Sigma models for quantum chaotic dynamics

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We review the construction of the supersymmetric sigma model for unitary maps, using the color-flavor transformation. We then illustrate applications by three case studies in quantum chaos. In two of these cases, general Floquet maps and quantum graphs, we show that universal spectral fluctuations arise provided the pertinent classical dynamics are fully chaotic (ergodic and with decay rates sufficiently gapped away from zero). In the third case, the kicked rotor, we show how the existence of arbitrarily long-lived modes of excitation (diffusion) precludes universal fluctuations and entails quantum localization.

I. INTRODUCTION

We review the supersymmetric sigma model for unitary quantum maps, including its derivation through the color-flavor transformation and present three case studies of applications to quantum chaos. Our intention is not to bring news to experts but rather to help newcomers getting acquainted with and possibly using the method. Therefore, we aim at a self-contained presentation. Facing a choice, we treat the simplest case and forgo generality. Following that motto we confine our study to two-point quantities. The language of physics rather than math will be spoken.

It has gradually transpired during the past decade that unitary maps allow for easier and cleaner treatment than autonomous flows. Zirnbauer's use of the color-flavor transformation to construct a supersymmetric sigma model was an important first step [1–3]. An advantage over previous treatments was that the Hubbard-Stratonovich transformation as well as saddle-point approximations could be avoided. Moreover, no regularization by external noise was necessary. A step ahead was the in-depth treatment, based on the sigma model, of the kicked rotor in all its ramifications [4]. On the semiclassical side, the periodic-orbit treatment of unitary maps could benefit from the existence of a rigorous variant of the 'Riemann-Siegel lookalike' as well as of a finite trace formula [5].

The sigma model is a field theoretic method expressing observables by integrals over supermatrix valued 'fields'. It allows to look for the sense in which and the conditions under which an individual quantum system can display universal behavior, in its spectral, transport or wave function characteristics. Even the study of system specific properties is feasible. Of course, averages over ensembles are possible as well.

A variant of the so called ballistic sigma model is accessible from the general form. Employing a suitable Hilbert-space representation for the fields involved, like Wigner or Husimi functions, we can classically approximate the single-step quantum map. That approximation is acceptable provided the strobe period underlying the map is small compared to the Ehrenfest time since only on that latter scale qualitative discrepancies between quantum and classical dynamics arise. (In fact, most maps even have Lyapounov times larger than the strobe period.) The effectively classical single-step propagation does not impair the quantum nature of the Wigner or Husimi functions as quasi-probabilities.

The three case studies to be discussed refer to individual quantum maps, rather than ensembles. For simplicity, they are all chosen without time-reversal invariance. Quantum localization is at issue for one of them (kicked rotor) while in the other two (general Floquet maps and quantum graphs) we look at spectral fluctuations.

Before going to work we would like to spell out what can be expected for spectral fluctuations of individual quantum systems. The two-point correlator of the density of levels is known not to be self-averaging unless subjected to suitable smoothing. A recent numerical investigation of smoothed versions of the two-point correlator $R(e)$ for the kicked top [6] has revealed self-averaging and fidelity to the pertinent RMT average (CUE) provided the quasi-energy variable e (reckoned in the mean level spacing as a unit) remains within periodically repeated windows of correlation decay. Within those windows a single spectrum makes for but relatively tiny noise; outside, however, noise prevails and even forbids distinction between CUE and GUE behavior. Two smoothing operations were shown to be equally useful in [6]. One is integrating as $\int_0^e de' R(e')$ with $\mathrm{Im}\, e = 0^+$. The other is complexification of the quasi-energy e with $\frac{1}{N} \ll \mathrm{Im}\, e \ll 1$; the noise strength (rms) is $\sim N^{-1/2}$ in the first case and $\sim (N\mathrm{Im}e)^{-1/2}$ in the second.

The numerical results just alluded to do set limits to what we can expect the sigma model to yield for individual dynamics. Most importantly, we must introduce a smoothing operation before the sigma model has a 'right' to a two-point correlator in any sense reminiscent of the universal one. All the sigma model can and does produce for the non-smoothed $R(e)$ of an individual system, is the correct train of Dirac deltas engendered by the spectrum of the unitary (Floquet) map U (see Appendix E). Such singular behavior is reflected in wild fluctuations of the sigma model fields. But smoothing of the 'observable' $R(e)$ renders ineffective the wildness and stabilizes the 'mean field' responsible for universal behavior, given full chaos. On the other hand, the aforementioned $\frac{1}{\sqrt{N}}$ noise survives the smoothing, both in the numerical treatment of Ref. [6] and in the sigma-model representation of the smoothed correlator $R(e)$. To capture such noise and its strength in analytic terms we would have to scrutinize, as usual, the square of the fluctuating quantity, going for a sigma-model based treatment of the four-point function. Such an extension of our present work appears quite feasable but will not be attempted here.

Our review is organized as follows. We first (Sect. II) construct the sigma model by representing a generating function $\mathcal{Z}$ (derivatives of which yield the correlator $R(e)$) by an integral over supermatrix valued fields $Z, \tilde{Z}$; the color-flavor transformation is the essential tool in replacing a certain center-phase average required in the definition of $\mathcal{Z}$ by the said superintegral. No approximation is involved in the construction. — Next (Sect. III), we give a pedestrian evaluation of the generating function of the special case of the 'zero dimensional' sigma model where the fields Z and $\tilde{Z}$ are stripped of any structure in the quantum Hilbert space accommodating the Floquet matrix U, to become 'mean fields'. Those mean fields afford the classical interpretation of ergodic equilibrium in phase space. The resulting generating function is the one known from the CUE of random-matrix theory. — We proceed (Sect. IV) to revealing that the general sigma model does indeed reduce to the zero dimensional one, provided the classical limit of the quantum map is 'fully chaotic': its spectrum of Ruelle-Pollicott resonances must exhibit a finite gap separating all resonances from the single eigenvalue pertaining to the ergodic equilibrium state; corrections turn out to be at most of the order $\frac{1}{N}$; under certain conditions on the resonance spectrum the corrections are even $\mathcal{O}(\frac{1}{N^2})$. At any rate, the corrections are way smaller than the $\frac{1}{\sqrt{N}}$ noise found in Ref. [6]. Our demonstration makes use of a strategy proposed by Kravtsov and Mirlin [7]. That strategy separates the fields Z and $\tilde{Z}$ into mean fields and fluctuations in a way fully respecting the invariances of the supermatrix manifold making up the sigma model. All of the work in Sect. IV requires protection by smoothing in the sense mentioned above. — In a short chapter (Sect. V) we argue that the sigma model for unitary maps can be used for autonomous flows, simply by describing the latter stroboscopically with a suitable strobe period. Some cleanliness is thus gained in treating individual dynamics inasmuch as the aforementioned drawbacks of previous 'ballistic sigma model' treatments are avoided. — The case study for the kicked rotor (Sect. VI) is meant to show how the sigma model can cope with chaotic dynamics without a gap in the resonance spectrum. We adhere to our motto of simplicity by limiting ourselves to strong chaos (beyond the KAM regime), excluding quantum resonances, and abstaining from including the well-known renormalization of the diffusion constant. The remaining frame still allows to show how arbitrarily long-lived diffusive modes make for quantum localization. — Directed quantum graphs are treated in Sect. VII. Here the restriction to incommensurate bond lengths makes for a drastic simplification: the center-phase average can be replaced by an N-fold average over the phases picked up by the quantum wave along each of the N bonds. Instead of the single color-flavor transformation (one color, N flavors) for general Floquet maps, the graphs in question allow for N such transformations (one color, one flavor) such that the fields Z and $\tilde{Z}$ become block diagonal, both with a 2×2 block for each bond. Again separating into mean field and fluctuations and treating the latter perturbatively we find the quantum dynamics taking a classical appearance: The bond-to-bond scattering matrix appears only with the squared moduli of its elements which afford interpretation in terms of a classical Markov process. We are again led to the validity of the mean-field treatment (equivalent to the CUE) for large N provided the spectrum of resonances is sufficiently gapped. Corrections to the mean-field behavior then turn out to be at most of the order $\propto \frac{1}{N\Delta_g^2}$ with Δ_g the said gap. As a special charm of the case, one gets the inkling that rigorous treatments may be close at hand. — The short section VIII wraps up what is achieved and what remains open.

Several appendices provide supplementary material and aim at making this review self-contained. In the first one (A), we present a derivation of the color-flavor transformation as a more pedestrian alternative to Zirnbauers beautiful original one [1, 3]. The second appendix (B) explains why the integration measure for our 'rational parametrization' in terms of the fields $Z, \tilde{Z}$ is flat and the third (C) expounds the origin of the general invariance of the rational parametrization. The fourth appendix (D) puts the sigma model into the persective of group theory and Riemann geometry. The fifth (E) is devoted to easing existence worries; we here show that the correct train of Dirac deltas arises for the non-smoothed correlator $R(e)$ when the superintegral over the fields $Z, \tilde{Z}$ is done by picking up the contributions of the so called standard and Andreev-Altshuler saddles. Three further appendices fill in some calculations with which we did not want to burden the main text, hoping to thus make for better readability.

II. SIGMA MODEL FOR FLOQUET MAPS, UNITARY SYMMETRY

Preliminaries: We consider periodically driven quantum systems. The time evolution of the wave function over one period of the driving is effected by a unitary $N \times N$ 'Floquet' matrix U without time reversal invariance nor any other symmetry. Powers U^n with $n = 1, 2, \dots$ give a stroboscopic evolution. The unimodular eigenvalues $\mathrm{e}^{-\mathrm{i}\varphi_\mu}$, $\mu = 1, 2, \dots, N$ are assumed, for simplicity, to fill the interval $[0, 2\pi]$ with the homogeneous mean density $N/2\pi$. We define a 'microscopic' density $\rho(\varphi) = \sum_\mu \delta_{2\pi}(\varphi - \varphi_\mu)$ with the help of the (2π)-periodic delta function $\delta_{2\pi}(\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\varphi}$. Fluctuations about the mean density are captured by the two-point correlator

$$R(e) = \left[\int_0^{2\pi} \frac{d\phi}{2\pi}\, \rho\Big(\phi + \frac{e}{N}\Big)\rho\Big(\phi - \frac{e}{N}\Big) - \Big(\frac{N}{2\pi}\Big)^2\right] \Big(\frac{2\pi}{N}\Big)^2, \tag{1}$$

a normalized covariance averaged over the center phase ϕ. The function $R(e)$ is the real part of a 'complex correlator' $C(e)$ which can be written in terms of the eigenphases as

$$C(e) = \frac{2}{N^2}\sum_{\mu,\nu} \frac{\mathrm{e}^{\mathrm{i}(2e/N - \Delta_{\mu\nu})}}{1 - \mathrm{e}^{\mathrm{i}(2e/N - \Delta_{\mu\nu})}}\,, \qquad \Delta_{\mu\nu} = \varphi_\mu - \varphi_\nu\,. \tag{2}$$

One is automatically led to $R = \mathrm{Re}\, C$ with the foregoing $C(e)$ when expressing $R(e)$ in terms of the eigenphases of U. The limit $\mathrm{Im}\, e \downarrow 0$ turns the real correlator $R(e) = \mathrm{Re}\, C(e)$ into a train of (2π)-periodic Dirac deltas, one for each eigenphase spacing $\Delta_{\mu\nu}$. Since the $\Delta_{\mu\nu}$ have a mean distance $\frac{2\pi}{N^2}$ an imaginary part $\mathrm{Im}\, e \gg \frac{1}{N}$ broadens those delta functions sufficiently to make the correlator structureless on the e-scale $\frac{1}{N}$.

We shall eventually have to contrast the quantum 'Floquet map' U with its classical counterpart, an area preserving map of the classical phase-space density brought about by the so called Perron-Frobenius operator $\mathcal{F}$. 'Iterations' $\mathcal{F}^n$ allow to follow how an initially smooth density approaches uniform equilibrium as the dimensionless time n grows, given ergodicity. The equilibration looks most simple under coarse graining to limited phase-space resolution. Exponential decay of smooth inhomogeneous densities then arises with rates called Ruelle-Pollicott resonances. It is also instructive to look at the classical map in the 'Newton picture' wherein a unique trajectory originates from any initial phase-space point. Two trajectories starting at initially close points separate, given chaos, exponentially with rates known as Lyapounov exponents. The smallest Lyapounov rate visible in the Newton picture corresponds to the smallest Ruelle-Pollicott resonance in the 'Liouville picture'. Unfortunately, the definitions of the correlators R and C do not lend themselves to direct use of the chaotic character of the dynamics.

Generating function: To get a chance to distinguish classically integrable and chaotic behavior we go towards the correlator through a generating function

$$\mathcal{Z}(a,b,c,d) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, \frac{\det(1 - c\mathrm{e}^{\mathrm{i}\phi}\, U)\det(1 - d\mathrm{e}^{-\mathrm{i}\phi}\, U^\dagger)}{\det(1 - a\mathrm{e}^{\mathrm{i}\phi}\, U)\det(1 - b\mathrm{e}^{-\mathrm{i}\phi}\, U^\dagger)} \tag{3}$$

which depends on four complex variables a, b, c, d; the moduli $|a|$, $|b|$ are taken (infinitesimally) smaller than unity while c, d remain unrestricted. The real 'center-phase' ϕ is averaged over.

The foregoing definition entails the identities

$$\mathcal{Z}(a,b,a,b) = 1 \qquad \text{and} \qquad \mathcal{Z}(a,b,c,d) = (cd)^N \mathcal{Z}\Big(a, b, \frac{1}{d}, \frac{1}{c}\Big) \tag{4}$$

which we shall refer to as normalization and Weyl symmetry. Moreover, the center-phase average has two consequences for any unitary operator U, irrespective of symmetries and character of the limiting classical dynamics. First, $\mathcal{Z}$ takes the value unity when either $a = c$ for any b, d or $b = d$ for any a, c [6], that is

$$\mathcal{Z} - 1 \propto (a - c)(b - d)\,; \tag{5}$$

second, $\mathcal{Z}$ depends on a, b, c, d only through three combinations which can be chosen as

$$a = b = \mathrm{e}^{\mathrm{i}e/N}\,, \quad \frac{c}{a} = \mathrm{e}^{\mathrm{i}\epsilon_+/N}\,, \quad \text{and} \quad \frac{d}{b} = \mathrm{e}^{\mathrm{i}\epsilon_-/N} \tag{6}$$

with $|ab| < 1$ and $\frac{c}{a}$, $\frac{d}{b}$ arbitrary complex. We can therefore extract the complex correlator from $\mathcal{Z}$ as

$$C(e) = \partial_c\partial_d \mathcal{Z}\, \frac{2ab}{N^2}\Big|_{a=b=c=d=\mathrm{e}^{\mathrm{i}e/N}} = -\partial_{\epsilon_+}\partial_{\epsilon_-} \mathcal{Z}\, 2ab\Big|_{a=b=\mathrm{e}^{\mathrm{i}e/N}, \epsilon_\pm=0} = \frac{\mathcal{Z} - 1}{(a-c)(b-d)}\, \frac{2ab}{N^2}\Big|_{a=b=c=d=\mathrm{e}^{\mathrm{i}e/N}}\,. \tag{7}$$

The generating function carries physically relevant information only for values of the variables $\epsilon_\pm$ close to zero, according to (7). Higher-order derivatives $\partial^m_{\epsilon_+}\partial^n_{\epsilon_-}\mathcal{Z}\big|_{\epsilon_\pm=0}$ lead to higher-order correlators of the level density as functions of the quasi-energy e but also involve no more than the $(\epsilon_\pm \approx 0)$-behavior. Moreover, nothing is lost by restricting $\epsilon_\pm$ to real values. According to the numerical results of Ref. [6] the generating function ceases to be faithful to RMT once $|\epsilon_\pm| \gtrsim 1$, so we need not bother to look there.

Even the variable e can eventually be taken as real, in the sense $\mathrm{Im}\, e \downarrow 0$. The latter limit brings forth delta function singularities in e for $\mathrm{Re}\, \mathcal{Z}$ and principal-value type $\frac{1}{e}$-singularities for $\mathrm{Im}\, \mathcal{Z}$. As already pointed out in the introduction, however, keeping an imaginary part of e as $\frac{1}{N} \ll \mathrm{Im} e \ll 1$ is one of the possible smoothings endowing the correlator $C(e)$ with the property of self-averaging and providing protection for the perturbative treatment of the sigma model we shall have to undertake.

It seems we are not any closer yet to distinguishing regular and chaotic motion. A reformulation of the definition of the generating function — to what is called the sigma model — will help.

Sigma model: To proceed towards the sigma model we represent the ratios of determinants in (3) by Gaussian integrals and write the generating function as

$$\begin{aligned}\mathcal{Z} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int d(\psi, \psi^*)\; \exp \sum_{k,l=1}^{N} \Big\{ &- \psi^*_{+,B,k}(\delta_{kl} - a\mathrm{e}^{\mathrm{i}\phi}\, U_{kl})\psi_{+,B,l} - \psi^*_{-,B,k}(\delta_{kl} - b\mathrm{e}^{-\mathrm{i}\phi}\, U^*_{lk})\psi_{-,B,l} \\ &- \psi^*_{+,F,k}\,(\delta_{kl} - c\mathrm{e}^{\mathrm{i}\phi}\, U_{kl})\psi_{+,F,l} - \psi^*_{-,F,k}\,(\delta_{kl} - d\mathrm{e}^{-\mathrm{i}\phi}\, U^*_{lk})\psi_{-,F,l}\Big\}\,;\end{aligned} \tag{8}$$

here, the $4N$ integration variables comprise $2N$ pairs of mutually complex conjugate ordinary (Bosonic) variables $\psi_{\pm,B,k}, \psi^*_{\pm,B,k}$ and $4N$ mutually independent Grassmannians $\psi_{\pm,F,k}, \psi^*_{\pm,F,k}$. The first index, $\lambda = +, -$, distinguishes the two 'columns' in the definition (3) of $\mathcal{Z}$ which can be associated with forward (+) and backward (−) time evolution; we shall speak of the two dimensional advanced (−)/retarded (+) space AR when referring to that index. The second index, $s = B, F$, distinguishes denominator and numerator in (3); since the two cases are respectively represented by Bosonic and Fermionic variables in the integral (8) we speak of a two dimensional Bose-Fermi (BF) space. Finally, the third index, $k = 1, 2, \ldots N$, pertains to the Hilbert space of the quantum dynamics (QD) wherein the Floquet matrix operates. The flat integration measure reads

$$d(\psi, \psi^*) = \prod_{k=1}^{N} \frac{d^2\psi_{+,B,k}}{\pi} \frac{d^2\psi_{-,B,k}}{\pi} \, d\psi_{+,F,k} \, d\psi^*_{+,F,k} \, d\psi_{-,F,k} d\,\psi^*_{-,F,k} \, . \tag{9}$$

The mentioned restrictions on a, b secure the existence of the Bosonic integrals in (8). We may lump the integration variables into four supervectors $\psi_\pm$ and $\psi^*_\pm$ and introduce the diagonal BF matrices

$$\hat{e}_+ = \begin{pmatrix} a & \\ & c \end{pmatrix}, \qquad \hat{e}_- = \begin{pmatrix} b & \\ & d \end{pmatrix} \tag{10}$$

and compact the superintegral (8) to

$$\mathcal{Z} = \int_0^{2\pi} \frac{d\phi}{2\pi} \int d(\psi, \psi^*) \, \exp\Big\{ -\psi_+^{*T}\big(1 - \mathrm{e}^{\mathrm{i}\phi}\hat{e}_+ U\big)\psi_+ - \psi_-^{*T}\big(1 - \mathrm{e}^{-\mathrm{i}\phi}\hat{e}_- U^\dagger\big)\psi_- \Big\} \, . \tag{11}$$

The center-phase average $\langle \cdot \rangle = \int_0^{2\pi} \frac{d\phi}{2\pi}(\cdot)$ can be traded against a supermatrix average, using the 'color-flavor transformation' [1–3] (for a derivation, see App. A). For arbitrary supervectors $\Psi_1, \Psi_2, \Psi_{1'}, \Psi_{2'}$ that transform reads

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \, \exp\big\{ \mathrm{e}^{\mathrm{i}\phi}\Psi_1^T\Psi_{2'} + \mathrm{e}^{-\mathrm{i}\phi}\Psi_2^T\Psi_{1'} \big\} = \int d(Z, \tilde{Z}) \, \mathrm{sdet}(1 - Z\tilde{Z}) \, \exp\big\{ \Psi_1^T\tilde{Z}\Psi_{1'} + \Psi_2^T Z \Psi_{2'} \big\} \, . \tag{12}$$

The integration measure $d(Z, \tilde{Z})$ is flat (see Appendix B), defined analogously to (9), and sdet denotes the superdeterminant (see e.g. [8]). The Bose-Bose and Fermi-Fermi blocks of the matrices $Z, \tilde{Z}$ are subject to the (convergence generating) conditions $\tilde{Z}_{BB} = Z^\dagger_{BB}$, $\tilde{Z}_{FF} = -Z^\dagger_{FF}$ and the integration domain is restricted as $|Z_{BB}Z^\dagger_{BB}| < 1$, that is, all eigenvalues of $Z_{BB}Z^\dagger_{BB}$ lie in the interval $[0, 1)$.

