7. Appendix

7.1. Further results relating to section 2 and proofs

7.1.1. Domains of attraction according to t_U and b

The domains of attraction may be usefully classified using a cross classification of b and t_U according to whether they are finite or infinite. This cross classification has implications for the behavior of $\mathcal{M}(s)$ as $s \uparrow b$ as noted in the table below.

Proposition 2. (**Classify by t_U and b**). *Suppose $Z_s \xrightarrow{w} Z$, where Z is non-degenerate, and $b \notin \mathcal{S}$. The following cross-classification for domain of attraction holds.*

	$b < \infty$	$b = \infty$
$t_U < \infty$	Empty	$\mathfrak{D}(-\mathcal{G}_\alpha)$ or $\mathfrak{D}(\mathcal{N})$ $\mathfrak{D}(-\mathcal{G}_\alpha) \iff \mathcal{RV}_\infty(-\alpha)$ $\mathfrak{D}(\mathcal{N}) \implies \mathcal{FTPD}$
$t_U = \infty$	$\mathfrak{D}(\mathcal{G}_\alpha)$ or $\mathfrak{D}(\mathcal{N})$ $\mathfrak{D}(\mathcal{G}_\alpha) \iff \mathcal{RV}_b(-\alpha)$ $\mathfrak{D}(\mathcal{N}) \implies \mathcal{FTPG}$	$\mathfrak{D}(\mathcal{N})$ $\mathfrak{D}(\mathcal{N}) \implies \mathcal{FTEG}$

(7.1)

In the above the following abbreviations have been used:
$\mathcal{FTPD}$: faster-than-power decay, meaning that for any $c > 0$, $\lim_{s\to\infty} s^c e^{-st_U}\mathcal{M}(s) = 0$;
$\mathcal{FTPG}$: faster-than-power growth, meaning that, for any $c > 0$, $\lim_{s\uparrow b}(b-s)^c\mathcal{M}(s) = \infty$;
$\mathcal{FTEG}$: faster-than-exponential growth, meaning for any $c > 0$, $\lim_{s\to\infty} e^{-cs}\mathcal{M}(s) = \infty$.
The shorthand in the table may be explained by considering e.g. the lower left cell, where $b < \infty$ and $t_U = \infty$: the statement "$\mathfrak{D}(\mathcal{G}_\alpha)$ or $\mathfrak{D}(\mathcal{N})$" means that this cell consists of distributions which are in the domain of attraction of the positive gamma with index α or the normal; the former case occurs if and only if the MGF of the distribution is regularly varying at b with index $-\alpha$; and whenever the latter occurs, the MGF has faster-than-polynomial growth at b.
The first row with $t_U < \infty$ is not relevant when X is integer-valued since in this case the limiting distribution Z is degenerate at t_U; the second row includes integer-valued cases.

Three examples of Proposition 2 are briefly mentioned without proof. The right tail of the noncentral χ^2 distribution is in $\mathfrak{D}(\mathcal{N})$ and occupies the lower left cell above. The left tail of the generalized inverse Gaussian distribution of Example 1 is in $\mathfrak{D}(\mathcal{N})$ and occupies the upper right cell. The right tail of the Poisson distribution is in the lower right cell.

Proof of Proposition 2. The upper cell in the table must be empty because if $t_U < \infty$ then necessarily $b = \infty$. If $t_U = \infty$, then it follows from Balkema et al. (2003 Prop. 4.1, (2)) that $X \in \mathfrak{D}(\mathcal{G})$ or $X \in \mathfrak{D}(\mathcal{N})$, and if $b = \infty$, then $X \in \mathfrak{D}(-\mathcal{G})$ or $X \in \mathfrak{D}(\mathcal{N})$ follows from Balkema et al. (2003 Prop. 4.1, (1)). The middle lines in the upper right and lower left cells are consequences of parts (b) and (c), respectively, of Proposition 1 and are restated to complete the table. The remainder of the proof, concerning the asymptotics in the three $\mathfrak{D}(\mathcal{N})$ cases, follows from results in Balkema et al. (1999b), especially the proof of Lemma 5.2 (op. cit.), and the fact that $X \in \mathfrak{D}(\mathcal{N})$ if and only if the CGF $\mathcal{K}(s)$ is asymptotically parabolic at b. In the proof of lemma 5.2 (op. cit.), it is noted that (using our notation) if $b < \infty$ then $\mathcal{K}''(s) >> (b-s)^{-2}$, which implies, after integrating twice and then exponentiating, that $\mathcal{M}(s) >> (b-s)^{-c} + O(1)$ when s is sufficiently close to b, for all fixed $c > 0$. Therefore, when $b < \infty$, $\mathfrak{D}(\mathcal{N})$ implies the property $\mathcal{FTPG}$, as stated in the lower left cell of the table. When $b = \infty$ and $\mathcal{K}(s) \to \infty$ as $s \to \infty$ then, from the proof of Lemma 5.2, $\mathcal{K}''(s) >> s^{-2}$, which implies that for any fixed $c > 0$, there exists an $s^*(c)$ such that $\mathcal{K}''(s) \geq cs^{-2}$ for $s > s^*(c)$. Integrating both sides from s_0 to s_1 and then integrating s_1 from s_2 to s and rearranging, we obtain

$$\mathcal{K}(s) \geq s\left\{\mathcal{K}'(s_0) + \frac{c}{s_0} - \frac{c\ln(s)}{s}\right\} + \mathcal{K}(s_2) - s_2\mathcal{K}'(s_0) - \frac{cs_2}{s_0} + c\ln(s_2), \tag{7.2}$$

for any $s \geq s_2 \geq s_0 \geq s^*(c)$. Choose a large $c_1 \in (0, \infty)$. Due to the assumption that $\mathcal{K}(s)$ is regular and therefore steep, for fixed c there exists an s_0 such that

$$\mathcal{K}'(s_0) + \frac{c}{s_0} - \frac{c\ln s}{s} > c_1,$$

for $s > s_0$. Exponentiating both sides of (7.2) with this choice of s_0, it is seen that $\mathcal{M}(s) \geq A(c, s_0, s_2)e^{c_1 s}$ for $s \geq s_2 \geq s_0 \geq s^*(c)$, where $A(c, s_0, s_2)$ is independent of s, and the property $\mathcal{FTEG}$ in the lower right cell of the table then follows easily. Finally, in the upper right cell, where $\mathcal{M}(\infty) = 0$, we have from Lemma 5.2 (op. cit.) that $-\mathcal{K}'(s) >> s^{-1}$. Integrating both sides, it follows that, for any fixed $c \in (0, \infty)$, $\mathcal{K}(s) \leq -c\ln(s)$ for s sufficiently large, from which the $\mathcal{FTPD}$ property in the upper right cell follows after exponentiating both sides. [Comment on notation: there is some inconsistency in the literature on the meaning of $g(x) >> f(x)$. In notation attributed to Vindogradov, it means $f(x) = O\{g(x)\}$, whereas for Balkema et al. and coauthors, and also for us, it means $f(x) = o\{g(x)\}$.] $\square$

7.1.2. Domains of attraction according to singularity type

Results from Balkema et al. (1999a,b, 2003) also permit categorization of domains of attraction according to the type of singularity that occurs at b.

Pole or algebraic branch point at b of order $\alpha > 0$. Suppose X is absolutely continuous (integer-valued) with MGF $\mathcal{M}$ (PGF $\mathcal{P}$) as given in (7.3), where L is slowly varying at ∞ ($L \in \mathcal{SV}_\infty$). Then $X \in \mathfrak{D}(\mathcal{G}_\alpha)$ follows since the form for $\mathcal{M}$ and $\mathcal{P}$ in (7.3) implies that they are in $\mathcal{RV}_b(-\alpha)$ and $\mathcal{RV}_r(-\alpha)$ respectively, i.e.

$$\mathcal{M}(z) = \frac{1}{(b-z)^\alpha} L\left(\frac{1}{b-z}\right) \qquad \text{and} \qquad \mathcal{P}(\omega) = \frac{1}{(r-\omega)^\alpha} L\left(\frac{1}{r-\omega}\right). \tag{7.3}$$

If α is not an integer, then the multi-function factors such as $(b-z)^{-\alpha}$ assume principal branch values which are real-valued for $z = s < b$ and make use of a branch cut along the positive real axis $[b, \infty]$.

Slowly varying at b. This category includes logarithmic branch points and their powers as well as iterated logarithmic branch points and their powers.

Proposition 3. (**MGF slowly varying at b**). *Suppose X is absolutely continuous with moment generating function $\mathcal{M}(s) = L\{1/(b-s)\}$ where $L \in \mathcal{SV}_\infty$ or X has PGF $\mathcal{P}(w) = L\{1/(r-w)\}$ where $L \in \mathcal{SV}_\infty$* (see §2 for definition) *and $r = e^b$. Then, Z_s does not converge weakly as $s \to b$.*

Proposition 3 covers logarithmic branch points, examples of which are given in Butler (2017, §8) and include the exponential integral density and the logarithmic series mass function.

Proof of Proposition 3. Since $\mathcal{M}$ and $\mathcal{P}$ are analytic on $a \leqq \mathrm{Re}(z) < b$ and $r_a \leqq |z| < r$, $L \in \mathcal{SV}_\infty$ and L is smooth. Let $u = 1/(b-s)$ and observe that $u \to \infty$ as $s \uparrow b$. Note that

$$\kappa_n = \frac{u^n L^{(n)}(u)}{L(u)} \to 0, \qquad s \uparrow b,$$

follows as a property of smoothly varying functions; see Bingham et al. (1987, §8). Then $\mathcal{K}'(s) = u\kappa_1$,

$$\mathcal{K}''(s) = 2u^2\kappa_1 + u^2\left(\kappa_2 - \kappa_1^2\right)$$
$$\mathcal{K}'''(s) = 6u^3\kappa_1 + 6u^3\left(\kappa_2 - \kappa_1^2\right) + u^3\left\{\kappa_3 - \kappa_2\kappa_1 - 2\kappa_1(\kappa_2 - \kappa_1^2)\right\}.$$

From this,

$$\frac{\mathcal{K}'''(s)}{\{\mathcal{K}''(s)\}^{3/2}} = \frac{6\kappa_1 + 6\left(\kappa_2 - \kappa_1^2\right) + \kappa_3 - \kappa_2\kappa_1 - 2\kappa_1(\kappa_2 - \kappa_1^2)}{\left(2\kappa_1 + \kappa_2 - \kappa_1^2\right)^{3/2}} \sim \frac{6\kappa_1 + 6\kappa_2 + \kappa_3}{\left\{2\kappa_1 + \kappa_2\right\}^{3/2}} \to \infty$$

as $u \to \infty$. Thus, by Proposition 1, there is no weak convergence. The proof for mass functions is the same. $\square$

Essential singularity at b. If $\mathcal{M}(z)$ is an entire function and not a finite polynomial, then it must have an essential singularity at $b = \infty$. Thus all such entire MGFs fall within this singularity type.

Proposition 4. (Essential singularities). *If b is an essential singularity and $Z_s \xrightarrow{w} Z$, then the following categorization holds.*

	$b < \infty$	$b = \infty$
$t_U < \infty$	Empty	$\mathfrak{D}(-\mathcal{G}_\alpha)$ or $\mathfrak{D}(\mathcal{N})^*$
$t_U = \infty$	$\mathfrak{D}(\mathcal{N})$	$\mathfrak{D}(\mathcal{N})$

(7.4)

Examples in the upper right cell for which $\mathfrak{D}(-\mathcal{G}_\alpha)$ holds, include the uniform distribution of Example 3 as well as the tail of a truncated distribution as in Example 4. The bottom left category includes a noncentral $\chi^2(n, \lambda)$ distribution with n degrees of freedom and noncentrality parameter λ, and the Poisson mass function occupies the lower right. No examples have been found for the starred case of $\mathfrak{D}(\mathcal{N})$ in the upper right cell but, so far as we are aware, it is a possibility that cannot be ruled out.

Proof of Proposition 4. If $b < \infty$, then suppose $X \in \mathfrak{D}(\mathcal{G}_\alpha)$. Then $(b - s)\mathcal{K}'(s) \sim \alpha$ as $s \uparrow b$ so that $\mathcal{K}(s) \sim -\alpha \ln(1 - s/b)$ and $\mathcal{M}(s) \sim (1 - s/b)^{-\alpha} = b^\alpha (b - s)^{-\alpha}$. However, $\mathcal{M}(z)$ has an essential singularity at b so that $\mathcal{M}(z)(b - z)^{\lfloor \alpha+1 \rfloor}$ still has an essential singularity at b. This contradicts the asymptotic order of $\mathcal{M}(s)$ as $s \uparrow b$ in which $\mathcal{M}(s)(b - s)^{\lfloor \alpha+1 \rfloor} \to 0$ as $s \uparrow b$, hence we reach a contradiction to the assumption that $X \in \mathfrak{D}(\mathcal{G}_\alpha)$. $\square$

The next result however, mostly rules out the starred case in practical applications.

Corollary 7. *The starred case of $\mathfrak{D}(\mathcal{N})$ in Proposition 4 cannot occur if the density f, when viewed as a function of complex variable z, satisfies the following condition: for some $\delta > -1$,*

$$f(z) = (t_U - z)^\delta h(z) \tag{7.5}$$

where h is analytic in a complex neighbourhood of t_U.

The rationale for condition (7.5) is to limit $f(t)$ to power-law behavior as $t \uparrow t_U$ but at the same time ensure it is still integrable on $(t_U - \varepsilon, t_U)$ by taking $\delta > -1$. Under this regime, f cannot have an essential singularity at t_U.

Proof of Corollary 7. Condition (7.5) is sufficient for showing that $X \in \mathfrak{D}(-\mathcal{G}_\alpha)$ for some $\alpha \geq \delta$. See the proof of Corollary 5 of §7.4 of Supplementary Materials. $\square$

7.1.3. Saddlepoint implications if $Z_s \xrightarrow{w} Z$

Various properties of the saddlepoint quantities $\hat{w}$ and $\hat{u}$ as $\hat{s} \to b$ are now developed based on the three possible weak distributional limits for Z.

Proposition 5. (Properties of $\hat{w}$ and $\hat{u}$). *Suppose $b \notin \mathcal{S}$.*

(a) *As $t \to \infty$,*

$$\hat{u} \to \begin{cases} \infty & \text{if} \quad X \in \mathfrak{D}(\mathcal{G}_\alpha) \quad \text{or} \quad X \in \mathfrak{D}(\mathcal{N}) \\ \sqrt{\alpha} & \text{if} \quad X \in \mathfrak{D}(-\mathcal{G}_\alpha) \end{cases} .$$

(b) *If X is such that Z_s converges weakly as $s \uparrow b$, then $\hat{w} \to \infty$ as $t \uparrow t_U$, $t_U \leq \infty$.*

(c) *If $X \in \mathfrak{D}(\mathcal{G}_\alpha)$ or $X \in \mathfrak{D}(-\mathcal{G}_\alpha)$ then $\hat{u}/\hat{w}^3 \to 0$ as $t \uparrow t_U \leq \infty$.*

Proof of Proposition 5. (a) For the case $X \in \mathfrak{D}(\mathcal{G}_a)$, we first show that $(b-\hat{s})^2\mathcal{K}''(\hat{s}) \to \alpha$ as $\hat{s} \to b$. Note that from Balkema et al. (2003, Prop. 4.1), $(b-s)X_s \xrightarrow{w}$ Gamma $(\alpha, 1)$ as $s \to b$ so that, by Balkema et al. (1999a, Theorem 3.6), we have pointwise convergence of the MGF of $(b-s)X_s$ to the MGF of Gamma (α), so that

$$\frac{\mathcal{M}\{s + (b-s)u\}}{\mathcal{M}(s)} \to \frac{1}{(1-u)^\alpha}, \qquad \mathrm{Re}(u) < 1.$$

This convergence ensures that the moments converge and the result follows by taking two derivatives in u and evaluating the limit at $u = 0$. Thus, $\hat{u} = \hat{s}\hat{\sigma} \sim b\sqrt{\alpha}/(b-\hat{s}) \to \infty$.

