Electron-hole coherent states for the Bogoliubov-de Gennes equation

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We construct a new set of generalized coherent states, *the electron-hole coherent states*, for a (quasi-)spin particle on the infinite line. The definition is inspired by applications to the Bogoliubov-de Gennes equations where the quasi-spin refers to electron- and hole-like components of electronic excitations in a superconductor. Electron-hole coherent states generally entangle the space and the quasi-spin degrees of freedom. We show that the electron-hole coherent states allow obtaining a resolution of unity and form minimum uncertainty states for position and velocity where the velocity operator is defined using the Bogoliubov-de Gennes Hamiltonian. The usefulness and the limitations of electron-hole coherent states and the phase space representations built from them are discussed in terms of basic applications to the Bogoliubov-de Gennes equation such as Andreev reflection.

I. INTRODUCTION

Coherent states in their multiple variants and generalizations have become an indispensable tool in various branches of quantum mechanics and quantum field theory. The standard coherent states are strongly connected to the quantum description of a one-dimensional harmonic oscillator. They were first mentioned by Schrödinger [1], who realised that they describe wave packets with minimal uncertainty that follow the classical dynamics without changing their shape. Glauber [2, 3] and Sudarshan [4] later rediscovered these states, gave a very detailed description of their properties and showed their relevance in quantum optics. Since then they have also become an important tool in quantum field theory and a central ingredient of phase space formulations of quantum mechanics and semiclassical approximations. The standard coherent states have been generalised along various lines. E.g., the group-theoretic approach generalizes the fact that standard coherent states can be obtained by a group action of the Heisenberg-Weyl group on the harmonic oscillator ground state. Generalized coherent states can then be obtained by taking a different initial state or a different group (and a Hilbert space which carries a representation of that group) [5].

There are various other approaches to generalize coherent states that are discussed in the literature with many interesting applications [6–9]. A property shared by most generalized coherent states is that they are labeled by continuous parameters and allow obtaining a formal resolution of unity (*via* a projector valued measure on the set of continuous parameters).

In this contribution we will consider a special case of a construction of coherent states on a Hilbert space which is a tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. There is an obvious canonical construction if each of the factor spaces already comes with a set of coherent states, say $|\alpha\rangle_1 \in \mathcal{H}_1$ and $|\beta\rangle_2 \in \mathcal{H}_2$ where α and β stands for the set of continuous parameters on that space. One may then define the coherent states on the tensor product $\mathcal{H}$ as the tensor product of coherent states in each factor $|\alpha, \beta\rangle := |\alpha\rangle_1 \otimes |\beta\rangle_2$. In the following we will show that the special case where $\mathcal{H}_1 \equiv L^2(\mathbb{R})$ and $\mathcal{H}_2 \equiv \mathbb{C}^2$ one may define generalized coherent states that are (for almost all values of the continuous parameters) entangled, i.e. not product states. Physically, the tensor coherent states are quite natural for describing a quantum point-particle on the line with spin 1/2 (a point particle in $\mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is described analogously by appropriate tensor-product coherent states).

It is however interesting to see that entangled coherent states, sharing many properties of the standard coherent states can be defined. We adopt the name *electron-hole coherent states* for our construction as they may be used in a natural way for electronic excitations in a superconductor or a hybrid superconducting-normalconducting device. These are described by the Bogoliubov-de Gennes equation and the factor $\mathbb{C}^2$ of the Hilbert space refers to electron-like and hole-like components of the wave function.

In Section II we review the main properties of standard coherent states, $SU(2)$ coherent states, and tensor product states of standard coherent states with the spin coherent states.

In Section III we construct electron-hole coherent states, derive their main properties, and show that they fit into the group theoretic approach to generalized coherent states if one drops the assumption that the group has to act linearly on the Hilbert space. We also give a brief account of the relation of electron-hole coherent states to the Bogoliubov-de

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Gennes equation
In Section IV we look at generalisations of various coherent state representations of operators and states. We also discuss some limitations of electron-hole coherent states when trying to generalize the phase space formulations of quantum mechanics that is based on Husimi functions. These limitations may be less severe in a semiclassical regime of the Bogoliubov-de Gennes equation.

II. A SHORT REVIEW OF COHERENT STATES

A. Standard (Schrödinger-Glauber-Sudarshan) coherent states

The construction of standard coherent states can be found in various textbooks. As they are one ingredient of the definition of electron-hole coherent states in Section III we will give a short summary of their construction and their basic properties. We will focus mainly on properties that may be used as alternative definitions – each of these has been used as a starting point for generalising the concept of coherent states. Usually focussing on one property of coherent states will imply that not all properties of standard coherent states will hold in the generalisation. In Section III we will analyse how these properties generalise to the electron-hole coherent states defined there (and what limitations arise).

1. *The unitary representation of the Heisenberg-Weyl group*

One may start from the (unique up to isomorphisms) irreducible unitary representation of the Heisenberg-Weyl group $H_3(\mathbb{R})$. In this representation the group is generated by the annihilation operator a, the creation operator $a^\dagger$ and the identity operator $\hat{1}$ (if c is a scalar we will usually write $c\hat{1} \equiv c$) with commutation relations

$$\left[a, a^\dagger\right] = \hat{1} \quad \text{and} \quad \left[\hat{1}, a\right] = \left[\hat{1}, a^\dagger\right] = 0. \tag{1}$$

In the unitary representation the creation and annihilation operators are Hermitian conjugates of each other (in general we will denote the Hermitian conjugate of an operator O as $O^\dagger$).
The unitary representation of the Heisenberg-Weyl group is obtained by taking the exponential of anti-Hermitian linear combinations of the generators, i.e. a (representation of a) group element is of the form

$$D(\alpha, \phi) = e^{\alpha a^\dagger - \alpha^* a + i\phi} = e^{-\frac{|\alpha|^2}{2} + i\phi} e^{\alpha a^\dagger} e^{-\alpha^* a}, \tag{2}$$

where $\alpha \in \mathbb{C}$ and $\phi \in \mathbb{R}$. The second equality can be established using the Baker-Campbell-Hausdorff formula which also allows one to find the multiplication law

$$D(\alpha_2, \phi_2) D(\alpha_1, \phi_1) = D(\alpha_1 + \alpha_2, \phi_1 + \phi_2 + \mathrm{Im}(\alpha_1^* \alpha_2)), \tag{3}$$

which is indeed the multiplication law of the Heisenberg-Weyl group $H_3(\mathbb{R})$. One also finds

$$D(\alpha, \phi)^{-1} = D(-\alpha, -\phi) = D(\alpha, \phi)^\dagger, \tag{4}$$

which shows that the representation is indeed unitary.
The infinite dimensional Hilbert space $\mathcal{H}_\infty \equiv \ell^2 \equiv L^2(\mathbb{R})$ of this irreducible representation is spanned by the orthogonal (and normalised) number state basis (aka Fock basis) $\{|n\rangle\}_{n=0}^{\infty}$ (where orthonormality implies $\langle n|n'\rangle = \delta_{nn'}$) in which the generators act as

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad \text{where } a|0\rangle = 0 \text{ for } n = 0, \tag{5a}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{5b}$$

$$\hat{1}|n\rangle = |n\rangle, \tag{5c}$$

$$a^\dagger a|n\rangle = n|n\rangle. \tag{5d}$$

The last equation shows that $|n\rangle$ is an eigenstate of the number operator $N := a^\dagger a$. These states are thus the energy eigenstates the harmonic oscillator which is described by the Hamiltonian $H = \hbar\omega\left(a^\dagger a + 1/2\right)$. For $n = 0$ the number state $|0\rangle$ will be called the vacuum.
It is well known that the abstract Hilbert space $\mathcal{H}_\infty$ together with the representation of the Heisenberg-Weyl generators are isomorphic to the Hilbert space of square-integrable functions on the line $L^2(\mathbb{R})$ which is used to describe quantum point particle in one dimension. The solution of the harmonic oscillator provides a standard way to identify the two Hilbert spaces and representations.

The standard coherent states on $\mathcal{H}_\infty$ can now be defined by acting with elements of the Heisenberg-Weyl group on the vacuum

$$|\alpha\rangle := D(\alpha,0)|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |n=0\rangle \ . \tag{6}$$

Note that $D(\alpha,\phi)|0\rangle = e^{i\phi}|\alpha\rangle$, so by setting $\phi = 0$ in our definition we fixed a phase convention. It is sometimes convenient to define coherent states that omit the normalisation factor $e^{-|\alpha|^2/2}$, that is

$$|\alpha) := e^{\alpha a^\dagger} |n=0\rangle = e^{|\alpha|^2/2} |\alpha\rangle \ . \tag{7}$$

While $|\alpha)$ is not normalised, it is analytic as a function of α. As a manifold the coherent states are equivalent to the complex plane $\mathbb{C}$ which is equivalent to the coset space $H_3(\mathbb{R})/U(1)$ (*via* the map $\alpha \mapsto D(\alpha,0)$). One may identify this plane with the classical phase space and the Heisenberg-Weyl group translates points in phase space, i.e.

$$D(\alpha_2,0)|\alpha_1\rangle = e^{i\mathrm{Im}(\alpha_1^*\alpha_2)}|\alpha_1+\alpha_2\rangle. \tag{8}$$

In the rest of the paper standard coherent states in the Hilbert space $\mathcal{H}_\infty$ will always be denoted with the letter α (sometimes with an additional index). Note that the vacuum is the coherent state with $\alpha = 0$, i.e. $|\alpha = 0\rangle = |n = 0\rangle$ (writing just $|0\rangle$ does thus not lead to ambiguities). A general coherent state is a superposition of number states

$$|\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad \text{or} \quad |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \ . \tag{9}$$

This implies that the probability p_n to find the n-th number state in a given coherent state is Poisson distributed $p_n = \frac{|\alpha|^2}{n!}e^{-|\alpha|^2}$. The probabilities take dominant values when n is comparable to $|\alpha|^2$. Indeed the maximal probability arises when n is the largest integer smaller than $|\alpha|^2$. It is also straightforward to find the expectation value of the number operator

$$\langle\alpha|N|\alpha\rangle = \sum_{n=0}^{\infty} np_n = |\alpha|^2. \tag{10}$$

This calculation becomes even simpler if one establishes that standard coherent states are eigenstates of the annihilation operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle \ . \tag{11}$$

This property can be used also as an alternative definition for standard coherent states. Using the identities

$$D(-\alpha,0)aD(\alpha,0) = a + \alpha, \tag{12a}$$
$$D(-\alpha,0)a^\dagger D(\alpha,0) = a^\dagger + \alpha^*, \tag{12b}$$

one can derive (11) and show that a definition based on (11) is equivalent to our definition (6) (up to choice of phase conventions).