For our study of the unitary symmetry class the supervectors in the integral transformations (12) have $2N$ components. Correspondingly, the supermatrices $Z, \tilde{Z}$ are $2N \times 2N$ and act in BF⊗QD. The transformation (12) entails

$$\mathcal{Z} = \int d(\psi, \psi^*) \, \exp\big(-\psi_+^{*T}\psi_+ - \psi_-^{*T}\psi_- \big) \int d(Z, \tilde{Z}) \, \mathrm{sdet}(1 - Z\tilde{Z}) \, \exp\big(\psi_+^{*T}\tilde{Z}\psi_- + \psi_-^{*T}U^\dagger\hat{e}_- Z \hat{e}_+ U \psi_+ \big) \, . \tag{13}$$

To compact even further we introduce the AR⊗BF⊗QD vectors $\psi = \binom{\psi_+}{\psi_-}$, $\psi^* = \binom{\psi_+^*}{\psi_-^*}$ and the matrix

$$M = 1 - \begin{pmatrix} 0 & U\tilde{Z}U^\dagger \\ \hat{e}_- Z \hat{e}_+ & 0 \end{pmatrix}_{\mathrm{AR}} . \tag{14}$$

Now, the Gaussian integral over ψ and ψ^* can be done,

$$\int d(\psi, \psi^*) \, \exp(-\psi^\dagger M \psi) = \big(\mathrm{Sdet}\, M\big)^{-1} = \exp\big(-\mathrm{Str} \ln M \big) \, , \tag{15}$$

where Str and Sdet refer to the $4N$ dimensional space AR⊗BF⊗QD (we reserve sdet and str to BF⊗QD). Expanding $\ln M$ in powers of $M - 1$, observing that the supertrace of odd powers of $M - 1$ vanishes and resumming one gets $\mathrm{Str} \ln M = \mathrm{str} \ln(1 - \tilde{Z}\hat{e}_- U^\dagger Z \hat{e}_+ U)$. We have arrived at the *sigma model* for unitary quantum maps from the unitary symmetry class,

$$\mathcal{Z} = \int d(Z, \tilde{Z}) \mathrm{e}^{-\mathcal{S}(Z, \tilde{Z})} \tag{16}$$

$$\mathcal{S}(Z, \tilde{Z}) = -\,\mathrm{str} \ln\big(1 - \tilde{Z}Z\big) + \mathrm{str} \ln\big(1 - U\tilde{Z}U^\dagger\hat{e}_- Z \hat{e}_+\big) \, ; \tag{17}$$

the action $\mathcal{S}$ is the central object of the theory. The measure is normalized as $\int d(Z,\tilde{Z})\mathrm{e}^{\mathrm{str}\ln\left(1-\tilde{Z}Z\right)}=1$. We are obviously free to change names as $Z\leftrightarrow\tilde{Z}$.

At this point we can see that the generating function depends only on three independent variables which we may choose as in (6). To reveal that property in the action $S(Z,\tilde{Z})$, we write the $2N\times 2N$ supermatrix $\hat{e}_-Z\hat{e}_+$ as 2×2 in BF (with the entries $N\times N$ in QD) and find

$$\hat{e}_-Z\hat{e}_+=\begin{pmatrix} abZ_{BB} & bcZ_{BF}\\ adZ_{FB} & cdZ_{FF}\end{pmatrix}=ab\begin{pmatrix} Z_{BB} & \frac{c}{a}Z_{BF}\\ \frac{d}{b}Z_{FB} & \frac{c}{a}\frac{d}{b}Z_{FF}\end{pmatrix}. \tag{18}$$

The action can be rewritten as a supertrace in AR⊗BF⊗QD. To do that we take the matrix M of (14) as 2×2 in AR with $(2\times 2)\otimes(N\times N)$ blocks in BF⊗QD. Some reshuffling (see App. F) gives

$$\mathcal{S}(Z,\tilde{Z})=\mathrm{Str}\ln\left[1+\Lambda(1-\hat{U})(1+\hat{U})^{-1}Q(Z,\tilde{Z})\right]+\mathrm{Str}\ln(1+\hat{U}), \tag{19}$$

$$Q(Z,\tilde{Z})=\begin{pmatrix}1 & Z\\ \tilde{Z} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & Z\\ \tilde{Z} & 1\end{pmatrix}^{-1}\equiv T\Lambda T^{-1}, \qquad \hat{U}=\begin{pmatrix}U\hat{e}_+ & \\ & U^\dagger\hat{e}_-\end{pmatrix}. \tag{20}$$

We note that the matrix Q arising from the 'trivial' configuration Λ by the transformation T obeys $\mathrm{Str}\,Q=0$ and $Q^2=1$. The manifold explored by Q as Z and $\tilde{Z}$ vary is a symmetric space whose properties are discussed in the Appendices B, C, and D.

The sigma model, be it in the form (16,17) or (19,20), does finally allow us to make explicit use of the chaos assumed for the classical dynamics, simply because the action $\mathcal{S}$ involves the Floquet matrix U only through the single-step time evolution $U\tilde{Z}U^\dagger$. Assuming the underlying period of the driving smaller than the smallest Lyapounov (or Ruelle-Pollicott) time we can classically approximate the single-step quantum evolution, quantum corrections vanishing like a power of $\hbar$ in the semiclassical limit $\hbar\sim\frac{1}{N}\to 0$. Only on the much larger Ehrenfest time scale would quantum features of the dynamics become dominant. With the said approximation we shall arrive at the *ballistic sigma model*.

The original representation (17) of the action is the natural starting point for studying the universal limit. It will also serve us fine for our investigation of deviations from universality; however, the alternative (19,20) will turn out indispensable for identifying the parts of $Z,\tilde{Z}$ which can be the basis of a systematic treatment of deviations.

We conclude our introduction of the sigma model by pointing out that not only the generating function $\mathcal{Z}$ but also the correlator $C(e)$ affords a representation by a superintegral. To see this we expand the action $\mathcal{S}$ as $\mathcal{S}=\mathcal{S}_0+(a-c)\mathcal{S}_++(b-d)\mathcal{S}_-+(a-c)(b-d)\mathcal{S}_{+-}+\dots$ and immediately proceed to the ensuing expansion for the generating function, $\mathcal{Z}=\int d(Z,\tilde{Z})\,\mathrm{e}^{-\mathcal{S}_0}\left[1-(a-c)S_+-(b-d)\mathcal{S}_-+(a-c)(b-d)(\mathcal{S}_+\mathcal{S}_--\mathcal{S}_{+-})\right]+\dots$. Now due to the identity (5) the zero-order term ('1' in the forgoing square bracket) integrates to unity while the first-oder terms integrate to zero. Finally, using the last member in (7) we arrive at the promised sigma-model representation of the correlator,

$$C(e)=\frac{2\mathrm{e}^{\mathrm{i}e/N}}{N^2}\int d(Z,\tilde{Z})\,\mathrm{e}^{-\mathcal{S}_0}\left(\mathcal{S}_+\mathcal{S}_+-S_{+-}\right). \tag{21}$$

The work to be presented could be simplified by right away forgetting about the generating function. However, we prefer to hold on to $\mathcal{Z}$ for a while, for the simple reason already mentioned above: there is physics in the higher-order derivatives $\partial^m_{\epsilon_+}\partial^n_{\epsilon_-}\mathcal{Z}\big|_{\epsilon_\pm=0}$. Only near the end of our various investigations of $\mathcal{Z}$ we shall resign to discussing no more than the two-point correlator and then benefit from the pertinent simplification.

III. THE UNIVERSAL LIMIT: ZERO DIMENSIONAL SIGMA MODEL

As a first application, we reduce the sigma model to the so called zero dimensional one and present an elementary evaluation of its generating function. We shall recover the universal spectral fluctuations as encapsulated in the Circular Unitary Ensemble of random-matrix theory [9].

A. Formal reduction

The reduction in question requires that the matrices $Z,\tilde{Z}$ act like unity in the QD Hilbert space. Given

$$Z\to \mathbb{1}_{\mathrm{QD}}\,B\,,\qquad \tilde{Z}\to\mathbb{1}_{\mathrm{QD}}\,\tilde{B} \tag{22}$$

with 2×2 supermatrices B, $\tilde{B}$, the Floquet operator disappears due to $UU^\dagger = 1$. This is how system independent universal behavior is invited. The supertrace in (17), over both BF and QD, then just yields a factor N from QD and only the BF supertrace remains. The generating function becomes an integral over the 2×2 supermatrices $B, \tilde{B}$

$$\mathcal{Z}_0 = \int d(B,\tilde{B})\, \mathrm{e}^{-N\mathcal{S}_0(B,\tilde{B})}\,, \qquad \mathcal{S}_0(B,\tilde{B}) = -\mathrm{str}\ln(1-\tilde{B}B) + \mathrm{str}\ln(1-\tilde{B}\hat{e}_-B\hat{e}_+)\,, \tag{23}$$

with the supertrace from this point on refering only to the two dimensional BF space. The subscript '0' on the generating function signals a certain zero dimensional character of the model due to the absence of Hilbert space structure; we shall drop that subscript in the remainder of the present section. At any rate, we have arrived at the so called zero dimensional sigma model for the circular unitary ensemble (CUE) of random matrices. The resulting explicit form of the generating function at finite N was rigorously determined as a CUE average in [10] and semiclassically constructed for individual dynamics in the framework of periodic-orbit theory [5].

B. Generating function

One may parametrize the supermatrices B and $\tilde{B}$ by their matrix elements,

$$B = \begin{pmatrix} s & \mu \\ \nu & t \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} s^* & \nu^* \\ \mu^* & -t^* \end{pmatrix}. \tag{24}$$

The integration ranges then are the unit disc for s and the whole complex t-plane, and the integration measure is flat. A more convenient set of integration variables is provided by the singular-value decomposition

$$B = W\begin{pmatrix} \sqrt{l_B} & \\ & \sqrt{-l_F} \end{pmatrix} V^{-1} \equiv W\underline{B}V^{-1}\,, \qquad \tilde{B} = V\begin{pmatrix} \sqrt{l_B} & \\ & -\sqrt{-l_F} \end{pmatrix} W^{-1} \equiv V\underline{\tilde{B}}W^{-1}\,; \tag{25}$$

here l_B and $-l_F$ both have positive numeric parts (see [8], p.535); the roots $\sqrt{l_B}$ and $\sqrt{-l_F}$ are meant positive. It follows that the products $B\tilde{B}$ and $\tilde{B}B$ are diagonalized as $W^{-1}B\tilde{B}W = V^{-1}\tilde{B}BV = \begin{pmatrix} l_B & \\ & l_F \end{pmatrix}$. The supermatrices W and V are given by

$$W = \begin{pmatrix} 1+\frac{\eta\eta^*}{2} & \eta \\ \eta^* & 1-\frac{\eta\eta^*}{2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi_B} & \\ & \mathrm{e}^{-\mathrm{i}\varphi_F} \end{pmatrix}, \qquad V = \begin{pmatrix} 1-\frac{\tau\tau^*}{2} & \tau \\ -\tau^* & 1+\frac{\tau\tau^*}{2} \end{pmatrix}, \tag{26}$$

$$W^{-1} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_B} & \\ & \mathrm{e}^{\mathrm{i}\varphi_F} \end{pmatrix}\begin{pmatrix} 1+\frac{\eta\eta^*}{2} & -\eta \\ -\eta^* & 1-\frac{\eta\eta^*}{2} \end{pmatrix}, \qquad V^{-1} = \begin{pmatrix} 1-\frac{\tau\tau^*}{2} & -\tau \\ \tau^* & 1+\frac{\tau\tau^*}{2} \end{pmatrix}. \tag{27}$$

The singular values $\sqrt{l_B}, \sqrt{-l_F}$, the two phases φ_B, φ_F, and the four independent Grassmannians $\eta, \eta^*, \tau, \tau^*$ can be expressed in terms of the matrix elements of $B, \tilde{B}$ [8]. Taking these eight new parameters (sometimes called 'polar coordinates') as integration variables we get the integration measure and the integration range [8]

$$d(B,\tilde{B}) = \frac{dl_B dl_F d\varphi_B d\varphi_F d\eta^* d\eta d\tau^* d\tau}{4\pi^2(l_B-l_F)^2}\,, \qquad 0<l_B\le 1\,, \quad -\infty<l_F<0\,, \quad 0\le\varphi_B,\varphi_F\le 2\pi\,. \tag{28}$$

Using the decomposition (25) we can write the two terms of the action in (23) as

$$\mathcal{S}_1 = -\,\mathrm{str}\ln(1-\tilde{B}B) = \ln\frac{1-l_F}{1-l_B}\,, \tag{29}$$

$$\mathcal{S}_2 = \mathrm{str}\ln(1-\tilde{B}\hat{e}_-B\hat{e}_+) = \mathrm{str}\ln\left(1-\underline{\tilde{B}}W^{-1}\hat{e}_-W\underline{B}V^{-1}\hat{e}_+V\right)\,. \tag{30}$$

To process the second term, we note

$$V^{-1}\hat{e}_+V = \hat{e}_+ + (a-c)\begin{pmatrix} -\tau\tau^* & \tau \\ \tau^* & -\tau\tau^* \end{pmatrix} \equiv \hat{e}_+ + (a-c)\Delta_+(\tau,\tau^*) \tag{31}$$

$$W^{-1}\hat{e}_-W = \hat{e}_- + (b-d)\begin{pmatrix} \eta\eta^* & e\eta \\ -e^*\eta^* & \eta\eta^* \end{pmatrix} \equiv \hat{e}_- + (b-d)\Delta_-(\eta,\eta^*)\,;$$

nilpotent matrices $\Delta_\pm$ occur here, and Δ_- contains the unimodular parameters $e \equiv \mathrm{e}^{\mathrm{i}(\varphi_B-\varphi_F)}$. These identities yield $\mathcal{S}_2 = \mathrm{str}\ln\Big(1-\underline{\tilde{B}}\big(\hat{e}_- + (b-d)\Delta_-\big)\underline{B}\big(\hat{e}_+ + (a-c)\Delta_+\big)\Big)$. Extracting the diagonal matrix $\begin{pmatrix} 1-abl_B & \\ & 1-cdl_F\end{pmatrix}$ as a factor from the argument of the foregoing logarithm we split $\mathcal{S}_2$ into a numerical and a nilpotent summand as

$$\mathcal{S}_2 = \ln\frac{1-abl_B}{1-cdl_F} + \mathrm{str}\ln(1-m)\,, \tag{32}$$

$$m(\tau,\tau^*,\eta,\eta^*) = \begin{pmatrix} (1-abl_B)^{-1} & \\ & (1-cdl_F)^{-1}\end{pmatrix}\big((a-c)\underline{\tilde{B}}\,\underline{B}\hat{e}_-\Delta_+ + (b-d)\underline{\tilde{B}}\Delta_-\underline{B}\hat{e}_+ + (a-c)(b-d)\underline{\tilde{B}}\Delta_-\underline{B}\Delta_+\big) \tag{33}$$

We shall have to look more closely at the nilpotent matrix m presently.

We can dispose of the unimodular variables e, e^*; according to (31) they appear only in the offdiagonal elements of the matrix Δ_- and disappear when we change integration variables as $e\eta \to \eta, e^*\eta^* \to \eta^*$. The integrals over the phases ϕ_B and ϕ_F then just give the factor $4\pi^2$. Collecting everything we write the generating function

$$\mathcal{Z} = \lim_{\epsilon\to 0}\int_\epsilon^1 dl_B\int_{-\epsilon}^{-\infty} dl_F\left[\frac{1}{(l_B-l_F)^2}\left(\frac{(1-l_B)(1-cdl_F)}{(1-l_F)(1-abl_B)}\right)^N\int d\eta^* d\eta d\tau^* d\tau\,\mathrm{e}^{-N\mathrm{str}\,\ln(1-m)}\right] \tag{34}$$

where from here on the matrix Δ_- entering m is understood purged of the said phase factors, i. e. , $e \to 1$.

We need to explain why we have cut an ϵ-neighborhood of $l_B = l_F = 0$ from the integration range and defined the integral through the limit $\epsilon \to 0$. According to the transformation (25), the 'point' $l_B = l_F = 0$ corresponds to $B = \tilde{B} = 0$ where the integrand has no singularity in terms of the parametrization (24), such that cutting an infinitly small neighborhood does not alter the integral. Moreover, the integral exists independently of the chosen integration variables and must therefore be correctly given by the limiting procedure in (34). On the other hand, the transformation (25) does bring in a singularity at $l_B = l_F = 0$. Indeed, the matrix m vanishes there while the factor $\frac{1}{(l_B-l_F)^2}$ diverges. As a consequence, the integral $\int_0^1 dl_B\int_0^{-\infty} dl_F[\ldots]$ differs from $\mathcal{Z} = \lim_{\epsilon\to 0}\int_\epsilon^1 dl_B\int_{-\epsilon}^{-\infty} dl_F[\ldots]$ by a finite term due to the singularity of the integrand in the 'polar coordinates'. We shall not have to labor to get the contribution of that singularity since it is automatically produced by the general property (5) as unity. We conclude

$$\mathcal{Z}-1 = \int_0^1 dl_B\int_0^{-\infty} dl_F\left[\frac{1}{(l_B-l_F)^2}\left(\frac{(1-l_B)(1-cdl_F)}{(1-l_F)(1-abl_B)}\right)^N\int d\eta^* d\eta d\tau^* d\tau\,\mathrm{e}^{-N\mathrm{str}\,\ln(1-m)}\right]. \tag{35}$$

We can even conclude that the factor $(a-c)(b-d)$ in $\mathcal{Z}-1$ must come from the Grassmann part of the foregoing superintegral. An elementary if somewhat lengthy calculation (see App. G) gives that Grassmann integral as

$$\mathcal{G} = \int d\eta^* d\eta d\tau^* d\tau\,\mathrm{e}^{-N\mathrm{str}\,\ln(1-m)} = (a-c)(b-d)\frac{N(l_B-l_F)\big(1-abcdl_Bl_F+N(abl_B-cdl_F)\big)}{(1-abl_B)^2(1-cdl_F)^2}\,. \tag{36}$$

The generating function $\mathcal{Z}$ can then be found without doing the remaining twofold integral over l_B and l_F, by just invoking the Weyl symmetry. We rewrite that symmetry for the quantity

$$J(ab,cd) = \frac{(1-ab)(1-cd)}{(a-c)(b-d)}(\mathcal{Z}-1) \tag{37}$$

which depends only on the two variables ab and cd, and confront the identity

$$\frac{(a-c)(b-d)}{(1-ab)(1-cd)}\Big(J(ab,cd) + (cd)^N J(ab,\tfrac{1}{cd})\Big) = (cd)^N - 1 - (cd)^N J(ab,\tfrac{1}{cd})\,. \tag{38}$$

Herein, the r. h. s. deppends only on the variables ab, cd while the l. h. s. is proportional to the differences $a-c$ and $b-d$. So both sides vanish. We conclude $J = 1-(cd)^N$ and arrive at the known generating function of the CUE

$$\mathcal{Z} = 1 + \Big(1-(cd)^N\Big)\frac{(a-c)(b-d)}{(1-ab)(1-cd)} = \mathcal{Z}_{\mathrm{CUE}}\,. \tag{39}$$

C. Directly to the correlator

A lot of work can be saved by directly going to the correlator $C(e)$. According to the last member of Eq. (7) we can set $a = b = c = d$ in the cofactor of $(a-c)(b-d)$ in the Grassmann integral $\mathcal{G}$ given in (36). To do the thus simplified