Suppose $X \in \mathfrak{D}(\mathcal{N})$. If $b = \infty$, then $1/\sigma_{\hat{s}}$ is self-neglecting in b by Corollary 5.4(a) of Balkema et al. (2003). Thus, by the discussion in paragraph 3 of Balkema et al. (2003), $1/\sigma_{\hat{s}} = o(\hat{s})$ so that $1/\hat{u} = o(1)$ as $\hat{s} \to \infty$ as required. When $b < \infty$, then Corollary 5.4(a) ensures that $\sigma_{\hat{s}} \to \infty$ and hence $\hat{u} \to \infty$ as $t \uparrow t_U$ as part of the condition AP3) in Balkema et al. (2003, p.91), which holds when there is convergence $E(Z_{\hat{s}}^3) \to 0$.

If $X \in \mathfrak{D}(-\mathcal{G}_a)$, then $s(X_s - t_U) \xrightarrow{w} -$ Gamma $(\alpha, 1)$ as $s \uparrow b$ as shown in Balkema et al. (2003, Prop. 4.1). The sequence of CGFs converges and the convergence of the second cumulant sequence gives $s^2\mathcal{K}''(s) \to \alpha$. Thus $\hat{u} = \hat{s}\hat{\sigma} \to \sqrt{\alpha}$.

(b) If $X \in \mathfrak{D}(\mathcal{G}_\alpha)$ then $b < \infty$ (Balkema et al., 2003, Prop. 4.1) so that $t_U = \infty$. By implicit differentiation, $\frac{1}{2}d\{\hat{w}(t)^2\}/dt = \hat{s}(t)$. Choose t_0 such that $\hat{s}(t_0) > 0$. With $\hat{w}_0 = \hat{w}(t_0)$, then

$$\tfrac{1}{2}\{\hat{w}(t)^2 - \hat{w}_0^2\} = \int_{t_0}^{t} \hat{s}(z)dz \geq (t-t_0)\hat{s}(t_0) \to \infty, \qquad t \to \infty, \tag{7.6}$$

when $X \in \mathfrak{D}(\mathcal{G}_\alpha)$.

For $X \in \mathfrak{D}(\mathcal{N})$, if $b < \infty$ then the argument used in (7.6) suffices since $t \to t_U = \infty$ as $\hat{s} \uparrow b$. If $b = \infty$, then, with $\hat{w}(t)$ given in (1.1), differentiate $\frac{1}{2}d\{\hat{w}(t)^2\}/d\hat{s} = \hat{s}\mathcal{K}''(\hat{s})$ so that

$$\tfrac{1}{2}\{\hat{w}(t)^2 - \hat{w}_0^2\} = \int_{\hat{s}(t_0)}^{\hat{s}(t)} y\mathcal{K}''(y)dy. \tag{7.7}$$

Since $1/\sqrt{\mathcal{K}''(y)} = o(y)$, then there exists $\varepsilon > 0$ such that $1/\{y^2\mathcal{K}''(y)\} < \varepsilon$ for all $y > y_0 > 0$. Hence, $y\mathcal{K}''(y) > 1/(\varepsilon y)$, and the integral in (7.7) is bounded below by

$$\int_{y_0}^{\hat{s}(t)} \frac{dy}{\varepsilon y} = \frac{1}{\varepsilon} \ln\{\hat{s}(t)/y_0\} \to \infty \qquad \hat{s}(t) \to \infty.$$

If $X \in \mathfrak{D}(-\mathcal{G}_a)$ then $t_U < \infty$ and $b = \infty$. Also, $y^2\mathcal{K}''(y) \to \alpha$ as $y \to \infty$ so that $y\mathcal{K}''(y) > (\alpha - \varepsilon)/y$ for some $\varepsilon \in (0, \alpha)$ and for all $y > y_1$. Using (7.7), then

$$\tfrac{1}{2}\hat{w}(t)^2 > \int_{y_1}^{\hat{s}(t)} y\mathcal{K}''(y)dy > \int_{y_1}^{\hat{s}(t)} \frac{\alpha - \varepsilon}{y} dy = (\alpha - \varepsilon)\ln\frac{\hat{s}(t)}{y_1} \to \infty \qquad \hat{s}(t) \to \infty.$$

(c) If $X \in \mathfrak{D}(\mathcal{G}_a)$, then $(b - \hat{s})t \to \alpha$. From part (a), $\hat{u} \sim b\sqrt{\alpha}/(b - \hat{s}) \sim bt/\sqrt{\alpha}$. From part (b), $\hat{w}^2 > 2(t - t_0)\hat{s}(t_0)$ so that

$$\frac{\hat{u}}{\hat{w}^3} < \frac{bt/\sqrt{\alpha}}{\{2(t - t_0)\hat{s}(t_0)\}^{3/2}} \to 0, \qquad t \to \infty.$$

If $X \in \mathfrak{D}(-\mathcal{G}_a)$, then $\hat{u}/\hat{w}^3 \sim \sqrt{\alpha}/\hat{w}^3 \to 0$ by part (b). □

7.2. Proofs for section 3

Proof of Theorem 1. The proof involves computing separate limits for the factors

$$\frac{1}{S(t)} \int_t^{t_U} \hat{f}(u)du \qquad \text{and} \qquad \frac{1}{\hat{S}(t)} \int_t^{t_U} \hat{f}(u)du. \tag{7.8}$$

The first term has the same limit as $\hat{f}(t)/f(t)$ and the second term has limit 1. Denote $r_\infty = \lim_{t \uparrow t_U} f(t)/\hat{f}(t) < \infty$. Then, for any $\varepsilon > 0$, $\sup_{t>T} |f(t)/\hat{f}(t) - r_\infty| < \varepsilon$ for T

sufficiently large. Starting from the first term in (7.8) we have

$$\begin{aligned}\left|\frac{1}{S(t)}\int_t^{t_U}\hat f(u)du-\frac{1}{r_\infty}\right| &= \frac{1}{S(t)}\left|\int_t^{t_U}\{\hat f(u)-\frac{1}{r_\infty}f(u)\}du\right|\\ &\le \frac{1}{S(t)}\int_t^{t_U}\left|\hat f(u)-\frac{1}{r_\infty}f(u)\right|du\\ &= \frac{1}{S(t)}\int_t^{t_U}\left|\frac{\hat f(u)}{f(u)}-\frac{1}{r_\infty}\right|f(u)du<\varepsilon,\end{aligned}$$

if $t>T$. Thus the limit of the first term in (7.8) is $1/r_\infty$. For the second term we use l'Hôpital's rule but in order to use this result, we must ensure that $\hat S(t)\to 0$ as $t\uparrow t_U$. Note from the form of $\hat S(t)$ given in (1.3) this happens as long as $\hat w\to\infty$ and $\liminf_{\hat s\uparrow b}\hat u>0$. By Proposition 5, both of these conditions hold under weak convergence of Z_s as $s\uparrow b$. Thus

$$\lim_{t\uparrow t_U}\frac{1}{\hat S(t)}\int_t^{t_U}\hat f(u)du=\lim_{t\uparrow t_U}\frac{\hat f(t)}{\hat f(t)}\left\{1+\frac{1}{\hat u^2}+\frac{\hat\kappa_3}{2\hat u}-\frac{\hat u}{\hat w^3}\right\} \tag{7.9}$$

where $\hat\kappa_3$ is the third standardized cumulant of the $\hat s$-tilted distribution, and the expression for $\hat S'(t)$ is given in Butler (2007, eqn. 2.82). If $X\in\mathfrak{D}(\mathcal{N})$ and we assume $\hat u/\hat w^3\to 0$, then the limit of the term in curly braces is 1 so the limit in (7.9) is 1. If $X\in\mathfrak{D}(\mathcal{G}_a)$ then $\hat u/\hat w^3\to 0$ and $\hat u\to\infty$ so the limit is 1 by Proposition 5. If $X\in\mathfrak{D}(-\mathcal{G}_a)$ then $\hat u/\hat w^3\to 0$ and

$$1+\frac{1}{\hat u^2}+\frac{\hat\kappa_3}{2\hat u}\to 1+\frac{1}{\alpha}+\frac{-2/\sqrt{\alpha}}{2\sqrt{\alpha}}=1.$$

Putting both results together, then $\lim_{t\uparrow t_U}S(t)/\hat S(t)=r_\infty$. □

Proof of Theorem 2. The relationship between the convergence in (2.1) and limiting ratio for the saddlepoint density is determined through the inversion formula for $f(t)$. Under the weak condition that the density f of X is locally of bounded variation (see condition (ii) in the statement of Theorem 2), f is determined by its MGF as

$$f(t)=\frac{1}{2\pi i}\int_{\hat s-i\infty}^{\hat s+i\infty}\mathcal{M}(z)e^{-zt}dz=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{M}(\hat s+iy)e^{-t(\hat s+iy)}dy$$

where $\hat s\in\mathcal{S}$ is the saddlepoint solving $\mathcal{K}'(\hat s)=t$. Substituting $\hat\mu=\mathcal{K}'(\hat s)=t$, then

$$\begin{aligned}f(t)&=\mathcal{M}(\hat s)e^{-t\hat s}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathcal{M}(\hat s+iy)}{\mathcal{M}(\hat s)}e^{-i\hat\mu y}dy\\ &=\frac{\mathcal{M}(\hat s)e^{-t\hat s}}{\hat\sigma}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathcal{M}(\hat s+iu/\hat\sigma)}{\mathcal{M}(\hat s)}e^{-i\hat\mu u/\hat\sigma}du\\ &=\hat f(t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\mathcal{M}(\hat s+iu/\hat\sigma)}{\mathcal{M}(\hat s)}e^{-i\hat\mu u/\hat\sigma}du,\end{aligned} \tag{7.10}$$

where the substitution $u = \hat{\sigma} y$ with $\hat{\sigma} = \sqrt{\mathcal{K}''(\hat{s})}$ has been made in the second line. The limit for the ratio $f(t)/\hat{f}(t)$ as $t \uparrow t_U$ is determined by letting $\hat{s} \to b$ in the inversion integral in (7.10) under the assumption of the convergence in (2.1). To justify this limiting operation we argue as follows. For large t, $f(t)$ is locally of bounded variation and continuous so the inversion theorem for $f(t)$ holds; see Doetsch (1974, Theorem 24.3). From (7.10) we take the limit

$$\lim_{t \uparrow t_U} \frac{f(t)}{\hat{f}(t)} = \frac{1}{\sqrt{2\pi}} \lim_{\hat{s} \uparrow b} \int_{-\infty}^{+\infty} \frac{\mathcal{M}(\hat{s} + iu/\hat{\sigma})}{\mathcal{M}(\hat{s})} e^{-iu\hat{\mu}/\hat{\sigma}} du. \tag{7.11}$$

Consider the case $X \in \mathfrak{D}(\mathcal{G}_\alpha)$ in which $(b - \hat{s})\mu_{\hat{s}} \to \alpha$ as $t \uparrow t_U$. Then $Z_{\hat{s}} \xrightarrow{w} Z \sim$ Gamma $(\alpha, \sqrt{\alpha}) - \sqrt{\alpha}$ and, for each fixed $u \in \mathbb{R}$,

$$\lim_{\hat{s} \uparrow b} \frac{\mathcal{M}(\hat{s} + iu/\hat{\sigma})}{\mathcal{M}(\hat{s})} e^{-iu\hat{\mu}/\hat{\sigma}} = \mathcal{M}_Z(iu) = \left(1 - \frac{iu}{\sqrt{\alpha}}\right)^{-\alpha} e^{-iu\sqrt{\alpha}}. \tag{7.12}$$

To show the limiting relative error ratio for the density, assume a dominating function D as in (3.2) exists. Then from (7.11), (7.12) and (3.2), we get

$$\begin{aligned}
\lim_{t \to t_U} \frac{f(t)}{\hat{f}(t)} &= \sqrt{2\pi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(1 - \frac{iu}{\sqrt{\alpha}}\right)^{-\alpha} e^{-iu\sqrt{\alpha}} du \\
&= \sqrt{2\pi} f_Z(0) = \sqrt{2\pi} f_{G(\alpha, \sqrt{\alpha})}(\sqrt{\alpha}) \\
&= \sqrt{2\pi} \frac{1}{\Gamma(\alpha)} \sqrt{\alpha}^{\alpha} \sqrt{\alpha}^{\alpha - 1} e^{-\sqrt{\alpha}\sqrt{\alpha}} = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)}.
\end{aligned}$$

Here, $f_{G(\alpha, \sqrt{\alpha})}$ refers to the density of a Gamma $(\alpha, \sqrt{\alpha})$ distribution.

Consider now the case $X \in \mathfrak{D}(-\mathcal{G}_\alpha)$. Then $Z_{\hat{s}} \xrightarrow{w} Z \sim -$ Gamma $(\alpha, \sqrt{\alpha}) + \sqrt{\alpha}$ so the convergence in (7.12) is now to

$$\mathcal{M}_Z(iu) = \left(1 + \frac{iu}{\sqrt{\alpha}}\right)^{-\alpha} e^{iu\sqrt{\alpha}} \qquad u \in \mathbb{R}.$$

Then

$$\lim_{t \uparrow t_U} \frac{f(t)}{\hat{f}(t)} = \sqrt{2\pi} f_Z(0) = \sqrt{2\pi} f_{-G(\alpha, \sqrt{\alpha})}(-\sqrt{\alpha}) = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)}.$$

For the case $X \in \mathfrak{D}(\mathcal{N})$, $Z_{\hat{s}} \xrightarrow{w} Z \sim$ Normal $(0, 1)$ and the limit is $\sqrt{2\pi} f_Z(0) = 1$. $\square$

7.3. Proofs for section 4

Proof of Theorem 3 using condition (4.2) with $j = 1$. As the proof of this result is quite long, we begin by giving a brief outline of the argument. In Step 1 we recall results given by Widder (2010) and Lukacs (1964) concerning Fourier inversion which apply in cases where the Fourier transform is not assumed to be absolutely integrable. In Step 2 we summarize the two key inequalities which follow from condition (4.2) when $j = 1$; see (7.17) and (7.18) below. Step 3, the key part of the proof, involves rewriting part of the inversion integral (the negligible part) over $(-\infty, \infty)$ as an integral over $[0, \pi]$ with a new integrand. Step 4 rewrites this integral as the sum of three terms using summation by parts, and Step 5 shows, using (7.17) and (7.18), that the contribution of each of the three parts is negligible in the limit. A major technical issue in the proof is that we are not able to assume that the inversion integrand is absolutely integrable, so detailed arguments are given.

It will be assumed for convenience that ε in condition (4.2) satisfies $\varepsilon \in (0, \alpha)$. There is no loss of generality in doing so, because if (4.2) holds for some $\varepsilon \geq \alpha$ then it must hold for all positive ε less then α.

Under the assumption that F is the CDF of an absolutely continuous distribution, of course F' is defined almost everywhere in the Lebesgue sense. In order to avoid uninteresting and trivial complications in the formulation of our results, we shall further assume that $f(t) = F'(t)$ at all points $t \in \mathbb{R}$ at which the derivative of F exists. How we define $f(t)$ at points on the measure-zero set where F is not differentiable is immaterial.

Step 1: Results on Fourier inversion. We recall two results that play a key role later. Define, for $A > 0$, the function $h_A(z)$ by

$$h_A(y) = \begin{cases} 1 - |y|/A & \text{if} \quad |y| \leq A \\ 0 & \text{if} \quad |y| > A. \end{cases}$$

Proposition 6 (Inversion Theorem). *(i) Let $f(t)$ denote a probability density function on $(-\infty, \infty)$ with MGF $\mathcal{M}(z)$. Then*

$$f(t) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} h_A(y)\mathcal{M}(iy)e^{-iyt}dy, \qquad y \in \mathbb{R}, \tag{7.13}$$

for almost all $t \in \mathbb{R}$ in the Lebesgue sense.

(ii) The procedure (7.13) gives the correct answer at every t that is a continuity point of $f = F'$ (and not just almost everywhere for such t).

