Vt)\right) + 2\gamma^2\sinh(\gamma t)}{\gamma^2+V^2}\right]^d, \tag{39}$$

whose short-time expansions are provided in the main text.

III. EVOLUTION OF TWO-BODY CORRELATIONS UNDER MANY-BODY DECAY

Here we briefly discuss the dynamics of two-body correlation functions and work out which difference arise between single-body and many-body dissipation. The effects are best observed in correlation functions of the type $\langle\sigma_k^+\sigma_m^-\rangle$, which involve coherences of the many-body state. For the sake of simplicity we consider a one-dimensional system, for which we obtain the following equations of motion for nearest and next-nearest neighbor correlations:

$$\begin{aligned}\frac{\partial}{\partial t}\langle\sigma_k^+\sigma_{k+1}^-\rangle &= iV\langle(n_{k-1}-n_{k+2})\sigma_k^+\sigma_{k+1}^-\rangle\\ &\quad -\gamma\left[\langle\sigma_k^+\sigma_{k+1}^-\rangle+\underline{\langle(n_{k-1}+n_{k+2})\,\sigma_k^+\sigma_{k+1}^-\rangle}\right]\end{aligned}\tag{40}$$

$$\begin{aligned}\frac{\partial}{\partial t}\langle\sigma_k^+\sigma_{k+1}^+\rangle &= i2V\langle\sigma_k^+\sigma_{k+1}^+\rangle+iV\langle(n_{k-1}+n_{k+2})\sigma_k^+\sigma_{k+1}^+\rangle\\ &\quad -\gamma\left[\langle\sigma_k^+\sigma_{k+1}^+\rangle+\underline{\langle(n_{k-1}+n_{k+2})\,\sigma_k^+\sigma_{k+1}^+\rangle}\right]\end{aligned}\tag{41}$$

$$\begin{aligned}\frac{\partial}{\partial t}\langle\sigma_{k-1}^+\sigma_{k+1}^-\rangle &= iV\langle(n_{k-2}-n_{k+2})\sigma_{k-1}^+\sigma_{k+1}^-\rangle\\ &\quad -\gamma\left[\langle\sigma_{k-1}^+\sigma_{k+1}^-\rangle+\underline{\langle(n_{k-2}+n_{k+2})\,\sigma_{k-1}^+\sigma_{k+1}^-\rangle}\right]\end{aligned}\tag{42}$$

$$\begin{aligned}\frac{\partial}{\partial t}\langle\sigma_{k-1}^+\sigma_{k+1}^+\rangle &= iV\langle(n_{k-2}+2n_k+n_{k+2})\sigma_{k-1}^+\sigma_{k+1}^+\rangle\\ &\quad -\gamma\left[\langle\sigma_{k-1}^+\sigma_{k+1}^+\rangle+\underline{\langle(n_{k-2}+n_k+n_{k+2})\,\sigma_{k-1}^+\sigma_{k+1}^+\rangle}\right]\end{aligned}\tag{43}$$

In each of the equations, we have underlined the parts which are absent when the (conventional) single-body decay is considered. Single-body decay leads to the decay of correlations at a rate γ. Many-body decay leads in general to the emergence of density-dependent terms that further accelerate this decoherence, i.e. a density-dependent dephasing. However, not all correlation functions are equally affected. For example, in comparison to the next-nearest neighbor correlation $\langle\sigma_{k-1}^+\sigma_{k+1}^+\rangle$ the equation of motion for $\langle\sigma_{k-1}^+\sigma_{k+1}^-\rangle$ does not possess a term proportional to n_k in the dissipative part. The reason for this difference is that the configurations $|\uparrow\rangle_{k-1}\,|\uparrow/\downarrow\rangle_k\,|\downarrow\rangle_{k+1}$ and $|\downarrow\rangle_{k-1}\,|\uparrow/\downarrow\rangle_k\,|\uparrow\rangle_{k+1}$ do not decohere relative to each other (due to many-body effects) since they are both located in the single excitation subspace, i.e., they are eigenstates (with eigenvalue 1) of the projector P_k^1.
At the level of quantum trajectories, emission events associated with projectors with rank larger than one, i.e., all those projectors P_k^ξ with $\xi\neq 0,2d$, can generate entanglement among spins which are in the neighborhood of the emitting one. This can be understood by focusing on the following simple case: lets consider a one-dimensional lattice with just three sites and take as initial state the one considered in the main text, $|\Psi_0\rangle=(1/2)^{3/2}\bigotimes_k\left[|\downarrow\rangle_k+|\uparrow\rangle_k\right]$. Imagine a trajectory in which the collective jump operator $P_2^1\sigma_2^-$ acts on this state. Then, after the emission occurred the state collapses onto $|\Psi_0'\rangle\propto P_2^1\sigma_2^-|\Psi_0'\rangle$, which is nothing but

$$|\Psi_0'\rangle=\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1\,|\downarrow\rangle_2\,|\downarrow\rangle_3+|\downarrow\rangle_1\,|\downarrow\rangle_2\,|\uparrow\rangle_3\right)\,.$$