Grassmann integral we simply decompose the matrix m as $m = (a-c)m_+ + (b-d)m_- + (a-c)(b-d)m_{+-}$, reading the summands from the definition (33). We then simplify the logarithm $\ln(1-m)$ by keeping only terms capable of contributing a term $\propto (a-c)(b-d)$ to $\mathcal{G}$. We thus get $\mathrm{e}^{-N\mathrm{str}\,\ln(1-m)} \to N(a-c)(b-d)\big\{\mathrm{str}\,(m_{+-} + m_+m_-) + N\mathrm{str}\,m_+\mathrm{str}\,m_-)\big\}$. The subsequent Grassmann integration is survived only by terms $\propto \eta\eta^*\tau\tau^*$ within the foregoing curly bracket; that fact is easily checked to allow for dropping the skew entries in $\Delta_\pm$. We thus immediately get

$$\mathcal{G} \to (a-c)(b-d)\frac{N(l_B - l_F)}{(1-a^2l_B)^2(1-a^2l_F)^2}\big\{1 - a^4l_Bl_F + Na^2(l_B - l_F)\big\} \tag{40}$$

and the correlator

$$C(e) = \frac{2a^2}{N}\int_0^1 dl_B \int_{-\infty}^0 dl_F\,\Big(\frac{1-l_B}{1-l_F}\Big)^N \frac{(1-a^2l_F)^{N-2}}{(1-a^2l_B)^{N+2}}\,\frac{1-a^4l_Bl_F + Na^2(l_B-l_F)}{l_B - l_F} \equiv \frac{2a^2}{N}(1 + a^2\partial_{a^2})\,h(a^2) \tag{41}$$

$$h(z) = \int_0^1 dl_B \int_{-\infty}^0 dl_F\,\Big(\frac{1-l_B}{1-l_F}\Big)^N\,\frac{(1-zl_F)^{N-1}}{(1-zl_B)^{N+1}}\,\frac{1}{l_B - l_F}\,,$$

where $z = a^2 = \mathrm{e}^{\mathrm{i}2e}$. The remaining twofold integration cannot be cut short by the Weyl symmetry but is easily done as follows. We expand the integrand in powers of z to obtain $h(z) = \sum_{n_2=0}^{N-1}\sum_{n_1=0}^{\infty} f_{n_1n_2}z^{n_1+n_2}$ and $(1+z\partial z)h(z) = \sum_{n_2=0}^{N-1}\sum_{n_1=0}^{\infty}(1+n_1+n_2)f_{n_1n_2}z^{n_1+n_2}$ with

$$(1+n_1+n_2)f_{n_1n_2} = (1+n_1+n_2)\begin{pmatrix}N-1\\ n_2\end{pmatrix}\begin{pmatrix}N+n_1\\ n_1\end{pmatrix}\int_0^1 dl_B\int_{-\infty}^0 dl_F\Big(\frac{1-l_B}{1-l_F}\Big)^N\frac{(-l_F)^{n_2}l_B^{n_1}}{l_B - l_F} = 1\,. \tag{42}$$

To check that the latter coefficients all equal unity we change integration variables, first as $l_F \to x = \frac{1-l_B}{1-l_F}$ and subsequently as $l_B \to (1-x)y$. Those transformations bring the foregoing double integral over l_B, l_F to the form of a product of two separate integrals like $\int_0^1 x^n(1-x)^m = \frac{m!n!}{(n+m+1)!}$, and the result $(1+n_1+n_2)f_{n_1n_2} = 1$ arises. The correlator thus becomes the product of two geometric series, $C = \frac{2z}{N}\sum_{n_2=0}^{N-1}\sum_{n_1=0}^{\infty} z^{n_1+n_2}$, producing the final form

$$C(e) = \frac{e^{2\mathrm{i}e} - 1}{2N^2\sin^2\frac{e}{N}} = C_{\mathrm{CUE}}(e)\,. \tag{43}$$

We face periodicity in the phase e with period πN. Much smaller is the phase scale in which $C(e)$ takes values independent of N, according to $N^2\sin^2\frac{e}{N} \sim e^2$ and $C(e) \sim \frac{e^{2\mathrm{i}e}-1}{2e^2}$, in coincidence with the correlator of the Gaussian unitary ensemble. That behavior arises in windows of 'correlation decay and revival', around $e = 0, N\pi, 2N\pi, \ldots$. Outside those windows $\sin^2\frac{e}{N}$ defies Taylor expansion and the correlator is of the order $\frac{1}{N^2}$.

IV. LIMIT OF VALIDITY OF THE ZERO DIMENSIONAL MODEL

A. Preparatory remarks

We have seen the zero dimensional sigma model to discard all QD structure of the matrices $\tilde{Z}, Z$, cf. (22). Now, we want to complement the 'mean field' $\mathbb{1}_{\mathrm{QD}}\,B$ by fluctuations. A quantitative treatment of that complement will reveal conditions under which deviations from universal spectral fluctuations are negligible.

A clue will be provided by a representation for $\tilde{Z}, Z$ allowing to implement the semiclassical limit, such that the single-step quantum evolution $U\tilde{Z}U^\dagger$ can be classically approximated. The error thus incurred is small provided the strobe period underlying the single-step Floquet map is small against the Ehrenfest time t_E, the time scale on which the quantum evolution acquires qualitative discrepancies from classical behavior.

The condition for universality of spectral fluctuations of a single quantum system will turn out to be that there must not be any arbirarily long-lived mode in the classical limit. Rather, all classical Frobenius-Perron rates must be finitely gapped away from the eigenvalue zero. The latter pertains to the unique ergodic equilibrium with uniform probability density covering (the accessible part of) the classical phase space. That ergodic equilibrium is the classical counterpart of the absence of any QD structure in the configurations (22) admitted by the zero dimensional sigma model. The absence of any conservation laws both for the quantum and the classical dynamics is implicitly assumed and in fact part of the conditions for universal fluctations in the quantum spectrum. The conditions for spectral universality just anticipated have also be found as constitutive for the periodic-orbit based theory of spectral fluctuations [5, 11].

The following *calculations must be done under the protection of sufficient smoothing*, or else the correlator and the generating function do not even exist as ordinary functions. Complexifying the quasienergy e suffices [6].

B. Perturbatively tractable fluctuating fields

For an intuitive generalization of the reduction (22) we split the matrices $\tilde{Z}, Z$ into mean fields and fluctuations as

$$Z = \mathbb{1}_{\mathrm{QD}}\, B + \delta Z\,, \qquad \tilde{Z} = \mathbb{1}_{\mathrm{QD}}\, \tilde{B} + \delta\tilde{Z}\,. \tag{44}$$

An alternative possibility pioneered by Kravtsov and Mirlin [7] is offered by the form (19,20) of the action S: we can therein split the transformation $T = \left(\begin{smallmatrix} 1 & Z \\ \tilde{Z} & 1 \end{smallmatrix}\right)$ — which generates the matrix $Q = T\Lambda T^{-1}$ out of the trivial configuration Λ — into two factors,

$$T = T_0\, T_{\mathrm{d}}\,, \qquad T_0 = \begin{pmatrix} 1 & B \\ \tilde{B} & 1 \end{pmatrix} \mathbb{1}_{\mathrm{QD}}\,, \qquad T_{\mathrm{d}} = \begin{pmatrix} 1 & Z_{\mathrm{d}} \\ \tilde{Z}_{\mathrm{d}} & 1 \end{pmatrix}. \tag{45}$$

The first factor, T_0, makes up the zero dimensional sigma model while T_{d} comprises all non-universal configurations.

The two options can be made equivalent. To that end we observe that the product $T_0 T_{\mathrm{d}} = \left(\begin{smallmatrix} 1+B\tilde{Z}_{\mathrm{d}} & B+Z_{\mathrm{d}} \\ \tilde{B}+\tilde{Z}_{\mathrm{d}} & 1+\tilde{B}Z_{\mathrm{d}} \end{smallmatrix}\right)$ has diagonal entries different from unity and thus seems to miss the required structure. We need not worry, however, since we may split off a block diagonal right factor, $T_0 T_{\mathrm{d}} = T' V$ with $V = \left(\begin{smallmatrix} 1+B\tilde{Z}_{\mathrm{d}} & \\ & 1+\tilde{B}Z_{\mathrm{d}} \end{smallmatrix}\right)$, such that $T' = \left(\begin{smallmatrix} 1 & (B+Z_{\mathrm{d}})\frac{1}{1+\tilde{B}Z_{\mathrm{d}}} \\ (\tilde{B}+\tilde{Z}_{\mathrm{d}})\frac{1}{1+B\tilde{Z}_{\mathrm{d}}} & 1 \end{smallmatrix}\right)$ does have unit diagonal entries. The block diagonal V, on the other hand, cancels from the action since it commutes with Λ. In other words, the product of two good transformations T is good again (see Appendix C). We conclude that the two options of capturing deviations from the zero dimensional sigma model are equivalent if we stipulate

$$B + \delta Z = (B + Z_{\mathrm{d}})\frac{1}{1+\tilde{B}Z_{\mathrm{d}}} \qquad \text{and} \qquad \tilde{B} + \delta\tilde{Z} = (\tilde{B} + \tilde{Z}_{\mathrm{d}})\frac{1}{1+B\tilde{Z}_{\mathrm{d}}}\,. \tag{46}$$

For reasons to be revealed presently, we now choose $Z_{\mathrm{d}}, \tilde{Z}_{\mathrm{d}}$ (rather than $\delta Z, \delta\tilde{Z}$) as basic representatives of 'fluctuating configurations' of $Z, \tilde{Z}$. Independence of the mean fields $\mathbb{1}_{\mathrm{QD}}\, B, \mathbb{1}_{\mathrm{QD}}\, \tilde{B}$ and the fluctuations Z_{d}, $\tilde{Z}_{\mathrm{d}}$ is secured by requiring the latter to be traceless in QD,

$$\mathrm{tr}_{\mathrm{QD}} Z_{\mathrm{d}} = \mathrm{tr}_{\mathrm{QD}} \tilde{Z}_{\mathrm{d}} = 0\,. \tag{47}$$

We shall expand the action in powers of the fluctuating fields $Z_{\mathrm{d}}, \tilde{Z}_{\mathrm{d}}$. Starting with in the form (17) for $\mathcal{S}$ we set $Z = B + \delta Z$ with δZ expressed in terms of Z_{d} as stipulated above in (46). Of course, the latter equivalence relation must be expanded as well, $\delta Z = (1 - B\tilde{B})\big(Z_{\mathrm{d}} - Z_{\mathrm{d}}\tilde{B}Z_{\mathrm{d}}\big) + \ldots$ and similarly for $\delta\tilde{Z}, \tilde{Z}_{\mathrm{d}}$. The reason we use the form (17) rather than (19,20) as a starting point is a considerable saving of labor and space.

Why prefer $Z_{\mathrm{d}}, \tilde{Z}_{\mathrm{d}}$ and thus insisting in the compatibility of the perturbation expansion to be expounded with the representation (19,20) of the action? The principal reason for our preference is that we so preserve the manifold (a symmetric space, see Appendix D) on which the supermatix Q lives. As a practical benefit of that choice we shall presently find our perturbation expansion, formally one in powers of the fluctuations, to actually go in powers of small parameters. Those latter will turn out the three phases $\frac{e}{N}, \frac{\epsilon_\pm}{N}$ representing the three combinations of a, b, c, d which the generating function has as its independent variables, cf. (6).

We shall be led to speaking of the fluctuating configurations $Z_{\mathrm{d}}, \tilde{Z}_{\mathrm{d}}$ as 'decaying modes' and would like to motivate that manner of speaking now. As already mentioned, it will prove helpful to choose a QD representation in which the fluctuations acquire a semiclassical meaning in the limit $\hbar \to 0$. Both the Husimi and Wigner function are suitable. Then the single-step Floquet map becomes, up to corrections vanishing as $\hbar \to 0$, the classical Frobenius-Perron map under which the 'traceless' fluctuation $\tilde{Z}_{\mathrm{d}}$ indeed decays, with rates known as resonances [12].

C. Formal expansion and identification of small parameters

We start with decomposing the action in three additive pieces, $\mathcal{S} = N\mathcal{S}_0 + \mathcal{S}_{\mathrm{d}} + \mathcal{S}_{\mathrm{c}}$. The first, $N\mathcal{S}_0 = \mathcal{S}|_{Z_{\mathrm{d}}=\tilde{Z}_{\mathrm{d}}=0}$, makes up the zero dimensional sigma model; the second, $\mathcal{S}_{\mathrm{d}} = \mathcal{S}|_{B=\tilde{B}=0}$, pertains to the free decaying modes, while the remainder $\mathcal{S}_{\rm c}$ refers to the coupling,

$$\begin{aligned}\mathcal{S}_{\rm c} =&\ -\operatorname{str}\left(\frac{1}{1-\tilde{B}\hat{B}}\,(1-\tilde{B}B)U\tilde{Z}_{\rm d}U^\dagger\hat{e}_-(1-\tilde{B}B)Z_{\rm d}\hat{e}_+\right)+\operatorname{str}U\tilde{Z}_{\rm d}U^\dagger\hat{e}_-Z_{\rm d}\hat{e}_+\\ &-\operatorname{str}\left(\frac{1}{1-\tilde{B}\hat{B}}\,(1-\tilde{B}B)\,U\tilde{Z}_{\rm d}U^\dagger\hat{B}\,\frac{1}{1-\tilde{B}\hat{B}}\,\tilde{B}\hat{e}_-\,(1-B\tilde{B})Z_{\rm d}\hat{e}_+\right)+\dots\\ =&\operatorname{str}U\tilde{Z}_{\rm d}U^\dagger\hat{e}_-Z_{\rm d}\hat{e}_+-\operatorname{str}\left(\frac{1}{1-\tilde{B}\hat{B}}\,(1-\tilde{B}B)\,U\tilde{Z}_{\rm d}U^\dagger\,\frac{1}{1-\hat{B}\tilde{B}}\,\hat{e}_-\,(1-B\tilde{B})Z_{\rm d}\hat{e}_+\right)+\dots\end{aligned}\tag{48}$$

with the shorthand $\hat{B}=\hat{e}_-B\hat{e}_+$. We have not bothered to write out the terms linear in $Z_{\rm d}$ and $\tilde{Z}_{\rm d}$ (they vanish due to the tracelessness in QD) as well as the ones $\propto Z_{\rm d}^2$, $\tilde{Z}_{\rm d}^2$ (they vanish in the mean coupling $\langle\mathcal{S}_{\rm c}\rangle$). In fact, on display are only the bilinear terms $\propto Z_{\rm d}\tilde{Z}_{\rm d}$; higher-order terms would have to be accounted for in higher orders of the perturbation expansion which it will not be necessary to go into. Now, *our coupling action vanishes as* $\hat{e}_\pm\to 1$*; that fact allows to argue that the small parameters underlying the expansion are the phases* $\frac{e}{N}$, $\frac{\epsilon_\pm}{N}$.

We are indeed free to take as small the phases $\epsilon_\pm$ (or, equivalently, the differences $a-c$ and $b-d$), since these source variables just serve to get the correlator from the generating function according to (7). We shall therefore save labor by carrying our perturbation expansion only as far as necessary to capture the term $\propto(a-c)(b-d)\propto\epsilon_+\epsilon_-$. On the other hand, assuming smallness of $\frac{e}{N}$ makes for a non-trivial restriction. Our perturbative procedure cannot cover the phase scale $e\sim N$ on which the correlator manifests, by its definition, periodicity with period $N\pi$. However, the range of correlation decay wherein the correlator is of non-negligible magnitude, say $C(e)\gg\frac{1}{N^2}$, is fully contained. We are not even limited to $\frac{e}{N}$ of order $\frac{1}{N}$; allowable is $e\sim N^\alpha$ with $\alpha<1$. If one wants to restrict e to the range where the correlator $C_{\rm CUE}$ is larger than any system specific noise the exponent α must be smaller yet, see [6]. For instance, if smoothing is done by $\operatorname{Im}e$ small compared to unity and independent of N one has $\alpha<\frac{1}{4}$. With that understanding we can regard $\frac{e}{N}$ as the small parameter making our perturbation expansion meaningful.

We should not silently pass over the question whether or not an expansion in powers of $\frac{e}{N}$ can capture the behavior of $R(e)$ on the Ehrenfest scale $e_E\sim\frac{N}{\ln N}$. Inasmuch as on that scale $\frac{e}{N}$ is still small, $\frac{e_E}{N}\sim\frac{1}{\ln N}$, one might hope for the expansion to remain marginally sensible. However, we do not indulge since any such correction is drowned by system specific noise.

The free-decay piece of the action becomes

$$\mathcal{S}_{\rm d}=\operatorname{str}(\tilde{Z}_{\rm d}Z_{\rm d}-U\tilde{Z}_{\rm d}U^\dagger\hat{e}_-Z_{\rm d}\hat{e}_+)\,;\tag{49}$$

again, we include only the leading contribution $\propto Z_{\rm d}\tilde{Z}_{\rm d}$. Terms proportional to both $Z_{\rm d}^n$ and $\tilde{Z}_{\rm d}^n$ with $n>1$ would accompany similar terms in $\mathcal{S}_{\rm c}$ in higher orders. We now represent the decaying modes by Wigner functions $Z_{\rm d}(x)$, $\tilde{Z}_{\rm d}(x)$, with x designating the pertinent phase-space variables. Then the tracelessness (47) is expressed by phase-space integrals, $\int dxZ_{\rm d}(x)=\int dx\tilde{Z}_{\rm d}(x)=0$, and the QD trace of a product of $Z_{\rm d}$ and $\tilde{Z}_{\rm d}$ becomes the integral $\operatorname{tr}_{\rm QD}A\tilde{Z}_{\rm d}BZ_{\rm d}=\int dxA\tilde{Z}_{\rm d}(x)BZ_{\rm d}(x)$ with A,B arbitrary BF matrices. The integration range for all x integrals is the phase space volume $\int dx=\Omega$. The stroboscopic time evolution can be accounted for by a propagator $\mathcal{F}$,

$$U\tilde{Z}_{\rm d}U^\dagger\to\mathcal{F}\tilde{Z}_{\rm d}(x)\,.\tag{50}$$

The stage is thus set for implementing the semiclassical limit. To within corrections of higher than first order in $\hbar$ the Wigner function propagator becomes the classical Frobenius-Perron operator [13]. Different types of classical decay are possible [12]. It may here suffice to look at the case of purely exponential decay, with complex frequencies λ_μ called Pollicott-Ruelle resonances. These come with normalizable biorthogonal right and left eigenfunctions of $\mathcal{F}$, to be denoted as $\tilde{c}_\mu(x)$ (right) and $c_\mu(x)$ (left). The eigenvalues of $\mathcal{F}$ are $\mathrm{e}^{-\lambda_\mu}$. We expand as $\tilde{Z}_{\rm d}(x)=\sum_\mu\tilde{z}_\mu\tilde{c}_\mu(x)$, the latter sum excluding the ergodic stationary eigenfunction of $\mathcal{F}$. The free-decay and the coupling parts of the action then become sums over resonances,

$$\mathcal{S}_{\rm c}=\sum_\mu\mathcal{S}_{{\rm c}\mu}=\sum_\mu\mathrm{e}^{-\lambda_\mu}\operatorname{str}\Big(\tilde{z}_\mu\hat{e}_-z_\mu\hat{e}_+-\frac{1}{1-\tilde{B}\hat{B}}\,(1-\tilde{B}B)\,\tilde{z}_\mu\frac{1}{1-\hat{B}\tilde{B}}\,\hat{e}_-\,(1-B\tilde{B})z_\mu\hat{e}_+\Big)\,,\tag{51}$$

$$\mathcal{S}_{\rm d}=\sum_\mu\mathcal{S}_{{\rm d}\mu}=\operatorname{str}\Big(\tilde{z}_\mu z_\mu-\mathrm{e}^{-\lambda_\mu}\tilde{z}_\mu\hat{e}_-z_\mu\hat{e}_+\Big)\,;\tag{52}$$

here and below the supertrace refers to BF only.