Proof. With cosmetic changes in notation, part (i) follows immediately from Theorem 6e in Widder (2010, p. 203). Part (ii) is due to Lukacs (1964). However, the statement of the result in Lukacs (1964) appears to be incomplete because an additional condition is required on the smoothing density, $q(t)$, to rule out pathological behavior. Close scrutiny of the Lukacs (1964) proof indicates that the following additional condition is sufficient for the Lukacs (1964) result to hold: for some fixed $\eta > 0$,

$$\sup_{|t|\geq T} q(t) = o(T^{-1-\eta}), \qquad T \to \infty. \tag{7.14}$$

From Widder (2010, p. 204), the smoothing density associated with the characteristic function $h_1(z)$ is given by $q(t) = 2\{\sin(t/2)\}^2/(\pi t^2)$, and this density clearly does satisfy the condition (7.14). □

Step 2: Implications of (4.2). We now show what condition (4.2) with $j = 1$ gives us. For any given real numbers $y_1 < y_2$, integrate $g'(z)/g(s)$ in $\mathbb{C}$ from $z = s + iy_1/\sigma_s$ to $z = s + iy_2/\sigma_s$. Thus, for any $0 \leq y_1 < y_2$,

$$\begin{aligned} \sup_{s\in[b^-,b)} \left| \frac{g(s+iy_2/\sigma_s)}{g(s)} - \frac{g(s+iy_1/\sigma_s)}{g(s)} \right| &= \sup_{s\in[b^-,b)} \left| \int_{s+iy_1/\sigma_s}^{s+iy_2/\sigma_s} \frac{g'(z)}{g(s)} dz \right| \\ = \sup_{s\in[b^-,b)} \left| \frac{i}{\sigma_s} \int_{y_1}^{y_2} \frac{g'(s+iy/\sigma_s)}{g(s)} dy \right| &\leq \int_{y_1}^{y_2} \sup_{s\in[b^-,b)} \left| \frac{1}{\sigma_s} \frac{g'(s+iy/\sigma_s)}{g(s)} \right| dy \\ \leq \int_{y_1}^{y_2} \frac{c_1}{(1+|y|)^{-\alpha+1+\varepsilon}} dy &= \frac{c_1}{\alpha-\varepsilon} \{(1+y_2)^{\alpha-\varepsilon} - (1+y_1)^{\alpha-\varepsilon}\}. \end{aligned} \tag{7.15}$$

It follows easily from (7.15) that, for any $y_1, y_2 \in \mathbb{R}$ which have the same sign and $|y_2| \geq |y_1|$, we have the bound

$$\sup_{s\in[b^-,b)} \left| \frac{g(s+iy_2/\sigma_s)}{g(s)} - \frac{g(s+iy_1/\sigma_s)}{g(s)} \right| \leq c_2 \left\{(1+|y_2|)^{\alpha-\varepsilon} - (1+|y_1|)^{\alpha-\varepsilon}\right\}, \tag{7.16}$$

and if y_1 and y_2 have different signs then the bound in (7.16) is modified by replacing the minus on the RHS with a plus. The key implications we shall need from (7.16) are the following:

$$\sup_{s\in[b^-,b)} \left| \frac{g(s+iy/\sigma_s)}{g(s)} - 1 \right| \leq c\,(1+|y|)^{\alpha-\varepsilon}, \qquad |y| \to \infty; \tag{7.17}$$

and for $y_1 = y$ and $y_2 = y + v$, where $v \in \mathbb{R}$ with $|v| \leq v_0$ for some fixed $v_0 > 0$,

$$\sup_{s\in[b^-,b)} \sup_{|v|\leq v_0} \left| \frac{g\{s+iy_2/\sigma_s\}}{g(s)} - \frac{g(s+iy_1/\sigma_s)}{g(s)} \right| \leq c\,(1+|y|)^{\alpha-1-\varepsilon}. \tag{7.18}$$

To obtain (7.17), put $y_1 = 0$ and $y_2 = y$ on the LHS of (7.16) and then modify the constant c_2. To obtain (7.18), focus initially on the case $0 < v < v_0 < y$ and $y > \delta v_0$ for some fixed $\delta > 1$. Then, using the exact form of Taylor's theorem with $\theta \in [0,1]$,

$$\begin{aligned}(1+y+v)^{\alpha-\varepsilon} - (1+y)^{\alpha-\varepsilon} &= (1+y)^{\alpha-\varepsilon}\left\{\left(1+\frac{v}{1+y}\right)^{\alpha-\varepsilon} - 1\right\} \\ &= (1+y)^{\alpha-\varepsilon}\left\{1+(\alpha-\varepsilon)\frac{v}{1+y}\left(1+\frac{\theta v}{1+y}\right)^{\alpha-\varepsilon-1} - 1\right\} \\ &\le c'(1+y)^{\alpha-\varepsilon-1},\end{aligned}$$

where $c' = c'(v_0)$. Note for future reference that we shall only require (7.18) for some fixed choice of $v_0 > 0$; we shall not require it to hold in the limit as $v_0 \to \infty$. The case where $y < -\delta v_0$ may be dealt with similarly. The cases where $|y| \le v_0$ follow from a suitable choice of the constant on the RHS of inequality (7.17) and inequality (7.18).

Step 3: Rearrangement of the inversion integral. In this step we rearrange the inversion integral by writing it as the integral of a different function over a finite region. Using the inversion formula in Step 1, define

$$\mathcal{I}(s,A) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{M}(s+iy/\sigma_s)}{\mathcal{M}(s)} h_A(y) e^{-iy\mu_s/\sigma_s}\, dy$$

We first perform all calculations with A fixed and then consider $A \to \infty$. Note that all calculations below are valid when A is finite. Define

$$\psi_A(y) = \left\{1 - \frac{iy}{\sigma_s(b-s)}\right)^{-\alpha}\left\{\frac{g(s+iy/\sigma_s)}{g(s)} - 1\right\} h_A(y), \tag{7.19}$$

where $h_A(y)$ is defined in Step 1. Then

$$\begin{aligned}\mathcal{I}(s,A) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{iy}{\sigma_s(b-s)}\right)^{-\alpha}\frac{g(s+iy/\sigma_s)}{g(s)} h_A(y) e^{-iy\mu_s/\sigma_s}\, dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{iy}{\sigma_s(b-s)}\right)^{-\alpha} h_A(y) e^{-iy\mu_s/\sigma_s}\, dy \qquad (7.20) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_A(y) e^{-iy\mu_s/\sigma_s}\, dy. \qquad (7.21)\end{aligned}$$

The term (7.20) is the inversion integral for the gamma density convolved with the time domain function for $h_A(y)$. In the limit as $A \to \infty$ for fixed $s \in [b^-, b)$ and $t > 0$, we have

$$\lim_{A\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{iy}{\sigma_s(b-s)}\right)^{-\alpha} h_A(y) e^{-iy\mu_s/\sigma_s}\, dy = f_{G\{\alpha,\sigma_s(b-s)\}}\left(\frac{\mu_s}{\sigma_s}\right),$$

where $f_{G(\alpha,\beta)}(t)$ is the density of the gamma (α,β) distribution at $t \in (0,\infty)$. In the above, we have used part (ii) of the inversion result stated in Step 1, combined with the fact that gamma densities are continuous when $t > 0$. Consequently, letting $s \uparrow b$ and using the saddlepoint density ratio in (7.10), we obtain the limit

$$\lim_{t \uparrow t_U} \frac{f(t)}{\hat{f}(t)} = \lim_{s \to b} \sqrt{2\pi} f_{G\{\alpha, \sigma_s(b-s)\}} \left(\frac{\mu_s}{\sigma_s} \right) = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)}.$$

Theorem 3 follows if we can show that the term (7.21) has a limit denoted by

$$R^{(s)} = \lim_{A \to \infty} \int_{-\infty}^{\infty} \psi_A(y) e^{-iy\mu_s/\sigma_s} dy, \tag{7.22}$$

that $R^{(s)}$ is bounded above and below over $s \in [b^-, b)$, and that $R^{(s)}$ satisfies

$$\lim_{s \uparrow b} R^{(s)} = 0. \tag{7.23}$$

We now rearrange the integral in (7.22) in such a way that we will then be able to prove that the limit in (7.22) is bounded for $s \in [b^-, b)$ and (7.23) holds. Focusing now on (7.21), with $\psi_A(y)$ defined as in (7.19), and writing $\rho_s = \sigma_s/\mu_s$, we obtain

$$\int_{-\infty}^{\infty} \psi_A(y) e^{-iy\mu_s/\sigma_s} dy = \rho_s \int_{-\infty}^{\infty} \psi_A(\rho_s y_1) e^{-iy_1} dy_1 = \rho_s \sum_{r=-\infty}^{\infty} \int_{(2r-1)\pi}^{(2r+1)\pi} \psi_A(\rho_s y_1) e^{-iy_1} dy_1 \tag{7.24}$$

$$= \rho_s \sum_{r=-\infty}^{\infty} \int_{-\pi}^{\pi} \psi_A \{\rho_s(2r\pi + v)\} e^{-i(2r\pi+v)} dv \tag{7.25}$$

$$= \rho_s \sum_{r=-\infty}^{\infty} \int_0^{\pi} \left[\psi_A \{\rho_s(2r\pi + v)\} e^{-iv} + \psi_A \{\rho_s(2r\pi + v - \pi)\} e^{-i(v-\pi)} \right] dv \tag{7.26}$$

$$= \rho_s \sum_{r=-\infty}^{\infty} \int_0^{\pi} [\psi_A \{\rho_s(2r\pi + v)\} - \psi_A \{\rho_s(2r\pi + v - \pi)\}] e^{-iv} dv$$

$$= \rho_s \int_0^{\pi} \sum_{r=-\infty}^{\infty} [\psi_A \{\rho_s(2r\pi + v)\} - \psi_A \{\rho_s(2r\pi + v - \pi)\}] e^{-iv} dv$$

$$= \rho_s \int_0^{\pi} F_A^{(s)}(v) e^{-iv} dv, \tag{7.27}$$

where $F_A^{(s)}(v)$ is defined by

$$F_A^{(s)}(v) = \sum_{r=-\infty}^{\infty} [\psi_A \{\rho_s(2r\pi + v)\} - \psi_A \{\rho_s(2r\pi + v - \pi)\}], \tag{7.28}$$

and $\psi_A(y)$ is defined in (7.19). Note that there are various interpretations of the function $F_A^{(s)}(v)$; e.g. it is derived by 'wrapping' $\mathbb{R}$ around a circle and then taking advantage of the natural differencing induced by the exponential with imaginary argument. Another interpretation of wrapping is that, after suitable rescaling of the period, we sum the integrand over terms with the same phase.

Step 4: Summation by parts. We now apply summation by parts to $F_A^{(s)}(v)$ in (7.28). Define

$$a_r(v,s) = \left\{1 - \frac{i\rho_s(2r\pi + v)}{\sigma_s(b-s)}\right\}^{-\alpha}, \qquad b_r(v,s) = \frac{g\{s + i\rho_s(2r\pi + v)/\sigma_s\}}{g(s)} - 1, \quad (7.29)$$

and

$$c_r(v,A) = h_A\left\{\rho_s(2r\pi + v)\right\} \quad (7.30)$$

Then, from the definitions,

$$F_A^{(s)}(v) = \sum_{r=-\infty}^{\infty} \left\{a_r(v,s)b_r(v,s)c_r(v,A) - a_r(v-\pi,s)b_r(v-\pi,s)c_r(v-\pi,A)\right\}. \quad (7.31)$$

So, using the identity

$$\begin{aligned} a_r(v,s)b_r(v,s)c_r(v,A) &- a_r(v-\pi,s)b_r(v-\pi,s)c_r(v-\pi,A) \\ &= \left\{a_r(v,s) - a_r(v-\pi,s)\right\} b_r(v-\pi,s)c_r(v-\pi,A) \\ &\quad + a_r(v,s)\left\{b_r(v,s) - b_r(v-\pi,s)\right\} c_r(v-\pi,A) \\ &\quad + a_r(v,s)b_r(v,s)\left\{c_r(v,A) - c_r(v-\pi,A)\right\}, \end{aligned}$$

then

$$|F_A^{(s)}(v)| \leq \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$$

where, due to the fact that $0 \leq h_A(y) \leq 1$,

$$\begin{aligned} \mathcal{R}_1 &= \left| \sum_{r=-\infty}^{\infty} \{a_r(v,s) - a_r(v-\pi,s)\} b_r(v-\pi,s)c_r(v-\pi,A) \right| \\ &\leq \sum_{r=-\infty}^{\infty} |a_r(v,s) - a_r(v-\pi,s)||b_r(v-\pi,s)||c_r(v-\pi,A)| \\ &\leq \sum_{r=-\infty}^{\infty} |a_r(v,s) - a_r(v-\pi,s)||b_r(v-\pi,s)| = \mathcal{R}_1'(v,s,A), \end{aligned} \quad (7.32)$$

$$
\begin{aligned}
\mathcal{R}_2 &= \left| \sum_{r=-\infty}^{\infty} a_r(v,s)\{b_r(v,s) - b_r(v-\pi,s)\}c_r(v-\pi,A) \right| \\
&\leq \sum_{r=-\infty}^{\infty} |a_r(v,s)||b_r(v,s) - b_r(v-\pi,s)||c_r(v-\pi,A)| \\
&\leq \sum_{r=-\infty}^{\infty} |a_r(v,s)||b_r(v,s) - b_r(v-\pi,s)| = \mathcal{R}_2'(v,s,A) \qquad (7.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_3 &= \left| \sum_{r=-\infty}^{\infty} a_r(v,s)b_r(v,s)\{c_r(v,A) - c_r(v-\pi,A)\} \right| \\
&\leq \sum_{r=-\infty}^{\infty} |a_r(v,s)b_r(v,s)||c_r(v,A) - c_r(v-\pi,A)| = \mathcal{R}_3'(v,s,A). \qquad (7.34)
\end{aligned}
$$

If we can show that, for $k = 1, 2, 3$,

$$\sup_{s\in[b^-,b)} \sup_{v\in[0,\pi]} \sup_{\{A:\, A\geq 1\}} \mathcal{R}_k'(v,s,A) < \infty \qquad (7.35)$$

and

$$\lim_{s\uparrow b} \lim_{A\to\infty} \mathcal{R}_k'(v,s,A) = 0, \qquad (7.36)$$

then (7.22) and (7.23) will follow from the bounded convergence theorem because the region of integration, $v \in [0,\pi]$, is bounded.