Apart from the algebraic structure given so far the probably most important property of coherent states is that they form a complete set in $\mathcal{H}_\infty$ that can be used in a similar way as a basis. Using either the expansion in number states (9) or the group multiplication (3) one can establish that coherent states have a non-vanishing overlap,

$$(\alpha_1|\alpha_2) = e^{\alpha_1^*\alpha_2}, \qquad \langle\alpha_2|\alpha_1\rangle = e^{-\frac{|\alpha_1-\alpha_2|^2}{2} - i\mathrm{Im}(\alpha_1^*\alpha_2)} \neq 0, \tag{13}$$

which implies that one cannot build an orthonormal basis from coherent states. Nevertheless, the coherent states are complete (indeed overcomplete) as they allow obtaining the resolution of unity in terms of projectors onto coherent states

$$\hat{1} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \ |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \ e^{-|\alpha|^2} \, |\alpha)\,(\alpha| \tag{14}$$

where $d^2\alpha = d\mathrm{Re}(\alpha)d\mathrm{Im}(\alpha)$. This resolution can be derived either by formally reducing it to the resolution $\hat{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$ in terms of the orthonormal basis of number states or (more elegantly but also by means of more advanced tools) using Schur's lemma from representation theory. From this identity it follows that we can calculate the trace of an operator O as

$$\operatorname{tr} O = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \ \langle\alpha|O|\alpha\rangle, \tag{15}$$

and that we may represent any abstract state $|\psi\rangle \in \mathcal{H}_\infty$ in terms of an explicit function $\psi(\alpha,\alpha^*) \equiv \langle\alpha|\psi\rangle$ on the plane. The scalar product of two states is then given by

$$\langle\psi_2|\psi_1\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \ \psi_2(\alpha,\alpha^*)^*\psi_1(\alpha,\alpha^*). \tag{16}$$

Note that $\psi(\alpha,\alpha^*)$ is not an analytic function of α. However $f(\alpha) = \psi(\alpha,\alpha)^* e^{|\alpha|^2/2} \equiv \langle\psi|\alpha)$ turns out to be analytic in α and this gives rise to the so-called Bargmann representation (see [5]).

3. *Coherent states as minimum uncertainty states*

As a and $a^\dagger$ are not Hermitian operators they do not qualify as quantum observables. We thus introduce the formal (dimensionless) position and momentum operators

$$Q = \frac{a + a^\dagger}{\sqrt{2}}, \tag{17a}$$

$$P = \frac{a - a^\dagger}{\sqrt{2}i}, \tag{17b}$$

which are Hermitian by definition and obey the commutation relation

$$i\,[P, Q] = \hat{1}\ . \tag{18}$$

The operators Q, P and $\hat{1}$ may be used as an alternative set of generators for the Heisenberg-Weyl group $H_3(\mathbb{R})$. Their expectation values in a coherent state are given by

$$\langle\alpha|Q|\alpha\rangle = \sqrt{2}\operatorname{Re}\alpha \quad \text{and} \quad \langle\alpha|P|\alpha\rangle = \sqrt{2}\operatorname{Im}\alpha, \tag{19}$$

which shows that we may think of the alpha plane as the classical phase space, such that for each point in phase space there is one coherent state with the corresponding expectation values for momentum and position.
The uncertainty of an observable O in the quantum state $|\psi\rangle$ may be measured by the variance $\operatorname{Var}[O] = \langle\psi|O^2|\psi\rangle - \langle\psi|O|\psi\rangle^2$ which vanishes if O takes a sharp value in the state $|\psi\rangle$. Quantum mechanics forbids that Q and P both take sharp values at the same time in any quantum state $|\psi\rangle$. Heisenberg's uncertainty principle states

$$\operatorname{Var}[P]\operatorname{Var}[Q] \geq \frac{1}{4} \tag{20}$$

for any state $|\psi\rangle$. Standard coherent states obey

$$\operatorname{Var}[P] = \operatorname{Var}[Q] = \frac{1}{2} \tag{21}$$

and thus minimize the uncertainty. It leads to an alternative definition of standard coherent states as the set of minimal uncertainty states with equal variances of Q and P. The minimal uncertainty of coherent states is one of the main reasons why coherent states have become such an important tool for investigating quantum-classical correspondence in the semiclassical regime, when a coherent state deforms mildly during the dynamics for sufficiently small times. If one starts the dynamics in the coherent state $|\alpha_0\rangle$ then for sufficiently short times the state is well approximated by a trajectory $|\psi(t)\rangle \approx e^{i\phi(t)}|\alpha(t)\rangle$ of coherent states, where $\alpha(t) = \frac{q(t)+ip(t)}{\sqrt{2}}$ is a classical trajectory in phase space (obtained from corresponding classical Hamiltonian dynamics). If the classical dynamics is not chaotic this approximation may work for quite long times. A special case is the harmonic oscillator $H = \frac{\omega}{2}\left(P^2 + Q^2\right) = \omega\left(a^\dagger a + \frac{1}{2}\right)$, where $e^{-iHt}|\alpha_0\rangle = e^{-i\omega t/2}|\alpha(t)\rangle$ remains exact for the classical trajectory $\alpha(t) = e^{-i\omega t}\alpha_0$. In the position representation this leads to a Gaussian function whose centre follows the classical oscillations in space without changing its shape otherwise. This is the form in which coherent states were used by Schrödinger to investigate quantum-classical correspondence for the harmonic oscillator [1].

B. $SU(2)$ or spin coherent states

One well-known generalisation of the standard coherent states replaces the Heisenberg-Weyl group by some other (physically relevant) group G whose generators are observables of a quantum system. This has been developed mainly by Gilmore [10] and Perelomov [5, 11]. In this case one takes an irreducible representation of G on an appropriate Hilbert space, chooses an appropriate reference state $|\mu\rangle$ and then defines the coherent states by $|g\rangle = g|\mu\rangle$ where $g \in G$ is an element of the group. If the reference state is chosen appropriately one may find coherent states which have analogues of almost all main properties of the standard coherent states (overcompleteness with an explicit resolution of unity, appropriately defined minimal uncertainties, relation to classical phase-space, etc.).

The most prominent example of this are the $SU(2)$ coherent states [12, 13], which are also known as spin coherent states or angular momentum coherent states. The generators of the $SU(2)$ group are the angular momentum or spin operators J_1, J_2 and J_3 with commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1 \quad \text{and}\, [J_3, J_1] = iJ_2. \tag{22}$$

The irreducible representations are characterised by the half-integer number j, such that the total angular momentum is $J_1^2+J_2^2+J_3^2 = j\,(j+1/2)$ and the dimension of the Hilbert space is $2j+1$. In the present context only the simplest case $j = 1/2$ is relevant, so we will focus our summary on these (see [5] for details). In that case the Hilbert space $\mathcal{H}_2 \equiv \mathbb{C}^2$ is spanned by two orthogonal states,

$$|+\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |-\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}, \tag{23}$$

and the $SU(2)$ generators are represented by the (Hermitian) Pauli matrices,

$$J_1 \equiv \frac{1}{2}\sigma_1 = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad J_2 \equiv \frac{1}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \text{and} \quad J_3 \equiv \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{24}$$

Let us introduce the combinations

$$J_+ = J_1 + iJ_2 \equiv \sigma_+ = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \text{and} \quad J_- = J_1 - iJ_2 \equiv \sigma_- = \begin{pmatrix}0&0\\1&0\end{pmatrix}. \tag{25}$$

Almost all elements of $SU(2)$ are covered by the parameterisation

$$U(\beta,\varphi) := \begin{pmatrix}\frac{e^{i\varphi}}{\sqrt{1+|\beta|^2}} & \frac{-\beta^* e^{-i\varphi}}{\sqrt{1+|\beta|^2}}\\ \frac{\beta e^{i\varphi}}{\sqrt{1+|\beta|^2}} & \frac{e^{-i\varphi}}{\sqrt{1+|\beta|^2}}\end{pmatrix} \equiv e^{\beta J_-} e^{-\log(1+|\beta|^2)J_3} e^{-\beta^* J_+} e^{i2\varphi J_3} \tag{26}$$

where $\beta \in \mathbb{C}$ and $0 \leq \phi \leq 2\pi$. The $SU(2)$ coherent states are now defined by

$$|\beta\rangle = U(\beta,0)|+\rangle = \frac{1}{\sqrt{1+|\beta|^2}}\left(|+\rangle + \beta|-\rangle\right) \equiv \begin{pmatrix}\frac{1}{\sqrt{1+|\beta|^2}}\\ \frac{\beta}{\sqrt{1+|\beta|^2}}\end{pmatrix} \tag{27}$$

to which one should add the coherent state $|\beta = \infty\rangle \equiv |-\rangle$. Again it is sometimes useful to define spin coherent states that are analytic in β but not normalised by

$$|\beta) = e^{\beta J_-}|+\rangle = |+\rangle + \beta|-\rangle\ . \tag{28}$$

As a manifold the coherent states form the sphere $S^2 = SU(2)/U(1)$ (see below for more details), which is the appropriate classical phase space for the dynamics of an angular momentum vector (with fixed length).