Turning to the generating function $\mathcal{Z}$ given by the integral (16) we see the integration variables fall into independent subsets pertaining to B, $\tilde{B}, Z_{\rm d}$, $\tilde{Z}_{\rm d}$. Expanding the integrand in powers of the coupling and dropping terms of higher than first order in $\mathcal{S}_\mathrm{c}$ we get

$$\mathcal{Z} = \int d(B,\tilde{B})\,\mathrm{e}^{-N\mathcal{S}_0} \int d(Z_\mathrm{d},\tilde{Z}_\mathrm{d})\,\mathrm{e}^{-\mathcal{S}_\mathrm{d}}\big(1-\mathcal{S}_\mathrm{c}+\dots\big) \equiv \mathcal{Z}_0\mathcal{Z}_\mathrm{d}\big(1-\langle\mathcal{S}_\mathrm{c}\rangle+\dots\big) \tag{53}$$

where $\mathcal{Z}_0$ is the generating function of the zero dimensional sigma model evaluated in the previous section, $\mathcal{Z}_\mathrm{d}$ the factor exclusively contributed by the decaying modes,

$$\mathcal{Z}_\mathrm{d} = \prod_\mu \left(1+\frac{\mathrm{e}^{-\lambda_\mu}(a-c)(b-d)}{(1-ab\mathrm{e}^{-\lambda_\mu})(1-cd\mathrm{e}^{-\lambda_\mu})}\right), \tag{54}$$

and $\langle\mathcal{S}_\mathrm{c}\rangle$ the relative correction due to the coupling. Like the full generating function $\mathcal{Z}$ and the mean-field one $\mathcal{Z}_0$, the free-decay part $\mathcal{Z}_\mathrm{d}$ is now seen to deviate from unity by a summand proportional to $(a-c)(b-d)$. The mean coupling must therefore obey

$$\langle\mathcal{S}_\mathrm{c}\rangle \propto (a-c)(b-d). \tag{55}$$

We can now harvest the fruits of the perturbative labor for the correlator $C(e)$, using (7). The forgoing results for the generating function entail

$$C(e) = C_0(e) + \tfrac{2\mathrm{e}^{\mathrm{i}2e/N}}{N^2}\partial_c\partial_d\mathcal{Z}_d\big|_{\epsilon_\pm=0} - \tfrac{2\mathrm{e}^{\mathrm{i}2e/N}}{N^2}\partial_c\partial_d\langle\mathcal{S}_\mathrm{c}\rangle\big|_{\epsilon_\pm=0} \tag{56}$$

where we have used $\langle\mathcal{S}_\mathrm{c}\rangle\big|_{\epsilon_\pm=0}=0$. The first of the two corrections to the universal form $C_0(e)$ is fixed by (54) as the sum over resonances

$$\tfrac{2\mathrm{e}^{\mathrm{i}2e/N}}{N^2}\partial_c\partial_d\mathcal{Z}_d\big|_{\epsilon_\pm=0} = \tfrac{2}{N^2}\sum_\mu \tfrac{\mathrm{e}^{\mathrm{i}2e/N-\lambda_\mu}}{(1-\mathrm{e}^{\mathrm{i}2e/N-\lambda_\mu})^2}\,. \tag{57}$$

The existence of a finite gap

$$\Delta_g = \min_{\lambda_\mu\neq 0}|1-\mathrm{e}^{-\lambda_\mu}| > 0 \tag{58}$$

is thus an indispensable requirement on the underlying classical dynamics. To further appreciate the correction in question we ask how many of the possibly infinitly many classical resonances can admitted as contributors? While a precise answer to that question is hard to give we can, for an order-of-magnitude argument, certainly say that the classical phase space consist of N Planck cells. The classical approximation for the single-step quantum evolution (50) can be trusted only inasmuch as it 'does not try' to resolve sub-Planck-cell structures. Now, the resonances come with 'eigenfunctions' whose typical phase-space scales decrease as $|\mathrm{e}^{-\lambda_\mu}|$ becomes smaller [14–16]. Admittable resonances are the ones associated with phase-space structures 'wider' than a Planck cell. It is hard to imagine their number to be larger than of the order N. Were we to take that imagination at face value we could estimate an upper limit to the modulus of our above correction as the number N times the maximal modulus (over all μ), thus arriving at the 'bound' $\frac{2}{N\Delta_g^2}\propto\frac{1}{N}$. We hurry to add that we do not face a bound in any strict sense since a 'common-sense' argument is at its base; for the quantum graphs to be treated in VII stronger statements will be possible. — That estimate could in some cases be unnessecarily restrictive. For instance, the moduli $|\mathrm{e}^{-\lambda_\mu}|$ (ordered such as to decrease for growing μ) may decrease so rapidly that the sum over μ in the correction (57) remains finite for $N\to\infty$, and for such systems the correction will be of the order $\frac{1}{N^2}$.

The same large-N behavior is met with in the second correction term in (56). Since no new ideas are needed to check this we defer the somewhat lengthy investigation to App. H.

We do not see the ratio $\frac{e}{N}$ in the leading-order correction because of the explicit factor $\frac{1}{N^2}$ in the relation (7) between generation function and correlator. We want to recall that our perturbative treatment is confined to the windows of correlation decay and revival, where $\frac{e\,\mathrm{mod}N\pi}{N}\to 0$ for $N\to\infty$; there, the CUE and GUE correlators agree. Given the $\frac{1}{N^2}$ corrections we conclude, as in the numerical investigation of Ref. [6], that CUE spectral fluctuations can, for an individual spectrum, be distinguished from GUE ones only by the periodicity of the CUE correlator.

We would like to underscore that we have introduced the 'ballistic sigma model' when replacing the quantum propagator $\mathcal{F}$ with the classical Perron-Frobenius operator. To bar all possibility of misunderstanding we note that our classical approximation for the *single-step propagation* does not amount to any prejudice for the difference between classical and quantum dynamics at large times. We are simply not led to iterations of the map (like $U^n\tilde{Z}(U^\dagger)^n$, $n=2,3,\dots$) which would no later than on the Ehrenfest scale $n_E=\lambda^{-1}\ln N$ defy classical approximation. As is well known, the ensuing effective equilibration looks rather different in classical and quantum perspective, the never ending classical stretching, squeezing, and folding being washed out by quantum fluctuations [17]. Moreover, inasmuch as the decaying fields Z_d and $\tilde{Z}_\mathrm{d}$ enter the action pieces (48,49) only bilinearly we are allowed to write the QD trace of the product $Z_\mathrm{d}\tilde{Z}_\mathrm{d}$ as a phase-space integral over the product of the Wigner functions. Higher orders of the perturbation expansion would, of course, bring in Moyal products for more than two Wigner functions [18].

V. SIGMA MODEL OF AUTONOMOUS FLOWS VIA STROBOSCOPIC DESCRIPTION

The color-flavor transformation employed in Sect. II for periodically driven systems can also be made availabe for autonomous flows, simply by resorting to a stroboscopic description with a suitably chosen strobe period τ and the pertinent Floquet operator $U = \mathrm{e}^{-\mathrm{i}H\tau/\hbar}$. Such a stroboscopic description of autonomous flows has already proven fruitful for the semiclassical treatment of spectral fluctuations [5].

We consider a (sub)spectrum of N consecutive energy levels $E_1 \leq E_2 \leq \ldots \leq E_N$ and the pertinent unimodular eigenvalues of U, $\mathrm{e}^{-\mathrm{i}E_m\tau/\hbar}, m = 1, 2, \ldots N$. The strobe period τ is then chosen such that the eigenphases alias quasienergies $\phi_m = E_m\tau/\hbar$ fill the 2π interval just once. The first and last energy levels thus become nearest-neighbor quasienergies whose angular distance can be fixed as the mean spacing $2\pi/N$. In the limit of large N the artificial nearest-neighbor relationship of ϕ_1 and ϕ_N looses any significance. Inasmuch as the strobe period can be considered a classical quantity independent of Planck's constant the extent of the spectrum becomes of order $\hbar$, i. e. $E_N - E_1 = \mathcal{O}(\hbar)$.

With the Floquet operator thus defined fed into the generating function (3) all considerations of the preceeding sections run through unchanged. Of course, the periodicity in the phases is now an artefact to be shed by changing variables as (6) and going to the limit $N \to \infty$ at constant e.

The stroboscopic description of autonomous flows and the ensuing construction of the 'ballistic' sigma model through the color-flavor transformation offers advantages. Neither Hubbard-Stratonovich transformation, nor saddle-point approximation, nor regularization by noise are needed, in contrast to previous versions of the ballistic sigma model [19–21]. Such advantages notwithstanding, it is appropriate to acknowledge that the transition to the ballistic limit is very similar now and in the previous treatments: where we replace the single-step quantum propagation $U\tilde{Z}U^\dagger$ by the classical (Fobenius-Perron) propagation of the Wigner function $\tilde{Z}(x)$, the previous procedure for autonomous flows was the same for an infinitesimal time interval, i. e., to replace $\frac{\mathrm{i}}{\hbar}[H, \tilde{Z}]$ by the action of the classical Liouville operator on the Wigner function. Of course, in every other respect, the quantum character of the 'fields' Z and $\tilde{Z}$ must be retained; as already mentioned, products of more than two such fields are to be understood as Moyal products [18]. It just so happened that we did not need to go to products of more than two fields in Sect. IV.

VI. KICKED ROTOR, QUANTUM LOCALIZATION

A. Introductory remarks

Quantum localization for the kicked rotor ('Chirikov's standard model') is nearly as old a topic as the whole field of quantum chaos [22–25]. Even the field theoretic treatment in terms of the ballistic sigma model was introduced more than a decade ago [26]. Strong evidence for the power of the sigma model has come from the recent extension recovering everything previously known about the rotor and even including higher dimensional cases [4, 27].

Our goal in this review is a comparatively modest one. Focussing on the simplest situation (a single classical degree of freedom, practically full chaos in the classical phase space, unitary symmetry class, absence of quantum resonances) we propose to develop the pertinent sigma model along a pedestrian path which should not be hard to follow for readers previously uninitiated.

Arbitrarily long lived excitations (slow diffusion of the angular momentum) then arise classically, forbidding a finite gap in the Frobenius-Perron spectrum. The quantum signature of the slow diffusion is localization (of the Floquet eigenfunctions in the angular momentum representation). The quasi-energy spectrum has Poissonian rather than CUE statistics, simply because Floquet eigenfunctions without overlap in angular momentum space provide no reason for their quasi-energies to repel.

One can start with a time dependent Hamiltonian, $\hat{H}(t) = \frac{(\hbar\hat{n})^2}{2I} + k\cos(\hat{\theta} + a)\sum_m \delta(t - mT)$, where $\hat{\theta}$ and $\hbar\hat{n} = -\mathrm{i}\hbar\partial_\theta$ are operators for the angular coordinate and angular momentum, obeying the canonical commutation relation $[\hat{\theta}, \hat{n}] = \mathrm{i}$; the momentum of inertia is denoted by I and T is the kicking period. A finite value of the phase a breaks invariance under the time reversal transformation $t, \theta, l \mapsto -t, -\theta, l$. We shall in the following employ the Floquet operator describing the time evolution over one period T (from right before a kick to right before the next),

$$\hat{U} = \left(\mathrm{e}^{-\mathrm{i}\int_{0^-}^{T^-} dtH(t)/\hbar}\right)_+ = \mathrm{e}^{-\frac{\mathrm{i}\tilde{\hbar}\hat{n}^2}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}K}{\tilde{\hbar}}\cos(\hat{\theta}+a)}\,, \tag{59}$$

with the dimensionless representative of Planck's constant $\tilde{h} = \frac{\hbar T}{I}$ and the dimensionless kicking strength $K = \frac{kT}{I}$. Of special interest for us is $K \gg 1$ since that limit produces strong chaos. To exclude the resonances mentioned before we can require $\tilde{h}/4\pi$ to be an irrational number. Equally acceptable and in a sense preferable is 'near irrationality', i.e., $\tilde{h} = 4\pi \frac{M}{N}$ with M, N both large primes. That latter case comes with the conservation laws $[T_N, U] = 0$ where T_N is a translation in angular momentum, $T_N|n\rangle = |n+N\rangle$ with $\hat{n}|n\rangle = n|n\rangle$ and integer n. We can then work in a Hilbert space of dimension N (the first Brillouin zone, in solid state language, see [4]). Grosso modo, the physics of localization will be unchanged if we assume N extremely large compared to the localization length.

For later use we note the matrix elements of the Floquet operator in the angular momentum representation,

$$U_{nm} = \langle n|\hat{U}|m\rangle = \mathrm{e}^{-\frac{\mathrm{i}\tilde{\hbar}n^2}{2}} \int_0^{2\pi} \frac{d\theta}{2\pi}\, \mathrm{e}^{-\mathrm{i}\frac{K}{\hbar}\cos(\theta+a)+\mathrm{i}(n-m)\theta}\,. \tag{60}$$

B. Inverse participation ratio

A convenient indicator of localization is the so called inverse participation ratio (IPR), a quantity intimately related to the localization length (the typical span over which an eigenfunction of U is extended in angular momentum space). The convenience of the IPR lies in the fact that it is a two-point quantity like the complex correlator $C(e)$ such that it is accessible through the previously used generating function, at least after slight technical modification, and in just that way the sigma model comes into play.

For an inidividual eigenvector $|\mu\rangle$ of U the inverse participation ratio is defined as $P_\mu = \sum_{n=1}^N |\langle n|\mu\rangle|^4$ where $\{|n\rangle,\, n = 0, 1, 2, \dots N\}$ are the angular momentum eigenstates which form the basis wherein we look for localization. Now, an extended eigenvector with $|\langle n|\mu\rangle| \sim \frac{1}{\sqrt{N}}$ would have $P_\mu \sim \frac{1}{N}$ and thus $P_\mu \to 0$ as $N \to \infty$. A finite IPR results, however, for an exponentially localized vector of width l_μ independent of N, namely $P_\mu \propto \frac{1}{l_\mu}$. In order to characterize the whole Floquet matrix rather than a single eigenvector, we employ the spectrally averaged IPR

$$P = \frac{1}{N} \sum_{\mu=1}^{N} \sum_{n=1}^{N} |\langle n|\mu\rangle|^4\,. \tag{61}$$

It is easy to check that the spectrally averaged IPR is indeed related to a two-point quantity. To that end we employ the matrix elements $G^{\pm}_{nn}(a,\phi) = \langle n|\frac{1}{1-a\mathrm{e}^{\pm\mathrm{i}\phi}U^{\pm1}}|n\rangle$ of the retarded/advanced Green function with $|a| = \mathrm{e}^{-\epsilon}$ at small positive ϵ. We propose to scrutinize the center-phase average of the product of these two matrix elements multiplied with $\frac{2\epsilon}{N}$,

$$\frac{2\epsilon}{N} \int_0^{2\pi} \frac{d\phi}{2\pi} G^+_{nn}(a,\phi) G^-_{nn}(a,\phi)\,, \tag{62}$$

in the limit as ϵ approaches 0. As long as $\epsilon > 0$ we can expand both Green functions in geometric series and afterwards do the ϕ-integral. Resumming the resulting new geometric series we get

$$\frac{2\epsilon}{N} \int_0^{2\pi} \frac{d\phi}{2\pi} G^+_{nn}(a,\phi) G^-_{nn}(a,\phi) = \frac{1}{N} \sum_{\mu,\nu=1}^{N} |\langle n|\mu\rangle|^2 |\langle n|\nu\rangle|^2 \frac{2\epsilon}{1 - a^2 \mathrm{e}^{\mathrm{i}(\phi_\nu - \phi_\mu)}}\,. \tag{63}$$

We are free to fix the source variable as $a = \mathrm{e}^{-\epsilon}$. Then only the diagonal terms of the double sum over Floquet eigenvectors survive the limit $\epsilon \downarrow 0$ and we indeed arrive at the spectrally averaged IPR after summing over n,

$$P = \lim_{\epsilon\downarrow 0} \sum_{n=1}^{N} \frac{2\epsilon}{N} \int_0^{2\pi} \frac{d\phi}{2\pi} G^+_{nn}(\mathrm{e}^{-\epsilon},\phi) G^-_{nn}(\mathrm{e}^{-\epsilon},\phi)\,. \tag{64}$$

C. Towards the sigma model

We can capture the center-phase average of the product $G^+_{nn}G^-_{nn}$ by a superintegral in the fashion of (8),

$$\begin{aligned}\int_0^{2\pi} \frac{d\phi}{2\pi} G^+_{nn}(a,\phi) G^-_{nn}(a,\phi) = \int_0^{2\pi} \frac{d\phi}{2\pi} &\int d(\psi,\psi^*) \psi^*_{+,B,n}\psi_{+,B,n}\psi^*_{-,B,n}\psi_{-,B,n} \\ &\times \exp \sum_{k,l=1}^{N} \Big\{ -\psi^*_{+,B,k}(\delta_{kl} - a\mathrm{e}^{\mathrm{i}\phi}\, U_{kl})\psi_{+,B,l} - \psi^*_{-,B,k}(\delta_{kl} - a\mathrm{e}^{-\mathrm{i}\phi}\, U^*_{lk})\psi_{-,B,l} \\ &\qquad - \psi^*_{+,F,k}\,(\delta_{kl} - a\mathrm{e}^{\mathrm{i}\phi}\, U_{kl})\psi_{+,F,l} - \psi^*_{-,F,k}\,(\delta_{kl} - a\mathrm{e}^{-\mathrm{i}\phi}\, U^*_{lk})\psi_{-,F,l} \Big\}\,.\end{aligned} \tag{65}$$

Note that we could simplify, relative to (8), by setting $a=b=c=d$. Indeed, then, the Gaussian integrals give just the left side of the foregoing equation.

As in Sect. II we can apply the color-flavor transformation (A1) to trade the center-phase average against an integral over supermatrices $Z,\tilde{Z}$. Instead of (13) we now get

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{2\pi}G^+_{nn}(a,\phi)G^-_{nn}(a,\phi) &= \int d(\psi,\psi^*)\psi^*_{+,B,n}\psi_{+,B,n}\psi^*_{-,B,n}\psi_{-,B,n}\exp\big(-\psi_+^{*T}\psi_+-\psi_-^{*T}\psi_-\big) \\ &\quad\times\int d(Z,\tilde{Z})\,\mathrm{sdet}(1-Z\tilde{Z})\exp\big(\psi_+^{*T}\tilde{Z}\psi_-+\psi_-^{*T}U^\dagger\hat{e}_-Z\hat{e}_+U\psi_+\big) \\ &= \int d(Z,\tilde{Z})\,\mathrm{sdet}(1-Z\tilde{Z})\int d(\psi,\psi^*)\psi^*_{+,B,n}\psi_{+,B,n}\psi^*_{-,B,n}\psi_{-,B,n}\,\mathrm{e}^{-\psi^\dagger M\psi}\end{aligned} \tag{66}$$

with the matrix M given in (14). Doing the integral over ψ,ψ^* with the help of Wick's theorem we arrive at

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{2\pi}G^+_{nn}(a,\phi)G^-_{nn}(a,\phi) &= \int d(Z,\tilde{Z})\,\mathrm{e}^{-\mathcal{S}}\Big(\big(M^{-1}\big)_{+Bn,+Bn}\big(M^{-1}\big)_{-Bn,-Bn}+\big(M^{-1}\big)_{+Bn,-Bn}\big(M^{-1}\big)_{-Bn,+Bn}\Big) \\ &\equiv \int d(Z,\tilde{Z})\,p_n\,\mathrm{e}^{-\mathcal{S}}\,.\end{aligned} \tag{67}$$

The action $\mathcal{S}$ as given in (17) reappears, now with $a=b=c=d=\mathrm{e}^{-\epsilon}$. We note the matrix in the prefactor p_n

$$M^{-1}=\begin{pmatrix}(1-a^2U\tilde{Z}U^\dagger Z)^{-1} & U\tilde{Z}U^\dagger(1-a^2ZU\tilde{Z}U^\dagger)^{-1}\\ a^2Z(1-a^2U\tilde{Z}U^\dagger Z)^{-1} & (1-a^2ZU\tilde{Z}U^\dagger)^{-1}\end{pmatrix}. \tag{68}$$

A finite value of the spectrally averaged IPV requires the foregoing superintegral to be $\propto\frac{1}{\epsilon}$ for $\epsilon\downarrow 0$. Anticipating that 'divergence' to come from the exponential factor $\mathrm{e}^{-\mathcal{S}}$ in the integrand we shall let $a\to 1$ in the prefactor.

D. Slow modes and diffusive sigma model

We propose to simplify the action $\mathcal{S}$, invoking the slow diffusive motion of the angular momentum in the classical limit (slow on the time scale given by the kicking period). It is these slow modes that preclude universal fluctuations in the quantum spectrum. In order to isolate the corresponding 'soft' quantum fluctuations we start with the action (17) which now depends only the single source variable $a=\mathrm{e}^{-\epsilon}=1-\epsilon+\dots$. Expanding in powers of ϵ we have

$$\mathcal{S}=-\mathrm{str}\ln(1-\tilde{Z}Z)+\mathrm{str}\ln(1-U\tilde{Z}U^\dagger Z)+\epsilon\,\mathrm{str}\frac{U\tilde{Z}U^\dagger Z}{1-U\tilde{Z}U^\dagger Z}+\mathcal{O}(\epsilon^2). \tag{69}$$

Our task simplifies considerably when we momentarily restrict the attention to the lowest order in $(\tilde{Z},Z)$,

$$\mathcal{S}=\mathrm{str}\big(\tilde{Z}Z-U\tilde{Z}U^\dagger Z\big)+\epsilon\,\mathrm{str}\,U\tilde{Z}U^\dagger Z+\dots\,; \tag{70}$$

in the end a symmetry argument will allow us to restore the full dependence on $Z,\tilde{Z}$.

Next, we employ the angular-momentum representation to write the supertrace over BF and QD as $\mathrm{str}(\cdot)=\mathrm{str}_{\mathrm{BF}}\sum_n\langle n|(\cdot)|n\rangle$. The matrix elements Z_{nm} (and similarly $\tilde{Z}_{nm}$) need not be treated in full generality. Rather, in our search for slow modes we can try to neglect off-diagonal terms, $Z_{nm}\sim\delta_{nm}Z(n)$. The following intuitive reasoning supports that ansatz. Off-diagonal elements of Z in the angular-momentum representation carry information about the direction of propagation in n-space (visible, for instance, in the kick-to-kick behavior of the Wigner function). A momentarily prevailing direction will be forgotten after a few kicks, simply since the angular momentum behaves diffusively. As the diffusion proceeds over many kicks off-diagonal elements will then indeed be negligible. (The situation is reminiscent of the unbiased diffusion of an electron on a 1D lattice which affords description by a density matrix diagonal w.r.t. the site label, on time scales larger than the scattering time.) Moreover, the diagonal elements $Z(n)$ can be expected to vary but slowly as the integer n changes. This is because the n-independent diagonal Z and $\tilde{Z}$ commute with U and are thus strictly stationary; slowly varying diagonal elements must thus make up the slow modes in search. To formalize the argument we represent the slow modes by a truncated Fourier integral,

$$Z_{nm}=\delta_{nm}\int_0^{\phi_0}\frac{d\phi}{2\pi}Z(\phi)\mathrm{e}^{-\mathrm{i}n\phi} \tag{71}$$

with a cut-off ϕ_0 to be specified presently, certainly with $\phi_0 \ll 1$. We may of course imagine the cut-off worked into $Z(\phi)$ such that $Z(\phi) \to 0$ as ϕ becomes larger than ϕ_0. With that understanding we extend the ϕ-integral in (71) and in what follows to the upper limit 2π.