Step 5: Proof of (7.35) and (7.36), and therefore (7.22) and (7.23). We first deal with $\mathcal{R}_3' = \sum_{r=-\infty}^{\infty} d_r$ with $\{d_r\}$ denoting its addends. Although $b_r(v,s)$ is not necessarily bounded as $|r| \to \infty$, it is the case that

$$\sup_{s\in[b^-,b)} \sup_{v\in[0,\pi]} \sup_{-\infty<r<\infty} |a_r(v,s)b_r(v,s)| < \infty \qquad (7.37)$$

and

$$\lim_{|r|\to\infty} \sup_{s\in[b^-,b)} \sup_{v\in[0.\pi]} |a_r(v,s)b_r(v,s)| \to 0. \qquad (7.38)$$

Statements (7.37) and (7.38) follow from the following two facts: from the definition of $a_r(v,s)$ in (7.29),

$$\sup_{s\in[b^-,b)} \sup_{v\in[0,\pi]} |a_r(v,s)| = O\{(1+|r|)^{-\alpha}\}; \qquad (7.39)$$

while from (7.17),

$$\sup_{s\in[b^-,b)} \sup_{v\in[0,\pi]} |b_r(v,s)| = O\{(1+|r|)^{\alpha-\varepsilon}\}. \qquad (7.40)$$

Then (7.37) and (7.38) follow after multiplying (7.39) and (7.40) together. Moreover, from the definition of $c_r(v, A)$ in (7.30),

$$\Delta c_r(v, A) := c_r(v, A) - c_r(v - \pi, A) = 0$$

if both

$$\rho_s|2r\pi + v| > A, \qquad \rho_s|2r\pi + v - \pi| > A; \tag{7.41}$$

if precisely one of the inequalities in (7.41) holds, then $|\Delta c_r(v, A)| \leq 1$; and $\Delta c_r(v, A) = \rho_s\pi/A$ otherwise, i.e. when neither of the inequalities in (7.41) holds. A key point to note is that, for each $v \in [0, \pi]$ and $A \geq 1$, there are at most two values of r for which precisely one of the inequalities in (7.41) holds. To see this, suppose that $r > 0$. Then, it follows from elementary arguments that, for precisely one of the inequalities to hold we must have $\rho_s(2r\pi + v) > A \geq \rho_s(2r\pi + v - \pi)$, from which it is easily deduced that

$$\frac{1}{2\pi}(\rho_s^{-1}A - v) < r \leq \frac{1}{2\pi}(\rho_s^{-1}A - v) + \frac{1}{2}, \tag{7.42}$$

and clearly there is at most one positive value of r, r_+ say, for which (7.42) holds. A similar argument shows that for negative values of r, there is at most one negative value of r, r_- say, for which precisely one of the inequalities in (7.41) holds. Consequently, since (when they exist) $r_- \to -\infty$ and $r_+ \to \infty$ as $A \to \infty$ uniformly for $v \in [0, \pi]$ and $s \in [b^-, b)$, it follows from (7.37) and (7.38) that the case in which precisely one of (7.41) holds makes a negligible contribution to $\mathcal{R}_3'(v, s, A)$ as $A \to \infty$. So we focus on the case in which neither of the inequalities in (7.41) holds which, by elementary calculations, corresponds to r-values given by

$$r_1 = -\left\lfloor \frac{1}{2\pi}(\rho_s^{-1}A + v) \right\rfloor \leq r \leq \left\lfloor \frac{1}{2\pi}(\rho_s^{-1}A - v) \right\rfloor = r_2.$$

Then

$$\begin{aligned}\mathcal{R}_3'(v, s, A) &= \sum_{r=-\infty}^{\infty} d_r = o(1) + \sum_{r=r_1}^{r_2} d_r \\ &= \frac{\rho_s\pi}{A} \sum_{r=r_1}^{r_2} |a_r(v, s)||b_r(v, s)| \to 0, \qquad \text{as} \quad A \to \infty,\end{aligned} \tag{7.43}$$

uniformly for $v \in [0, \pi]$ and $s \in [b^-, b)$. since $|r_1| = O(A)$ and $r_2 = O(A)$ imply that, apart from negligible terms, the sum (7.43) is a linear combination of two Cesáro sums, each of which has the same limit as $a_r(v, s)b_r(v, s)$ as $|r| \to 0$, which by (7.38) is zero. We conclude that $\mathcal{R}_3'(v, s, A)$ satisfies (7.35) and (7.36).

We now consider $\mathcal{R}_2'$. Using the fact that

$$b_r(v, s) - b_r(v - \pi, s) = \frac{g\{s + i\rho_s(2r\pi + v)/\sigma_s\}}{g(s)} - \frac{g\{s + i\rho_s(2r\pi + v - \pi)/\sigma_s\}}{g(s)},$$

it follows from (7.18), taking $y_1 = \rho_s(2r\pi + v - \pi)/\sigma_s$ and $y_2 = \rho_s(2r\pi + v)/\sigma_s$, that

$$\sup_{v\in[0,\pi]}\sup_{s\in[b^-,b)} |b_r(v,s) - b_r(v-\pi,s)| \leq c(1+|r|)^{\alpha-1-\varepsilon}).$$

Consequently, since $|a_r(v,s)|$ and $|a_r(v-\pi,s)|$ are both $O\{(1+|r|)^{-\alpha}$ uniformly for $s \in [b^-,b)$ and $v \in [0,\pi]$, it follows that

$$|a_r(v-\pi,s)||b_r(v,s) - b_r(v-\pi,s)|$$

decays as

$$O\{(1+|r|)^{-\alpha}\}O\{(1+|r|)^{\alpha-1-\varepsilon}\} = O(\{(1+|r|)^{-1-\varepsilon}\},$$

and therefore $\mathcal{R}_2'$ satisfies (7.35) and (7.36). Similar considerations show that

$$\sup_{s\in[b^-,b)}\sup_{v\in[0,\pi]} |a_r(v,s) - a_r(v-\pi,s)| \leq c(1+|r|)^{-\alpha-1},$$

and we know from (7.17) that $b_r(v,\pi) = O\{(1+|r|)^{\alpha-\varepsilon}\}$ uniformly for $v \in [0,\pi]$ and $s \in [b^-,b)$. Consequently, $|a_r(v,s) - a_r(v-\pi,s)||b_r(v,s)|$ decays as

$$O\{(1+|r|)^{-\alpha-1}\}O\{(1+|r|)^{\alpha-\varepsilon}\} = O\{(1+|r|)^{-1-\varepsilon}\},$$

and hence $\mathcal{R}_1'$ also satisfies (7.35) and (7.36). Therefore, it follows from bounded convergence that (7.22) and (7.23) hold and so Theorem 3 is proved under condition (4.2) with $j=1$. □

Proof of Theorem 3 under condition (4.2) with $j=2$. The structure of the proof in this case is similar but the technicalities are somewhat more cumbersome, the main change being that second differences rather than first differences arise. Brief details of the required modifications are now given. Step 1 is unchanged, while Step 2 gives us bound (7.17) with exponent $\alpha+1-\varepsilon$ instead of $\alpha-\varepsilon$, bound (7.18) with exponent $\alpha-\varepsilon$ instead of $\alpha-1-\varepsilon$, and the bound

$$\begin{aligned}\sup_{s\in[b^-,b),\,|\delta|\leq\delta_0,\,|\eta|\leq\eta_0}&\left|\frac{g\{s+i(y+\delta+\eta)\}}{g(s)} - \frac{g\{s+i(y+\delta)\}}{g(s)} - \frac{g\{s+i(y+\eta)\}}{g(s)} + \frac{g(s+iy)}{g(s)}\right| \\ &\leq c(1+|y|)^{\alpha-1-\varepsilon},\end{aligned}$$

for some fixed $\delta_0, \eta_0 > 0$, and $\delta, \eta \in \mathbb{R}$. In the modified Step 3, the complex exponential e^{-iv} is split into sine and cosine terms, so as to exploit second differencing. Using subscripts R and I to denote real and imaginary parts, it is seen that the left side of the real part of the integral (7.24) may be written as

$$\rho_s\left\{\int_{-\infty}^{\infty}\psi_{R,A}(\rho_s y)\cos(y)dy + \int_{-\infty}^{\infty}\psi_{I,A}(\rho_s y)\sin(y)dy\right\}.$$

Then, using steps similar to those in (7.27), we obtain

$$\int_{-\infty}^{\infty} \psi_{R,A}(\rho_s y)\cos(y)dy = \int_0^{\pi/2} \bar{F}_{R,A}^{(s)}(v)\cos(v)dv, \tag{7.44}$$

where

$$\bar{F}_{R,A}^{(s)}(v) = \sum_{r=-\infty}^{\infty}\Big[\psi_{R,A}\{\rho_s(2r\pi+v)\} - \psi_{R,A}\{\rho_s(2r\pi+v-\pi)\} \\ - \psi_{R,A}\{\rho_s(2r\pi+\pi-v)\} + \psi_{R,A}\{\rho_s(2r\pi-v)\}\Big]. \tag{7.45}$$

We briefly explain how to derive the above. When moving from (7.25) to (7.26), we pair the two terms corresponding terms v and $v-\pi$, where $v \in [0,\pi]$, where $v=\pi$ gives the complex exponential a coefficient $+1$, and $v-\pi$ gives it a coefficient -1. In contrast, when deriving (7.44) and (7.45), we combine the four terms corresponding to v, $-v$, $\pi-v$ and $v-\pi$, where now $v \in [0,\pi/2]$. For the first two terms, corresponding to $\pm v$, the coefficient of the cosine is $+1$, and for the other two terms, corresponding to $\pm(\pi-v)$, the coefficient of the cosine is -1. This leads to (7.45).

A similar, but slightly modified argument shows that

$$\int_{-\infty}^{\infty} \psi_{I,A}(\rho_s y)\sin(y)dy = \int_0^{\pi/2} \bar{F}_{I,A}^{(s)}(v)\sin(v)dv, \tag{7.46}$$

where

$$\bar{F}_{I,A}^{(s)}(v) = \sum_{r=-\infty}^{\infty}\Big[\psi_{I,A}\{\rho_s(2r\pi+(\pi/2)+v)\} - \psi_{I,A}\{\rho_s(2r\pi+(\pi/2)+v-\pi)\} \\ - \psi_{I,A}\{\rho_s(2r\pi+(\pi/2)-v+\pi)\} + \psi_{I,A}\{\rho_s(2r\pi+(\pi/2)-v)\}\Big]. \tag{7.47}$$

To derive (7.46), we first of all apply the transformation $y \mapsto y-\pi/2$ to the variable of integration on the left side of (7.46). This has the affect of changing $\sin(y)$ to $\cos(y)$ in the integrand, but we also need to add $\pi/2$ to $2r\pi$ each time the latter appears. We then apply the same argument that was used to derive (7.44) and (7.45); note that (7.47) has the same structure as (7.45), but with $\pi/2$ added to $2r\pi$ each time the latter appears.

The imaginary part of the integral in (7.24) is zero. In Step 4, we do a second order summation by parts. As a generic example of a second order summation by parts, just including two factors to give the general idea even though we actually have three factors, consider

$$\begin{aligned}
a_r(v,s)b_r(v,s) - a_r(v-\pi,s)b_r(v-\pi,s) - a_r(\pi-v,s)b_r(\pi-v,s) + a_r(-v,s)b_r(-v,s) \\
= \{a_r(v,s) - a_r(v-\pi,s) - a_r(\pi-v,s) + a_r(-v,s)\}\, b_r(v,s) \\
+ a_r(v-\pi,s)\,\{b_r(v,s) - b_r(v-\pi,s) - b_r(\pi-v,s) + b_r(-v,s)\} \\
+ \{a_r(-v,s) - a_r(\pi-v,s)\}\,\{b_r(\pi-v,s) - b_r(v,s)\} \\
+ \{a_r(\pi-v,s) - a_r(v-\pi,s)\}\,\{b_r(-v,s) - b_r(\pi-v,s)\}\,,
\end{aligned}$$

where each term on the right is either a second difference or a product of two first differences. A similar but more complicated formula can be derived when three factors are present. With $a_r(v,s)$, $b_r(v,s)$ and $c_r(v,A)$ given similar but slightly modified definitions to those in (7.29) and (7.30), using real and imaginary parts as appropriate, and noting that these terms will be defined slightly differently in the sine and cosine integrals, the modified version of Step 5 follows, after using the bounds in the modified version of Step 2 and using the fact that

$$\sup_{s\in[b^-,b),\, v\in[0,\pi/2]} |a_r(v,s) - a_r(v-\pi,s) - a_r(\pi-v,s) + a_r(-v,s)| \le c(1+|r|)^{-\alpha-2}. \quad \square$$

Proof of Corollary 1. In this case, using property (4.6), and noting that as a consequence $\inf_{s\in[b^-,b)}\{g(s)\}^{-1} > 0$,

$$\begin{aligned}
\sup_{s\in[b^-,b)} \left| \frac{1}{\sigma_s} \frac{g'(s+iy/\sigma_s)}{g(s)} \right| &\le \sup_{s\in[b^-,b)} \left[\frac{1}{\sigma_s} \frac{1}{g(s)} \frac{c}{(1+|y|/\sigma_s)^{1+\delta}} \right] \\
&\le c_1 \sup_{s\in[b^-,b)} \frac{1}{\sigma_s} \frac{1}{(1+|y|/\sigma_s)^{1+\delta}},
\end{aligned}$$

for some $\delta > 0$. Using elementary calculus, it is straightforward to check that if the above has a stationary maximum over $s \in [b^-, b)$ for fixed y then it occurs when $\sigma_s = \delta|y|$, in which case the maximum is $\delta|y|^{-1}(1+\delta^{-1})^{-1-\delta}$; otherwise, for $|y|$ sufficiently small, the maximum will be non-stationary and will occur at $\inf_{s\in[b^-,b)} \sigma_s > 0$, in which case the maximum is bounded by a constant. We can combine these bounds into a global bound (in y) of the form $c(1+|y|)^{-1}$, and consequently it is seen that (4.2) with $j = 1$ is satisfied by any $\varepsilon \in (0, \alpha)$, and the result is proved. $\square$

Proof of Corollary 2. Here, the MGF has a simple pole at $z = b$ so in this case $\alpha = 1$. Without loss of generality, it is assumed that $\delta \in (0, 1)$. We are required to prove that, when the MGF's $\mathcal{M}_{ij}(z)$ satisfy condition (4.6), then the form of $g(z)$ in the semi-Markov case implies that g satisfies (4.2) with $j = 1$. Write

$$A(z) = \frac{(-1)^{m+1}|\mathbf{\Psi}_{m1}(z)|}{\mathcal{F}_{1m}(0)} \qquad B\,(z) = |\mathbf{\Psi}_{mm}(z)| \qquad C(z) = |\mathbf{\Psi}_{mm}(z)|(b-z)^{-1}$$

so that $g(z) = A(z)/C(z)$. We need to bound $g'(s+iy/\sigma_s)/\{\sigma_s g(s)\}$ over $s \in [b^-, b)$. Differentiating, we obtain

$$\frac{i}{\sigma_s}\frac{g'(s+iy/\sigma_s)}{g(s)} = \frac{i}{\sigma_s}\left\{\frac{A'(s+iy/\sigma_s)C(s)}{A(s)C(s+iy/\sigma_s)} - \frac{A(s+iy/\sigma_s)C(s)C'(s+iy/\sigma_s)}{A(s)\{C(s+iy/\sigma_s)\}^2}\right\} \quad (7.48)$$

Since in the present context $g(s)$ satisfies

$$0 < \inf_{s\in[b^-,b)} g(s) < \sup_{s\in[b^-,b)} g(s) < \infty, \quad (7.49)$$

it follows from (7.48) that we just need to show that, for each fixed $y \in \mathbb{R}$,

$$\sup_{s\in[b^-,b)}\left|\frac{1}{\sigma_s}\frac{A'(s+iy/\sigma_s)}{C(s+iy/\sigma_s)}\right| \quad \text{and} \quad \sup_{s\in[b^-,b)}\left|\frac{1}{\sigma_s}\frac{A(s+iy/\sigma_s)C'(s+iy/\sigma_s)}{\{C(s+iy/\sigma_s)\}^2}\right| \quad (7.50)$$

are bounded above by $c(1+|y|)^{\alpha-1-\delta} = c(1+|y|)^{-\delta}$ for some constants $c > 0$ and $\delta > 0$.