Thus, the neighbors of the emitting spin have been projected onto a Bell-pair state. This simple example, which can be generalized to jump operators associated with projectors onto any degenerate subspace, demonstrates that the collective jump operators can lead to entanglement in quantum trajectories. Nonetheless, the system (when not driven by a laser) converges to the state with all spins in $|\downarrow\rangle$ and thus entanglement is not possible asymptotically, not even in single trajectories.

IV. MEAN FIELD EQUATIONS

In general, starting from a master equation in Lindblad form with operator $\mathcal{L}$ [1], one can derive a system of equations of motion for an observable A with:

$$\left\langle\frac{\partial}{\partial t}A(t)\right\rangle=\left\langle\mathcal{L}^\dagger A\right\rangle\,.\tag{44}$$

In the analysis of our system, the average number of excitations in the system, normalized by the number of particles N, is of interest. We define this quantity as the excitation density $\langle n\rangle$ of the system with:

$$\langle n\rangle = \frac{1}{N}\sum_{k=1}^{N} n_k\,. \tag{45}$$

Based on the master equations we have obtained, for collective and single-atom dissipation, the equations of motion for $\langle n_k\rangle$ as well as for $\langle \sigma^k_{x,y}\rangle$ can be derived. Assuming translation-invariance in the system, i.e., $\langle n_k\rangle = \langle n_{k'}\rangle = n$ and $\langle \sigma^k_{x,y}\rangle = \left\langle \sigma^{k'}_{x,y}\right\rangle = s_{x,y}$ results in just three coupled equations:

$$\begin{aligned} \dot n &= \quad \Omega s_y - \gamma n \\ \dot s_x &= -\Delta s_y - \frac{\gamma}{2}(\underline{4dn}+1)s_x - 2dVns_y \\ \dot s_y &= -\Delta s_x - \frac{\gamma}{2}(\underline{4dn}+1)s_y + 2dVns_x - \Omega(4n-2)\,. \end{aligned} \tag{46}$$

Only when using the collective jump operators, the term $4dn$ is present, which is underlined in the above equations. In order to calculate the data in Fig. 2, we have put the left-hand side of these equations to zero and solved for the stationary density. This yields $n^c_{\rm ss}$ when the underlined terms are included and $n^s_{\rm ss}$ when the underlined terms are not included.