Given the foregoing preparation we can write (the contribution of the slow modes to) the action as

$$\mathcal{S} = \sum_{nm} \mathrm{str}_{\mathrm{BF}} Z(n)\tilde{Z}(m)\big(\delta_{nm} - (1-\epsilon)|U_{nm}|^2\big) = \int \frac{d\phi}{2\pi}\int \frac{d\phi'}{2\pi}\, k(\phi,\phi')\, \mathrm{str}_{\mathrm{BF}} Z(\phi)\tilde{Z}(\phi') \tag{72}$$

with the kernel

$$k(\phi,\phi') = \sum_{nm} \big(\delta_{nm} - (1-\epsilon)|U_{nm}|^2\big)\, \mathrm{e}^{-\mathrm{i}(n\phi+m\phi')}. \tag{73}$$

We here insert the matrix elements (60) of the Floquet operator and use the identity $\sum_n \mathrm{e}^{\mathrm{i}n\phi} = 2\pi\delta(\phi)$ to get

$$k(\phi,\phi') = 2\pi\delta(\phi+\phi')\left(1 - (1-\epsilon)\int_0^{2\pi} \frac{d\theta}{2\pi}\, \mathrm{e}^{-\mathrm{i}\frac{K}{\tilde{\hbar}}[\cos(\theta+a)-\cos(\theta+a-\phi)]}\right). \tag{74}$$

The parameter a breaking the time reversal invariance is here seen to disappear, simply by changing the integration variable θ; the localization length of the rotor, to be determined in what follows, is thus independent of a. The restriction $0 \le \phi < \phi_0 \ll 1$ allows to expand $\cos\theta - \cos(\theta-\phi) = -\phi\sin\theta + \ldots$. Further restricting the cut-off as

$$\phi_0 \ll \max(1, \tilde{\hbar}/K) \tag{75}$$

we can expand the exponential $\mathrm{e}^{\frac{\mathrm{i}K}{\tilde{\hbar}}\phi\sin\theta}$ in powers of the small quantity $\frac{K\phi}{\tilde{\hbar}}$,

$$k(\phi,\phi') \simeq 2\pi\delta(\phi+\phi')\Big(\epsilon + \left(\tfrac{K}{2\tilde{\hbar}}\right)^2\phi^2\Big)\Big(1 + \mathcal{O}(\epsilon, \phi^2, (K\phi/\tilde{\hbar})^2)\Big). \tag{76}$$

Finally returning to the angular-momentum representation we approximate $\sum_n \simeq \int dn$ (recall slow fields are smooth). The back transformation $Z(\phi) \simeq \int dn Z(n)\mathrm{e}^{\mathrm{i}n\phi}$ gives

$$\mathcal{S} = \int dn\, \mathrm{str}_{\mathrm{BF}}\Big(\epsilon Z(n)\tilde{Z}(n) + \left(\tfrac{K}{2\tilde{\hbar}}\right)^2 \partial_n Z(n)\partial_n\tilde{Z}(n)\Big). \tag{77}$$

Here, we had to preserve consistency and recall that the restriction of $Z(\phi)$ to small ϕ allows to do the ϕ-integral over the interval $[0,2\pi]$ such that the orthogonality $\int_0^{2\pi}\frac{d\phi}{2\pi}\mathrm{e}^{\mathrm{i}(n-m)\phi} = \delta_{nm}$ arises. The slow-mode action can be said to be small, in the following sense: the first term is proportional to the infinitesimal ϵ while the second cannot get large due to the smoothness condition (75).

We may now invoke the promised symmetry argument to shed the restriction to the quadratic action (70). Employing the matrix $Q = T\Lambda T^{-1}$ with $T = 1 - \left(\begin{smallmatrix} & Z \\ \tilde{Z} & \end{smallmatrix}\right)$ as defined in (20) we note that transformations T infinitesimally close to unity entail $\mathrm{Str}(\partial_n Q)^2 \sim -8\,\mathrm{str}(\partial_n Z)(\partial_n\tilde{Z})$ and $\mathrm{Str}\, Q\Lambda \sim 4\,\mathrm{str} Z\tilde{Z}$ and thus

$$\mathcal{S} = \frac{1}{4}\int dn\, \mathrm{Str}\left(\epsilon Q(n)\Lambda - \tfrac{1}{2}\left(\tfrac{K}{2\tilde{\hbar}}\right)^2\partial_n Q(n)\partial_n Q(n)\right) = \frac{1}{4}\sum_n \mathrm{Str}\left(\epsilon Q(n)\Lambda - \tfrac{1}{2}\left(\tfrac{K}{2\tilde{\hbar}}\right)^2\Big(Q(n+1)-Q(n)\Big)^2\right), \tag{78}$$

where the supertrace now is over AR and BF. At this point we may drop the restriction of the transformation, arguing as follows. The original matrix $Q = T\Lambda T^{-1}$ allows for general transformations $T = 1 - \left(\begin{smallmatrix} & Z \\ \tilde{Z} & \end{smallmatrix}\right)$ with $\tilde{Z}_{BB} = Z_{BB}^\dagger$, $\tilde{Z}_{FF} = -Z_{FF}^\dagger$ and $|Z_{BB}Z_{BB}^\dagger| < 1$ as the only conditions. In the present context where each 'site' n has its separate $Q(n)$ and we can demand $Q(n) = T(n)\Lambda T^{-1}(n)$ with separate transformations $T(n) = 1 - \left(\begin{smallmatrix} & Z(n) \\ \tilde{Z}(n) & \end{smallmatrix}\right)$. These single-site transformations $T(n)$ need not be close to the identity. The respective matrices $Z(n)$ and $\tilde{Z}(n)$, all 2×2 in BF, are restricted as before except that Z_{BB} and Z_{FF} are now complex numbers. Letting the latter numbers range freely otherwise we are no longer confined to small $Z, \tilde{Z}$. We have arrived at what is called the (action of the) diffusive 1D sigma model.

It remains to scrutinize the prefactor p_n of $\mathrm{e}^{-\mathcal{S}}$ in (67). As already mentioned above we may put $a = \mathrm{e}^{-\epsilon} \to 1$ there since the overall factor ϵ in the IPR cannot be compensated by the prefactor. Moreover, the slow modes are negligible since in contrast to their role in the action they are but small corrections of relative order $\big(K\phi_0/\tilde{\hbar}\big)^2$ in p_n. So we can simply put $U \to 1$. The thus simplified matrix M^{-1} differs from $\frac{1}{2}(Q\Lambda - 1)$ only by swapped diagonal AR blocks, $(\cdot)_{++} \leftrightarrow (\cdot)_{--}$, and likewise for the off-diagonal blocks, $(\cdot)_{+-} \leftrightarrow (\cdot)_{-+}$. That double swap does not change the prefactor in (67) whose structure is $(\cdot)_{++}(\cdot)_{--} + (\cdot)_{+-}(\cdot)_{-+}$ whereupon we can write

$$4p_n = -\big(Q(n)_{+B,+B} - 1\big)\big(Q(n)_{-B,-B} + 1\big) - Q(n)_{+B,-B}Q(n)_{-B,+B}\,. \tag{79}$$

With both the action $\mathcal{S}$ and the prefactor p_n expressed in terms of the single-site matrix $Q(n)$ we have constructed the diffusive 1D sigma model for the IPR,

$$P = \lim_{\epsilon\to 0}\sum_{n=1}^{N}\frac{2\epsilon}{N}\int d(Z,\tilde{Z})p_n \mathrm{e}^{-\mathcal{S}}. \tag{80}$$

We shall not bother to retrace the further evaluation of the IPR which is well documented in the literature [8, 28]. It is worth stating, though, that in the limit $N \to \infty$, that is for irrational values of $\tilde{h}$, the foregoing expression can hardly be imagined to produce anything else than, up to numeric factors, $P \propto \frac{1}{l} \propto (\frac{\tilde{h}}{K})^2$. That expectation is suggested by the appearance of the $(K/\tilde{h})^2$ as the single parameter characterizing the rotor, accompanying the term $[\partial_n Q(n)][\partial_n Q(n)]$ in the slow-mode action.

A critical remark on our somewhat cavalier construction of the slow-mode action is in order. It is to be complemented by checking stability against configurations outside the slow-mode 'sector'. We shall forgo that analysis which has been performed in Ref. [4]. The 'massive' modes we have neglected here turn out to renormalize the diffusion constant to a weak further dependence on $\tilde{h}$ and K (see also [22]).

VII. QUANTUM GRAPHS

A. Preliminary remarks

Ever since Kottos and Smilansky derived an exact trace formula akin to the Gutzwiller one 15 years ago [29] quantum graphs have been a paradigmatic model of quantum chaos [30]. Graphs were the first systems for which a microscopic derivation of the universal two-point function could be achieved without any average over system parameters [31, 32]. This derivation was based on the supersymmetry method using the color-flavor transformation. The approach was later generalized to universal wave function statistics [33, 34], chaotic scattering [35, 36], and spectral correlators of all orders [37]. We here focus on directed graphs where the method finds its clearest and simplest realization.

B. Directed graphs and their spectra

A quantum graph G consists of a metric graph and a wave equation in the form of an eigenequation for a self-adjoint operator. We shall allow for V vertices and N edges (often called bonds) such that each edge e is attached to one vertex at either side and is assigned a length L_e. We only consider connected graphs which cannot be divided into two or more unconnected subgraphs. For each edge a coordinate $0 \le x_e \le L_e$ and a wave function $\phi_e(x_e)$ with $e = 1, 2, \ldots N$ are introduced. We adopt the first-order wave equation

$$-\mathrm{i}\phi_e'(x_e) = k\phi_e(x_e) \tag{81}$$

with local solution $\phi_e(x_e) = a_e \mathrm{e}^{\mathrm{i}kx_e}$ where a_e are (at this point) undetermined constants. Unlike the free Schrödinger equation the first-order equation (81) is not invariant under switching coordinates as $x_e \mapsto \tilde{x}_e = L_e - x_e$. Fixing a direction on each edge we can speak of incoming and outgoing edges for each vertex. A self-adjoint momentum operator on a directed metric graph can be defined [38] if each vertex has as many incoming as outgoing edges and if each vertex is assigned a unitary scattering matrix that expresses the amplitudes of the wave functions of the outgoing edge as a linear superposition of the corresponding incoming amplitudes. It is then straightforward to see that solutions to the wave equation (81) with the above matching conditions only exist for a discrete set of wave numbers – the spectrum of the momentum operator. One may characterize that spectrum by the condition

$$\zeta(k) = 0 \tag{82}$$

for the characteristic function

$$\zeta(k) = \det(1 - U(k))\,, \qquad U(k) = ST(k) \tag{83}$$

where $U(k)$, S, and $T(k)$ are three unitary $N \times N$ matrices acting on the wave amplitudes on each edge. The diagonal matrix $T(k)_{ee'} = \delta_{ee'} \mathrm{e}^{\mathrm{i}kL_e}$ contains the phase difference at the two ends of the edge, $\phi_e(L_e) = \mathrm{e}^{\mathrm{i}kL_e}\phi_e(0)$, while the scattering matrix S harbors all matching conditions: if there is a vertex such that e' is an incoming edge and e an outgoing edge then $S_{ee'}$ is the element of the scattering matrix at that vertex; otherwise, if the two edges are not connected through a vertex we have $S_{ee'} = 0$ – thus S encodes all matching conditions and also the connectivity of the graph. The matrix $U(k)$ is called the *quantum map* and contains all relevant information of the quantum graph (connectivity, scattering amplitudes at vertices, and phases for the transport along edges). Via (82) the quantum map determines the momentum spectrum. The condition has the clear physical interpretation that k belongs to the spectrum if there is an eigenfunction such that $\phi_e(0) = \sum_{e'} U(k)_{ee'}\phi_{e'}(0) = \sum_{e'} S_{ee'} e^{ikL_{e'}}\phi_{e'}(0)$ – i.e. if there is a stationary vector under the quantum map.

Let $\left\{\mathrm{e}^{\mathrm{i}\theta_\ell(k)}\right\}_{\ell=1}^{N}$ be the set of the unimodular eigenvalues of the quantum map. The phases $\theta_\ell(k)$ are continuous functions of k. It is easy to see that these functions are strictly increasing $\frac{d\theta_\ell(k)}{dk} > 0$, and that the spectrum of the graph is given by $k_{\ell,n}$ such that $\theta_\ell(k_{\ell,n}) = 2\pi n$ ($n \in \mathbb{Z}$). Note that we are now dealing with two different spectra: the infinite momentum spectrum of the quantum graph $\{k_{\ell,n}\}_{n\in\mathbb{Z},\ell=1,\dots,N}$, and the parametric N-dimensional quasi-energy spectrum $\{\theta_\ell(k)\}$ of the quantum map which depends on k as a parameter.

It is in order to break time reversal invariance that we treat directed graphs and, in addition, do not allow for loops (edges starting and ending at the same vertex) or 'parallel' edges (i.e. there is at most one directed edge between any two vertices) and choose all edge lengths rationally independent.

To highlight the analogy of the quantum map for directed graphs to the Floquet maps of Sect. II we shall speak of the N dimensional habitat of U, S, T as QD.

C. Spectral statistics

One can show [30] that the spectral statistics of the quantum graph under study is equivalent to the k-averaged spectral statistics of the quantum map – in certain situations even with complete rigour [39]. We shall here focus on the technically simpler case of the k-averaged spectrum of the quantum map. As in the preceeding sections we describe two-point correlations in terms of a generating function,

$$\mathcal{Z}(a,b,c,d) = \left\langle \frac{\det\left(1 - cU(k)\right)\det\left(1 - dU(k)^\dagger\right)}{\det\left(1 - aU(k)\right)\det\left(1 - bU(k)^\dagger\right)} \right\rangle_k ; \tag{84}$$

the k-average now employed,

$$\langle F(k)\rangle_k := \lim_{K\to\infty} \frac{1}{K}\int_{-K/2}^{K/2} F(k)dk\,, \tag{85}$$

contains the center-phase average previously used in the Floquet map setting. However, the k-average is much stronger than just center phase since $\mathcal{Z}(a,b,c,d)$ depends on the parameter k via N unimodulars $\{\mathrm{e}^{\mathrm{i}kL_e}\}_{e=1}^{N}$. In fact, the map $k \mapsto (\mathrm{e}^{\mathrm{i}kL_1}, \dots, \mathrm{e}^{\mathrm{i}kL_N})$ describes a 'trajectory' on an N-dimensional torus, k playing the role of a time. If the trajectory is ergodic on the torus with uniform measure we may replace the k-average (formally a time average) by an average over the torus (the corresponding phase-space average). The trajectory is indeed ergodic if the edge lengths L_e ('frequencies') are rationally independent. Assuming that scenario we can replace $kL_e \mapsto \varphi_e$ and

$$\langle(\cdot)\rangle_k \to \int \frac{d^N\varphi}{(2\pi)^N}(\cdot)\,. \tag{86}$$

The generating function thus involves an N-fold phase average,

$$\mathcal{Z}(a,b,c,d) = \int \frac{d^N\varphi}{(2\pi)^N}\,\frac{\det\left(1 - cT(\boldsymbol{\varphi})S\right)\det\left(1 - dS^\dagger T(-\boldsymbol{\varphi})\right)}{\det\left(1 - aT(\boldsymbol{\varphi})S\right)\det\left(1 - bS^\dagger T(-\boldsymbol{\varphi})\right)}\,, \tag{87}$$

where $T(\boldsymbol{\varphi})_{ee'} = \delta_{ee'}\mathrm{e}^{\mathrm{i}\varphi_e}$ is now a diagonal matrix of random phases. We can even consider the matrices $T(\boldsymbol{\varphi})S$ for a fixed matrix S as a random ensemble, and that ensemble is guaranteed to have spectral correlations identical to the ones of the original individual quantum graph. Such ensembles have been called unistochastic [40]. We also note that some system parameters (the edge lengths) have disappeared; this is a first glimpse of universality. In order to see universality in the sense of the BGS conjecture we now move on and look for the conditions for the unistochastic ensemble $T(\boldsymbol{\varphi})S$ to be equivalent to the CUE.

D. Sigma model

The construction of the sigma model proceeds as in Sect. II and again leads to

$$\mathcal{Z}(a,b,c,d) = \int d(Z,\tilde{Z})\,\mathrm{sdet}\left(1 - Z\tilde{Z}\right)\mathrm{sdet}^{-1}\left(1 - S\tilde{Z}S^{\dagger}\hat{e}_{-}Z\hat{e}_{+}\right) = \int d(Z,\tilde{Z})\,\mathrm{e}^{-\mathcal{S}(Z,\tilde{Z})} \tag{88}$$

$$\mathcal{S}(Z,\tilde{Z}) = -\mathrm{str}\ln\left(1 - \tilde{Z}Z\right) + \mathrm{str}\ln\left(1 - S\tilde{Z}S^{\dagger}\hat{e}_{-}Z\hat{e}_{+}\right)\,. \tag{89}$$

We see the graph scattering matrix S taking the role of the Floquet matrix U; the matrix $T(\varphi)$ has disappeared through the phase average (86). The only other difference to Floquet maps which in fact amounts to a considerable simplification is hidden in the compact notation.

For quantum maps, the supermatrices $Z, \tilde{Z}$ were full matrices and the integral involved $4N^2$ real commuting (and as many anti-commuting) degrees of freedom. For quantum graphs, the supermatrices $Z, \tilde{Z}$ are block-diagonal and we are left with only $4N$ real commuting (and as many anti-commuting) degrees of freedom in the remaining integral. The origin the simplification lies in the N-fold phase average (86) in the generating function (87). Each edge on the graph has a separate phase average and thus affords its own color-flavor transformation (single flavor) with 2×2 supermatrices Z_e and $\tilde{Z}_e$ in BF. All edges together give rise to the block diagonal $Z_{ee'} = \delta_{ee'}Z_e$ and $\tilde{Z}_{ee'} = \delta_{ee'}\tilde{Z}_e$.

In order see how classical dynamics sneaks in and universal behavior arises we introduce the mean fields

$$B_{ss'} = \frac{1}{N}\sum_{e=1}^{N} Z_{e,ss'}\,, \qquad \tilde{B}_{ss'} = \frac{1}{N}\sum_{e=1}^{N}\tilde{Z}_{e,ss'} \tag{90}$$

and deviations therefrom through

$$Z_{b,ss'} = B_{ss'} + \delta Z_{b,ss'}\,, \qquad Z_{b,ss'} = \tilde{B}_{ss'} + \delta\tilde{Z}_{b,ss'}\,. \tag{91}$$

Clearly, the 'mean-field approximation' which neglects the fluctuations gives the zero dimensional sigma model.

For the same reasons as in Sect. IV we now choose the 'decaying fields' Z_{d} and $\tilde{Z}_{\mathrm{d}}$ rather than the 'fluctuations' δZ and $\delta\tilde{Z}$ as basic representatives of non-universal corrections, using the relations (46) and requiring the tracelessness (47) of the decaying fields. The further treatment of the non-universal corrections then parallels the one given for general Floquet maps, with the graph specific simplifications due to the block diagonality of Z_{d} and $\tilde{Z}_{\mathrm{d}}$. In particular, the coupling part of the action (48) reduces to

$$\mathcal{S}_{\mathrm{c}} = \sum_{ee'} |S_{ee'}|^2\,\mathrm{str}\Big(\tilde{Z}_{\mathrm{d}e}\hat{e}_{-}Z_{\mathrm{d}e'}\hat{e}_{+} - \frac{1}{1-\tilde{B}\hat{B}}\,(1-\tilde{B}B)\,\tilde{Z}_{\mathrm{de}}\,\frac{1}{1-\hat{B}\tilde{B}}\,\hat{e}_{-}\,(1-B\tilde{B})Z_{\mathrm{d}e'}\hat{e}_{+}\Big) + \ldots\,. \tag{92}$$

As a most remarkable consequence of the block diagonality of the fluctuations the quantum dynamics is represented by the edge-to-edge transition probabilities

$$|S_{ee'}|^2 = |U(k)_{ee'}|^2 = \mathcal{F}_{ee'} \tag{93}$$

which are independent of the wave number k. The dynamics thus acquires a classical appearance, without approximation. (Recall that for general Floquet maps classical dynamics resulted from the approximation (50).)