First of all, by assumption, (4.6) holds for all $(i,j) \in S_{\backslash m} \times S$ with a common minimal $\delta > 0$. By integrating both sides of the bound in (4.6), we obtain

$$|\mathcal{M}_{ij}(s+iy)| = o\{(1+|y|)^{-\delta}\}, \qquad (i,j) \in S_{\backslash m} \times S.$$

Since $A(z)$ is the determinant of a non-principal minor of $\mathbf{I}_m - \mathbf{T}(z)$, it follows that $A(z)$ is a sum of terms which are finite products of $\mathbf{T}(z)$ entries with no constant term. Each of these addends must contain at least one factor in $S_{\backslash m} \times S$. All other factors in each addend are analytic for $\mathrm{Re}(z) \in [b^-, b]$ and achieve their maximal modulus on the real axis, so that

$$\sup_{y\in\mathbb{R}} |\mathcal{M}_{ij}(s+iy)| = \mathcal{M}_{ij}(s) < \infty, \qquad s \in [b^-, b].$$

Consequently, it follows that, for $y \in \mathbb{R}$,

$$\sup_{s\in[b^-,b)} |A(s+iy)| \le c(1+|y|)^{-\delta}. \quad (7.51)$$

Let us now consider the derivative of $A(z)$. From (4.6), it is seen that ,

$$|\mathcal{M}'_{ij}(s+iy)| \le c_1(1+|y|)^{-1-\delta} \qquad (i,j) \in S_{\backslash m} \times S,$$

and so

$$\sup_{s\in[b^-,b)} |A'(s+iy)| \le c_1'(1+|y|)^{-1-\delta},$$

for some constant $c_1' \in (0, \infty)$. It follows that for $s \in [b^-, b)$ and all $y \in \mathbb{R}$,

$$\sigma_s^{-1} |A'(s + iy/\sigma_s)| \le c_2 \sigma_s^{-1} (1 + |y|/\sigma_s)^{-1-\delta}. \tag{7.52}$$

We now consider $C(s + iy/\sigma_s)$. First note that, using Henrici (1977, p. 277, Theorem 10.7a), we have

$$\sup_{s \in [b^-, b]} |\mathcal{M}_{ij}(s + iy)| \to 0, \qquad |y| \to \infty, \tag{7.53}$$

and so, due to the structure of B noted above,

$$\sup_{s \in [b^-, b)} |B(s + iy) - 1| \to 0, \qquad |y| \to \infty. \tag{7.54}$$

Also, using the fact that $B(s+iy) = (b-s-iy)C(s+iy)$, we have $C(s+iy) \sim 1/(b-s-iy)$ as $|y| \to \infty$, uniformly for $s \in [b^-, b)$. In view of the above, and using the fact that $g(s+iy)$ is analytic when $s \in [b^-, b)$, with no singularities on the line $b + iy$, $y \in \mathbb{R}$, it follows we can find a $c_3 \in (0, \infty)$ such that

$$\sup_{s \in [b^-, b),\, |y| \le c_3} \frac{1}{|C(s + iy)|} \le c_4 \tag{7.55}$$

and

$$\sup_{s \in [b^-, b),\, |y| > c_3} \frac{1}{|C(s + iy)|} \le c_5 + c_6 |y|. \tag{7.56}$$

Therefore it follows from (7.55) and (7.56) that, for $s \in [b^-, b)$ and $y \in \mathbb{R}$,

$$|C(s + iy/\sigma_s)|^{-1} \le c_7 + c_6 |y|/\sigma_s \le c_8 (1 + |y|/\sigma_s). \tag{7.57}$$

After multiplying the inequalities (7.52) and (7.57) together, we obtain

$$\frac{1}{\sigma_s} \left| \frac{A'(s + iy/\sigma_s)}{C(s + iy/\sigma_s)} \right| \le \frac{c_9}{\sigma_s} (1 + |y|/\sigma_s)^{-1-\delta} (1 + |y|/\sigma_s) = \frac{c_9}{\sigma_s} (1 + |y|/\sigma_s)^{-\delta}. \tag{7.58}$$

where $s \in [b^-, b)$ and $y \in \mathbb{R}$. Since we have assumed without loss of generality that $\delta \in (0, 1)$, it follows from elementary calculus that the supremum of the right hand side of (7.58) is achieved at $\sigma^* = \inf_{s \in [b^-, b)} \sigma_s$. Since $\sigma^* > 0$, it follows that the right hand side of (7.58) has an upper bound of the form $c(1 + |y|)^{-\delta}$. Thus condition (4.2) with $j = 1$ is satisfied with $\alpha = 1$ and $\varepsilon = \delta$, which gives the desired bound for the first term on the right of (7.48).

We conclude by showing that the second term in (7.50) also satisfies (4.2) with $j = 1$. Since $C(s + iy)$ is analytic when $s \in [b^-, b]$ and $y \in \mathbb{R}$ it follows that its derivative is also analytic in the same region. Moreover, given the relation $B(z) = (b - z)C(z)$, the derivative of $C(z)$ may be written

$$C'(s + iy) = \frac{B'(s + iy)}{b - s - iy} + \frac{B(s + iy)}{(b - s - iy)^2}.$$

Now when we differentiate $B(z)$ we lose the constant term, which is 1. Moreover, using reasoning similar to that to bound $A'(s+iy)$, it is straightforward to establish that

$$|B'(s+iy)| \leq c_{10}(1+|y|)^{-1-\delta}.$$

Consequently, using (7.54) and recalling that $\delta \in (0,1)$, we obtain the following bound:

$$\frac{|C'(s+iy)|}{|C(s+iy)|^2} \leq c_{11} + c_{12}(1+|y|)^2(1+|y|)^{-2} \leq c_{13},$$

for constants c_{11}, $c_{12}, c_{13} \in (0,\infty)$. Therefore, using (7.51),

$$\left|\frac{1}{\sigma_s}\frac{A(s+iy/\sigma_s)C'(s+iy/\sigma_s)}{\{C(s+iy/\sigma_s)\}^2}\right| \leq \frac{c_{14}}{\sigma_s}(1+|y|/\sigma_s)^{-\delta}. \tag{7.59}$$

Similar arguments to those given above show that the left side of (7.59) has an upper bound of the form $c(1+|y|)^{-\delta}$, and the proof is now complete. □

Proof of Corollary 3. Here, $g(z) = G\{\mathcal{M}_0(z)\}\{C(z)\}^{-\alpha}$, and the main new feature is the presence of G. However, using (7.53), $\sup_{s\in[b^-,b)}|\mathcal{M}_0(s+iy)| \to 0$ as $|y| \to \infty$. Moreover, $G(z)$ is analytic when z is in the unit circle and $G(z) \sim r^\alpha p_1 z$ as $z \to 0$, where p_1 is the probability of 1 in the compounding distribution. Therefore for $|y|$ large, $|G\{\mathcal{M}_0(s+iy)\}| \sim r^\alpha p_1 |\mathcal{M}_0(s+iy)|$. Now

$$\begin{aligned}\frac{1}{\sigma_s}\frac{g'(s+iy/\sigma_s)}{g(s)} &= \frac{iC(s)^\alpha}{\sigma_s G(\mathcal{M}_0(s))}\left[\frac{\mathcal{M}_0'(s+iy/\sigma_s)G'\{\mathcal{M}_0(s+iy/\sigma_s)\}}{\{C(s+iy/\sigma_s)\}^\alpha}\right.\\ &\qquad \left. - \frac{\alpha G\{\mathcal{M}_0(s+iy/\sigma_s)\}}{\{C(s+iy/\sigma_s)\}^{\alpha-1}}\frac{C'(s+iy/\sigma_s)}{\{C(s+iy/\sigma_s)\}^2}\right]\end{aligned} \tag{7.60}$$

Using a similar argument to that which led to (7.57), we obtain the bound

$$|C(s+iy)|^{-\beta} \leq c(1+|y|)^\beta, \qquad \beta = \alpha - 1,\, \alpha, \qquad y \in \mathbb{R};$$

using (4.6) and $G(z) \sim r^\alpha p_1 z$,

$$|\mathcal{M}_0'(s+iy/\sigma_s)| \leq c_1(1+|y|/\sigma_s)^{-1-\delta}, \qquad |G'\{\mathcal{M}_0(s+iy/\sigma_s)\}| < c_1,$$

and

$$|G\{\mathcal{M}_0(s+iy/\sigma_s)\}| \leq c_2(1+|y|/\sigma_s)^{-\delta};$$

inequalities (7.49) hold; and, using arguments similar to those in the proof of Corollary 2, we have

$$\left|\frac{C'(s+iy/\sigma_s)}{\{C(s+iy/\sigma_s)\}^2}\right| \leq c_3.$$

Applying the above inequalities to the terms in (7.60), it is seen that (4.2) with $j = 1$ holds with $\varepsilon = \delta$. □

Proof of Corollary 4. If (4.9) holds for some $\varepsilon_0 > 0$ then necessarily it holds for all $\varepsilon \in (0, \varepsilon_0)$, so if (4.9) holds it must hold for some $\varepsilon \in (0, \alpha)$. Then

$$\begin{aligned}\sup_{s\in[b^-,b)}\left|\frac{1}{\sigma_s}\frac{g'(s+iy/\sigma_s)}{g(s)}\right| &= \sup_{s\in[b^-,b)}\left|\frac{1}{\sigma_s}\frac{1}{(b-s-iy/\sigma_s)^2}\frac{L'\{1/(b-s-iy/\sigma_s)\}}{L\{1/(b-s)\}}\right| \\ &\le c\sup_{s\in[b^-,b)}\left|\frac{1}{\{\sigma_s(b-s)-iy\}}\right|(1+|y|)^{\alpha-\varepsilon} \\ &\le c_1(1+|y|)^{\alpha-1-\varepsilon}, \end{aligned} \tag{7.61}$$

since, for $X \in \mathfrak{D}(\mathcal{G}_\alpha)$, $\sigma_s(b-s)$ stays bounded away from 0 as $s \uparrow b$ by Proposition 1. Therefore (4.2) with $j = 1$ holds with the same choice of ε. □

7.4. Proofs for section 5

Proof of Theorem 4. To show $X \in \mathfrak{D}(-\mathcal{G}_\alpha)$, we first compute σ_s. Since the expansion for $\mathcal{M}(z)$ is uniform over a sector containing the real line, a uniform asymptotic expansion for $\mathcal{M}'(z)$ can be obtained by differentiating term-by-term the expansion for $\mathcal{M}(z)$. For justification of this, see Copson (1965, (vii), pp. 11-12). Iterating upon this leads to another expansion for $\mathcal{M}''(z)$ and higher derivatives, so that

$$\mathcal{M}^{(k)}(z) = \frac{(-1)^k\Gamma(\alpha+k)}{\Gamma(\alpha)}\frac{c_1}{z^{\alpha+k}} + O\left(\frac{1}{z^{\alpha+k+\eta}}\right) \qquad k \ge 0, \quad |\arg(z)| \le \frac{\pi}{2} - \varepsilon. \tag{7.62}$$

The variance is obtained when $k = 2$ by substituting $z = s$, leading to

$$\sigma_s^2 = \mathcal{K}''(s) = \left\{\frac{\mathcal{M}''(s)}{\mathcal{M}(s)}\right\} - \left\{\frac{\mathcal{M}'(s)}{\mathcal{M}(s)}\right\}^2 \sim \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\frac{1}{s^2} - \left\{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\frac{1}{s}\right\}^2 = \frac{\alpha}{s^2}.$$

This gives the asymptotic expansion

$$\frac{\mathcal{M}(s+u/\sigma_s)}{\mathcal{M}(s)} \sim \frac{c_1(s+us/\sqrt{\alpha})^{-\alpha}}{c_1 s^{-\alpha}} = \left(1+\frac{u}{\sqrt{\alpha}}\right)^{-\alpha} \qquad s \to \infty. \tag{7.63}$$

This is the MGF for a $-$Gamma $(\alpha, \sqrt{\alpha})$ distribution so $X \in \mathfrak{D}(-\mathcal{G}_\alpha)$.

We show (5.2) for the survival approximation by directly applying the inversion formula for the survival function of X and then mimicking the proof of Theorem 2. Now

$t \uparrow 0$ as $\hat{s} \to \infty$ and so the inversion formula is

$$
\begin{aligned}
S(t) &= \frac{1}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} \frac{\mathcal{M}(z)}{z} e^{-zt} dz = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathcal{M}(\hat{s}+iy)}{\hat{s}+iy} e^{-t(\hat{s}+iy)} dy \\
&= \frac{\mathcal{M}(\hat{s})e^{-\hat{s}t}}{\hat{s}} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(1+\frac{iy}{\hat{s}}\right)^{-1} \frac{\mathcal{M}(\hat{s}+iy)}{\mathcal{M}(\hat{s})} e^{-iy\hat{\mu}} dy \\
&= \frac{\hat{f}(t)}{\hat{s}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \left(1+\frac{iu}{\hat{s}\hat{\sigma}}\right)^{-1} \frac{\mathcal{M}(\hat{s}+iu/\hat{\sigma})}{\mathcal{M}(\hat{s})} e^{-iu\hat{\mu}/\hat{\sigma}} du, \qquad (7.64)
\end{aligned}
$$

where substitution $u = \hat{\sigma} y$ has been used in the last step.

The leading term in (7.64) as $t \uparrow 0$ is $\hat{f}(t)/\hat{s} \sim \hat{S}(t)$ as we now show. The continued fraction expansion of Mill's ratio in Abramowitz and Stegun (1972, 26.2.14) leads to the bounds

$$
\frac{\phi(t)}{t+1/t} \le 1-\Phi(t) < \frac{\phi(t)}{t} \qquad t>0.
$$

Therefore, from (1.3),

$$
\frac{\phi(\hat{w})}{\hat{w}+1/\hat{w}} - \phi(\hat{w})\left(\frac{1}{\hat{w}}-\frac{1}{\hat{u}}\right) \le \hat{S}(t) < \phi(\hat{w})\frac{1}{\hat{u}}.
$$

Thus,

$$
\frac{1}{\hat{s}}\left\{1-\frac{\hat{u}}{\hat{w}^3}\frac{1}{1+\hat{w}^{-2}}\right\} \le \frac{\hat{S}(t)}{\hat{f}(t)} < \frac{1}{\hat{s}} \qquad (7.65)
$$

and, since by Proposition 5, $\hat{u}/\hat{w}^3 \to 0$ and $\hat{w} \to \infty$, then $\hat{S}(t) \sim \hat{f}(t)/\hat{s}$ as $t \to 0$.

From (7.64),

$$
\lim_{t\uparrow 0}\frac{S(t)}{\hat{S}(t)} = \lim_{\hat{s}\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(1+\frac{iy}{\hat{s}\hat{\sigma}}\right)^{-1}\frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})} e^{-i\hat{\mu}y/\hat{\sigma}} dy. \qquad (7.66)
$$

We now determine a dominating bound for the integrand in (7.66) so that the limit as $\hat{s} \to \infty$ may be passed through the integral. Note that there is a $c_1 > 0$ such that the expansion of $\mathcal{M}$ in (7.62) has the remainder term $|O(1/\hat{s}^{\alpha+\eta})| < c_1/|\hat{s}|^{\alpha+\eta}$ for $|\hat{s}|$ sufficiently large. Thus,

$$
\sup_{\hat{s}\ge b^-}\left|\frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})}\right| \le \sup_{\hat{s}\ge b^-}\left|1+\frac{iy}{\hat{s}\hat{\sigma}}\right|^{-\alpha}\frac{1+c_2|1+iy/(\hat{s}\hat{\sigma})|^{-\eta}}{1-c_2|\hat{s}|^{-\eta}} < c_3\left|1+\frac{iy}{\sqrt{\alpha}^+}\right|^{-\alpha} \qquad (7.67)
$$

for some constants $c_2, c_3 > 0$ where $\sqrt{\alpha}^+ > \sqrt{\alpha}$. Also, by Proposition 5, $\hat{u} \to \sqrt{\alpha} < \infty$ when $X \in \mathfrak{D}(-\mathcal{G}_a)$, so that

$$
\sup_{\hat{s}\ge b^-}\left|\left(1+\frac{iy}{\hat{u}}\right)^{-1}\frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})}\right| \le c_4\left(1+\frac{|y|}{\sqrt{\alpha}^+}\right)^{-1-\alpha}. \qquad (7.68)
$$

The integrability of the upper bound in (7.68) allows the limit in (7.66) to be passed inside. Since $\hat{s}\hat{\mu} \to -\alpha$ and $\hat{s}\hat{\sigma} \to \sqrt{\alpha}$, then $\hat{\mu}/\hat{\sigma} \to -\sqrt{\alpha}$. The inversion of the limiting integrand in (7.66) is therefore

$$\begin{aligned}\sqrt{2\pi} f_{-G(\alpha+1,\sqrt{\alpha})}(-\sqrt{\alpha}) &= \sqrt{2\pi} f_{G(\alpha+1,\sqrt{\alpha})}(\sqrt{\alpha}) \\ &= \sqrt{2\pi}\frac{(\sqrt{\alpha})^{\alpha+1}(\sqrt{\alpha})^{\alpha} e^{-\sqrt{\alpha}\sqrt{\alpha}}}{\Gamma(\alpha+1)} = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)}. \end{aligned} \tag{7.69}$$