In analogy to the standard coherent states the $SU(2)$ coherent states can be used as an overcomplete basis in the Hilbert space $\mathcal{H}_{2j+1} = \mathbb{C}^{2j+1}$ of the irreducible $SU(2)$ representation with spin j. For $j = 1/2$ it is straightforward to evaluate the overlap

$$(\beta_2|\beta_1) = 1 + \beta_2^*\beta_1 \quad \text{or} \quad \langle\beta_2|\beta_1\rangle = \frac{1+\beta_2^*\beta_1}{\sqrt{(1+|\beta_2|^2)\,(1+|\beta_1|^2)}} \tag{29}$$

of two coherent states and find an explicit resolution of unity as

$$\hat{1} = \frac{2}{\pi}\int_{\mathbb{C}} d^2\beta \frac{1}{(1+|\beta|^2)^2}|\beta\rangle\langle\beta| = \frac{2}{\pi}\int_{\mathbb{C}} d^2\beta \frac{1}{(1+|\beta|^2)^3}\,|\beta)\,(\beta| \tag{30}$$

where the integration over $\mathbb{C}$ with measure $d^2\beta \frac{1}{(1+|\beta|^2)^2}$ may be considered as an intregral over a unit sphere where β is the stereographic projection of a point on the sphere onto the complex plane attached at the north pole. Note that the sphere consitutes the classical phase space of a spin of fixed length. This allows us to represent states in $\mathcal{H}_2$ as complex functions on phase space in analogy to the standard coherent states described above.
One may see that the coherent states span a manifold equivalent to a sphere by considering the expectation values

$$\begin{aligned} \langle\beta|\sigma_+|\beta\rangle &= \frac{\beta}{1+|\beta|^2}\ , & \langle\beta|\sigma_-|\beta\rangle &= \frac{\beta^*}{1+|\beta|^2}\ , & & \\ \langle\beta|\sigma_1|\beta\rangle &= \frac{2\mathrm{Re}(\beta)}{1+|\beta|^2}\ , & \langle\beta|\sigma_2|\beta\rangle &= \frac{2\mathrm{Im}(\beta)}{1+|\beta|^2}\ , & \langle\beta|\sigma_3|\beta\rangle &= \frac{1-|\beta|^2}{1+|\beta|^2}\ . \end{aligned} \tag{31}$$

The three expectation values $\langle\beta|\sigma_j|\beta\rangle$ $(j = 1, 2, 3)$ build a vector in $\mathbb{R}^3$ of unit length, i.e. $\sum_{j=1}^3 \langle\beta|\sigma_j|\beta\rangle^2 = 1$ and it is clear that mapping coherent states to points on the unit sphere with (31) is one-to-one (including the coherent state $|\beta = \infty\rangle$ which maps to the south pole).
To conclude this section let us comment on the minimum uncertainty properties of $SU(2)$ coherent states. For the $j = 1/2$ representation every normalised state is a coherent state (up to an irrelevant phase), so coherent states are not distinguished by any minimal uncertainty. For other irreducible representations $j \geq 1$ the $SU(2)$ coherent states do fulfill a minimal uncertainty relation (see [5, 14, 15]).

C. Product coherent states

It is straightforward to define coherent states on the tensor product space $\mathcal{H}_\otimes = \mathcal{H}_\infty \otimes \mathcal{H}_2$ by taking tensor product of coherent states in the two factors. If $|\alpha\rangle \in \mathcal{H}_\infty$ is a standard coherent state and $|\beta\rangle \in \mathcal{H}_2$ is a spin coherent state then one defines a (tensor-)product coherent state as

$$|\alpha \otimes \beta\rangle := |\alpha\rangle \otimes |\beta\rangle = [D(\alpha, 0) \otimes U(\beta, 0)]\ |0\rangle \otimes |+\rangle\ . \tag{32}$$

The analytic but non-normalised variant is defined as $|\alpha \otimes \beta) := e^{\alpha a^\dagger} e^{\beta J_-} |0\rangle \otimes |+\rangle = |\alpha) \otimes |\beta)$. The product coherent states are associated with the group $H_3(\mathbb{R}) \times SU(2)$, which acts naturally on $\mathcal{H}_\otimes$, as each factor is equipped with an appropriate irreducible representation. This means that acting with any element of $H_3(\mathbb{R}) \times SU(2)$ on a product coherent state gives a product coherent state up to an additional scalar phase factor.

Product coherent states of this or analogous forms are often very useful in the analysis of quantum systems that involve tensor product spaces. For instance, the states $|\alpha \otimes \beta\rangle$ are often a natural choice to analyse equations of the form

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \left(H_0(P, Q) + \prod_{j=x,y,z} H_j(P, Q)\sigma_j \right) |\Psi(t)\rangle, \tag{33}$$

where $H_0(P, Q)$ and $H_j(P, Q)$ are Hermitian operators on $\mathcal{H}_\infty$ lifted naturally to $\mathcal{H}_\otimes$ (such standard abuse of notation will be used frequently). One important example in this class is the Pauli equation where $H_0(P, Q) = \frac{1}{2m}(P - eA(Q))^2 + V(Q)$ with the magnetic potential $A(Q)$, and $H_j(Q, P) = \hbar\mu B_j(Q)$ is the magnetic field that couples to the spin variables σ_j. In this setting the product coherent states are traditionally referred to as spin coherent state (somewhat ambiguously as the same term may also refer to $SU(2)$ coherent states). In this case in the leading order of the semiclassical asymptotics (formally $\hbar \to 0$) the spin degrees of freedom decouple from the space degrees of freedom and an initial coherent state $|\alpha_0 \otimes \beta_0\rangle$ will approximately remain a coherent state $|\alpha(t) \otimes \beta(t)\rangle$ such that $\alpha(t)$ follows the corresponding classical dynamics generated by H_0 and the spin rotates along the trajectory [16, 17].

It is interesting to note that (32) is not the only way to define a continuous overcomplete basis on $\mathcal{H}_\otimes$ that one may build from the standard coherent states in the factor $\mathcal{H}_\infty$, as we will show in the next section.

III. ELECTRON-HOLE COHERENT STATES AND THEIR PROPERTIES

Let us define *electron-hole coherent states* in $\mathcal{H}_\otimes = \mathcal{H}_\infty \otimes \mathcal{H}_2$ as

$$|\alpha \bowtie \beta\rangle = \frac{1}{\sqrt{1+|\beta|^2}}|\alpha\rangle \otimes |+\rangle + \frac{\beta^*}{\sqrt{1+|\beta|^2}}|\alpha^*\rangle \otimes |-\rangle \tag{34}$$

where $|\alpha\rangle$ is a standard coherent state in the factor $\mathcal{H}_\infty$. The corresponding non-normalised variant

$$|\alpha \bowtie \beta) = |\alpha) \otimes |+\rangle + \beta^* |\alpha^*) \otimes |-\rangle \tag{35}$$

will also be used as many formulas turn out to be more compact than by using the normalised variants.
For the normalised states we formally add the point $\beta = \infty$ by defining $|\alpha \bowtie \infty\rangle = |\alpha^*\rangle \otimes |-\rangle$.

Note that the states are generally entangled (unless $\beta = 0$ or $\beta = \infty$). They do not seem to arise in an obvious way from a linear group action. We will show, however, that they possess many properties associated to coherent states. E.g., they form an overcomplete basis that allows obtaining a straightforward resolution of unity with associated coherent state representations of the Hilbert space and its operators. We will also show that it is indeed associated with a group action – though not a linear one, as will be explained in Section III B.

The name electron-hole coherent states refers to possible applications in superconducting systems described by the Bogoliubov-de Gennes equation – in that setting the basis $\{|+\rangle, |-\rangle\}$ of the factor $\mathcal{H}_2$ refers to electron-like and hole-like excitations in a superconductor.

A. Properties of electron-hole coherent states

It is straightforward to evaluate the overlap between two different electron-hole coherent states or the overlap of an electron-hole coherent state with a product coherent state

$$\begin{aligned}(\alpha_1 \bowtie \beta_1|\alpha_2 \bowtie \beta_2) =& e^{\alpha_1^*\alpha_2} + \beta_1\beta_2^* e^{\alpha_1\alpha_2^*}\\ (\alpha_1 \otimes \beta_1|\alpha_2 \bowtie \beta_2) =& e^{\alpha_1^*\alpha_2} + \beta_1^*\beta_2^* e^{\alpha_1^*\alpha_2^*}\end{aligned} \tag{36}$$

for the non-normalised states. The corresponding overlaps for the normalised coherent states follow straightforwardly.

The electron-hole states allow obtaining the following resolutions of unity

$$\begin{aligned}\hat{1} =& \frac{2}{\pi^2}\int d^2\alpha\; d^2\beta\; \frac{1}{(1+|\beta|^2)^2}\,|\alpha \bowtie \beta\rangle\,\langle \alpha \bowtie \beta|\\ =& \frac{2}{\pi^2}\int d^2\alpha\; d^2\beta\; \frac{e^{-|\alpha|^2}}{(1+|\beta|^2)^3}\,|\alpha \bowtie \beta)\,(\alpha \bowtie \beta|\,,\end{aligned} \tag{37}$$

which can be shown straightforwardly by first integrating out β and then applying the resolution of unity of standard coherent states. The resolution of unity allows us to represent states and operators in the overcomplete basis of electron-hole coherent states. We will explore such representations in sections (IV B) and (IV C).

1. *Expectation values in electron-hole coherent states*

As we have reviewed above, standard coherent states can be viewed as a mapping from classical phase space to minimal uncertainty states in a quantum Hilbert space, such that expectation values of position and momentum give back the original phase space point. In this section we want to discuss the expectation values of the central quantities and see to which extent analogous properties hold for electron-hole coherent states. It is straightforward to evaluate the expectation values of the fundamental observables. Writing $\alpha = \frac{q+iv}{\sqrt{2}}$ where q and v are real one obtains

$$\langle \alpha \bowtie \beta | a | \alpha \bowtie \beta \rangle = \frac{q}{\sqrt{2}} + i \frac{1-|\beta|^2}{1+|\beta|^2} \frac{v}{\sqrt{2}} \tag{38a}$$

$$\langle \alpha \bowtie \beta | a^\dagger | \alpha \bowtie \beta \rangle = \frac{q}{\sqrt{2}} - i \frac{1-|\beta|^2}{1+|\beta|^2} \frac{v}{\sqrt{2}} \tag{38b}$$

$$\langle \alpha \bowtie \beta | Q | \alpha \bowtie \beta \rangle = q \tag{38c}$$

$$\langle \alpha \bowtie \beta | P | \alpha \bowtie \beta \rangle = \frac{1-|\beta|^2}{1+|\beta|^2} v \tag{38d}$$

$$\langle \alpha \bowtie \beta | \sigma_+ | \alpha \bowtie \beta \rangle = \frac{\beta^* e^{-v^2 - iqv}}{1+|\beta|^2} \tag{38e}$$

$$\langle \alpha \bowtie \beta | \sigma_- | \alpha \bowtie \beta \rangle = \frac{\beta e^{-v^2 + iqv}}{1+|\beta|^2} \tag{38f}$$

$$\langle \alpha \bowtie \beta | \sigma_1 | \alpha \bowtie \beta \rangle = \frac{2e^{-v^2} \mathrm{Re}\left(\beta e^{iqv}\right)}{1+|\beta|^2} \tag{38g}$$

$$\langle \alpha \bowtie \beta | \sigma_2 | \alpha \bowtie \beta \rangle = -\frac{2e^{-v^2} \mathrm{Im}\left(\beta e^{iqv}\right)}{1+|\beta|^2} \tag{38h}$$

$$\langle \alpha \bowtie \beta | \sigma_3 | \alpha \bowtie \beta \rangle = \frac{1-|\beta|^2}{1+|\beta|^2} \,. \tag{38i}$$