Interestingly, the classical dynamics ruling the small fluctuations on our connected directed graph is stochastic rather than Hamiltonian. We are actually facing a Markov process since the matrix $\mathcal{F}$ is bistochastic, i.e., $\sum_e \mathcal{F}_{ee'} = \sum_{e'}\mathcal{F}_{ee'} = 1$. The uniform probability distribution $P_e = 1/N$ is invariant under the action of the map $\mathcal{F}$, that is $\sum_{e'}\mathcal{F}_{ee'}P_{e'} = N^{-1}\sum_{e'}\mathcal{F}_{ee'} = 1/N = P_e$. If we assume that all processes $e \to e'$ allowed by the connectivity of the graph have some finite probability, the map $\mathcal{F}$ is even ergodic, and the uniform distribution is the unique stationary mode with its eigenvalue 1 of $\mathcal{F}$ non-degenerate. All other eigenvalues $\mathrm{e}^{-\lambda_\mu}$ have modulus below one, $|\mathrm{e}^{-\lambda_\mu}| < 1$; the possibly complex rates thus have positive real parts, $\mathrm{Re}\,\lambda_\mu > 0$. The time scale on which ergodicity becomes apparent in the dynamics is given by the inverse of the classical gap Δ_g introduced in (58) analogous to the gap of the ergodic Frobenius-Perron operator in the Floquet map setting (which latter, however, does not depend on N). As already found for Floquet maps, the gap Δ_g is the central quantity that governs the strength of deviations from universality in spectral correlations of quantum graphs. We should note that only the decaying eigenmodes are admitted in $Z_{\mathrm{d}}, \tilde{Z}_{\mathrm{d}}$, the stationary one being barred by the tracelessness of the decaying fields.

The final result (56) for the complex correlator $C(e)$ of our treatment of general Floquet maps can now be taken over but requires a slightly refined interpretation. The gap Δ_g need not stay fixed and may even close for a sequence of quantum graphs (in contrast to a Floquet map) with growing N. Moreover, the number of resonances is exactly

N (again in contrast to a Floquet map where the number of resonances is ignorant of N). In particular, the first of the two correction terms in (56) can be estimated as

$$\Big|\frac{2\mathrm{e}^{\mathrm{i}2e/N}}{N^2}\partial_c\partial_d\mathcal{Z}_d\big|_{\epsilon_\pm=0}\Big| \le \frac{2}{N}\frac{1}{\Delta_g^2}\,. \tag{94}$$

To obtain the upper limit we have replaced the sum over resonances by the factor N. Now a gap remaining finite gives a correction of the order $\frac{1}{N}$, rather than the $\frac{1}{N^2}$ correction found for Floquet maps (see (57)). In the graph case, the correction becomes even larger when the gap Δ_g closes as $N \to \infty$, say according to the power law $\Delta_g = N^{-\nu}$. The correction under discussion then vanishes asymptotically only as long as $\nu < 1/2$. Again, we should recall that our analysis is valid only within the windows of correlation decay and revival where $C(e)$ remains finite for $N \to \infty$.

The second term in the correction (57) can be estimated similarly.

VIII. OPEN ENDS

We hope to have presented a convincing case for the usefulness and flexibility of the sigma model for unitary maps. Put in a nutshell, we have shown in an at least self-consistent manner that (i) spectral fluctuations (as captured in the two-point correlator of the level density) of individual dynamics become universal for $N \to \infty$, given full chaos and a gapped set of classical resonances; and (ii) the absence of a gap in the resonance spectrum of the kicked rotor is accompanied by quantum localization whose effect on wavefunction statistics is captured in a nonlinear sigma model that includes all soft modes.

The range of applicability is far from being exhausted in the existing literature. To mention but a few examples of worthwhile studies, the kicked top and its approach to the kicked rotor in a certain limit for its control parameters appears quite doable and so might even be rigorous treatments of the simplest chaotic quantum graphs.

The lowest-order corrections ($\sim \frac{1}{N^2}$ or $\frac{1}{N}$, see Sect. IV C) to the CUE spectral fluctuations provided by the perturbative treatment of decaying modes are smaller than the $\frac{1}{\sqrt{N}}$ noise found numerically for individual dynamics (kicked top). As already stated in the introduction, an analytic explanation could be found in extending our work to the four-point correlator of the level density. We conjecture that there, too, the decaying modes make for a negligible correction to universal behavior which latter implies a $\frac{1}{\sqrt{N\mathrm{Im}e}}$ typical deviation of the single-spectrum two-point correlator from its CUE mean [6]. For quantum graphs, that behavior is in fact implied by results reported in Ref. [37].

A caveat is in order. Before starting to work with the sigma model for some physical system (or class of systems) one must be sure to know all of its symmetries and set up the model accordingly. For instance, when time reversal invariance reigns in the sense of the orthogonal or symplectic symmetry classes, the space harboring the matrices $Z, \tilde{Z}, Q, \dots$ must be enlarged relative to the one appropriate for the unitary symmetry class treated in this review. Systems with other symmetries require their proper adjustments of the model as well. A well known example of symmetries that make for non-universal spectral fluctuations even though the classical behavior is fully chaotic, are the Hecke symmetries making for quantum conservation laws without classical counterparts. Finding the proper setup for the sigma model in such cases of 'arithmetic chaos' [45] is an interesting challenge. Equally interesting would be a sigma model treatment of the exceptional behavior of the cat map [46].

We believe that the existing literature on the kicked rotor can be a good guide in further applications. The rotor exhibits a wealth of behavior, like renormalization of the diffusion constant by massive modes, different types of resonances, accelerator modes, Anderson transitions in dimensions larger than one, which one after the other have been captured in successive refinements of the sigma model, accounting for the respective properties of the Floquet operator.

We would like to finally remark that for universal spectral fluctuations, the sigma model is an alternative to Gutzwiller's periodic-orbit theory [41]. A sum over closely packed bunches of periodic orbits there arises which is equivalent to a perturbation expansion of the zero dimensional sigma model [5, 11, 42–44]. While periodic-orbit theory retains the charm of revealing practicability and power of Gutzwiller's semiclassical quantization of chaotic dynamics, the sigma model can now perhaps be seen as a more economic procedure. Incidentally, the periodic-orbit sum giving universal spectral fluctuations for the correlator $C(e)$ neglects *noisy* contributions of other orbit sets (not from closely packed bunches) which are dispatched as 'interfering destructively'. The pertinent noise strength could be captured as $\propto \frac{1}{\sqrt{N}}$ by studying the variance of the single-system correlator in a periodic-orbit expansion, similarly as in the sigma-model approach.

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Appendix A: Color-flavor transformation

Zirnbauer has first proven the color-flavor transformation (12) using coherent-state techniques [1–3]. The alternative to be presented here makes full use of the invariance of the flat measure $d(Z, \tilde{Z})$ under the transformation

$$Z \to Z' = (AZ + B)(CZ + D)^{-1}\,, \qquad \tilde{Z} \to \tilde{Z}' = (D\tilde{Z} + C)(B\tilde{Z} + A)^{-1} \tag{A1}$$

with supermatrices A, B, C, D constrained by invertibility of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. On the other hand, the foregoing transformation makes for an invariance of the manifold accommodating the supermatrix Q (see Appendix C). The left-hand side in the color-flavor transformation (12) is an integral representation of a Bessel function

$$\mathrm{lhs} = \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\left\{\mathrm{e}^{\mathrm{i}\phi}\Psi_1^T\Psi_{2'} + \mathrm{e}^{-\mathrm{i}\phi}\Psi_2^T\Psi_{1'}\right\} = \sum_{n=0}^{\infty} \frac{y^n}{n!^2} = J_0(2\mathrm{i}\sqrt{y}) \qquad \text{with} \qquad y = (\Psi_1^T\Psi_{2'})(\Psi_2^T\Psi_{1'})\,. \tag{A2}$$

The above invariance of the measure is needed to reduce the right-hand side of (12),

$$\mathrm{rhs} = \int d(Z, \tilde{Z})\,\mathrm{sdet}(1 - Z\tilde{Z})\exp\{\Psi_1^T\tilde{Z}\Psi_{1'} + \Psi_2^T Z\Psi_{2'}\} \equiv \Big\langle \mathrm{sdet}(1 - Z\tilde{Z})\exp\{\Psi_1^T\tilde{Z}\Psi_{1'} + \Psi_2^T\tilde{Z}\Psi_{2'}\}\Big\rangle\,, \tag{A3}$$

to that same Bessel function.

We start with noting that the special case $B = C = 0$ of (A1), $Z' = AZD^{-1}$, $\tilde{Z}' = D\tilde{Z}A^{-1}$, leaves the determinant $\mathrm{sdet}(1-Z\tilde{Z})$ unchanged. Therefore, that special transformation can be used to simplify the accompanying exponential. We can always find invertible supermatrices A and D filling up the 'Bosonic basis vector' $\mathbf{e} = (1, 0, \ldots, 0)^T$ as

$$\Psi_{2'} = D\mathbf{e}\,, \qquad\qquad \Psi_2^T = \mathbf{e}A^{-1}\,. \tag{A4}$$

Now using just such A, D we first write the quantity y defined in (A2) as $y = (\mathbf{e}^T A^{-1}\Psi_{2'})(\Psi_2^T D\mathbf{e})$ and, second, replace $Z \to AZD^{-1}$ and $\tilde{Z} \to D\tilde{Z}A^{-1}$ to get

$$\mathrm{rhs} = \Big\langle \mathrm{sdet}(1 - Z\tilde{Z})\exp\{\Psi_1^T D\tilde{Z}A^{-1}\Psi_{1'} + Z_{BB,11}\}\Big\rangle\,. \tag{A5}$$

Here, the exponent only contains one single matrix element from the Bose-Bose block of Z while all elements of $\tilde{Z}$ still appear. However, we may write $\Psi_1^T D\tilde{Z}A^{-1}\Psi_{1'} = yZ^*_{BB,11} + (\ldots)$ and thus

$$\mathrm{rhs} = \Big\langle \mathrm{sdet}(1 - Z\tilde{Z})\exp\{Z_{BB,11} + yZ^*_{BB,11} + \ldots\}\Big\rangle\,; \tag{A6}$$

we have used $\tilde{Z}_{BB} = Z^\dagger_{BB}$ and lumped all elements of $\tilde{Z}$ except for the one in the upper left corner, $Z^*_{BB,11}$, into $(\ldots)$. All these latter elements can in fact be set to zero in the exponent since they cannot contribute to the integral over $(Z, \tilde{Z})$, due to the remaining freedom in choosing the matrices A and D. In particular, we can still set $A \to A\mathrm{e}^{\mathrm{i}\varphi}$, $D \to D\mathrm{e}^{\mathrm{i}\varphi}$ with φ a diagonal matrix with arbitrary real Bosonic and arbitrary Fermionic diagonal entries. All terms in $(\ldots)$ then acquire arbitrary phase factors. No particular value of any of these phases being favored against any other we may average over the phases and so indeed find all of $(\ldots)$ annulled. Expanding the remaining exponential and realizing that only powers of $|Z_{BB,11}|^2$ can survive the subsequent integral we employ binomial formulas to get

$$\mathrm{rhs} = \sum_n \frac{y^n}{n!^2}\Big\langle \mathrm{sdet}(1 - Z\tilde{Z})|Z_{BB,11}|^{2n}\Big\rangle\,. \tag{A7}$$

For the Bessel function $J_0(2\mathrm{i}\sqrt{y})$ to arise the identity

$$\Big\langle \mathrm{sdet}(1 - Z\tilde{Z})|Z_{BB,11}|^{2n}\Big\rangle = 1 \tag{A8}$$

must hold for all naturals n. The invariance of the measure indeed implies this identity, as is seen by applying the shift $Z \to (Z-B)(1-CZ)^{-1}$ and $\tilde{Z} \to (\tilde{Z}-C)(1-B\tilde{Z})^{-1}$ to the normalization of the measure, $\left\langle \mathrm{sdet}(1-Z\tilde{Z})\right\rangle = 1$. Some algebra (see further below) shows

$$\mathrm{sdet}\big(1-(Z-B)(1-CZ)^{-1}(\tilde{Z}-C)(1-B\tilde{Z})^{-1}\big) = \frac{\mathrm{sdet}(1-Z\tilde{Z})\,\mathrm{sdet}(1-BC)}{\mathrm{sdet}(1-B\tilde{Z})\,\mathrm{sdet}(1-CZ)}\,. \tag{A9}$$

We conclude

$$\left\langle \frac{\mathrm{sdet}(1-Z\tilde{Z})}{\mathrm{sdet}(1-B\tilde{Z})\,\mathrm{sdet}(1-CZ)} \right\rangle = \frac{1}{\mathrm{sdet}(1-BC)} \tag{A10}$$

and use that identity for the special case where B and C only contain a single non-vanishing entry,

$$B_{BB,11} = \beta\,, \qquad C_{BB,11} = \gamma\,, \tag{A11}$$

such that we have

$$\mathrm{sdet}(1-BC) = 1-\beta\gamma\,, \qquad \mathrm{sdet}(1-B\tilde{Z}) = 1-\beta Z^*_{BB,11}\,, \qquad \mathrm{sdet}(1-CZ) = 1-\gamma Z_{BB,11}\,. \tag{A12}$$

The identity (A10) thus becomes

$$\left\langle \frac{\mathrm{sdet}(1-Z\tilde{Z})}{(1-\beta Z^*_{BB,11})(1-\gamma Z_{BB,11})} \right\rangle = \frac{1}{1-\beta\gamma}\,. \tag{A13}$$

Expanding in powers of β and γ we confirm (A8) and thus the color-flavor transformation (12).

It remains to fill in the algebra proving (A9). After multiplication of both sides with $\mathrm{sdet}(1-B\tilde{Z})$ the assertion to be proven reads

$$\mathrm{sdet}\big(1-B\tilde{Z}-(Z-B)(1-CZ)^{-1}(\tilde{Z}-C)\big) \overset{?}{=} \frac{\mathrm{sdet}(1-Z\tilde{Z})\,\mathrm{sdet}(1-BC)}{\mathrm{sdet}(1-CZ)}\,. \tag{A14}$$

In the left-hand-side determinant we write $B\tilde{Z} = B\frac{1-ZC}{1-ZC}\tilde{Z}$ and multiply out the numerator brackets. Then the term $B\frac{1}{1-CZ}\tilde{Z}$ cancels such that the left-hand side becomes

$$\begin{aligned}
&\mathrm{sdet}\left(1+BCZ\frac{1}{1-CZ}\tilde{Z}-Z\frac{1}{1-CZ}\tilde{Z}+Z\frac{1}{1-CZ}C-B\frac{1}{1-CZ}C\right) \\
&=\mathrm{sdet}\left(\frac{1+ZC}{1-ZC}+BC\frac{1}{1-ZC}Z\tilde{Z}-\frac{1}{1-ZC}Z\tilde{Z}+\frac{1}{1-ZC}ZC-BC\frac{1}{1-ZC}\right) \\
&=\mathrm{sdet}\left(\frac{1}{1-ZC}+BC\frac{1}{1-ZC}Z\tilde{Z}-\frac{1}{1-ZC}Z\tilde{Z}-BC\frac{1}{1-ZC}\right) \\
&=\mathrm{sdet}\left((1-BC)\frac{1}{1-ZC}-(1-BC)\frac{1}{1-ZC}Z\tilde{Z}\right) = \mathrm{Sdet}\left((1-BC)\frac{1}{1-ZC}(1-Z\tilde{Z})\right).
\end{aligned} \tag{A15}$$

We have indeed arrived at the right-hand side of (A14) and proven that assertion. Our construction of the color-flavor transformation is thus completed.

Appendix B: Flat measure $d(Z,\tilde{Z})$

For the sake of completeness, the flatness of the measure $d(Z,\tilde{Z})$ should be explained. Following standard practice (see, e.g. [8]) we get that measure form the metric in the superspace accommodating the matrix $Q = T\Lambda T^{-1}$,

$$\mathrm{Str}\, dQ^2 = \mathrm{Str}\,[T^{-1}dT,\Lambda]^2\,. \tag{B1}$$

For our 'rational parametrization' of the transformation, $T = \left(\begin{smallmatrix} 0 & Z \\ \tilde{Z} & 0\end{smallmatrix}\right)$, we have

$$[T^{-1}dT,\Lambda] = \begin{pmatrix} 0 & -2(1-Z\tilde{Z})^{-1}dZ \\ 2(1-\tilde{Z}Z)^{-1}d\tilde{Z} & 0 \end{pmatrix} \tag{B2}$$

and therefore

$$\mathrm{Str}\, dQ^2 \propto \mathrm{str}\,(1-Z\tilde{Z})^{-1}dZ(1-\tilde{Z}Z)^{-1}d\tilde{Z}\,; \tag{B3}$$

here we have dropped a numerical factor which can be fixed in the end by normalizing the measure. It is now convenient to block diagonalize as

$$(1-Z\tilde{Z})^{-1} = ARA^{-1} \qquad \text{and} \qquad (1-\tilde{Z}Z)^{-1} = DRD^{-1}\,. \tag{B4}$$

We here meet an analogy with the singular-value decomposition (25) which in the present case makes $A^{-1}ZD$ and $D^{-1}\tilde{Z}A$ block diagonal. Not fortuitously, the present block diagonalizing transformation is a special case of (A1) (with $B=C=0$). We should add that A, D, and R are 2×2 in BF and $R = \left(\begin{smallmatrix} R_{BB} & \\ & R_{FF}\end{smallmatrix}\right)$ with the $N\times N$ blocks R_{BB}, R_{FF} in QD. The 'squared length element' for dQ then becomes

$$\mathrm{Str}\, dQ^2 \propto \mathrm{str} RA^{-1}dZDRD^{-1}d\tilde{Z}A \equiv \mathrm{str} RdY Rd\tilde{Y} \tag{B5}$$

or, after writing out the supertrace in BF explicitly,

$$\begin{aligned}\mathrm{Str}\, dQ^2 \propto \mathrm{tr}_{\mathrm{QD}}\Big(& R_{BB}dY_{BB}R_{BB}d\tilde{Y}_{BB} + R_{BB}dY_{BF}R_{FF}d\tilde{Y}_{FB} \\ & - R_{FF}dY_{FB}R_{BB}d\tilde{Y}_{BF} - R_{FF}dY_{FF}R_{FF}d\tilde{Y}_{FF}\Big)\,.\end{aligned} \tag{B6}$$

Next, we introduce four-component vectors whose elements are $N\times N$ in QD to write

$$\begin{pmatrix} dW_{BB} \\ dW_{FF} \\ dW_{BF} \\ dW_{FB}\end{pmatrix} \equiv \begin{pmatrix} R_{BB} & 0 & 0 & 0 \\ 0 & -R_{FF} & 0 & 0 \\ 0 & 0 & R_{BB} & 0 \\ 0 & 0 & 0 & -R_{FF}\end{pmatrix}\begin{pmatrix} dY_{BB} \\ dY_{FF} \\ dY_{BF} \\ dY_{FB}\end{pmatrix}\,. \tag{B7}$$

Momentarily employing a short hand with dW and dY four-component supervectors, $\hat{R}$ a 4×4 supermatrix, and all entries $N\times N$ in QD, we can write the foregoing transformation from dY to dW as $dW = \hat{R}dY$. Obviously now, that transformation has the Berezinian unity since $\mathrm{Sdet}\hat{R}=1$. Analogously, we get $d\tilde{W} = \hat{R}d\tilde{Y}$ with unit Berezinian.

Proceeding to the transformations $dY = A^{-1}dZD$ and $d\tilde{Y} = D^{-1}\tilde{Z}A$ we propose to reveal the (nearly obvious) fact that the respective Berezinians are reciprocal such that the combination of these two transformation has unit Berezinian. The automatic conclusion will then be the flatness of the measure $d(Z,\tilde{Z})$ since in the sequence of transformations $dQ \to dWd\tilde{W} \to dYd\tilde{Y} \to dZd\tilde{Z}$ flatness is preserved step by step.

So we look at $dY = A^{-1}dZD$ and write that transformation as a sequence of two, $dY = dZ'D$ with $Z' = A^{-1}dZ$. Writing the first step more explicitly, $dY_{ij} = \sum_k dZ'_{ik}D_{kj}$ where the indices comprise BF and QD labels we face $2N$ transformations, one for each 'value' of i. The overall Berezinian is $\big(\mathrm{Sdet}D\big)^{2N}$. Arguing similarly we find that the second step has the Berezinian $\big(\mathrm{Sdet}A^{-1}\big)^{2N}$ such that their combination brings about the product $\big(\frac{\mathrm{Sdet}D}{\mathrm{Sdet}A}\big)^{2N}$. The transformation $d\tilde{Y} = B^{-1}\tilde{Z}A$ has A and D swapped such that it indeed has the reciprocal Berezinian. We are done.