We now show the same limit applies to the saddlepoint density ratio by applying Theorem 2. If $\alpha > 1$ the right-hand bound in (7.67) suffices as a dominating bound so that condition (3.2) holds and the saddlepoint density ratio achieves the limit in (7.69). For $\alpha \leq 1$, a different argument is needed which uses integration by parts in conjunction with Theorem 2 arguments. For the inversion integral in (7.11), take $u = \mathcal{M}_0(\hat{s}+iy/\hat{\sigma})/\mathcal{M}_0(\hat{s})$ and $dv = e^{-iy\hat{\mu}/\hat{\sigma}}dy$ so that for $t < 0$,

$$\frac{f(t)}{\hat{f}(t)} = \frac{-\hat{\sigma}}{\sqrt{2\pi} i\hat{\mu}}\frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})}e^{-iy\hat{\mu}/\hat{\sigma}}\Bigg|_{y=-\infty}^{y=\infty} + \frac{1}{\sqrt{2\pi}\hat{\mu}}\int_{-\infty}^{\infty}\frac{\mathcal{M}'(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})}e^{-iy\hat{\mu}/\hat{\sigma}}dy. \tag{7.70}$$

The first term in (7.70) is 0 by the Riemann-Lebesgue lemma. To pass to the limit as $\hat{s} \to \infty$ inside the integral in (7.70), a dominating function is needed. Since $\mathcal{M}'(z) = -\alpha c_1/z^{\alpha+1} + O(1/z^{\alpha+1+\eta})$ where $|O(1/z^{\alpha+1+\eta})| < c_5/|z|^{\alpha+1+\eta}$ for $|z|$ sufficiently large, then

$$\begin{aligned}\frac{\mathcal{M}'(\hat{s}+iy/\hat{\sigma})}{\hat{\mu}\mathcal{M}(\hat{s})} &= \frac{-\alpha c_1/(\hat{s}+iy/\hat{\sigma})^{\alpha+1} + O\{1/(\hat{s}+iy/\hat{\sigma})^{\alpha+1+\eta}\}}{\hat{\mu}c_1/\hat{s}^{\alpha} + \hat{\mu}\, O(1/\hat{s}^{\alpha+\eta})} \\ &= \frac{1}{\{1+iy/(\hat{s}\hat{\sigma})\}^{\alpha}}\frac{(-\alpha)}{(\hat{\mu}\hat{s}+iy\hat{\mu}/\hat{\sigma})}\frac{1+O\{1/(\hat{s}+iy/\hat{\sigma})^{\eta}\}}{1+\hat{\mu}\, O(1/\hat{s}^{\eta})}. \end{aligned} \tag{7.71}$$

Since $\hat{s}\hat{\sigma} < \sqrt{\alpha}^{+}$, $-\hat{\mu}\hat{s}/\alpha > 1^{-} \in (0,1)$, and $-\hat{\mu}/(\hat{\sigma}\sqrt{\alpha}) > 1^{-}$ for sufficiently large $\hat{s}$, then

$$\left|\frac{\mathcal{M}'(\hat{s}+iy/\hat{\sigma})}{\hat{\mu}\mathcal{M}(\hat{s})}\right| < \frac{1}{|1+iy/(\sqrt{\alpha}^{+})|^{\alpha}}\frac{1^{+}}{|1+iy/\sqrt{\alpha}^{+}|}\frac{1+c_5/|\hat{s}|^{\eta}\}}{1-c_2/|\hat{s}|^{\eta}}$$

for $\hat{s}$ sufficiently large. Thus, assumption (5.1) ensures that a dominating bound, as required in condition (3.2) of Theorem 2, can always be found so the theorem applies. From (7.71), the pointwise limit is

$$\frac{\mathcal{M}'(\hat{s}+iy/\hat{\sigma})}{\sqrt{2\pi}\hat{\mu}\mathcal{M}(\hat{s})}e^{-iy\hat{\mu}/\hat{\sigma}} \to \frac{1}{\sqrt{2\pi}}\frac{1}{\{1+iy/\sqrt{\alpha}\}^{1+\alpha}}e^{iy\sqrt{\alpha}},$$

and therefore

$$\lim_{t\uparrow 0}\frac{f(t)}{\hat{f}(t)} = \sqrt{2\pi} f_{-G(1+\alpha,\sqrt{\alpha})}(-\sqrt{\alpha}) = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)},$$

as required. □

Proof of Corollary 5. Apply Watson's lemma (Copson, 1965, §22) to determine an expansion for $\mathcal{M}(z)$ based upon the assumptions in (5.4). The only condition for using Watson's lemma that is not explicitly specified is showing that $|h(z)|$ is of exponential order, i.e. there exists $c > 0 < d$ such that $|h(t)| < ce^{dt}$. This must hold since $\mathcal{M}(s)$ is convergent for all $s \geq 0$. Thus, for any $\varepsilon > 0$,

$$\begin{aligned}\mathcal{M}(z) &\sim \sum_{k=0}^{\infty} h_k \frac{\Gamma(k+1-\beta)}{z^{k+1-\beta}} \qquad |z| \to \infty,\ |\arg(z)| < \frac{\pi}{2} - \varepsilon < \frac{\pi}{2} \\ &= h_m \frac{\Gamma(m+1-\beta)}{z^{m+1-\beta}} + O\left(\frac{1}{z^{m+2-\beta}}\right),\end{aligned}$$

and Watson's lemma ensures that the expansion is uniform. □

7.5. Proofs for section 6

Proof of Theorem 5. Follow the approach used for continuous densities. The inversion integral for the mass function is

$$p(n) = \frac{1}{2\pi i} \int_{\hat{s}-\pi i}^{\hat{s}+\pi i} \mathcal{M}(z) e^{-zn} dz = \hat{p}(n) \frac{1}{\sqrt{2\pi}} \int_{-\pi\hat{\sigma}}^{\pi\hat{\sigma}} \frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})} e^{-i\hat{\mu}y/\hat{\sigma}} dy. \tag{7.72}$$

If $X \in \mathfrak{D}(\mathcal{G}_\alpha)$, then $\hat{\sigma} \sim \sqrt{\alpha}/(b-\hat{s}) \to \infty$ and, assuming (3.2), the saddlepoint ratio from (7.72) has limit

$$\lim_{n\to\infty} \frac{p(n)}{\hat{p}(n)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - \frac{iy}{\sqrt{\alpha}}\right)^{-a} e^{-iy\sqrt{\alpha}} dy = \frac{\hat{\Gamma}(\alpha)}{\Gamma(\alpha)}.$$

If $X \in \mathfrak{D}(\mathcal{N})$, then the limit is 1. □

Proof of Theorem 6. From (6.1) we may write

$$\mathcal{M}(z) = (b-z)^{-\alpha} g(z) \qquad \text{where} \qquad g(z) = \frac{(b-z)^\alpha}{e^{b\alpha}\left\{1 - e^{-(b-z)}\right\}^\alpha} G(e^z).$$

Consequently,

$$\begin{aligned}\frac{g(s+iy/\sigma_s)}{g(s)} &= \left\{1 - \frac{iy}{\sigma_s(b-s)}\right\}^\alpha \left\{\frac{1-e^{-(b-s)+iy/\sigma_s}}{1-e^{-(b-s)}}\right\}^{-\alpha} \frac{G(e^{s+iy/\sigma_s})}{G(e^s)} \\ &=: \{A_s(y)\}^\alpha \{B_s(y)\}^{-\alpha} C_s(y).\end{aligned} \tag{7.73}$$

Moreover, $B_s(y)$, may be written

$$B_s(y) = \frac{1 - e^{-(b-s)+iy/\sigma_s}}{1 - e^{-(b-s)}} = \frac{1 - e^{-(b-s)} + e^{-(b-s)} - e^{-(b-s)+iy/\sigma_s}}{1 - e^{-(b-s)}}$$
$$= 1 + \frac{e^{-(b-s)}}{1 - e^{-(b-s)}}(1 - e^{iy/\sigma_s}) = 1 + h(s)T(y/\sigma_s)y,$$

where

$$h(s) = e^{-(b-s)} \left\{ \frac{b-s}{1 - e^{-(b-s)}} \frac{1}{\sigma_s(b-s)} \right\}$$

has a removable singularity at $s = b$, and is both positive and bounded away from 0 when $s \in [b^-, b)$; and $T(v) = (1 - e^{iv})/v$ is finite and bounded away from 0 for $|v| \leq \pi$. Therefore

$$\sup_{s\in[b^-,b)} \frac{1}{\sigma_s} \left| \frac{g'(s + iy/\sigma_s)}{g(s)} \right| \leq \sup_{s\in[b^-,b)} \left| \alpha A'_s(y) \{A_s(y)\}^{\alpha-1} \{B_s(y)\}^{-\alpha} C_s(y) \right|$$
$$+ \sup_{s\in[b^-,b)} \left| -\alpha B'_s(y) \{A_s(y)\}^{\alpha} \{B_s(y)\}^{-\alpha-1} C_s(y) \right|$$
$$+ \sup_{s\in[b^-,b)} \left| \{A_s(y)\}^{\alpha} \{B_s(y)\}^{-\alpha} C'_s(y) \right|. \tag{7.74}$$

We shall need the following facts, which can be checked directly from the definitions of the relevant functions. Since, from Section 2, $\sigma_s(b-s)$ is bounded away from 0 and ∞ when $s \in [b^-, b)$, using the definition of $A_s(y)$ we have

$$\sup_{s\in[b^-,b)} |A_s(y)|^\beta = \sup_{s\in[b^-,b)} \left| 1 - \frac{iy}{\sigma_s(b-s)} \right|^\beta \leq c_1(1 + |y|)^\beta \tag{7.75}$$

for $\beta = \alpha - 1$ and $\beta = \alpha$, and

$$\sup_{s\in[b^-,b)} |A'_s(y)| = \sup_{s\in[b^-,b)} \left| -\frac{i}{\sigma_s(b-s)} \right| \leq c_1. \tag{7.76}$$

Now consider $B_s(y)$. Assuming that $s, y \in \mathbb{R}$, the imaginary part of $B_s(y)$ is given by $-h(s)\{\sin(y/\sigma_1)/(y/\sigma_s)\}y$ and so, for $|y| \leq \sigma_s\pi$, there exists a constant $c_1 > 0$ for which

$$\inf_{s\in[b^-,b)} |B_s(y)| \geq c_1|y|.$$

Moreover, since $B_s(0) = 1$ for all $s \in [b^-, b)$, it follows that $\inf_{s\in[b^-,b)} |B_s(y/\sigma_s)| > 0$ for $|y|$ sufficiently small (in fact, positivity holds for all $|y|$), and therefore, for sufficiently small $c_2 > 0$,

$$|B_s(y)| \geq c_2(1 + |y|),$$

from which we conclude that

$$|B_s(y)|^{-\beta} \le c_3(1+|y|)^{-\beta} \tag{7.77}$$

for $\beta = \alpha - 1$ and $\beta = \alpha$, when $c_3 < \infty$ is sufficiently large. Similar arguments show that

$$\sup_{s\in[b^-,b)} |B_s'(y)| \le c_4. \tag{7.78}$$

Finally, we focus on $C_s(y)$. We know from condition (6.2) that

$$\sup_{s\in[b^-,b)} |C_s'(y)| = \sup_{s\in[b^-,b)} \left| e^{s+iy/\sigma_s} \frac{i}{\sigma_s} \frac{G'(e^{s+iy/\sigma_s})}{G(e^s)} \right| \le c_5(1+|y|)^{\alpha-1-\varepsilon}, \tag{7.79}$$

and integrating the bound (6.2) as in Step 2 of the proof of Theorem 3, we conclude that

$$\sup_{s\in[b^-,b)} |C_s(y)| = \sup_{s\in[b^-,b)} \left| \frac{G(e^{s+iy/\sigma_s})}{G(e^s)} \right| \le c_6(1+|y|)^{\alpha-\varepsilon}. \tag{7.80}$$

Using (7.75)-(7.80) to bound the three terms on the right side of (7.74), it is seen that the left side of (7.74) satisfies condition (4.2) that was used in Theorem 3. Moreover, from the second form of the inversion integral in (7.72),

$$\begin{aligned} \frac{p(n)}{\hat{p}(n)} &= \frac{1}{\sqrt{2\pi}} \int_{-\pi\sigma_s}^{\pi\sigma_s} \frac{\mathcal{M}(s+iy/\sigma_s)}{\mathcal{M}(s)} e^{-i\mu_s y/\sigma_s} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathcal{M}(s+iy/\sigma_s)}{\mathcal{M}(s)} 1_{\{y\in\mathbb{R}:\, |y|\le\pi\sigma_s\}} e^{-i\mu_s y/\sigma_s} dy. \end{aligned} \tag{7.81}$$

Apart from the indicator function in the second integral in (7.81), the integrand is of exactly the same form as in the continuous case and so exactly the same argument can be used as in the continuous case to justify the limiting argument. □

Proof of Corollary 6. Since $G(\omega)$ is assumed analytic on $|\omega| \le r$, it follows that $G'(\omega)$ is also analytic when $|\omega| \le r$. Therefore

$$\inf_{s\in[b^-,b]} \inf_{|y|\le\pi\sigma_s} |G'(e^{s+iy/\sigma_s})| \le c_1,$$

for some finite positive constant c_1. Also, since $\inf_{s\in[b^-,b)} \sigma_s > 0$, and $|y| \le \pi\sigma_s$, it follows that $\sigma_s^{-1} \le c_2(1+|y|)^{-1}$ for some constant $c_2 \in (0,\infty)$. Finally, since $G(r) \ne 0$ by assumption, $\inf_{s\in[b^-,b]} G(e^s) = c_3^{-1} > 0$. Therefore

$$\sup_{s\in[b^-,b)} \left| \frac{1}{\sigma_s} \frac{G'(e^{s+iy/\sigma_s})}{G(e^s)} \right| \le c_1c_2c_3(1+|y|)^{-1} \le c_4(1+|y|)^{\alpha-1-\varepsilon},$$

for some positive constant c_4, and any $\alpha > 0$, whenever $\varepsilon \in (0, \alpha)$. So (6.2) and therefore (6.3) both hold. □

Preliminary results needed for the proof of Theorem 7. Our next result is needed in the proof of Theorem 7. Based on Table 1, define

$$\hat{p}_j(n) = \phi(\hat{w}_j)/\sqrt{\mathcal{K}''(\hat{s}_j)} \qquad j = 1, 2, 3, 4.$$

Thus $\hat{p}_1(n) = \hat{p}_4(n) = \hat{p}(n)$, the ordinary saddlepoint mass function of Theorems 5 and 6. The approximations $\hat{p}_2(n) = \hat{p}_3(n)$ are based on the offset saddlepoint $\hat{s}^-$ and are only defined because they are used in proving Theorem 7. Let $\hat{s}_j(n+1)$ be the saddlepoint for $p(n+1)$ using the jth approximation.

Proposition 7. (Offset saddlepoint mass functions). *Suppose $b \notin \mathcal{S}$,* either $X \in \mathfrak{D}(\mathcal{G}_\alpha)$, or $X \in \mathfrak{D}(\mathcal{N})$ *and (6.5) does not hold. Then as $n \to \infty$,*

$$\frac{\hat{p}_2(n)}{\hat{p}(n)} \sim \eta^{1/4} e^{\hat{s}_2/2} \qquad \text{and} \qquad \frac{\hat{p}_j(n+1)}{\hat{p}_j(n)} \sim \eta e^{-\hat{s}_j(n+1)}, \qquad j = 1, 2.$$

Here, $\eta = 1$ if X does not satisfy (6.4) and $\eta = \exp(1/(2\kappa_\infty))$ otherwise.