The expectation values of P, σ_1 and σ_2 do not suggest a clear connection to a phase space description. Replacing the momentum operator P by the operator

$$V = P\sigma_3 \tag{39}$$

makes the picture somewhat nicer as

$$\langle \alpha \bowtie \beta | V | \alpha \bowtie \beta \rangle = v \,. \tag{40}$$

We will see later in Section IV A, that V has the interpretation of a velocity operator in the context of the Bogoliubov-de Gennes equation. Note that $P = V\sigma_3$ and $P^2 = V^2$. We thus regard $\alpha = \frac{q+iv}{\sqrt{2}}$ as a point in the phase space spanned by position and velocity, rather than position and momentum. The expectation values of the Pauli-matrices σ_1 and σ_2 imply

$$R^2 := \sum_{j=1}^{3} \langle \alpha \bowtie \beta | \sigma_j | \alpha \bowtie \beta \rangle^2 = 1 - (1 - e^{-2v^2}) \frac{4|\beta|^2}{(1+|\beta|^2)^2} \leq 1 \tag{41}$$

which, in general, does not describe a unit sphere. If we allow the sphere to deform into an ellipsoid we find indeed

$$e^{2v^2} \left(\langle \alpha \bowtie \beta | \sigma_1 | \alpha \bowtie \beta \rangle^2 + \langle \alpha \bowtie \beta | \sigma_2 | \alpha \bowtie \beta \rangle^2 \right) + \langle \alpha \bowtie \beta | \sigma_3 | \alpha \bowtie \beta \rangle^2 = 1 \,. \tag{42}$$

Altogether this implies that the phase space underlying electron-hole coherent states can be viewed as the plane $\mathbb{R}^2 \equiv \mathbb{C}$ spanned by position and velocity expectation values, and an ellipsoid of revolution with two axes of length e^{-v^2} and the third one of the unit length, attached at each point of the plane.

2. *Minimum uncertainty*

Let us now show that electron-hole coherent states are minimal uncertainty states with respect to position operator Q and velocity operator V. It is straightforward to show that coherent states have uncertainties

$$\mathrm{Var}_{|\alpha \bowtie \beta\rangle}[V] = \mathrm{Var}_{|\alpha \bowtie \beta\rangle}[Q] = \frac{1}{2} \tag{43}$$

and thus

$$\mathrm{Var}_{|\alpha \bowtie \beta\rangle}[V] \mathrm{Var}_{|\alpha \bowtie \beta\rangle}[Q] = \frac{1}{4} \,. \tag{44}$$

Now the standard uncertainty relation for two operators A and B and an arbitrary state $|\Psi\rangle$ in quantum mechanics reads $\mathrm{Var}_{|\Psi\rangle}[A]\mathrm{Var}_{|\Psi\rangle}[B] \geq \left|\langle\Psi|[A,B]|\Psi\rangle\right|^2/4$. In our case $[V,Q] = -i\sigma_3$ what results in a lower bound $|\langle\Psi|\sigma_3|\Psi\rangle|^2/4$ taking any value between zero and a quarter. The stricter uncertainty relation

$$\mathrm{Var}_{|\Psi\rangle}[V]\mathrm{Var}_{|\Psi\rangle}[Q] \geq \frac{1}{4} \tag{45}$$

can be obtained by a slight modification of the standard derivation by first writing (using $V = \sigma_3 P$ and $\sigma_3^2 = \hat{1}$ and defining $v = \langle\Psi|V|\Psi\ \rangle$ and $q = \langle\Psi|Q|\Psi\ \rangle$)

$$\mathrm{Var}_{|\Psi\rangle}[V]\mathrm{Var}_{|\Psi\rangle}[Q] = \langle\Psi|\left(V-v\right)^2|\Psi\rangle\langle\Psi|\left(Q-q\right)^2|\Psi\rangle = \langle\Psi|\left(P-v\sigma_3\right)^2|\Psi\rangle\langle\Psi|\left(Q-q\right)^2|\Psi\rangle\ . \tag{46}$$

We may now introduce operators $\Delta P = P - \sigma_3 v$ and $\Delta Q = Q - q$ and proceed as in the standard derivation, i.e. using Hermiticity and Schwartz's inequality we write

$$\langle\Psi|\Delta P^2|\Psi\rangle\langle\Psi|\Delta Q^2|\Psi\rangle\ = \langle\Delta P\Psi|\Delta P\Psi\rangle\langle\Delta Q\Psi|\Delta Q\Psi\rangle \geq \left|\langle\Psi|\Delta P\Delta Q|\Psi\rangle\right|^2\ . \tag{47}$$

Now, as for any complex number z one has $|z|^2 \geq (\mathrm{Im}\ z)^2$ and

$$\mathrm{Im}\langle\Psi|\Delta P\Delta Q|\Psi\rangle = \frac{1}{2i}\langle\Psi|[\Delta P,\Delta Q]|\Psi\rangle = \frac{1}{2}, \tag{48}$$

and the uncertainty (45) follows.

While we have found that electron-hole coherent states obey an uncertainty relation in V and Q, they do not obey any uncertainty relation for the quasi-spin variables σ_j. With $\Delta\sigma_j = \sigma_j - \langle\Psi|\sigma_j|\Psi\rangle$, the uncertainty relation obeyed by spin coherent states is $\sum_{j=1}^3 \sum\langle\Psi|\Delta\sigma_j^2|\Psi\rangle \geq 2$. This lower bound for the sum over uncertainties is equivalent to the upper bound $\sum_{j=1}^3 \langle\Psi|\sigma_j|\Psi\rangle^2 \leq 1$ and we have seen above that the quasi-spin expectation values do not lie on the unit sphere but on an ellipsoid inside the sphere. The failure of electron-hole coherent states to obey a minimum uncertainty relation is related to entanglement between the two factor spaces $\mathcal{H}_\infty$ and $\mathcal{H}_2$ as we will discuss next.

3. *Entanglement of electron-hole coherent states*

Let us first consider how quasi-spin expectation values can measure the amount of mixing for quasi-spin states in $\mathcal{H}_2$. For this note that all pure (normalised) states $|\xi\rangle \in \mathcal{H}_2$ obey $\sum_{j=1}^3\langle\xi|\sigma_j|\xi\rangle^2 = 1$. A mixed states in $\mathcal{H}_2$ is described by a density matrix

$$\rho = p_1|\xi_1\rangle\langle\xi_1| + p_2|\xi_2\rangle\langle\xi_2| \tag{49}$$

where $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$. The expectation values of a quasi-spin matrix σ_j are $\mathrm{tr}\,\rho\sigma_j = \sum_{m=1}^2 p_m\langle\xi_m|\sigma_j|\xi_m\rangle$. Without loss of generality we may assume that $|\xi_1\rangle$ and $|\xi_2\rangle$ are orthogonal (as ρ is Hermitian). The state is pure if $\rho^2 = \rho$, that is either $p_1 = 0$ or $p_1 = 1$. We want to show that the quantity

$$R^2 := \sum_{j=1}^3 \left(\mathrm{tr}\ \rho\sigma_j\right)^2 \tag{50}$$

equals unity if and only if the state is pure, and $R^2 < 1$ otherwise. Since R^2 is a rotation invariant quantity we may assume $|\xi_1\rangle = |+\rangle$ and $|\xi_2\rangle = |-\rangle$ and thus obtain

$$R^2 = (p_1 - p_2)^2 = (1-2p_1)^2\ . \tag{51}$$

So $R^2 = 1$ if and only if $p_1 = 0$ or $p_1 = 1$ which is what we wanted to show. Moreover $R^2 < 1$ for $0 < p_1 < 1$ with a minimum at $p_1 = 1/2$ where $R^2 = 0$. We can thus use R^2 to measure the amount of mixing in a state ρ.

Now let us come back to the product space $\mathcal{H}_\otimes = \mathcal{H}_\infty \otimes \mathcal{H}_2$. A pure state $|\Psi\rangle \in \mathcal{H}_\otimes$ is called separable if $|\Psi\rangle = |\phi\rangle \otimes |\xi\rangle$ with $|\phi\rangle \in \mathcal{H}_\infty$ and $|\xi\rangle \in \mathcal{H}_2$. A state $|\Psi\rangle \in \mathcal{H}_\otimes$ is called entangled if it is not separable. The product coherent states $|\alpha\otimes\beta\rangle$ discussed in section II C are seperable by definition. The electron-hole coherent states $|\alpha \bowtie \beta\rangle$ are separable if either $\beta = 0$ (and α arbitrary) or $\beta = \infty$ (and α arbitrary) or $\mathrm{Im}(\alpha) = v/\sqrt{2} = 0$ (and β arbitrary) – otherwise they are entangled.

We may measure the amount of entanglement of a pure state $|\Psi\rangle \in \mathcal{H}_\otimes$ by considering the reduced density matrix

$$\rho_{\mathrm{red}} = \mathrm{tr}_{\mathcal{H}_\infty}|\Psi\rangle\langle\Psi| \tag{52}$$

where the trace is taken over the factor $\mathcal{H}_\infty$, so that $\rho_{\rm red}$ is an operator in $\mathcal{H}_2$. It is easy to see that $|\Psi\rangle$ is separable if and only if $\rho_{\rm red} = \rho_{\rm red}^2$ describes a pure state. The quantity $R^2 \geq 0$ defined in (50) applied to $\rho_{\rm red}$ can thus be used as a measure of entanglement such that $R^2 = 1$ for separable states and $R^2 < 1$ for entangled states with maximal entanglement for $R^2 = 0$. A short calculations shows

$$R^2 = \sum_{j=1}^{3} \langle\Psi|\sigma_j|\Psi\rangle^2, \tag{53}$$

which we have given for electron-hole coherent states in equation (41). For a fixed value $\alpha = (q + iv)/\sqrt{2}$ electron-hole coherent states obey $1 \geq R^2 \geq e^{-2v^2} > 0$, where the lower bound (maximal entanglement) is obtained when $|\beta| = 1$ (the upper bound is obtained for $\beta = 0$ and $\beta = \infty$).
We have seen above that electron-hole coherent states obey minimum uncertainty relations in position and velocity. So they have semiclassical properties in the space degrees of freedom. The entanglement of electron-hole coherent states is a nonclassical property that stems from the additional quasi-spin degrees of freedom. A somewhat similar duality has recently been discussed in the context of nonlinear coherent states on a noncommutative manifold [18].