Appendix C: Invariances of the Q manifold

We had already seen in Sect. IV B that the product of two good transformations $T = \left(\begin{smallmatrix} 1 & Z \\ \tilde{Z} & 1\end{smallmatrix}\right)$ is again good. We here extend the previous argument with the goal of establishing the transformation (A1) as the general invariance of the manifold $Q = T\Lambda T^{-1}$. We start with the product

$$\begin{pmatrix} A & C \\ B & D\end{pmatrix}\begin{pmatrix} 1 & Z \\ \tilde{Z} & 1\end{pmatrix} = \begin{pmatrix} A+B\tilde{Z} & AZ+B \\ C+D\tilde{Z} & CZ+D\end{pmatrix} \equiv T'\begin{pmatrix} (A+B\tilde{Z})^{-1} & \\ & (CZ+D)^{-1}\end{pmatrix}\,. \tag{C1}$$

The block diagonal matrix factored out on the right of T' provides T' with unit diagonal blocks as required for a good transformation placing $T'\Lambda T'^{-1}$ into the manifold of Q's. Inasmuch as the right cofactor of T' commutes with Λ we may say that T' and $\left(\begin{smallmatrix} A & C \\ B & D\end{smallmatrix}\right)T$ generate the same Q. Moreover, we have

$$T' = \begin{pmatrix} 1 & Z' \\ \tilde{Z}' & 1\end{pmatrix} \tag{C2}$$

with Z' and $\tilde{Z}'$ related to Z and $\tilde{Z}$ by (A1).

We hurry to add that the matrices A, B, C, D, all $2N\times 2N$ in BF$\otimes$QD, are not completely arbitrary. As already mentioned they must allow invertibility of the AR supermatrix $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$. They are further restricted such that Z' and $\tilde{Z}'$ retain the properties $\tilde{Z}'_{BB} = {Z'}^{\dagger}_{BB}$, $\tilde{Z}'_{FF} = -{Z'}^{\dagger}_{FF}$, and $|{Z'}_{BB}{Z'}^{\dagger}_{BB}| < 1$.

Appendix D: Q manifold as a symmetric space

We here sketch how group theory provides a natural setting for everything said in the preceeding three appendices. The group $GL(2N/2N)$ consists of invertible $4N \times 4N$ supermatrices (over complex numbers) which we write as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \tag{D1}$$

each block is a $2N \times 2N$ supermatrix which can in turn be written with $N \times N$ blocks, e.g. $A = \left(\begin{smallmatrix} A_{BB} & A_{BF} \\ A_{FB} & A_{FF} \end{smallmatrix}\right)$. Here A_{BB} and A_{FF} contain commuting numbers (complex numbers and even elements of a Grassmann algebra with complex coefficients), while A_{BF} and A_{FB} contain anti-commuting numbers (odd elements of a Grassmann algebra with complex coefficients). Otherwise the blocks A, B, C and D are only restricted by invertibility of the matrix g.

We define the set of supermatrices of the form

$$Q(g) = g\Lambda g^{-1} \qquad \text{with} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{D2}$$

the coset space $\mathcal{M}_N = GL(2N/2N)/GL(N/N)^2$. Indeed, the matrix Λ commutes with all block diagonal $4N \times 4N$ supermatrices whose diagonal blocks are $2N \times 2N$ and belong to the subgroup $GL(N/N)$. Moreover, the manifold in question is a $GL(2N/2N)$-invariant space. These properties reveal $\mathcal{M}_N$ as what is called a symmetric space. Introducing a 'coordinate chart' which parametrizes almost all matrices in $\mathcal{M}_N$ we write

$$Q(Z, \tilde{Z}) = T(Z, \tilde{Z})\Lambda T(Z, \tilde{Z})^{-1} \qquad \text{and} \qquad T = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \tag{D3}$$

where Z and $\tilde{Z}$ are independent $2N \times 2N$-supermatrices (over complex numbers).

For further clarification we look at the numeric (non-nilpotent) part of $Q(Z, \tilde{Z})$ by setting all Grassmannian generators to zero. This implies $Z_{FB} = Z_{BF} = \tilde{Z}_{BF} = \tilde{Z}_{FB} = 0$, while the remaining entries become purely numeric (complex). The ensuing matrix $T(Z, \tilde{Z})^{(\mathrm{num})}$ contains two uncoupled subblocks. We thus see that the manifold of matrices $Q(g)^{(\mathrm{num})}$ where g runs over $GL(2N/2N)$ is equivalent to $GL(2N, \mathbb{C})^2/GL(N, \mathbb{C})^4$. In the color-flavor transformation an integral over the symmetric space $\mathcal{M}_N$ is used. The integration requires a choice of the integration range for the numeric part $Q(g)^{(\mathrm{num})}$. In our case (one color and N flavors) the right choice is the product of two symmetric spaces, the non-compact $U(N, N)/U(N)^2$ for the Bose sector and the compact $U(2N)/U(N)^2$ for the Fermi sector. The non-compact part is spanned by Z_{BB} and $\tilde{Z}_{BB}$ with restrictions $Z_{BB}^\dagger = \tilde{Z}_{BB}$ and all eigenvalues of $Z_{BB}^\dagger Z_{BB}$ strictly inside the unit circle. The compact part is spanned by Z_{FF} and $\tilde{Z}_{FF}$ with the restriction $Z_{FF}^\dagger = -\tilde{Z}_{FF}$.

Generalizing the standard Riemann geometry of symmetric spaces spanned by ordinary matrices to supermatrices [1] we find the flat integration measure as shown above in Appendix B.

Appendix E: Existence worries

Zirnbauer has argued [3] that the sigma model has N 'zero modes', the projectors onto the eigenstates of U, forbidding preponderance of supermatrices $Z, \tilde{Z}$ with smooth Wigner functions in the semiclassical limit. Even though there is one smooth stationary Wigner function, the one corresponding to the classical ergodic equilibrium, the $N-1$ others do have quantum fine structure. Moreover, according to Zirnbauer, the latter states make for 'neutral directions' at $Z = \tilde{Z} = 0$ allowing for huge fluctuations.

To appreciate the problem we momentarily employ the eigenrepresentation of U for the supermatrices $Z, \tilde{Z}$ in the sigma model integral (16,17). In that representation the potentially dangerous zero modes can be scrutinized most easily. If we could do the superintegral exactly we would expect to get the formally exact expression (2) for the correlator in terms of the eigenvalues of U, no more and no less. While that expression becomes a distribution for real e, an imaginary part $\frac{1}{N} \ll \operatorname{Im} e \ll 1$ turns the correlator into a smooth self-averaging function [6]. So protected, the superintegral for $\mathcal{Z}$ should and will in fact turn out to be well behaved.

We begin our work with the quadratic part of the action, thus picking up fluctuations about the *standard saddle* $Z = \tilde{Z} = 0$ at which, according to (19), $Q = \Lambda$ and $\mathcal{S} = 0$. Abbreviating the eigenphase spacings as $\Delta_{\mu\nu} = \phi_\mu - \phi_\nu$

and accounting for $\tilde{Z}_{BB} = Z_{BB}^\dagger$, $\tilde{Z}_{FF} = -Z_{FF}^\dagger$ we get

$$\begin{aligned}\mathcal{S}^{(\mathrm{stand})} &= \mathrm{str}\Big(\tilde{Z}Z - U\tilde{Z}U^\dagger \big(\begin{smallmatrix} b & \\ & d\end{smallmatrix}\big) Z \big(\begin{smallmatrix} a & \\ & c\end{smallmatrix}\big)\Big) \\ &= \sum_{\mu,\nu=1}^{N} \Big(|Z_{BB\nu\mu}|^2 (1 - ab\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}) + |Z_{FF\nu\mu}|^2 (1 - cd\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}) \\ &\qquad + \tilde{Z}_{BF\mu\nu} Z_{FB\nu\mu} (1 - ad\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}) - \tilde{Z}_{FB\mu\nu} Z_{BF\nu\mu} (1 - bc\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}) \Big)\,.\end{aligned} \tag{E1}$$

The pertinent Gaussian integral $\mathcal{Z}^{(\mathrm{stand})} = \int d(Z,\tilde{Z})\, \mathrm{e}^{-S^{(\mathrm{stand})}}$ gives

$$\mathcal{Z}^{(\mathrm{stand})} = \prod_{\mu,\nu=1}^{N} \frac{\big(1 - ad\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)\big(1 - bc\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)}{\big(1 - ab\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)\big(1 - cd\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)} = \prod_{\mu,\nu=1}^{N} \left(1 + \frac{(a-c)(b-d)\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}}{\big(1 - ab\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)\big(1 - cd\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\nu}}\big)}\right). \tag{E2}$$

The general structure $\mathcal{Z} - 1 \propto (a-c)(b-d)$ is obviously respected. More importantly, we have $cd = ab\frac{c}{a}\frac{d}{b}$ with $|\frac{c}{a}| = |\frac{d}{b}| = 1$, $ab = \mathrm{e}^{\mathrm{i}2e/N}$ and can conclude that the integral exists as long as $\mathrm{Im}\, e > 0$.

To simplify the study of the limit $\mathrm{Im}\, e \downarrow 0$ we extract the Gaussian approximation to the correlator,

$$C^{(\mathrm{stand})}(e) = \frac{2a^2}{N^2} \frac{\mathcal{Z}^{(2)} - 1}{(a-b)(c-d)}\Big|_{a=b=c=d=\mathrm{e}^{\mathrm{i}e/N}} = \frac{2}{N^2} \sum_{\mu,\nu=1}^{N} \frac{\mathrm{e}^{\mathrm{i}(2e/N - \Delta_{\mu\nu})}}{\big(1 - \mathrm{e}^{\mathrm{i}(2e/N - \Delta_{\mu\nu})}\big)^2}\,. \tag{E3}$$

Instead of producing the train of Dirac deltas characteristic for the real part of the exact correlator, the limit $\epsilon \downarrow 0$ here gives derivatives of Dirac deltas. Only the third primitive is finite and continuous. The zero modes ($\mu = \nu$) are no more 'unruly' than the others. For $\mathrm{Im} e \sim \frac{1}{N}$ the contribution of the standard saddle is well behaved.

Other saddles than the standard Λ can contribute. Varying the action (19) w.r.t. Z and $\tilde{Z}$ one finds saddle-point equations with many more solutions obtained by permuting elements of Λ such that a number of entries $+1$ are shifted from the retarded block to the advanced one and the same number of entries -1 from the advanced to the retarded block, the shifts possibly in different 'eigensectors' μ. One is thus led to the so called *Andreev-Altshuler saddles* [47]. Of lesser interest for us are saddles contributing finitely to $\mathcal{Z}$ but not to the correlator $C(e)$ – due to the appearance of higher powers of $(a-c)$ and/or $(b-d)$. We focus on the particlar ones relevant for $C(e)$, and these differ from the standard Λ by just a single swap of a $+1$ and a -1 between the retarded and the advanced block, either between two different eigensectors μ and μ' or within a single eigensector μ. We shall see that the swaps in question happen only within the Fermi sector such that the pertinent permutations are

$$R = P_B + P_F \otimes \big((1-P) + P \otimes \tau_1\big) \qquad \text{with} \qquad R^2 = 1\,; \tag{E4}$$

here P_B and P_F respectively project onto the Bose and Fermi sector while P projects onto the subspace with fixed μ and μ', either coinciding ($\mu = \mu'$) or different. We refrain from labeling R and P with these indices. In the end, we shall sum over the contributions of all N^2 such pairs. In the Fermi sector of the $(\mu\mu')$-subspace a two dimensional AR space is left and therein the swap we are after is realized by the Pauli matrix $\tau_1 = \big(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\big)$. Each such permutation R turns the standard saddle Λ into its $R\Lambda R$.

Fluctuations about these non-standard saddles are conveniently parametrized by 'permuting from the outside'. Working with the form (19) of the action we parametrize as $Q^{(R)} = RQR = RT\Lambda T^{-1}R$ with T as before – rather than with $T^{(R)}R\Lambda R(T^{(R)})^{-1}$ and a new transformation $T^{(R)}$. Upon substituting the new saddle RQR into the action (19) we obtain

$$\mathcal{S}(Z,\tilde{Z}) = \mathrm{Str}\ln\Big(1 + \hat{X}^{(R)} Q(Z,\tilde{Z})\Big) + \mathrm{Str}\ln(1 + \hat{U}), \qquad \hat{X}^{(R)} \equiv R\hat{X}R^{-1}, \qquad \hat{X} = \Lambda(1-\hat{U})(1+\hat{U})^{-1}\,. \tag{E5}$$

Inasmuch as we are working with the eigenrepresentation of the Floquet matrix U, the matrix $\hat{X}^{(R)}$ is diagonal in AR$\otimes$BF$\otimes$QD. Indeed, writing $R = 1 + PP_F(-1+\tau_1)$ we have $\hat{X}^{(R)} = \hat{X} + PP_F(-\hat{X} + \tau_1 X \tau_1)$; now since $\hat{X} = \big(\begin{smallmatrix} X_+ & \\ & X_-\end{smallmatrix}\big)$ is diagonal, so is $\hat{X}^{(R)} = \big(\begin{smallmatrix} X_+ & \\ & X_-\end{smallmatrix}\big) + PP_F\big(\begin{smallmatrix} X_- - X_+ & \\ & X_+ - X_-\end{smallmatrix}\big)$. To capture fluctuations we expand the action (E5) to the order $Z\tilde{Z}$. A straightforward calculation yields

$$\mathcal{S}^{(R)} = \mathcal{S}_0^{(R)} + \mathrm{str}\big(Z\tilde{Z} - \hat{Y}_+^{(R)} Z \hat{Y}_-^{(R)} \tilde{Z}\big) \tag{E6}$$

where $\mathcal{S}_0^{(R)} = \mathrm{Str}\ln\big(1+\hat{X}^{(R)}\Lambda\big) + \mathrm{Str}\ln\big(1+\hat{U}\big)$ is the action at the saddle and $\hat{Y}^{(R)} = 2\big((\hat{X}^{(R)})^{-1}\Lambda+1\big)^{-1}-1$ a matrix inheriting diagonality from $\hat{X}^{(R)}$; the retarded/advanced blocks $\hat{Y}_\pm^{(R)}$ of that matrix appear in the foregoing 'quadratic' action. Evaluating the quantity $\mathcal{S}_0^{(R)}$ is an easy matter, again due to the diagonality of $\hat{U}$ and $\hat{X}^{(R)}$,

$$\mathcal{S}_0^{(R)} = -\ln\big(cd\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)\,. \tag{E7}$$

We now integrate over $Z,\tilde{Z}$ as for the standard saddle and get

$$\mathcal{Z}^{(R)} = \big(cd\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)\prod_{\rho,\sigma=1}^{N}\frac{\big(\hat{Y}^{(R)}_{+B\rho}\hat{Y}^{(R)}_{-F\sigma}-1\big)\big(\hat{Y}^{(R)}_{+F\rho}\hat{Y}^{(R)}_{-B\sigma}-1\big)}{\big(\hat{Y}^{(R)}_{+B\rho}\hat{Y}^{(R)}_{-B\sigma}-1\big)\big(\hat{Y}^{(R)}_{+F\rho}\hat{Y}^{(R)}_{-F\sigma}-1\big)} \equiv \big(cd\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)\prod_{\rho,\sigma=1}^{N}\Xi_{\rho\sigma}\,. \tag{E8}$$

This result formally contains the standard saddle, as one can check by letting R be the identity whereupon one comes back to $\mathcal{S}_0=0$ and $\hat{Y} = 2\big((\hat{X}^{-1}\Lambda+1\big)^{-1}-1$; the entries in the diagonal matrix $\hat{Y}$ are easily found and (E2) is recovered. But now we are interested in the non-standard saddles generated by the permutation (E4) for which only two of the diagonal elements of $\hat{Y}^{(R)}$ differ from the ones of $\hat{Y}$,

$$\hat{Y}^{(R)}_{+F\mu} = -\tfrac{1}{d}\,\mathrm{e}^{-\mathrm{i}\phi_{\mu'}} \qquad\text{and}\qquad \hat{Y}^{(R)}_{-F\mu'} = -\tfrac{1}{c}\,\mathrm{e}^{\mathrm{i}\phi_{\mu}}\,; \tag{E9}$$

relative to the elements of $\hat{Y}$ the change is $c\to\frac{1}{d},\ d\to\frac{1}{c}$, and $\mu\leftrightarrow\mu'$. The factor $\Xi_{\mu\mu'}$ in (E8) thus becomes

$$\Xi_{\mu\mu'} = -\frac{(a-c)(b-d)\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}}{\big(1-ab\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)\big(1-cd\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)}\,, \tag{E10}$$

indeed proportional to $(a-c)(b-d)$. We can therefore set $a=c$ and $b=d$ in all other factors $\Xi_{\rho\sigma}$ in (E8) which thereby all take the value 1. Summing over μ and μ' in (E8) to pick up the contributions of all these saddles we get

$$\mathcal{Z}^{(AA)} = -\sum_{\mu\mu'}\frac{(a-c)(b-d)\mathrm{e}^{-2\mathrm{i}\Delta_{\mu\mu'}}}{\big(1-ab\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)\big(1-cd\,\mathrm{e}^{-\mathrm{i}\Delta_{\mu\mu'}}\big)} + \dots\,, \tag{E11}$$

where the dots refer to contributions $\propto (a-c)^m(b-d)^n$ with $m+n>1$; the latter are uncapable of influencing the correlator for which reason we do not bother to treat the pertinent further saddles.

For the correlator we finally get

$$C(e) = \frac{2a^2}{N^2}\,\frac{\mathcal{Z}^{(\mathrm{stand})}+\mathcal{Z}^{(AA)}-1}{(a-c)(b-d)}\Big|_{a=b=c=d=\mathrm{e}^{\mathrm{i}e/N}} = \frac{2}{N^2}\sum_{\mu\mu'}\frac{\mathrm{e}^{\mathrm{i}(2e/N-\Delta_{\mu\mu'})}}{1-\mathrm{e}^{\mathrm{i}(2e/N-\Delta_{\mu\mu'})}}\,, \tag{E12}$$

the exact expression (2) in terms of the eigenrepresentation of U. The formal exactness of the saddle-point treatment is somewhat of a surprise and may be worth further thought.

Most importantly in our present context, one need not worry about existence of the sigma model. As long as $\mathrm{Im}\,e>0$ there are no massless modes invalidating expansions to higher orders in $Z,\tilde{Z}$. Smoothing by $\mathrm{Im}\,e>0$ is necessary anyway, to turn the correlator into a smooth function. Moreover, in the train of Dirac deltas arising for $\mathrm{Im}e\downarrow 0$ the formally massless diagonal modes ($\mu=\mu'$) are in no way distinguished over the formally massive off-diagonal ones. Unfortunately, the U-representation is useless inasmuch as it yields no clues as to why and under what conditions the smoothed correlator behaves universally. The foregonig investigation in fact lends support to endeavors involving representations which in contrast to the U-representation invite implementation of the semiclassical limit.

Appendix F: Derivation of Eq. 19

The action in (17) can be rewritten as

$$S = -\,\mathrm{Str}\ln\left(1-\begin{pmatrix}0 & \tilde{Z}\\ Z & 0\end{pmatrix}\right) + \mathrm{Str}\ln\left(1-\begin{pmatrix}U\hat{e}_+ & 0\\ 0 & U^\dagger\hat{e}_-\end{pmatrix}\begin{pmatrix}0 & \tilde{Z}\\ Z & 0\end{pmatrix}\right) \tag{F1}$$

due to $\mathrm{Str}\begin{pmatrix}0 & Z\\ \tilde{Z} & 0\end{pmatrix}^{2n+1}=0$ and the Taylor expansion $\ln(1-x) = -\sum\frac{x^n}{n}$. We now use the freedom to rename as $Z\leftrightarrow\tilde{Z}$ and then employ the identity

$$1-\begin{pmatrix}0 & Z\\ \tilde{Z} & 0\end{pmatrix} = 2\,(1+Q\Lambda)^{-1}\,,\quad \Lambda = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,, \tag{F2}$$

which is easily verified upon inserting the definition of Q given in (19) and the matrix inverse

$$\begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -Z \\ -\tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-Z\tilde{Z}} & 0 \\ 0 & \frac{1}{1-\tilde{Z}Z} \end{pmatrix} . \tag{F3}$$

The action thus becomes

$$S = -\operatorname{Str}\ln \frac{2}{Q\Lambda+1} + \operatorname{Str}\ln\left(1 - \begin{pmatrix} U\hat{e}_+ & 0 \\ 0 & U^\dagger \hat{e}_- \end{pmatrix} \frac{Q\Lambda - 1}{Q\Lambda+1}\right) = \operatorname{Str}\ln\Big((1-\hat{U})Q\Lambda + (1+\hat{U})\Big) . \tag{F4}$$

Employing the shorthand $\hat{U}$ introduced in (20) and extracting the factor $1+\hat{U}$ from the argument of the logarithm we arrive at $S(Z,\tilde{Z}) = \operatorname{Str}\ln\Big[1 + (1-\hat{U})(1+\hat{U})^{-1}Q\Lambda\Big] \ + \ \operatorname{Str}\ln(1+\hat{U})$, as given in (19).