V. NUMERICAL COMPUTATION OF THE RYDBERG EXCITATION DENSITY IN THE STATIONARY STATE

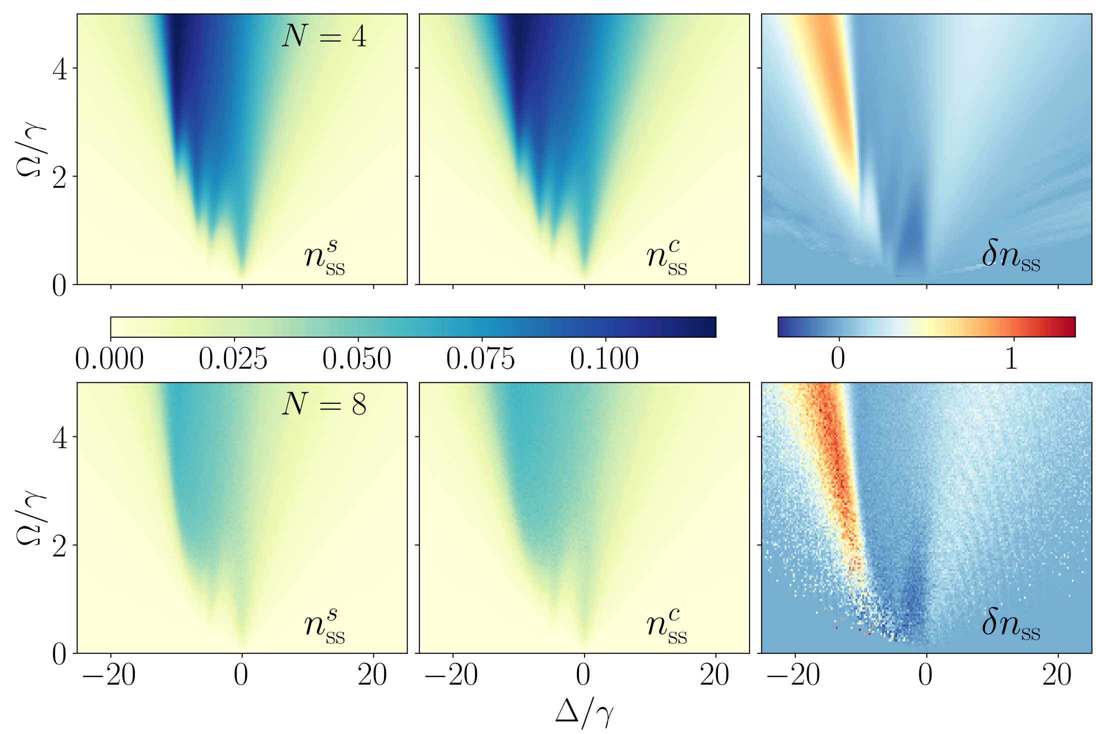

FIG. 1. **Stationary state of a $d = 1$ chain with periodic boundaries.** Shown is the stationary density of Rydberg excitations as a function of the laser detuning Δ and the Rabi frequency Ω with $V = 10\gamma$ in the presence of single-body decay ($n^s_{\rm ss}$) and collective many-body decay ($n^c_{\rm ss}$). In the rightmost column we show the relative difference between the two densities, $\delta n_{\rm ss} = (n^c_{\rm ss} - n^s_{\rm ss})/n^s_{\rm ss}$. Significant deviations are visible for negative detunings in the region where the mean field analysis predicts bistable behavior. In addition to Fig. 3 from the main text we extended the range of Δ/γ to $[-25, 25]$. Further information on the simulation methods used can be found below.

To obtain the results of Fig. 3 of the main text and of Fig. 1 we have solved Eq. (5) with the Hamiltonian given in Eq. (6) numerically. To this end, we utilized the Python library QuTiP (version 4.7.0) [5, 6] which provides a powerful and easy-to-use framework for simulating open quantum systems. For $N = 4$ the dimension of the Lindblad generator is small enough that we can use exact-diagonalization methods to calculate the steady state. This is done with the *steady-state-solver* of QuTiP. For $N = 8$, due to performance reasons we instead simulate quantum jump trajectories exploiting the continuous-time Monte-Carlo algorithm [7]. In QuTiP this algorithm is implemented in the function *mcsolve*. To estimate the steady state we average over 100 linearly spaced data points in the interval $[4.75\,\gamma t,\, 5.00\,\gamma t]$ for 3000 trajectories. The oscillations in Fig. 1 visible in the region $\Delta/\gamma > 0$ are a finite time effect.

VI. THE ROLE OF LASER PHASE NOISE

Throughout the paper, we have assumed that the atomic excitation is achieved through an ideal laser. Here, we briefly address the role of phase noise and how the latter affects our findings. Following, Ref. [8] phase noise can be modelled through the dissipator

$$\mathcal{D}_{\text{phase}}[\rho] = \sum_m \sum_{\alpha=x,y,z} \gamma_\alpha \left[\sigma_\alpha^m \rho \sigma_\alpha^m - \rho\right], \tag{47}$$

where the "rates" γ_α are in general time-dependent. This dissipator, when acting alone, destroys quantum coherence and leads to an infinite temperature, or completely mixed, state. This can be seen by inspecting the evolution equation of the operators (considering solely laser phase noise)

$$\dot{n}_k\,|_{\text{phase}} = \mathcal{D}^\dagger_{\text{phase}}[n_k] = -\left(\gamma_x + \gamma_y\right)\left(2n_k - 1\right) \tag{48}$$

$$\dot{\sigma}^x_k\,|_{\text{phase}} = \mathcal{D}^\dagger_{\text{phase}}[\sigma^x_k] = -2\left(\gamma_y + \gamma_z\right)\sigma^x_k \tag{49}$$

$$\dot{\sigma}^y_k\,|_{\text{phase}} = \mathcal{D}^\dagger_{\text{phase}}[\sigma^y_k] = -2\left(\gamma_x + \gamma_z\right)\sigma^y_k, \tag{50}$$

whose stationary state solution is $n_k = 1/2$, $\sigma^x_k = \sigma^y_k = 0$. In order to assess whether phase noise affects our findings in a qualitative fashion, we assume that the γ_α are constants and much smaller than the decay rate γ of the Rydberg states. Only in this regime, i.e. with sufficiently weak phase noise, it is possible to speak about coherent laser excitations. Note, that in experiment the impact of phase noise can be controlled, e.g., through filtering, as shown in Ref. [9].