B. The group theoretic approach to electron-hole coherent states

Though there is no obvious group that transforms one electron-hole coherent state into another *via a linear representation* in Hilbert space we will demonstrate here that there is a *non-linear* action of the product group $H_3(\mathbb{R}) \times SU(2)$. We have seen in Section (II C) that the product coherent states are spanned by the action of the linear (tensor-product) representation of this group acting on the state $|0\rangle \otimes |+\rangle \in \mathcal{H}_\infty \otimes \mathcal{H}_2$. Repeated linear actions of the group $H_3(\mathbb{R}) \times SU(2)$ transform then product coherent states to product coherent states (up to a phase factor). A small twist of this action will render the group action non-linear and lead to an analogous behaviour for electron-hole coherent states.

Let us first define an anti-unitary time-reversal operator $\mathcal{T}$ by its action on an arbitrary state in $\mathcal{H}_\infty \otimes \mathcal{H}_2$

$$\mathcal{T} \sum_{n=0}^{\infty} \left(a_{n,+}|n\rangle \otimes |+\rangle + a_{n,-}|n\rangle \otimes |-\rangle\right) = \sum_{n=0}^{\infty} \left(a_{n,+}^*|n\rangle \otimes |+\rangle + a_{n,-}^*|n\rangle \otimes |-\rangle\right), \tag{54}$$

where $|n\rangle$ is the number basis in $\mathcal{H}_\infty$. Anti-linearity and anti-unitarity follow straightforwardly from the definition. Acting with $\mathcal{T}$ on a product coherent state

$$\mathcal{T}|\alpha \otimes \beta\rangle = |\alpha^* \otimes \beta^*\rangle, \tag{55}$$

gives just the product coherent state with complex conjugate parameters α and β ($\beta = \infty$ should be considered as a real number for this purpose).
Now let us define the operator

$$\mathcal{Z} = \hat{1}_\infty \otimes |+\rangle\langle+| + \mathcal{T}\left(\hat{1}_\infty \otimes |-\rangle\langle-|\right) \tag{56}$$

in $\mathcal{H}_\infty \otimes \mathcal{H}_2$, where $\hat{1}_\infty$ is the identity operator in the first factor. This is neither a linear nor an anti-linear operator as it behaves linearly in one subspace but anti-linearly in the other. It is also clear from the definition that $\mathcal{Z}$ is its own inverse,

$$\mathcal{Z}^2 = \hat{1} \qquad \Rightarrow \qquad \mathcal{Z}^{-1} = \mathcal{Z}\ . \tag{57}$$

Moreover, it leaves the norm of any state $|\psi\rangle \in \mathcal{H}_\infty \otimes \mathcal{H}_2$ invariant,

$$\langle\mathcal{Z}\psi|\mathcal{Z}\psi\rangle = \langle\psi|\psi\rangle. \tag{58}$$

However, in general one has $\langle\mathcal{Z}\psi_1|\mathcal{Z}\psi_2\rangle \neq \langle\psi_1|\psi_2\rangle$, so $\mathcal{Z}$ cannot be considered as a unitary operation. Moreover there is no way to define a generalised adjoint to $\mathcal{Z}$, such an operation would need to satisfy at least $|\langle\psi_1|\mathcal{Z}\psi_2\rangle| = |\langle\mathcal{Z}^\dagger\psi_1|\psi_2\rangle|$ for two arbitrary states, and this leads to contradictions.
Our main interest here stems from the fact that $\mathcal{Z}$ transforms product coherent states into electron-hole coherent states and *vice versa*,

$$\mathcal{Z}|\alpha \otimes \beta\rangle = |\alpha \bowtie \beta\rangle \quad \Leftrightarrow \quad \mathcal{Z}|\alpha \bowtie \beta\rangle = |\alpha \otimes \beta\rangle\ , \tag{59}$$

which implies

$$|\alpha \bowtie \beta\rangle = \mathcal{Z}\left[D(\alpha,0)\otimes U(\beta)\right]\mathcal{Z}|0\rangle \otimes |+\rangle. \tag{60}$$

We thus define the action of the group $H_3(\mathbb{R}) \times SU(2)$ in $\mathcal{H}_\infty \otimes \mathcal{H}_2$ by the operator $\mathcal{Z}\left[D(\alpha,\phi)\otimes U(\beta,\varphi)\right]\mathcal{Z}$, where $D(\alpha,\phi)$ and $U(\beta,\varphi)$ are the corresponding unitary representations of the two groups in the corresponding spaces. It is easy to check that this group action leaves the set of electron-hole coherent states invariant (up to phase factors). Note that the conjugation of $D(\alpha,\phi)\otimes U(\beta,\varphi)$ with the nonlinear operator $\mathcal{Z}$ can no longer be written in terms of two factors acting independently in the two factor spaces. This is indeed necessary to produce the entanglement of electron-hole coherent states that we have discussed in Section III A 3.

Finally note that group elements that are represented by a simple phase factor in the linear unitary representation may be less trivial in the nonlinear representation (and vice versa). For instance $D(0,\phi)\otimes U(0,0)$ acting directly on any state is just scalar multiplication with $e^{i\phi}$ while

$$\mathcal{Z}\left[D(0,\phi)\otimes U(0,0)\right]\mathcal{Z} = D(0,0)\otimes U(0,\phi) \tag{61}$$

happens to be unitary and the angle ϕ has swapped from the $H_3(\mathbb{R})$ to the $SU(2)$ part. Similarly one has

$$\mathcal{Z}\left[D(0,0)\otimes U(0,\phi)\right]\mathcal{Z} = D(0,\phi)\otimes U(0,0)\ . \tag{62}$$

IV. SOME APPLICATIONS OF ELECTRON-HOLE COHERENT STATES

Let us now discuss some applications of electron-hole coherent states. We will focus on quasi-particle excitations in a superconductor which can be described in the Hilbert space $\mathcal{H}_\infty \otimes \mathcal{H}_2$ where the first factor carries the observables that correspond to space (here taken to be just the line), and the second factor is spanned by electron-like excitations represented by the basis state $|+\rangle$ and hole-like excitations represented by $|-\rangle$.

A. The Bogoliubov-de Gennes equation and Heisenberg Dynamics of a Coherent State

Quasiparticle excitations in a one-dimensional superconductor are described by the Bogoliubov-de Gennes equation [19, 20] which can be written formally as a Schrödinger equation

$$i\frac{d}{dt}\left|\Psi(t)\right\rangle = H\left|\Psi(t)\right\rangle \tag{63}$$

with the Hamilton operator

$$H = \left(\frac{1}{2}P^2 - \mu\right)\sigma_3 + \Delta(Q)\sigma_1\ . \tag{64}$$

Here, $\mu > 0$ is the Fermi energy and $\Delta(q)$ is the real-valued pair potential (replacing the operator Q by a real eigenvalue q) that couples electron- and hole-like excitations. We have here assumed the simplest setting, generalizations such as adding a magnetic field or taking a complex-valued pair potential are straightforward to implement but will not be discussed here. Note that we assume $\hbar = 1$.

Let us discuss the dynamics of the Bogoliubov-de Gennes equation in the Heisenberg picture for either an electron-hole or a product coherent state. The Heisenberg equations of motion for the fundamental operators are easily found to be

$$\frac{d}{dt}Q_H = P_H\ \sigma_{H,3} \equiv V_H \tag{65a}$$

$$\frac{d}{dt}P_H = -\Delta'(Q_H)\sigma_{H,1} \tag{65b}$$

$$\frac{d}{dt}V_H = \left(P_H\Delta(Q_H) + \Delta(Q_H)P_H\right)\sigma_{H,2} \tag{65c}$$

$$\frac{d}{dt}\sigma_1 = -2\left(\frac{1}{2}P_H^2 - \mu\right)\sigma_{H,2} \tag{65d}$$

$$\frac{d}{dt}\sigma_2 = 2\left(\frac{1}{2}P_H^2 - \mu\right)\sigma_{H,1} - 2\Delta(Q_H)\sigma_{H,3} \tag{65e}$$

$$\frac{d}{dt}\sigma_3 = 2\Delta(Q_H)\sigma_{H,2} \tag{65f}$$

where we have added the index H to refer to the time dependent operators in the Heisenberg pictures. We have included here the operator V_H – equation (65a) shows the origin of the label 'velocity operator' that we have used before.
At time $t = 0$ we start the dynamics with the standard representations of the corresponding operators (i.e $Q_H(0) \equiv Q$ and so on for all other operators). Now let the system be either in the electron-hole coherent state $|\alpha \bowtie \beta\rangle$ or the product coherent state $|\alpha \otimes \beta\rangle$ with $\alpha = \frac{q+iv}{\sqrt{2}}$. Both states are localized in configuration space at $t = 0$ near q and we will focus on how the localization properties change for short time. We will assume here that any variation of $\Delta(q)$ can be neglected over the extent of the coherent state and its vicinity and replace it by a constant Δ_0 and neglect all terms $\Delta'(Q)$. Using a short-hand $\langle Q\rangle_\bowtie \equiv \langle \alpha \bowtie \beta|Q|\alpha \bowtie \beta\rangle$ for electron-hole and $\langle Q\rangle_\otimes \equiv \langle \alpha\otimes\beta|Q|\alpha\otimes\beta\rangle$ for product coherent states we then have the following expectation values at $t = 0$,

$$\langle Q\rangle_\bowtie|_{t=0} = q, \qquad \langle Q\rangle_\otimes|_{t=0} = q, \tag{66a}$$

$$\langle P\rangle_\bowtie|_{t=0} = v\frac{1-|\beta|^2}{1+|\beta|^2}, \qquad \langle P\rangle_\otimes|_{t=0} = v, \tag{66b}$$

$$\langle V\rangle_\bowtie|_{t=0} = v, \qquad \langle V\rangle_\otimes|_{t=0} = v\frac{1-|\beta|^2}{1+|\beta|^2}. \tag{66c}$$