Appendix G: Evaluation of the Grassmann integral (36)

An economic way of doing the Grassmann integral $\mathcal{G}$ is by differentiation, $\int d\eta^* d\eta d\tau^* d\tau(\cdot) = \partial_{\eta^*}\partial_\eta \partial_{\tau^*}\partial_\tau(\cdot)$, since the chain and product rules prove helpful. It is worthwhile to spell out the product rule since a peculiarity of Grassmann calculus must be pointed at. Due to the anticommutativity of Grassmannians we must distinguish between "even" and "odd" functions $A(\tau,\tau^*,\eta,\eta^*)$. Even ones additively contain only bilinear terms and/or the quadruple of the arguments while odd functions contain only linear or trilinear terms. The product rule can then be written as

$$\partial_\tau\Big[A(\tau,\tau^*,\eta,\eta^*)B(\tau,\tau^*,\eta,\eta^*)\Big] = A_\tau(\tau^*,\eta,\eta^*)B(\tau^*,\eta,\eta^*) \pm A(\tau^*,\eta,\eta^*)B_\tau(\tau^*,\eta,\eta^*) \tag{G1}$$

where the plus (minus) sign refers to even (odd) A. The absence of the argument τ on the right-hand side of the forgoing rule and the alternative $\pm$ constitute the announced peculiarity. We promise to pedantically adhere to the notation thus introduced, while evaluating $\mathcal{G}$: the absence of any of the four arguments from any function means either that the argument has been removed by differentiation (as indicated by the corresponding index) or that a cofactor is differentiated (whereupon in the function in question the argument must be replaced with 0 as well).

Writing $D(\tau,\tau^*,\eta,\eta^*) = \operatorname{Sdet}\big(1 - m(\tau,\tau^*,\eta,\eta^*)\big)$ we have $\mathcal{G} = \partial_{\eta^*}\partial_\eta\partial_{\tau^*}\partial_\tau D(\tau,\tau^*,\eta,\eta^*)^{-N}$. In differentiating we respect that the superdeteminant $D(\tau,\tau^*,\eta,\eta^*)$ is even, to get

$$\begin{aligned} \mathcal{G} =& \partial_{\eta^*}\partial_\eta\partial_{\tau^*}\big[-ND_\tau(\tau^*,\eta,\eta^*)D(\tau^*,\eta,\eta^*)^{-(N+1)}\big] \\ =& \partial_{\eta^*}\partial_\eta\big[-ND_{\tau^*\tau}(\eta,\eta^*)D(\eta,\eta^*)^{-(N+1)} - N(N+1)D_\tau(\eta,\eta^*)D_{\tau^*}(\eta,\eta^*)D(\eta,\eta^*)^{-(N+2)}\big] ; \end{aligned} \tag{G2}$$

in the last step we have used that $D_\tau(\tau^*,\eta,\eta^*)$ is odd. Continuing in this vein we get a sum of 15 terms, all with the four derivatives distributed over up to four superdeterminants. However, eleven of these terms vanish due to $D_\tau = D_{\tau^*} = D_\eta = D_{\eta^*} = 0$. Indeed, a glance at the definitions of the matrices $\Delta_\pm$ in (31) reveals that upon setting three of the four Grassmannians to zero, one of these matrices must vanish while the other becomes purely off-diagonal, such that the supertrace $\operatorname{Str}\ln(1-m)$ vanishes; moreover, that supertrace is devoid of the bilinear summands $\tau\eta$ and $\tau^*\eta^*$ and we conclude $D_{\eta\tau} = D_{\eta^*\tau^*} = 0$. We finally use $D = D(0,0,0,0) = 1$ and are left with

$$\mathcal{G} = -ND_{\eta^*\eta\tau^*\tau} + N(N+1)\big(D_{\tau^*\tau}D_{\eta^*\eta} + D_{\eta^*\tau}D_{\eta\tau^*}\big) . \tag{G3}$$

We must now inspect the 2×2 superdeterminant $D(\tau,\tau^*,\eta,\eta^*)$. Momentarily dropping arguments we write

$$D = \operatorname{Sdet}\begin{pmatrix} 1-m_{BB} & -m_{BF} \\ -m_{FB} & 1-m_{FF} \end{pmatrix} = \frac{1-m_{BB}}{1-m_{FF}} - \frac{m_{BF}m_{FB}}{(1-m_{FF})^2} . \tag{G4}$$

By its definition (33), the matrix m has even diagonal entries and odd off-diagonal ones. Furthermore, all four entries are nilpotent. In particular, $m_{FF}^n = 0$ for $n > 2$, and therefore we can expand as

$$D - 1 = -m_{BB} + m_{FF} - m_{BB}m_{FF} + m_{FF}^2 - m_{BF}m_{FB}(1 + 2m_{FF}) . \tag{G5}$$

That expansion must equal $D(\tau,\tau^*,\eta,\eta^*) = 1 + D_{\tau^*\tau}\tau\tau^* + D_{\eta^*\eta}\eta\eta^* + D_{\eta^*\tau}\tau\eta^* + D_{\eta\tau^*}\tau^*\eta + D_{\eta^*\eta\tau^*\tau}\tau\tau^*\eta\eta^*$, with the coefficients appearing in (G3); note that we are back to the promised pedantery. To determine these coefficients we need to inspect the matrix m in detail. Momentarily writing $\delta_+ = a - c$, and $\delta_- = b - d$ we have

$$\begin{aligned} m_{BB}(\tau,\tau^*,\eta,\eta^*) &= (1-abl_B)^{-1}\Big(l_B\big(-\delta_+ b\tau\tau^* + \delta_- a\eta\eta^* - \delta_+\delta_-\tau\tau^*\eta\eta^*\big) - \delta_+\delta_-\sqrt{-l_Bl_F}\,\tau^*\eta\Big) \\ m_{FF}(\tau,\tau^*,\eta,\eta^*) &= (1-cdl_F)^{-1}\Big(l_F\big(-\delta_+ d\tau\tau^* + \delta_- c\eta\eta^* - \delta_+\delta_-\tau\tau^*\eta\eta^*\big) - \delta_+\delta_-\sqrt{-l_Bl_F}\,\tau\eta^*\Big) \\ m_{BF}(\tau,\tau^*,\eta,\eta^*) &= (1-abl_B)^{-1}\Big(l_B\delta_+\big(b+\delta_-\eta\eta^*\big)\tau + \sqrt{-l_Bl_F}\,\delta_-\big(c-\delta_+\tau\tau^*\big)\eta\Big) \\ m_{FB}(\tau,\tau^*,\eta,\eta^*) &= (1-cdl_F)^{-1}\Big(l_F\delta_+\big(d+\delta_-\eta\eta^*\big)\tau^* + \sqrt{-l_Bl_F}\,\delta_-\big(a-\delta_+\tau\tau^*\big)\eta^*\Big)\,. \end{aligned} \tag{G6}$$

Only the diagonal elements and the product $m_{BF}m_{FB}$ of the off-diagonals contain terms bilinear in the Grassmannians. We read out the second derivatives

$$D_{\tau^*\tau} = \frac{(a-c)(bl_B - dl_F)}{(1-abl_B)(1-cdl_F)}\,, \quad D_{\eta^*\eta} = -\frac{(b-d)(al_B - cl_F)}{(1-abl_B)(1-cdl_F)}\,, \quad D_{\eta^*\tau} = -D_{\eta\tau^*} = -\frac{(a-c)(b-d)\sqrt{-l_Bl_F}}{(1-abl_B)(1-cdl_F)} \tag{G7}$$

and their combination

$$D_{\tau^*\tau}D_{\eta^*\eta} + D_{\eta^*\tau}D_{\eta\tau^*} = -\frac{(a-c)(b-d)(abl_B - cdl_F)(l_B - l_F)}{(1-abl_B)^2(1-cdl_F)^2}\,. \tag{G8}$$

The forth derivative, in turn, receives contributions from all terms on the right-hand side of the expansion (G5),

$$D_{\eta^*\eta\tau^*\tau} = \frac{(a-c)(b-d)(1+cdl_F)(l_B-l_F)}{(1-abl_B)(1-cdl_F)^2}\,. \tag{G9}$$

Putting together Eqs. (G8) and (G9) we get the announced result (36) for the Grassmann integral $\mathcal{G}$.

Appendix H: Checking $\partial_c\partial_d\langle\mathcal{S}_c\rangle\big|_{\epsilon_\pm=0} \propto \frac{1}{N^2}$

We must invoke the easily proven contraction rule

$$\begin{aligned} \big\langle \mathrm{str}_{\mathrm{BF}} X\tilde{z}_\mu Y z_\mu\big\rangle_{\mathrm{d}} &= X_{BB}\left(\frac{Y_{BB}}{1-ab\mathrm{e}^{-\lambda_\mu}} - \frac{Y_{FF}}{1-ad\mathrm{e}^{-\lambda_\mu}}\right) - X_{FF}\left(\frac{Y_{BB}}{1-bc\mathrm{e}^{-\lambda_\mu}} - \frac{Y_{FF}}{1-cd\mathrm{e}^{-\lambda_\mu}}\right) &\text{(H1)} \\ &= \frac{\mathrm{str}_{\mathrm{BF}}X\,\mathrm{str}_{\mathrm{BF}}Y}{1-ab\mathrm{e}^{-\lambda_\mu}} + \frac{(a-c)b\mathrm{e}^{-\lambda_\mu}}{(1-ab\mathrm{e}^{-\lambda_\mu})^2}X_{FF}\,\mathrm{str}_{\mathrm{BF}}Y + \frac{(b-d)a\mathrm{e}^{-\lambda_\mu}}{(1-ab\mathrm{e}^{-\lambda_\mu})^2}Y_{FF}\,\mathrm{str}_{\mathrm{BF}}X &\text{(H2)} \\ &\quad + (a-c)(b-d)\,\frac{\mathrm{e}^{-\lambda_\mu}\,(1+ab\mathrm{e}^{-\lambda_\mu})}{(1-ab\mathrm{e}^{-\lambda_\mu})^3}X_{FF}Y_{FF} + \ldots \end{aligned}$$

with 2×2 matrices X, Y in BF; the dots refer to terms of higher order in $a-c$ and $b-d$.

Easy to evaluate is the average $\langle \mathrm{str}\,\hat{e}_+\tilde{z}_\mu\hat{e}_-z_\mu\rangle = \mathcal{Z}_0^{-1}\int d(B,\tilde{B})\mathrm{e}^{-N\mathcal{S}_0}\langle\mathrm{str}\hat{e}_+\tilde{z}_\mu\hat{e}_-z_\mu\rangle_{\mathrm{d}}$. The mean-field average herein trivially gives the factor unity since the quantity to be averaged is independent of $B, \tilde{B}$. The contraction rule gives the partial average over the decaying modes. Due to $\mathrm{str}\,\hat{e}_+ = a - c = \delta_+$ and $\mathrm{str}\,\hat{e}_- = b - d = \delta_-$ we immediately get an overall factor $\delta_+\delta_-$. In the remaining cofactor we may right away set $a = b = c = d$ since we are interested in $\partial_c\partial_d\langle\mathcal{S}_c\rangle\big|_{a=b=c=d}$. In particular, we can set $\mathcal{Z}_0^{-1} = 1$. A few steps of elementary algebra then lead to $\langle \mathrm{str}\,\hat{e}_+\tilde{z}_\mu\hat{e}_-z_\mu\rangle = \delta_+\delta_-(1+a^2\mathrm{e}^{-\lambda_\mu})(1-a^2\mathrm{e}^{-\lambda_\mu})^{-3}$, independent of N. The pertinent correction to the correlator therefore is of the order $\frac{1}{N^2}$.

We now turn to the average of the second summand under the supertrace in (51),

$$\left\langle \mathrm{str}\,\hat{e}_+\frac{1}{1-\tilde{B}\hat{B}}(1-\tilde{B}B)\tilde{z}_\mu\frac{1}{1-\hat{B}\tilde{B}}\,\hat{e}_-(1-B\tilde{B})z_\mu\right\rangle \equiv \langle \mathrm{str}\,X\tilde{z}_\mu\,Y z_\mu\rangle\,. \tag{H3}$$

The mean-field average herein is conveniently done with the help of the singular-value decomposition (25),

$$\langle \mathrm{str}X\tilde{z}_\mu Y z_\mu\rangle = \mathcal{Z}_0^{-1}\int_0^1 dl_B\int_{-\infty}^0 dl_F\frac{1}{(l_B-l_F)^2}\left(\frac{(1-l_B)(1-a^2l_F)}{(1-l_F)(1-a^2l_B)}\right)^N\int d\eta^*d\eta d\tau^*d\tau\mathrm{e}^{-N\mathrm{str}\ln(1-m)}\langle\mathrm{str}X\tilde{z}_\mu Y z_\mu\rangle_{\mathrm{d}}\,. \tag{H4}$$

We know already that the foregoing superintegral is proportional to $(a-c)(b-d)$. Moreover, for the correlator no terms of higher order in $a-c$ and $b-d$ are needed. Therefore, we can set $\mathcal{Z}_0 \to 1$ and $a=b=c=d$, except in the resulting factor $(a-c)(b-d)$.

Being after the large-N limit we confine ourselves to small $\frac{e}{N}$ such that $a = 1 + \mathrm{i}\frac{e}{N} + \ldots$. Then the base under the exponent N becomes proportional to $\frac{1}{N}$ such that we can use $(1+\frac{x}{N})^N = \mathrm{e}^x(1-\frac{x^2}{2N}+\ldots)$. For our purposes, we can drop the $\frac{1}{N}$ correction and write $\Big(\frac{(1-l_B)(1-a^2 l_F)}{(1-l_F)(1-a^2 l_B)}\Big)^N = \exp\big\{2\mathrm{i}e\big(\frac{l_B}{1-l_B} - \frac{l_F}{1-l_F}\big)\big\}$ and thus

$$\langle \mathrm{str} X \tilde{z}_\mu Y z_\mu \rangle \sim \int_0^1 dl_B \int_{-\infty}^0 dl_F \frac{\exp\big\{2\mathrm{i}e\big(\frac{l_B}{1-l_B} - \frac{l_F}{1-l_F}\big)\big\}}{(l_B-l_F)^2} \int d\eta^* d\eta d\tau^* d\tau \mathrm{e}^{-N \mathrm{str}\ln(1-m)} \langle \mathrm{str} X \tilde{z}_\mu Y z_\mu \rangle_{\mathrm{d}} \,. \tag{H5}$$

To continue our quest for the large-N asymptotics we use $\hat{e}_+ = a-(a-c)P$, $\hat{e}_- = b-(b-d)P$ with the BF projector $P = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to expand the matrices $(1-\tilde{B}\hat{B})^{-1}$ and $(1-\hat{B}\tilde{B})^{-1}$ in powers of $\delta_+ = (a-c)$ and $\delta_- = (b-d)$,

$$\begin{aligned} X &= \hat{e}_+ \frac{1}{1-\tilde{B}\hat{B}}(1-\tilde{B}B) = X_0 + \delta_+ X_+ + \delta_- X_- + \delta_+\delta_- X_{+-} + \ldots \,, \\ Y &= \frac{1}{1-\hat{B}\tilde{B}}\, \hat{e}_-(1-B\tilde{B}) = Y_0 + \delta_+ Y_+ + \delta_- Y_- + \delta_+\delta_- Y_{+-} + \ldots \,, \end{aligned} \tag{H6}$$

where the dots stand for higher-order terms incapable of contributing to the correlator. The matrices $X_0, X_\pm, X_{+-}$ and $Y_0, Y_\pm, Y_{+-}$ all have finite vanishing limits as $N \to \infty$.

We proceed to the average over the decaying modes with the help of the contraction rule (H2). The shorthand $f_\mu = 1 - ab\mathrm{e}^{-\lambda_\mu}$ allows to write

$$\begin{aligned} \langle \mathrm{str} X \tilde{z}_\mu Y z_\mu \rangle_{\mathrm{d}} =& f_\mu \mathrm{str} X_0 \mathrm{str} Y_0 \\ &+ \delta_+ f_\mu \big[\mathrm{str} X_+ \mathrm{str} Y_0 + \mathrm{str} X_0 \mathrm{str} Y_+ + b f_\mu \mathrm{e}^{-\lambda_\mu} X_{0FF} \mathrm{str} Y_0\big] \\ &+ \delta_- f_\mu \big[\mathrm{str} X_- \mathrm{str} Y_0 + \mathrm{str} X_0 \mathrm{str} Y_- \\ &+ \delta_+\delta_- f_\mu \Big[\mathrm{str} X_0 \mathrm{str} Y_{+-} + \mathrm{str} X_{+-} \mathrm{str} Y_0 + \mathrm{str} X_+ \mathrm{str} Y_- + \mathrm{str} X_- \mathrm{str} Y_+ \\ &\qquad + \mathrm{e}^{-\lambda_\mu} f_\mu \big(a(\mathrm{str} X_+) Y_{0FF} + a(\mathrm{str} X_0) Y_{+FF} + b X_{-FF} \mathrm{str} Y_0 + b X_{0FF} \mathrm{str} Y_-\big) \\ &\qquad + \mathrm{e}^{-\lambda_\mu}(1+ab\mathrm{e}^{-\lambda_\mu}) f_\mu^2 X_{0FF} Y_{0FF}\Big] + \ldots \,. \end{aligned} \tag{H7}$$

where we should and henceforth shall set $a=b=c=d=\mathrm{e}^{\mathrm{i}e/N}$ except in the explicit factors $\delta_\pm$.

For the first term in (H7), the one of zero order in $\delta_\pm$, we need $X_0 = a\frac{1-\tilde{B}B}{1-a^2\tilde{B}B} \sim a(1-(2\mathrm{i}e/N)\tilde{B}B(1-\tilde{B}B)^{-1})$ and $\mathrm{str} X_0 = \mathrm{str} Y_0 \sim -\frac{2\mathrm{i}e}{N}\frac{l_B-l_F}{(1-l_B)(1-l_F)}$. The full average of the first term thus reads

$$\langle f_\mu \mathrm{str} X_0 \mathrm{str} Y_0 \rangle = -f_\mu \left(\frac{2e}{N}\right)^2 \int_0^1 dl_B \int_{-\infty}^0 dl_F \left(\frac{l_B-l_F}{(1-l_B)(1-l_F)}\right)^2 \mathcal{G} \frac{1}{(l_B-l_F)^2} \exp\Big\{2\mathrm{i}e\Big(\frac{l_B}{1-l_B} - \frac{l_F}{1-l_F}\Big)\Big\} \,. \tag{H8}$$

Then to leading order in N, the Grassmann integral (40) becomes $\mathcal{G} \sim N^2\delta_+\delta_-(\frac{l_B}{1-l_B} - \frac{l_F}{1-l_F})^2$. Changing integration variables as $x = \frac{l_B}{1-l_B}$, $y = \frac{l_F}{1-l_F}$ we get for the average under study

$$\frac{\langle f_\mu \mathrm{str} X_0 \mathrm{str} Y_0 \rangle}{(a-c)(b-d)} \sim -4e^2 f_\mu \int_0^\infty dx \int_{-1}^0 dy (x-y)^2 \exp\{2\mathrm{i}e(x-y)\} \,, \tag{H9}$$

i.e. independence of N. We conclude the corresponding correction to the correlator to be $\propto (\frac{e}{N})^2$.

For the full average of the second term of (H7), the one linear in δ_+, we need to set $\delta_+ = 0$ in the nilpotent matrix m (such that $m \to \delta_- m_-$) and expand as $\mathrm{e}^{-N \mathrm{str}\ln(1-\delta_- m_-)} = 1 - N\delta_- \mathrm{str}\, m_-$; the summand 1 does not survive the Grassmann integration since the square bracket in the second term of (H7) contains no quadrilinear product $\tau\tau^*\eta\eta^*$, such that under the Grassmann integral we have $\mathrm{e}^{-N \mathrm{str}\ln(1-\delta_- m_-)} = -N\delta_- \mathrm{str}\, m_- = -\eta\eta^* N(l_B - l_F)$. The factor N here appearing is compensated by the $\frac{e}{N}$ appearing in the traces $\mathrm{str} X_0 = \mathrm{str} Y_0$ such that again the average $\langle \delta_- f_\mu[\cdot]\rangle$ under study is asymptotically independent of N and thus incapable of contributing more than a $\frac{1}{N^2}$ correction to the correlator. Analogous reasoming gives the same result for the third term in (H7).

Finally, the fourth term in (H7), the one bilinear in δ_+ and δ_-, cannot produce anything larger either since here the nilpotent matrix m vanishes for $\delta_\pm \to 0$ such that no factor N comes from $\mathrm{e}^{-N \mathrm{str}\ln(1-\delta_- m)}$ while the square bracket in the fourth term of (H7) is asymptotically independent of N; indeed, one easily checks that the terms $\mathrm{str}X_+\mathrm{str}Y_- + \mathrm{str}X_-\mathrm{str}Y_+$ do not vanish in the limit $N \to \infty$.

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