In Fig. 2 we show the same plot as in Fig. 1 but in the presence of phase noise. As can be seen, the deviations between the simulations with single-body and many-body decay become smaller as the strength of the phase noise increases. This is consistent since the larger the phase noise, the more the stationary state will tend to the infinite temperature state, no matter whether single-body or many-body decay is considered. Qualitatively similar behavior can be observed in the mean field phase diagram, which is shown in Fig. 3. Phase noise decreases the size of the parameter region in which bistable solutions exist and can even lead to its complete removal. Indeed, in Ref. [10] it was shown that strong dephasing can remove the phase transition in dissipative Rydberg gases. Note, however, that bistable behavior has been observed experimentally [11, 12]. This corroborates our theoretical findings, that laser phase noise may lead to qualitative changes but does not destroy collective phenomena such as bistable behavior.

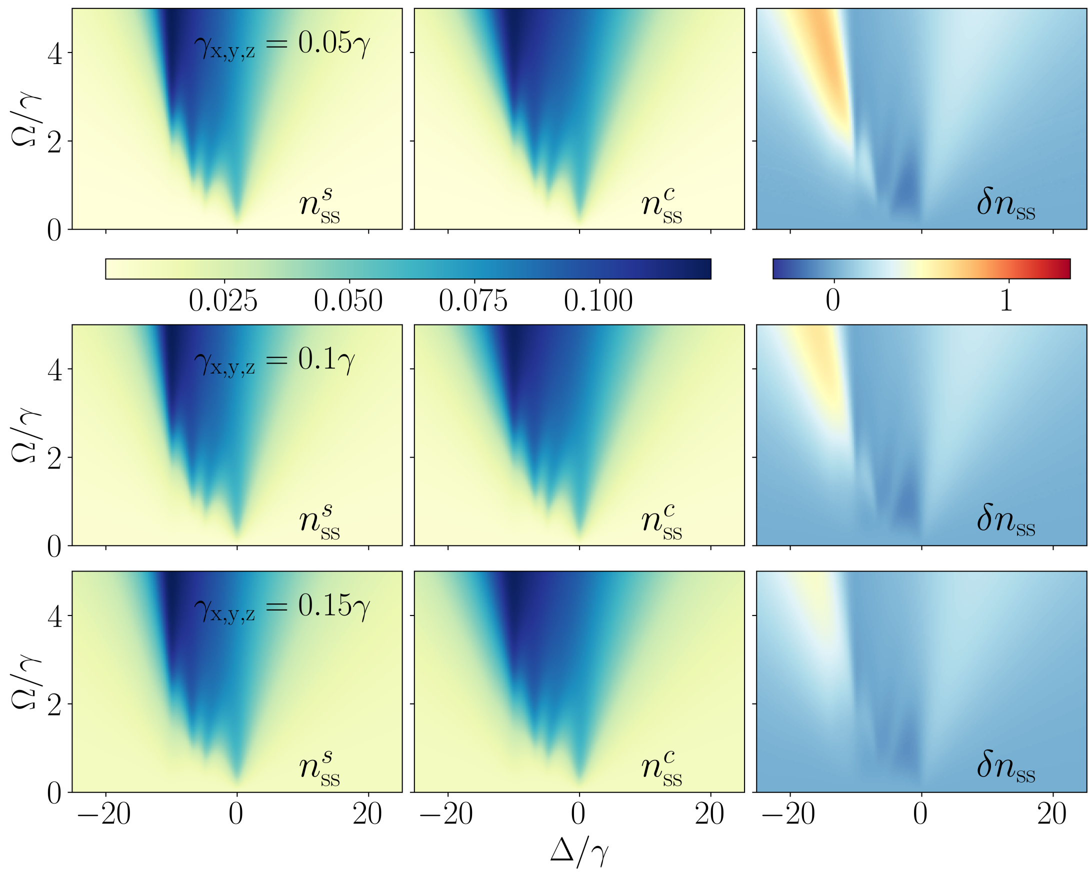

FIG. 2. **Stationary state of a $d = 1$ chain with periodic boundaries in the presence of laser phase noise.** Shown is the stationary density of Rydberg excitations as a function of the laser detuning Δ and the Rabi frequency Ω with $V = 10\gamma$ in the presence of single-body decay (n^s_{ss}) and collective many-body decay (n^c_{ss}). From top to bottom we increase the strength of laser phase noise, which is modelled according to Eq. (47). In the rightmost column we show the relative difference between the two densities, $\delta n_{\mathrm{ss}} = (n^c_{\mathrm{ss}} - n^s_{\mathrm{ss}})/n^s_{\mathrm{ss}}$. Increasing the laser phase noise decreases the magnitude, i.e, the absolute values, of δn_{ss}.