Their corresponding changes at $t = 0$ are

$$\frac{d}{dt}\langle Q\rangle_\bowtie\Big|_{t=0} = v = \langle V\rangle_\bowtie|_{t=0}\,, \qquad \frac{d}{dt}\langle Q\rangle_\otimes\Big|_{t=0} = v\frac{1-|\beta|^2}{1+|\beta|^2} = \langle V\rangle_\otimes|_{t=0} = \langle P\rangle_\otimes\langle\sigma_3\rangle_\otimes|_{t=0}\,, \tag{67a}$$

$$\frac{d}{dt}\langle P\rangle_\bowtie\Big|_{t=0} = 0, \qquad \frac{d}{dt}\langle P\rangle_\otimes\Big|_{t=0} = 0, \tag{67b}$$

$$\frac{d}{dt}\langle V\rangle_\bowtie\Big|_{t=0} = 0, \qquad \frac{d}{dt}\langle V\rangle_\otimes\Big|_{t=0} = 2i\Delta_0 v\frac{\beta-\beta*}{1+|\beta|^2} = 2\Delta_0\ \langle P\rangle_\otimes\langle\sigma_2\rangle_\otimes|_{t=0}\ . \tag{67c}$$

For an electron-hole coherent state this implies that the expectation values for short time follow a free trajectory with velocity v in the phase space spanned by position and velocity – irrespective of the parameter β that describes the quasi-spin. For product coherent states the expectation value of the position also follows the free motion but the corresponding velocity can be anything between $-v$ and v depending on quasi-spin configuration. The structure of electron-hole coherent states thus seems more adapted to applications in the Bogoliubov-de Gennes equation. This statement is supported more profoundly by looking at the variance in position after a short time δt. For an electron-hole coherent state this variance is

$$\mathrm{Var}_\bowtie\left[Q_H(0)\right] = \frac{1}{2}, \qquad \mathrm{Var}_\otimes\left[Q_H(0)\right] = \frac{1}{2}, \tag{68a}$$

$$\mathrm{Var}_\bowtie\left[Q_H(\delta t) = Q + \delta t V\right] = \frac{1}{2}\left(1+\delta t^2\right), \qquad \mathrm{Var}_\otimes\left[Q_H(\delta t)\right] = \frac{1}{2}\left(1+\left(1+v^2\frac{8|\beta|^2}{(1+|\beta|^2)^2}\right)\delta t^2\right)\ . \tag{68b}$$

Thus product coherent states generally disperse much quicker than electron-hole coherent states. The latter follow the short-time dispersion for a free scalar wave packet. The origin of the quick dispersion is obvious, the electron-like component of the product coherent state moves with velocity v while the hole-like component moves with opposite velocity $-v$ and such that the total quasi-spinor wave packet breaks up. This does not happen for the electron-hole coherent state, where both components have the same velocity v.
The time scale connected to quasi-spin rotations is $\sim 1/\Delta_0$. One may expect that this is the time scale where the electron-hole coherent state may change its character and, for instance, ceases to be well localised in the position-velocity phase space. A more detailed analysis of the wave packet dynamics on intermediate and longer time scales in the Schrödinger picture reveals an interesting interplay between space and spinor evolutions for electron-hole coherent states where the velocity v is close to the Fermi velocity $v_F = \sqrt{2\mu}$ (or momentum close to the Fermi momentum $p_F = \sqrt{2\mu}$ which is equivalent to the Fermi velocity in value as we have used units such that the mass is unity) [21].

B. Electron-hole coherent state representation

The resolution of unity (37) implies that we can use electron-hole coherent states to represent states in $\mathcal{H}_\otimes$ as functions of two complex parameters α and β

$$|\Psi\rangle \mapsto \Psi(\alpha,\alpha^*,\beta,\beta^*) = \langle\alpha \bowtie \beta|\Psi\rangle\ . \tag{69}$$

The Hilbert space spanned by such functions $\Psi(\alpha,\alpha^*,\beta,\beta^*)$ under the above map consists of functions of the form

$$\Psi(\alpha,\alpha^*,\beta,\beta^*) = \frac{e^{-|\alpha|^2/2}}{1+|\beta|^2}\left(u_{|\Psi\rangle}(\alpha^*) + \beta v_{|\Psi\rangle}(\alpha)\right) \tag{70}$$

where $u_{|\Psi\rangle}(\alpha^*)$ is analytic in α^*, $v_{|\Psi\rangle}(\alpha)$ analytic in α and both integrals $\int_{\mathbb{C}} d^2\alpha |u_{|\Psi\rangle}(\alpha^*)|^2 e^{-|\alpha|^2}$, and $\int_{\mathbb{C}} d^2\alpha |v_{|\Psi\rangle}(\alpha)|^2 e^{-|\alpha|^2}$ exist. The scalar product in this representation is

$$\langle\Psi_1|\Psi_2\rangle = \frac{2}{\pi^2}\int d^2\alpha d^2\beta \frac{1}{(1+|\beta|^2)^2}\Psi_1(\alpha,\alpha^*,\beta,\beta^*)^*\Psi_2(\alpha,\alpha^*,\beta,\beta^*). \tag{71}$$

It is more convenient to use the functions

$$f_{|\Psi\rangle}(\alpha,\alpha^*,\beta) \equiv u_{|\Psi\rangle}(\alpha^*) + \beta v_{|\Psi\rangle}(\alpha) \tag{72}$$

with a rescaled scalar product

$$\langle\Psi_1|\Psi_2\rangle = \frac{2}{\pi^2}\int d^2\alpha d^2\beta \, \frac{e^{-|\alpha|^2}}{(1+|\beta|^2)^3} f^*_{|\Psi_1\rangle} f_{|\Psi_2\rangle} = \frac{1}{\pi}\int d^2\alpha \, \left(u_{|\Psi_1\rangle}(\alpha^*)^* u_{|\Psi_2\rangle}(\alpha^*) + v_{|\Psi_1\rangle}(\alpha)^* v_{|\Psi_2\rangle}(\alpha)\right) \,. \tag{73}$$

Using

$$a\,|\alpha \bowtie \beta) = \left(\alpha\left(1-\beta^*\frac{\partial}{\partial\beta^*}\right) + \alpha^*\beta^*\frac{\partial}{\partial\beta^*}\right)|\alpha \bowtie \beta) \tag{74a}$$

$$a^\dagger\,|\alpha \bowtie \beta) = \left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right)|\alpha \bowtie \beta) \tag{74b}$$

$$\sigma_1\,|\alpha \bowtie \beta) = \left(\beta^*\left(1-\beta^*\frac{\partial}{\partial\beta^*}\right) + \frac{\partial}{\partial\beta^*}\right)|\alpha^* \bowtie \beta) = \left(\beta^*\left(1-\beta\frac{\partial}{\partial\beta}\right) + \frac{\partial}{\partial\beta}\right)|\alpha^* \bowtie \beta^*) \tag{74c}$$

$$\sigma_2\,|\alpha \bowtie \beta) = \left(-i\beta^*\left(1-\beta^*\frac{\partial}{\partial\beta^*}\right) + i\frac{\partial}{\partial\beta^*}\right)|\alpha^* \bowtie \beta) = \left(-i\beta^*\left(1-\beta\frac{\partial}{\partial\beta}\right) + i\frac{\partial}{\partial\beta}\right)|\alpha^* \bowtie \beta^*) \tag{74d}$$

$$\sigma_3\,|\alpha \bowtie \beta) = \left((1-\beta^*\frac{\partial}{\partial\beta^*}) - \beta^*\frac{\partial}{\partial\beta^*}\right)|\alpha \bowtie \beta) \tag{74e}$$

one may then rewrite the Bogoliubov-de Gennes equation as a nonlocal partial differential equation

$$\begin{aligned} i\frac{\partial}{\partial t} f_{|\Psi\rangle}(\alpha,\alpha^*,\beta) =& \frac{1}{4}\left[\left(1-\beta\frac{\partial}{\partial\beta}\right)\left(1-\alpha^{*2}+2\alpha^*\frac{\partial}{\partial\alpha^*}-\frac{\partial^2}{\partial\alpha^{*2}}\right)\right] f_{|\Psi\rangle}(\alpha,\alpha^*,\beta) \\ &- \left[\beta\frac{\partial}{\partial\beta}\left(1-\alpha^2+2\alpha\frac{\partial}{\partial\alpha}-\frac{\partial^2}{\partial\alpha^2}\right)\right] f_{|\Psi\rangle}(\alpha,\alpha^*,\beta) \\ &- \mu\left[1-2\beta\frac{\partial}{\partial\beta}\right] f_{|\Psi\rangle}(\alpha,\alpha^*,\beta) \\ &+ \Delta\left[\beta\left(1-\beta\frac{\partial}{\partial\beta}\right)+\frac{\partial}{\partial\beta}\right] f_{|\Psi\rangle}(\alpha^*,\alpha,\beta) \end{aligned} \tag{75}$$

for a scalar (complex-valued) function on phase space (with constant pair potential $\Delta(Q) = \Delta_0$). The nonlocal character of (75) can be seen from the different order of arguments in the last term (where α and α^* have been interchanged).
This representation in terms of functions $f_{|\Psi\rangle}$ has similarities to the well-known Bargmann representations – with the drawback that $f_{|\Psi\rangle}$ is not an analytic function of α.