Proof of Proposition 7. For the ratio

$$\frac{\hat{p}_2(n)}{\hat{p}(n)} = \sqrt{\frac{\mathcal{K}''(\hat{s})}{\mathcal{K}''(\hat{s}_2)}} \exp\left\{\mathcal{K}(\hat{s}_2) - \hat{s}_2 n^- - \mathcal{K}(\hat{s}) + \hat{s}n\right\}, \tag{7.82}$$

let K_n denote the exponent. We show $K_n - \hat{s}_2/2 - \ln\eta/4 \to 0$ as $n \to \infty$. Expand $\mathcal{K}(\hat{s}_2)$ about $\hat{s}$ so that

$$\begin{aligned} K_n &= -\hat{s}_2 n^- + \hat{s}n + \mathcal{K}'(\hat{s})(\hat{s}_2 - \hat{s}) + \frac{1}{2}\mathcal{K}''(\hat{s})(\hat{s}_2 - \hat{s})^2 + \frac{1}{6}\mathcal{K}'''(\tilde{s}_1)(\hat{s}_2 - \hat{s})^3 \\ &= -\hat{s}_2 n^- + \hat{s}n + n(\hat{s}_2 - \hat{s}) + \frac{1}{2}\mathcal{K}''(\hat{s})(\hat{s}_2 - \hat{s})^2 + \frac{1}{6}\mathcal{K}'''(\tilde{s}_1)(\hat{s}_2 - \hat{s})^3 \\ &= \frac{\hat{s}_2}{2} + \frac{1}{2}\mathcal{K}''(\hat{s})(\hat{s}_2 - \hat{s})^2 + \frac{1}{6}\mathcal{K}'''(\tilde{s}_1)(\hat{s}_2 - \hat{s})^3, \end{aligned} \tag{7.83}$$

with $\tilde{s}_1 \in (\hat{s}_2, \hat{s})$. To determine the order of $\hat{s} - \hat{s}_2$, subtract the equalities $n = \mathcal{K}'(\hat{s})$ and $n^- = \mathcal{K}'(\hat{s}_2)$ to get

$$\frac{1}{2} = \mathcal{K}'(\hat{s}) - \mathcal{K}'(\hat{s}_2) = \mathcal{K}''(\tilde{s}_2)(\hat{s} - \hat{s}_2), \qquad \tilde{s}_2 \in (\hat{s}_2, \hat{s}). \tag{7.84}$$

If $\mathcal{K}''(\tilde{s}_2) \to \infty$, as occurs when the distribution of X satisfies neither (6.4) nor (6.5), then from (7.84), $\hat{s}_2 - \hat{s} = o(1)$. Combining (7.83) and (7.84),

$$\begin{aligned} K_n - \frac{\hat{s}_2}{2} &= \frac{1}{8}\frac{\mathcal{K}''(\hat{s})}{\{\mathcal{K}''(\tilde{s}_2)\}^2} - \frac{1}{48}\frac{\mathcal{K}'''(\tilde{s}_1)}{\{\mathcal{K}''(\tilde{s}_2)\}^3} \\ &\sim \frac{1}{8\mathcal{K}''(\tilde{s}_2)} + O\left(\frac{1}{\{\mathcal{K}''(\tilde{s}_2)\}^{3/2}}\right) \to 0, \qquad n \to \infty, \end{aligned} \tag{7.85}$$

since $\mathcal{K}''(\hat{s})/\mathcal{K}''(\tilde{s}_2) \sim 1$, as shown in the next paragraph, and $\mathcal{K}'''(\tilde{s}_1)/\{\mathcal{K}''(\tilde{s}_1)\}^{3/2} = O(1)$ by weak convergence of the standardized tilted distribution Z_s as $s \uparrow b$. Otherwise, if $\mathcal{K}''(\tilde{s}_2)$ stays bounded, then from (7.84) $\hat{s} - \hat{s}_2$ stays bounded. Thus, if (6.4) holds, then $K_n - \hat{s}_2/2 - \ln\eta/4 \to 0$. If (6.5) holds, then $K_n - \hat{s}_2/2$ diverges but remains bounded. This explains the dominant exponential term.

The remainder of the argument concerns the leading ratio in (7.82) which can be ignored since it converges to 1. If $X \in \mathfrak{D}(\mathcal{G}_\alpha)$, then $\mathcal{K}''(\hat{s}) \sim \alpha(b-\hat{s})^{-2}$ and $\mathcal{K}'(\hat{s}) \sim \alpha(b-\hat{s})^{-1}$ so

$$R := \sqrt{\frac{\mathcal{K}''(\hat{s}_2)}{\mathcal{K}''(\hat{s})}} \sim \frac{b-\hat{s}}{b-\hat{s}_2} \sim \frac{\mathcal{K}'(\hat{s}_2)}{\mathcal{K}'(\hat{s})} = \frac{n^-}{n} \to 1.$$

If $X \in \mathfrak{D}(\mathcal{N})$, then $\psi(s) = 1/\sqrt{\mathcal{K}''(s)}$ is self-neglecting and

$$R = \frac{\psi(\hat{s})}{\psi(\hat{s}_2)} = \frac{\psi\{\hat{s}_2 + x\psi(\hat{s}_2)\}}{\psi(\hat{s}_2)}, \qquad x = \sqrt{\mathcal{K}''(\hat{s}_2)}/\{2\mathcal{K}''(\tilde{s}_2)\}.$$

If X satisfies (6.4) then x is clearly bounded; if $\mathcal{K}''(\hat{s}_2) \to \infty$ then it is also bounded due to $X \in \mathfrak{D}(\mathcal{N})$ and convergence of the third standardized cumulant to 0. To see this, note for sufficiently large n that $|\mathcal{K}'''(\tilde{s}_2)| < \{\mathcal{K}''(\tilde{s}_2)\}^{3/2}$ so that

$$|\mathcal{K}''(\hat{s}_2) - \mathcal{K}''(\tilde{s}_2)| < (\tilde{s}_2 - \hat{s}_2)c_1\{\mathcal{K}''(\tilde{s}_2)\}^{3/2} < c_2\{\mathcal{K}''(\tilde{s}_2)\}^{3/2},$$

and

$$x = \frac{\sqrt{\mathcal{K}''(\hat{s}_2)}}{2\mathcal{K}''(\tilde{s}_2)} < \frac{1}{2\mathcal{K}''(\tilde{s}_2)}\sqrt{\mathcal{K}''(\tilde{s}_2) + c_2\{\mathcal{K}''(\tilde{s}_2)\}^{3/2}} = O\left\{\{\mathcal{K}''(\tilde{s}_2)\}^{-1/4}\right\}.$$

Thus, $R \to 1$ since ψ is locally uniform self-neglecting.

The argument for the ratio $\hat{p}_j(n+1)/\hat{p}_j(n)$ is the same argument but now

$$1 = \mathcal{K}''(\tilde{s}_2)\{\hat{s}_j(n+1) - \hat{s}_j(n)\} \qquad \tilde{s}_2 \in (\hat{s}_j(n), \hat{s}_j(n+1)),$$

and

$$K_n + \hat{s}_j(n+1) = \frac{1}{2}\frac{\mathcal{K}''(\tilde{s}_1)}{\{\mathcal{K}''(\tilde{s}_2)\}^2} \to 0$$

if neither of (6.4) or (6.5) holds. If (6.4) holds, then $K_n \sim -\hat{s}_j(n+1) + \ln\eta$. □

Proof of Theorem 7. The proof uses Proposition 7 and roughly follows the same approach as that for Theorem 1. However there is considerably more subtlety to the arguments due to the continuity corrections. We first note that

$$\hat{S}_j(n) \sim \frac{\hat{p}_j(n)}{\hat{v}_j}, \qquad n \to \infty; \qquad \hat{v}_j = \frac{\hat{u}_j}{\sqrt{\mathcal{K}''(\hat{s}_j)}}, \qquad j = 1,2,3,4. \tag{7.86}$$

Here, $\hat{p}_j(n) = \hat{p}_1(n)$ for $j = 1,4$ and $\hat{p}_j(n) = \hat{p}_2(n)$ for $j = 2,3$, and, for example, $\hat{v}_1 = 1 - e^{-\hat{s}}$. These results follow from the same arguments used to derive the bounds (7.65) in the proof of Theorem 4.

For the first continuity correction with $j = 1$, define

$$A_{1n} := \frac{1}{S(n)} \sum_{k=n}^{\infty} \hat{p}_1(k) \qquad \text{and} \qquad B_{1n} := \hat{S}_1(n) / \sum_{k=n}^{\infty} \hat{p}_1(k).$$

For any $\varepsilon > 0$, $\sup_{k \geq N} |\hat{p}_1(k)/p(k) - 1/l| < \varepsilon$ for sufficiently large N, where $l = \hat{\Gamma}(\alpha)/\Gamma(\alpha)$ if $X \in \mathfrak{D}(\mathcal{G}_a)$ and $l = 1$ if $X \in \mathfrak{D}(\mathcal{N})$. Thus,

$$|A_{1n} - 1/l| = \frac{1}{S(n)} \left| \sum_{k=n}^{\infty} \{\hat{p}_1(k) - p(k)/l\} \right| \leq \frac{1}{S(n)} \sum_{k=n}^{\infty} \left| \frac{\hat{p}_1(k)}{p(k)} - 1/l \right| p(k) < \varepsilon$$

for $n > N$. Thus, $A_{1n} \to 1/l$. For B_{1n}, we use the Stolz-Cesàro theorem given below in Lemma 1 with $a_n = \hat{S}_1(n)$ and $b_n = \sum_{k=n}^{\infty} \hat{p}_1(k) \downarrow 0$. The sequence $a_n \to 0$ if $\hat{w}_1 \to \infty$ and $\liminf_{n\to\infty} \hat{u}_1 > 0$ and these conditions hold under weak convergence as $s \uparrow b$ of the standardized tilted distribution Z_s by Proposition 5. Therefore it suffices to consider the limit of

$$B_{1n} \sim \frac{\hat{S}_1(n+1) - \hat{S}_1(n)}{-\hat{p}_1(n)} \sim \frac{-\hat{p}_1(n+1)}{\hat{v}_1(n+1)\hat{p}_1(n)} + \frac{1}{\hat{v}_1(n)} \sim \frac{-\eta e^{-\hat{s}_1(n+1)}}{1 - e^{-\hat{s}_1(n+1)}} + \frac{1}{1 - e^{-\hat{s}_1(n)}},$$

where (7.86) has been used and the last expression holds for all cases except (6.5). For $b < \infty$ or for $b = \infty$ and $\mathcal{K}''(s) \to \infty$ as $s \to \infty$, then the limit is 1 giving $j = 1$ results for parts (a) and (b) of the theorem. Part (c) is the case (6.4) where the limit is also 1. For the case (6.5), the ratio $\hat{p}_1(n+1)/\hat{p}_1(n)$ stays bounded but the Stolz-Cesàro theorem does not allow conclusions to be drawn.

For the second continuity correction with $j = 2$, we let

$$A_{2n} := \frac{1}{S(n)} \sum_{k=n}^{\infty} e^{-\hat{s}_2(k)/2} \hat{p}_2(k) \qquad \text{and} \qquad B_{2n} := \hat{S}_2(n) / \sum_{k=n}^{\infty} e^{-\hat{s}_2(k)/2} \hat{p}_2(k)$$

and we first show that $A_{2n} - \eta^{1/4} A_{1n} \to 0$ as $n \to \infty$. Choose N_1 so that $A_{1n} < 1/l^-$ for all $n \geq N_1$ with $l^- \in (0, l)$. For arbitrarily small $\varepsilon > 0$, suppose, by Proposition 7,

$$\left| \frac{e^{-\hat{s}_2(k)/2} \hat{p}_2(k)}{\eta^{1/4} \hat{p}_1(k)} - 1 \right| < l^- \varepsilon / \eta^{1/4} \qquad k \geq N_2.$$

Then

$$\begin{aligned}|A_{2n} - \eta^{1/4} A_{1n}| &= \frac{1}{S(n)} \left| \sum_{k=n}^{\infty} \left\{ e^{-\hat{s}_2(k)/2} \hat{p}_2(k) - \eta^{1/4} \hat{p}_1(k) \right\} \right| \\ &\leq \frac{\eta^{1/4}}{S(n)} \sum_{k=n}^{\infty} \left| \frac{e^{-\hat{s}_2(k)/2} \hat{p}_2(k)}{\eta^{1/4} \hat{p}_1(k)} - 1 \right| \hat{p}_1(k) \\ &< \frac{l^{-}\varepsilon}{S(n)} \sum_{k=n}^{\infty} \hat{p}_1(k) = l^{-}\varepsilon A_{1n} < \varepsilon, \qquad n \geq \max\{N_1, N_2\}.\end{aligned}$$

Thus $A_{2n} \sim \eta^{1/4} A_{1n} \to \eta^{1/4}/l$ as $n \to \infty$. Also using the Stolz-Cesàro theorem along with (7.86) and Proposition 7,

$$B_{2n} \sim \frac{\hat{S}_2(n+1) - \hat{S}_2(n)}{-e^{-\hat{s}_2(n)/2} \hat{p}_2(n)} \sim \frac{-\eta e^{-\hat{s}_2(n+1)}}{e^{-\hat{s}_2(n)/2} 2 \sinh\{\hat{s}_2(n+1)/2\}} + \frac{e^{\hat{s}_2(n)/2}}{2 \sinh\{\hat{s}_2(n)/2\}} \to 1$$

for $b < \infty$, or for $b = \infty$ and $\mathcal{K}''(s) \to \infty$, as $s \to \infty$. This is also the limit for the case (6.4). Thus the overall limit for $\hat{S}_2(n)/S(n)$ is $\eta^{1/4}/l$ and this specializes to the results in parts (a)–(c) for $j = 2$.

For the third continuity correction, $A_{3n} = A_{2n} \to 1/(l\eta^{1/4})$. Also B_{3n} is the same, with $\hat{S}_3(n)$ replacing $\hat{S}_2(n)$, so that

$$B_{3n} \sim \frac{\hat{S}_3(n+1) - \hat{S}_3(n)}{-e^{-\hat{s}_2(n)/2} \hat{p}_2(n)} \sim \frac{-\eta e^{-\hat{s}_2(n+1)}}{e^{-\hat{s}_2(n)/2} \hat{s}_2(n+1)} + \frac{e^{\hat{s}_2(n)/2}}{\hat{s}_2(n)}.$$

For $b < \infty$, B_{3n} converges to $2\sinh(b/2)/b$ providing results for cases (a) and (b) when $b < \infty$. For $b = \infty$, B_{3n} converges to ∞ so $S(n)/\hat{S}_3(n) \to 0$ for case (b) with $b = \infty$ and also for case (c).

For the fourth approximation, $A_{4n} = A_{1n} \to 1/l$. Also

$$B_{4n} = \hat{S}_4(n) / \sum_{k=n}^{\infty} \hat{p}_1(k) \sim \frac{\hat{S}_4(n+1) - \hat{S}_4(n)}{-\hat{p}_1(n)} \sim \frac{-\eta e^{-\hat{s}_1(n+1)}}{\hat{s}_1(n+1)} + \frac{1}{\hat{s}_1(n)},$$

which converges to $(1 - e^{-b})/b$ for $b < \infty$ providing results for cases (a) and (b) when $b < \infty$. If $b = \infty$, then the limit is 0, hence $S(n)/\hat{S}_4(n) \to \infty$ for case (b) with $b = \infty$ and also case (c). □

Lemma 1. (Stolz-Cesàro). *Let $\{a_n\}$ and $\{b_n\}$ be sequences that converge to zero, and assume that $\{b_n\}$ is strictly decreasing for large n. If*

$$\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \gamma \qquad \text{(finite or infinite)},$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \gamma.$$

Proof. See Huang, X.-C. (1988).

Mid-p-values. We consider four approximation to the mid-p-value $S^-(n) := S(n) - p(n)/2$. The first two survival approximations are modified to use

$$\hat{S}_j^-(n) := \hat{S}_j(n) - \hat{p}(n)/2 \qquad j = 1, 2.$$

The third $\hat{S}_3(n)$ and fourth $\hat{S}_4(n)$ are not modified and are used as they are. In practical applications, $\hat{S}_4(n)$ has been found to provide quite accurate approximations as discussed in Paige et al. (2011).

When there is a need to distinguish a saddlepoint at n from the one at $n+1$, we use the notation $\hat{s}_j(n)$ and $\hat{s}_j(n+1)$ for both the ordinary saddlepoint ($j = 1$) and the offset saddlepoint ($j = 2$).