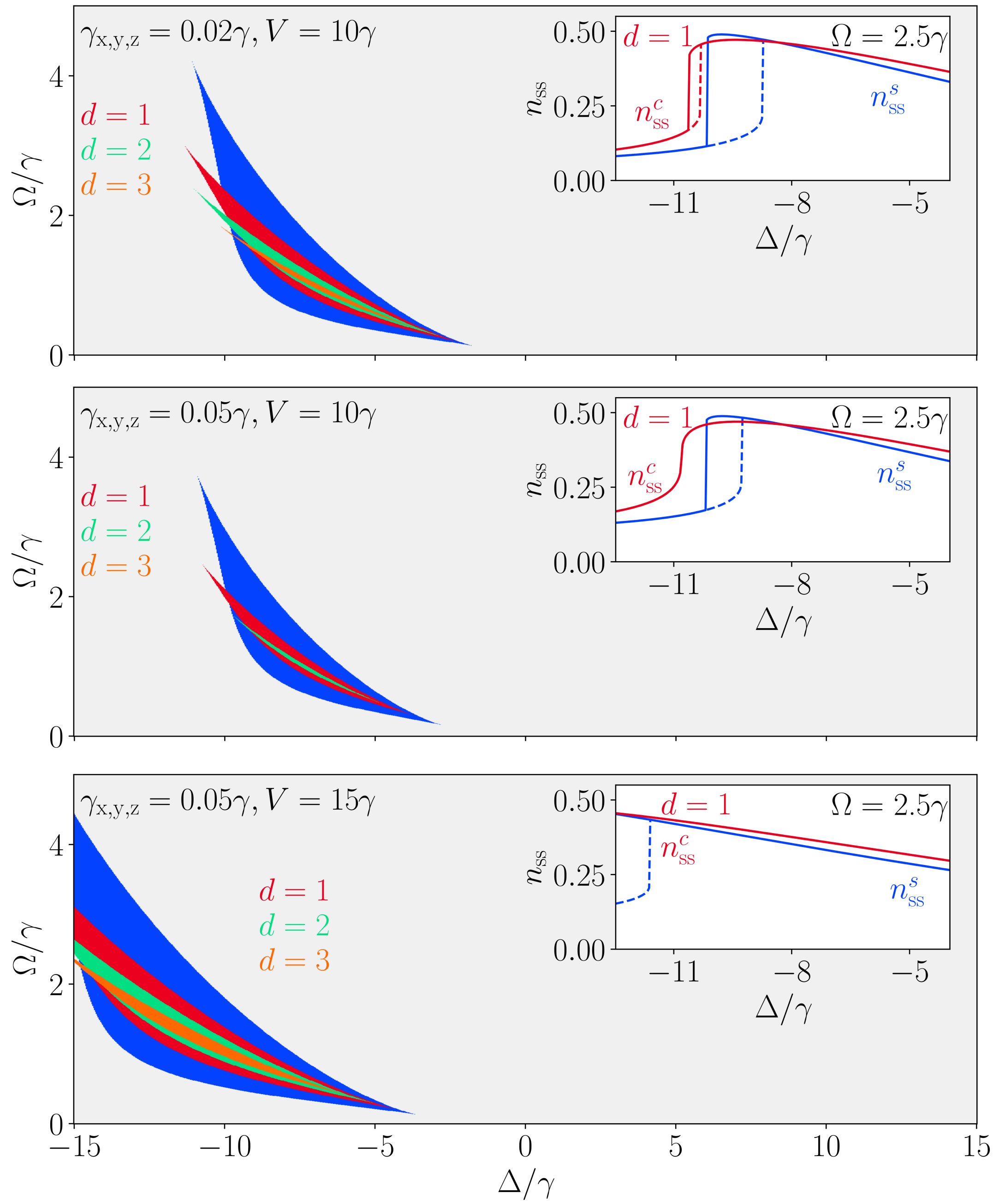

FIG. 3. **Mean field phase diagram in the presence of dephasing noise.** When the rates γ_α [see Eq. (47)] are increased — a situation which corresponds to an increase in laser phase noise — the region, in the $\Delta - \Omega$-plane, in which bistability is observed shrinks (top to middle panel). This can be compensated by increasing the interaction strength V (middle to bottom panel).

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