C. The generalized Husimi function and its limitations

The standard coherent states $\{|\alpha\rangle\}$ allow constructing a phase space representations of quantum mechanics, where a state of the system ρ (mixed or pure) is represented by the Husimi function

$$h_\rho(\alpha,\alpha^*) = \langle\alpha|\rho|\alpha\rangle \tag{76}$$

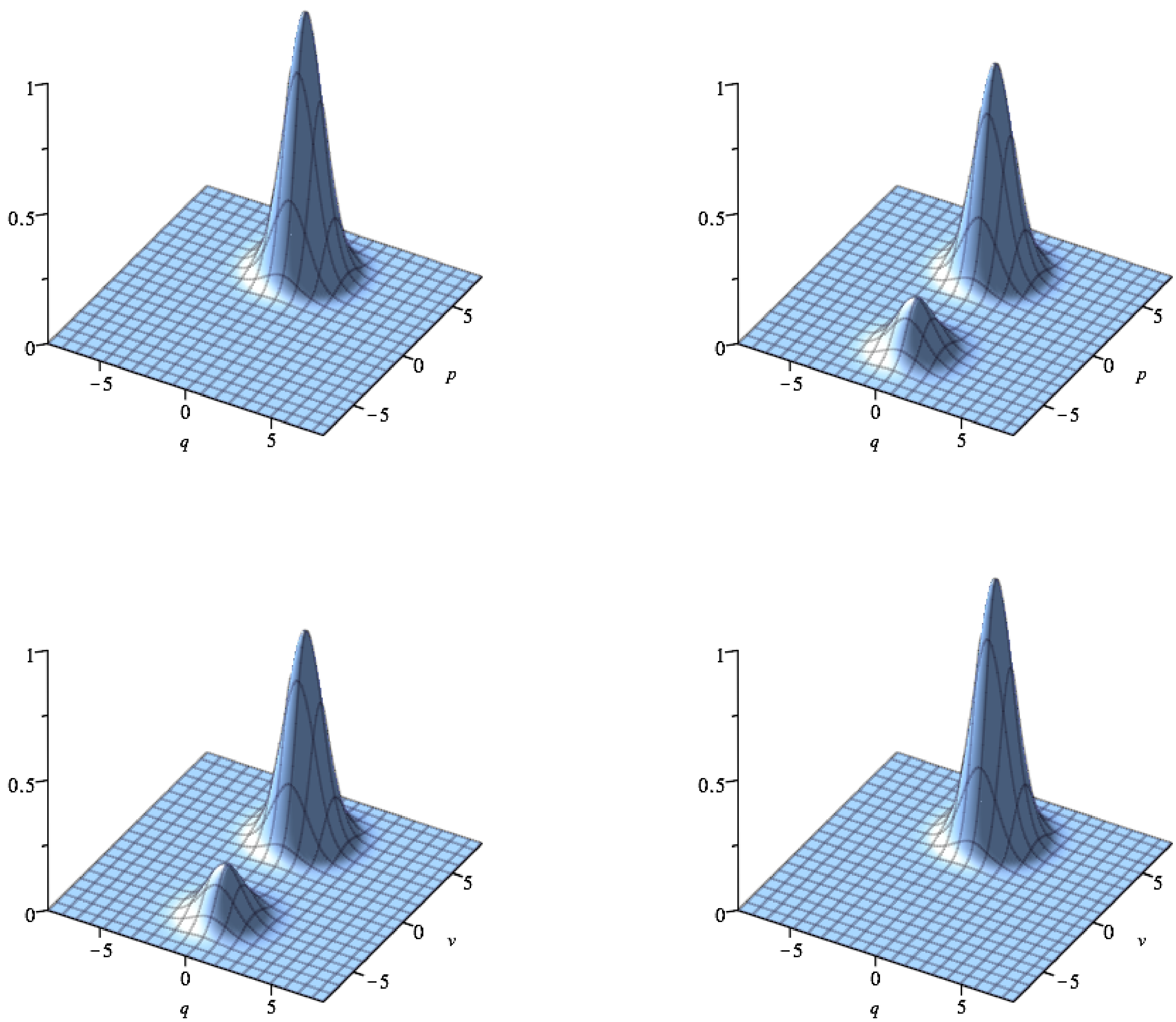

FIG. 1. Reduced phase space functions $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*)$ (top figures with $\alpha = (q+ip)/\sqrt{2}$) and $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ (bottom figures with $\alpha = (q+iv)/\sqrt{2}$) for product coherent state $\rho = |\alpha_0 \otimes \beta_0\rangle\langle\alpha_0 \otimes \beta_0|$ (left figures) and an electron-hole coherent state $\rho = |\alpha_0 \bowtie \beta_0\rangle\langle\alpha_0 \bowtie \beta_0|$ (with $\alpha_0 = 4i/\sqrt{2}$ and $\beta_0 = 1/2$). The top left (top right) figure is identical to the bottom right (bottom left) figure apart from the axis description which changes from momentum p to velocity v.

which has all formal properties of a classical probability density function (non-negative and normalised). Analogously an operator A is represented by its symbol

$$\mathcal{A}(\alpha,\alpha^*) = \langle\alpha|A|\alpha\rangle \ . \tag{77}$$

There is a one-to-one correspondence between operators and their symbols, or quantum states and their Husimi functions. The dynamics of the state and evaluation of any expectation values can be expressed entirely in terms of symbols and the Husimi function (by adding some further structure such as a non-commutative product for functions on the phase space). Many generalizations of coherent states (e.g. the $SU(2)$ coherent states) allow similar constructions. On the tensor product space $\mathcal{H}_\infty \otimes \mathcal{H}_2$ the product coherent states lead to an analogous one-to-one correspondence between operators on $\mathcal{H}_\infty \otimes \mathcal{H}_2$ and their functions on the phase space $\mathbb{C}^2 \times S^2$. For an operator A

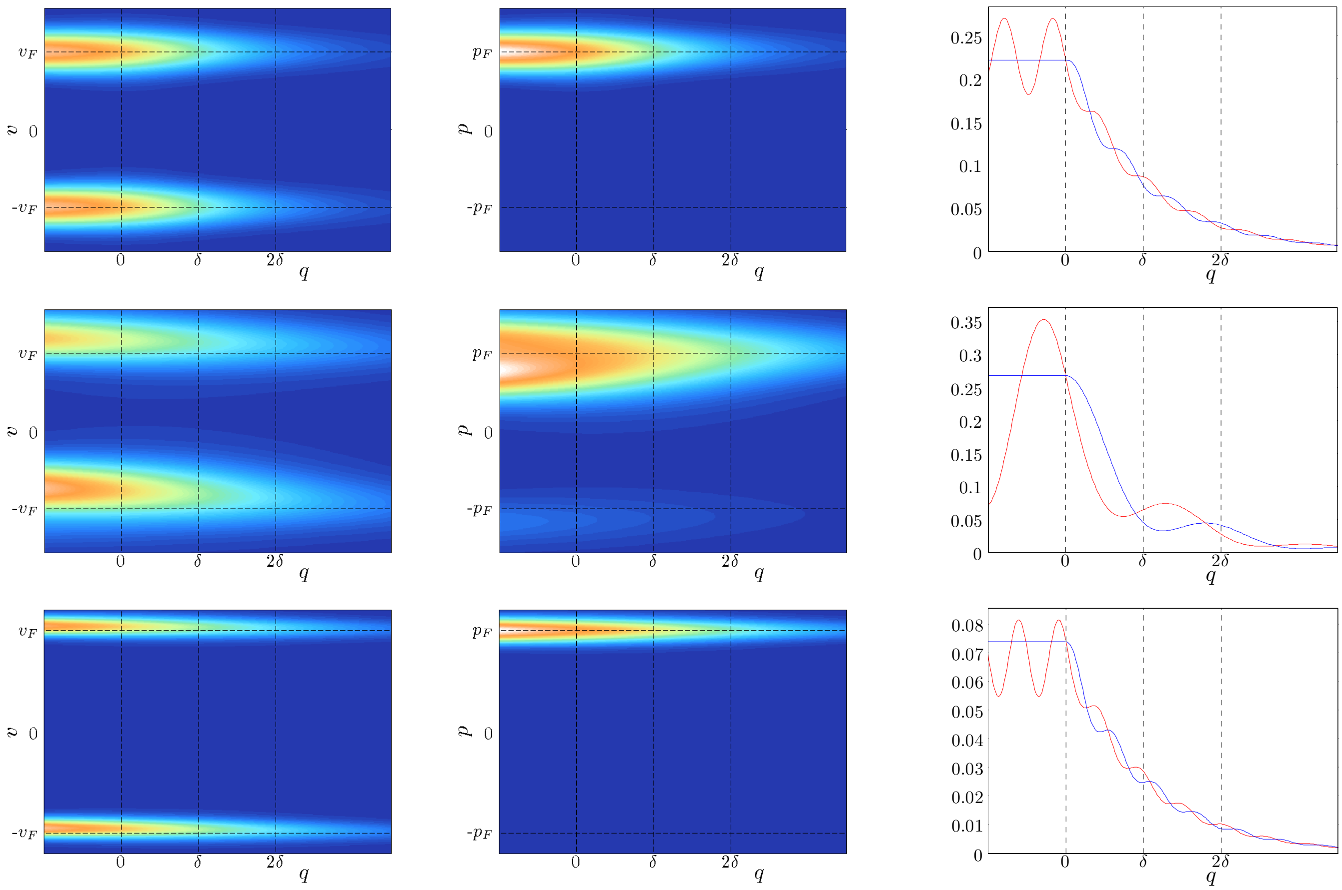

FIG. 2. Andreev reflection from a pair potential $\Delta(q) = \Delta_0\theta(q)$. Stationary scattering states at energy E for an incoming electron wave are shown in terms of the reduced position-velocity phase space function $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha, \alpha^*)$ (left column), the reduced position-momentum phase space function $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha, \alpha^*)$ (middle column), and their spinor wave functions (right column, red: absolute value squared of electron component, blue: absolute value squared of hole component). The scale δ is the penetration depth defined in (84).
Upper row: $E = 0$, $\Delta_0 = 2$, $\mu = 10$; middle row: $E = 5$, $\Delta_0 = 8$, $\mu = 10$; middle row: $E = 10$, $\Delta_0 = 20$, $\mu = 100$.

the corresponding function is again known as the symbol of A and is defined as

$$\mathcal{A}_{\otimes}(\alpha, \alpha^*, \beta, \beta^*) = \langle \alpha \otimes \beta | A | \alpha \otimes \beta \rangle. \tag{78}$$

If $\rho \in \mathcal{H}_\infty \otimes \mathcal{H}_2$ is a (possibly mixed) state of the system then the corresponding Husimi function defined via product coherent states is

$$h_{\otimes,\rho}(\alpha, \alpha^*, \beta, \beta^*) = \langle \alpha \otimes \beta |\, \rho \,| \alpha \otimes \beta \rangle\,. \tag{79}$$

If one tries to repeat these constructions with electron-hole coherent states one finds that there is *no one-to-one correspondence* between operators and the corresponding 'symbols' in electron-hole coherent states. For instance the two Hermitian operators $\sigma_1 \frac{a^{\dagger 2} + a^2}{2}$ and $\sigma_1 a^\dagger a$ would have the same symbol

$$\langle \alpha \bowtie \beta |\, \sigma_1 a^\dagger a \,| \alpha \bowtie \beta \rangle = \frac{\beta^* \alpha^{*2} e^{\alpha^{*2}} + \beta \alpha^2 e^{\alpha^2}}{1 + |\beta|^2} e^{-|\alpha|^2} = \langle \alpha \bowtie \beta |\, \sigma_1 \frac{a^2 + a^{\dagger 2}}{2} \,| \alpha \bowtie \beta \rangle \tag{80}$$

and this also equals the would-be symbol of the non-Hermitian operators $\sigma_1 a^2$ and $\sigma_1 a^{\dagger 2}$.
Analogously the 'Husimi function'

$$h_{\bowtie,\rho}(\alpha, \alpha^*, \beta, \beta^*) = \langle \alpha \bowtie \beta |\, \rho \,| \alpha \bowtie \beta \rangle \tag{81}$$

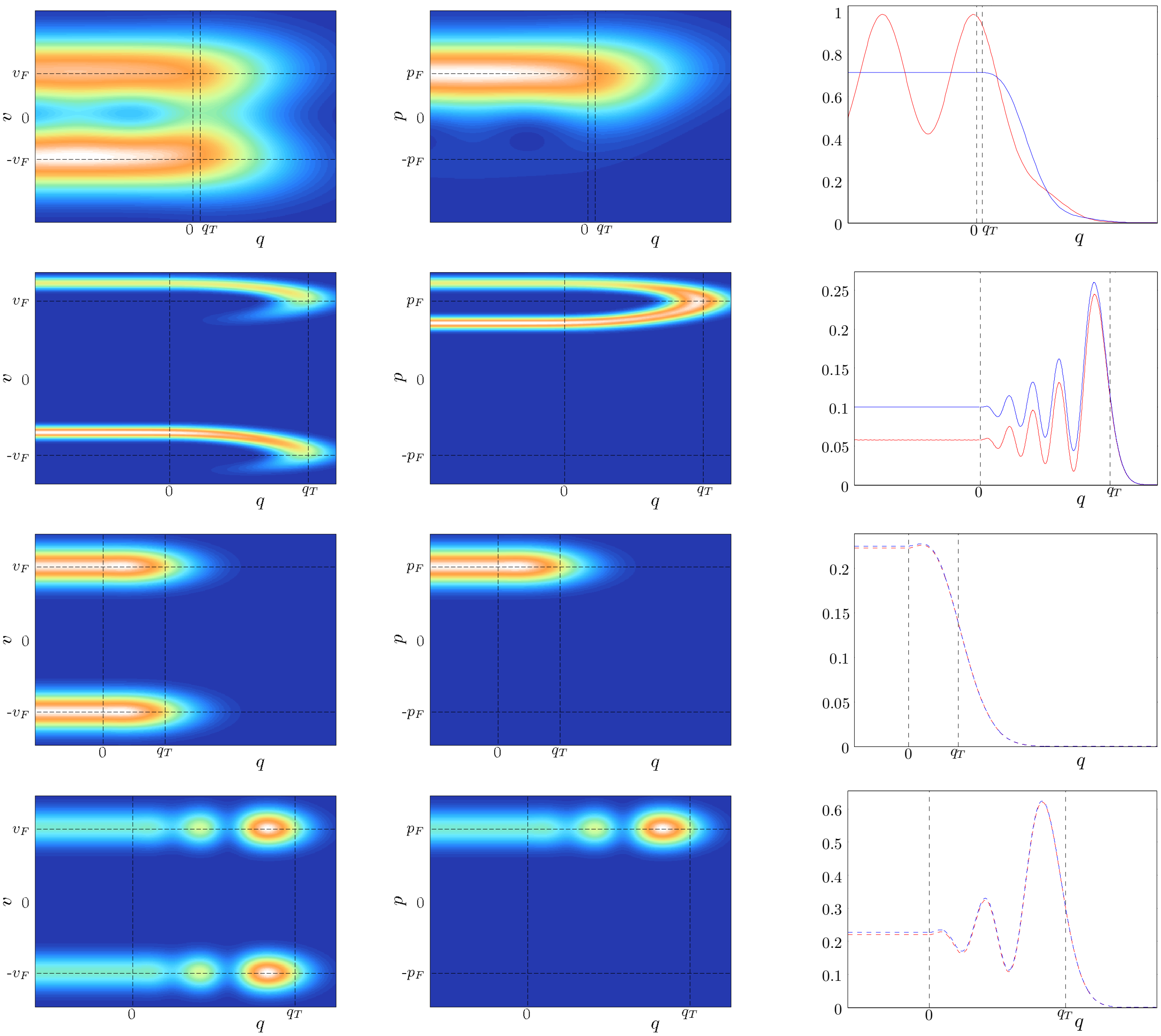

FIG. 3. Andreev reflection from a pair potential $\Delta(q) = \nu x\theta(x)$. Stationary scattering states at energy E for an incoming electron wave are shown in terms of the reduced position-velocity phase space function $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ (left column), the reduced position-momentum phase space function $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*)$ (middle column), and their spinor wave functions (right column, red: absolute value squared of electron component, blue: absolute value squared of hole component). The scale is given in terms of the turning point $q_T = \nu/E$.
First row: $E = 0.1$, $\mu = 1$, $\nu = 0.75$; second row: $E = 50$, $\mu = 100$, $\nu = 10$; third row: $E = 0.1$, $\mu = 10$, $\nu = 0.002$; fourth row: $E = 0.3$, $\mu = 10$, $\nu = 0.002$.

does not give a complete description of the state. It can still be useful in order to analyse visually how a given state is distributed in the position-velocity phase space. One may compare this to the distribution in the position-momentum phase space *via* the corresponding Husimi function $h_{\otimes,\rho}(\alpha,\alpha^*,\beta,\beta^*)$ in product coherent states (giving a complete description). For this one may use the reduced phase space functions

$$h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*) = \frac{2}{\pi}\int \frac{d^2\beta}{(1+|\beta|^2)^2} h_{\otimes,\rho}(\alpha,\alpha^*,\beta,\beta^*), \tag{82}$$

$$h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*) = \frac{2}{\pi}\int \frac{d^2\beta}{(1+|\beta|^2)^2}\; h_{\bowtie,\rho}(\alpha,\alpha^*,\beta,\beta^*), \tag{83}$$

where the quasi-spin variable is integrated out.

Figure 1 shows such reduced phase space distributions if the state is in a particular electron-hole coherent state $\rho_\otimes = |\alpha_0 \otimes \beta_0\rangle\langle\alpha_0 \otimes \beta_0|$ or a particular product coherent state $\rho_\bowtie = |\alpha_0 \bowtie \beta_0\rangle\langle\alpha_0 \bowtie \beta_0|$. The quasi-spin position of the product state $\rho_\otimes$ cannot be seen in the reduced function $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*)$ in position-momentum phase space. Comparing it to the reduced function $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ in position-velocity phase space reveals that the original state has both an electron and a hole component. For an electron-hole coherent state $\rho_\bowtie$ the roles of the two reduced phase space functions is interchanged.

An interesting physical phenomenon that may be analysed visually in the phase space is the Andreev reflection at a boundary between a normalconducting region (where the pair potential vanishes $\Delta(q) = 0$) and a superconducting region ($\Delta(q) \neq 0$). An incoming electron-like state with energy E close to the Fermi energy μ is then reflected as a hole-like state with (almost) opposite velocity while the momentum has hardly changed.

Figure 2 shows reduced phase space functions for the stationary scattering states at energy E from a staircase pair potential $\Delta(q) = \Delta_0\theta(q)$ together with the corresponding spinor wave function. For $q > 0$ the (envelope of the) intensity of the spinor wave function decays exponentially $\propto e^{-q/\delta}$ where

$$\delta^{-1} = 2\mathrm{Im}\sqrt{2(\mu + i\sqrt{\Delta_0^2 - E^2})}. \tag{84}$$

For these states the reduced representation $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ in the position-velocity phase space separates the incoming electron and the reflected hole amplitudes and gives a more detailed picture of the dynamics. However this does not necessarily imply that the electron-hole coherent state representation should always be used on its own. For instance in the middle row of Figure 2 parameters have been chosen such that the incoming electron has a velocity that is somewhat above the Fermi velocity and the velocity of the reflected hole is (in absolute value) somewhat below the Fermi velocity. This is well resolved in the position-velocity phase space function $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ built from electron-hole coherent states but not in the position-momentum phase space function $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*)$ built on product coherent states, where the incoming and outgoing contributions overlap strongly. On the other side the same parameters also imply an appreciable electron-electron reflection where the reflected electron has opposite momentum to the incoming. This weak effect can only be seen in the position-momentum representation (the weak stripe near $p = -p_F$).

The Andreev reflection from an inhomogeneous superconductor may be modelled by a pair potential of the form $\Delta(q) = \nu x\theta(x)$. The stationary solutions at energy E are oscillatory for $|E| < |\Delta(q)|$ and decay for $|E| > |\Delta(q)|$. We will choose $E \geq 0$ and $\nu > 0$. Then the corresponding turning point is at $q_T = \nu/E$. Figure 3 shows reduced phase space distributions for stationary scattering states for an incoming electronic wave. While generally the phase space function $h^{(\mathrm{red})}_{\bowtie,\rho}(\alpha,\alpha^*)$ in position-velocity space gives more details than the position-momentum function $h^{(\mathrm{red})}_{\otimes,\rho}(\alpha,\alpha^*)$. However, the position-velocity representation does not offer nice semiclassical descriptions in terms of trajectories. As can be seen in the second row of Figure 3 these only come about in the position-momentum representation.

V. CONCLUSION

We have constructed electron-hole coherent states as minimum uncertainty states for position and velocity for the Bogoliubov-de Gennes equation. We derived and analysed their main properties: they entangle space and quasi-spin degrees of freedom and they form an overcomplete set with an explicit resolution of unity. Basic applications to stationary scattering in superconductors revealed both their usefulness and some limitations. In spite of these limitations we have shown that electron-hole coherent states have a potential of describing the dynamics of electron-hole excitations in a superconductor as remain localized in position-velocity phase space for a certain time. A more detailed description of the dynamics of these states may lead to new insights on well-known effects in superconductors and at normalconducting-superconducting interfaces.

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