Corollary 8. (Mid-p-values). *Assume the conditions of Theorem 7 for cases (a)–(c). For approximations $\hat{S}_1^-(n)$ and $\hat{S}_2^-(n)$ the ratios $S^-(n)/\hat{S}_j^-(n)$ have the same limits as in (6.6) and (6.7) for cases (a) and (b). For case (c), in which X* satisfies (6.4), *the limit is* 1 *for $\hat{S}_1^-(n)$ and $(2\eta^{1/4} - 1)^{-1}$ for $\hat{S}_2^-(n)$.*

For $j = 3$, the limits depend upon whether b is finite or infinite with

$$\lim_{n\to\infty} \frac{S^-(n)}{\hat{S}_3(n)} = \begin{cases} l\frac{1}{2}be^{-b/2}\coth(b/2) & \text{if} \quad b < \infty \\ 0 & \text{if} \quad b = \infty, \end{cases} \tag{7.87}$$

where $l = \hat{\Gamma}(\alpha)/\Gamma(\alpha)$ if $X \in \mathfrak{D}(\mathcal{G}_a)$ and $l = 1$ if $X \in \mathfrak{D}(\mathcal{N})$. Likewise, for $j = 4$,

$$\lim_{n\to\infty} \frac{S^-(n)}{\hat{S}_4(n)} = \begin{cases} l\frac{1}{2}b\coth(b/2) & \text{if} \quad b < \infty \\ \infty & \text{if} \quad b = \infty. \end{cases}$$

When $b = \infty$ then certainly $\hat{S}_1^-(n)$ and $\hat{S}_2^-(n)$ are asymptotically better than $\hat{S}_3(n)$ and $\hat{S}_4(n)$. For $b < \infty$, $\hat{S}_1^-(n)$ and $\hat{S}_2^-(n)$ dominate $\hat{S}_3(n)$, since

$$\tfrac{1}{2}be^{-b/2}\coth(b/2) < 1$$

for $b > 0$. The comparison of $\hat{S}_1^-(n)$ and $\hat{S}_2^-(n)$ with $\hat{S}_4(n)$ for $b < \infty$ is more complicated since $\frac{1}{2}b\coth(b/2) \geq 1$ and $\frac{1}{2}b\coth(b/2) \to 1$ as $b \to 0$. Consider the example in which $\alpha = 1$ and $X \in \mathfrak{D}(\mathcal{G}_1)$. In the range $b \in (0, 1.45)$ which covers many practical examples, the limiting ratio for $\hat{S}_4$ is closer to 1 than for $\hat{S}_1^-(n)$ and $\hat{S}_2^-(n)$ but the latter are closer for $b > 1.45$.

Proof of Corollary 8. Consider the case $j = 2$. Then

$$\frac{S^-(n)}{\hat{S}_2(n) - \hat{p}_1(n)/2} = \frac{S(n)}{\hat{S}_2(n)} \frac{1 - \frac{1}{2}h(n)}{1 - \frac{1}{2}\hat{p}_1(n)/\hat{S}_2(n)}, \tag{7.88}$$

where $h(n)$ is the hazard function, and

$$\frac{\hat{p}_1(n)}{\hat{S}_2(n)} = \frac{\hat{p}_1(n)}{\hat{p}_2(n)} \frac{\hat{p}_2(n)}{\hat{S}_2(n)} \sim \eta^{-1/4} e^{-\hat{s}_2/2} 2\sinh(\hat{s}_2/2) \to \eta^{-1/4}\left(1 - e^{-b}\right).$$

This is $1 - e^{-b}$ if $b < \infty$ and $\eta^{-1/4}$ if $b = \infty$. Since

$$\frac{1}{h(n)} \frac{\hat{p}_1(n)}{\hat{S}_2(n)} = \frac{\hat{p}_1(n)}{p(n)} \frac{S(n)}{\hat{S}_2(n)} \to \eta^{-1/4},$$

then $h(n) \sim 1 - e^{-b}$ for $b < \infty$ and $h(n) \sim 1$ for $b = \infty$. Thus, (7.88) has the limit

$$\eta^{-1/4} \frac{1 - \left(1 - e^{-b}\right)/2}{1 - \eta^{-1/4}\left(1 - e^{-b}\right)/2} = \frac{1 - \left(1 - e^{-b}\right)/2}{\eta^{1/4} - \left(1 - e^{-b}\right)/2}, \tag{7.89}$$

which is 1 when $\eta = 1$ and $(2\eta^{1/4} - 1)^{-1}$ when (6.4). The same argument applies in case $j = 1$, with $\hat{p}_1(n)/\hat{S}_1(n) \to 1 - e^{-b}$. For $j = 3$, then

$$\frac{S^-(n)}{\hat{S}_3(n)} \sim \frac{S(n)}{\hat{S}_3(n)} \left(1 - \frac{h(n)}{2}\right) \sim l \frac{b}{2\sinh(b/2)} \left(1 - \frac{1 - e^{-b}}{2}\right) = l \frac{be^{-b/2}}{2} \coth(b/2), \tag{7.90}$$

when $b < \infty$. If $b = \infty$, then $h(n) \to 1$ and the limit for (7.90) is 0.

7.6. Limiting relative errors and complex regular variation

Sufficient conditions are now presented for (4.3) to hold in terms of complex regular and slow variation; see Vuilleumier (1976) and Bingham et al. (1987, Appendix 1). First, we define what is meant by complex slowly varying at ∞. Define the sector

$$V_\vartheta = \{z \in \mathbb{C} : |z| > 0 \text{ and } |\arg(z)| < \vartheta\} \qquad \text{for some } \vartheta \in (0, \pi).$$

A function $L : \mathbb{C} \to \mathbb{C}$ is *complex slowly varying at infinity in sector* V_ϑ or, equivalently in abbreviated form, $L \in \mathcal{CSV}_\infty(\vartheta)$, if it is analytic and has no zeros in sector V_ϑ, and for all $\lambda > 0$, $L(\lambda z)/L(z) \to 1$ as $|z| \to \infty$ uniformly for $|\arg(z)| \leq \lambda < \vartheta$.

A function f is said to be *complex regularly varying* at $\xi \in \mathbb{C}$ with index $\alpha \in \mathbb{R}$, if it is of the form

$$f(z) = (\xi - z)^{\alpha} L\{1/(\xi - z)\},$$

where $L \in \mathcal{CSV}_{\infty}(\vartheta)$. In abbreviated form we write $f \in \mathcal{CRV}_{\xi}(\alpha, \vartheta)$.

Corollary 9. (Complex regular variation). *Suppose the MGF $\mathcal{M}(z)$ is complex regularly varying at b, or $\mathcal{M} \in \mathcal{CRV}_b(-\alpha, \vartheta)$* for some $\vartheta \in (\pi/2, \pi)$ and $\alpha > 0$. *Thus, $\mathcal{M}$ takes the form (4.1) with $g(z) = L\{1/(b - z)\}$, $z \in \mathbb{C}$, where $L \in \mathcal{CSV}_{\infty}(\vartheta)$ for $\vartheta \in (\pi/2, \pi)$. Additionally, suppose L satisfies the following conditions:*

$$z\frac{L'(z)}{L(z)} = O(1) \qquad \mathbb{C}^{+} \ni z \to 0, \qquad \sup_{s \in [b^-, b)} \left| \frac{g\left(s + iy/\sigma_s\right)}{g\left(s\right)} \right| < c_1 (1 + |y|)^{\delta} \tag{7.91}$$

for some $\delta \in (0, \alpha)$ and $c_1 > 0$, where $\mathbb{C}^{+} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$. Then condition (4.2) holds with $j = 1$, $X \in \mathfrak{D}(\mathcal{G}_a)$, and the limiting saddlepoint ratios in (4.3) follow.

Proof of Corollary 9. Factorize the expression in condition (4.2) so that

$$\frac{1}{\sigma_s}\frac{g'\left(s + iy/\sigma_s\right)}{g\left(s\right)} = \frac{1}{\sigma_s}\frac{g'\left(s + iy/\sigma_s\right)}{g\left(s + iy/\sigma_s\right)}\frac{g\left(s + iy/\sigma_s\right)}{g\left(s\right)} =: \frac{1}{\sigma_s} R_1(y, s) R_2(y, s).$$

By assumption, $\sup_{s \in [b^-, b)} |R_2(y, s)| < c_1 (1 + |y|)^{\delta}$ for some $\delta \in (0, \alpha)$.

We now determine a bound for $\sup_{s \in [b^-, b)} |R_1(y, s)/\sigma_s|$. Differentiation gives

$$\frac{1}{\sigma_s} R_1(y, s) = \frac{1}{\sigma_s}\frac{L'\left(\frac{1}{b - s - iy/\sigma_s}\right)}{L\left(\frac{1}{b - s - iy/\sigma_s}\right)}\frac{1}{(b - s - iy/\sigma_s)^2} = \frac{1}{\sigma_s(b - s) - iy} Q\left\{\frac{1}{b - s - iy/\sigma_s}\right\}, \tag{7.92}$$

where $Q(z) = zL'\left(z\right)/L\left(z\right)$. We now show that the Q factor in (7.92) is uniformly bounded in y. Since $\sigma_s \sim \sqrt{\alpha}/(b - s)$, we can write $1/\sigma_s = \tau_s (b - s)/\sqrt{\alpha}$ where $\tau_s \to 1$ as $s \uparrow b$. The argument of Q is

$$\frac{1}{b - s - iy/\sigma_s} = \frac{1}{(b - s)(1 - i\tau_s y/\sqrt{\alpha})} = t^{-1} r_{y,s} e^{i\theta_{y,s}}$$

where $t = b - s \to 0$, and $r_{y,s} e^{i\theta_{y,s}}$ is the polar form for $(1 - i\tau_s y/\sqrt{\alpha})^{-1}$. The inversion changes the sign of the argument and $\theta_{y,s} \in (-\pi/2, \pi/2)$ with $r_{y,s} = (1 + \tau_s^2 y^2/\alpha)^{-1/2} \leq 1$. Thus,

$$\sup_{s \in [b^-, b)} \left| Q\left\{t^{-1} r_{y,s} e^{i\theta_{y,s}}\right\}\right| \leq \sup_{|z| \in [R(y), 1]} \ \sup_{|\arg(z)| \leq \pi/2} \left| Q\left(t^{-1} z\right)\right|,$$

where $R(y) = (1 + y^2/\alpha^-)^{-1/2}$ for $\alpha^- \in (0, \alpha)$. Note that $R(y) \to 0$ as $|y| \to \infty$. Using Vuilleumier (1976, Theorem 1, (2.2)), with uniform convergence over $\theta_{y,s} \in [-\pi/2, \pi/2]$, then there exists sufficiently large $R_0 > 0$ such that

$$\sup_{|z|/t > R_0} \sup_{|\arg(z)| \leq \pi/2} \left| Q\left(t^{-1}z\right) \right| < c_2.$$

By assumption, $|Q(z/t)| < c_3$ for $0 < |z|/t < \varepsilon_1$ for some $\varepsilon_1 > 0$. Since $Q(z/t)$ is analytic on the remaining range for z/t in the compact half-disc $\{z/t \in \mathbb{C} : \varepsilon_1 \leq |z|/t \leq R_0, |\arg(z)| \leq \pi/2\}$, then $|Q(z/t)| < c_4$ on this set. Putting all three bounds together, then the Q factor in (7.92) is uniformly bounded in y such that

$$\sup_y \sup_{s \in [b^-, b)} \left| Q\left\{ \frac{1}{b - s - iy/\sigma_s} \right\} \right| \leq \sup_{|z|/t > 0} \sup_{|\arg(z)| \leq \pi/2} \left| Q\left(\frac{z}{t}\right) \right| < c_5.$$

The first factor of (7.92) is of order $O\{(\sqrt{\alpha} - iy)^{-1}\}$ making $R_1 R_2/\sigma_s$ of order $O\{(1 + |y|)^{-1+\delta}\}$. Thus, condition (4.2) with $j = 1$ holds with any $\varepsilon \in (0, \alpha - \delta)$. □

The traditional smoothly regularly varying and slowly varying functions which arise in practical applications, such as powers, powers of logarithms and powers of iterated logarithms, admit analytic continuations which are $\mathcal{CRV}_b(-\alpha, \vartheta)$ and $\mathcal{CSV}_\infty(\vartheta)$ in sector V_ϑ for any $\vartheta \in (\pi/2, \pi)$ and also satisfy the conditions in (7.91).

The assumption that $\mathcal{M} \in \mathcal{CRV}_b(-\alpha, \vartheta)$ for $\vartheta \in (\pi/2, \pi)$ is stronger than $\mathcal{RV}_b(-\alpha)$ and sufficiently so that, together with conditions (7.91), they imply condition (4.2) with $j = 1$ and guarantee that saddlepoint ratios attain the limits in (4.3). Recall from Proposition 1(b) that $\mathcal{M} \in \mathcal{RV}_b(-\alpha)$ for $\alpha > 0$ is a necessary and sufficient condition for $X \in \mathfrak{D}(\mathcal{G}_a)$, but it is not sufficient to guarantee the limits in (4.3). The key to the $\mathcal{CRV}_b(-\alpha, \vartheta)$ assumption is the requirement that $\vartheta > \pi/2$. This places conditions on the analytic continuation of $\mathcal{M}$ within sector $\{z \in \mathbb{C} : |\arg(z - b)| \in (\pi - \vartheta, \pi/2]\}$. As a result, condition (4.2) with $j = 1$ follows from the local uniformity of convergence of L on the subsector $b + \mathbb{C}^+ := \{z \in \mathbb{C} : |\arg(z - b)| \leq \pi/2\}$ and the conditions in (7.91). Note that it is not sufficient to assume $\mathcal{CRV}_b(-\alpha, \pi/2)$ so as to avoid assumptions in the analytic continuation of $\mathcal{M}$ since then local uniformity of a $\mathcal{CSV}_\infty(\pi/2)$ function would only hold on proper subsectors $\{z \in \mathbb{C} : |\arg(z - b)| \leq \varsigma\}$ for $\varsigma \in (0, \pi/2)$ and this is not sufficient to ensure condition (4.2).

7.7. Some further examples and details

Here we provide details for the generalized inverse Gaussian distribution and consider seven more examples. Examples 6 and 7 cover logarithmic singularities, Example 8 considers a finite mixture, and Example 9 gives an example with an oscillatory density.

Example 10 considers $X \sim$ Pareto and also the distribution of $\exp(-X)$ with $X \sim$ Pareto. The final examples are Poisson (Example 11) and a discretized normal mass function (Example 12).

Example 1. (Further details for the Generalized inverse Gaussian distribution).

(Right tail). (Case $p > 0$). From (4.4) with $p > 0$, it follows that

$$\mathcal{M}(s) = (\beta - 2s)^{-p} g(s)$$

where

$$g(s) = c_0\{(\beta - 2s)\gamma\}^{p/2} K_p\{\sqrt{(\beta - 2s)\gamma}\},$$

and $c_0 = (\beta/\gamma)^{p/2}/K_p(\sqrt{\beta\gamma})$. Note that, from Abramowitz and Stegun (1972, 9.6.9, p. 375), $g(s)$ stays bounded away from 0 as $s \uparrow b = \beta/2$, so that we may ignore the denominator $g(s)$ in (4.2). It follows from formulae in Abramowitz and Stegun (1972, p. 376) that, for $j = 1, 2$,

$$g^{(j)}(s) = c_0 \gamma^j \{(\beta - 2s)\gamma\}^{(p-j)/2} K_{p-j}\{\sqrt{(\beta - 2s)\gamma}\};$$

this is an easy consequence of the result

$$\frac{d}{dy}\{y^p K_p(y)\} = -y^p K_{p-1}(y),$$

which follows in turn from classical Bessel function identities, combined with an application of the chain rule to $y = \{\gamma(\beta - 2s)\}^{1/2}$.

Moreover, from Abramowitz and Stegun (1972, 9.7.2, p. 378),

$$K_\nu(z) \sim e^{-z}\sqrt{\frac{\pi}{2z}}\{1 + O(|z|^{-1})\}, \qquad z \in \mathbb{C},$$

where the error term is uniform as $|z| \to \infty$ for $|\arg(z)| < 3\pi/2$. Consequently, putting $z = \sqrt{(\beta - 2s - 2iy)\gamma}$, it follows that

$$\sup_{s\in[b^-,b)} |g^{(1)}(s + iy)| \le c_1(1 + |y|)^{(p-1)/2}(1 + |y|)^{-1/4} = c_1(1 + |y|)^{(2p-3)/4},$$

and so

$$\sup_{s\in[b^-,b)} \frac{1}{\sigma_s}\left|\frac{g^{(1)}(s + iy/\sigma_s)}{g(s)}\right| = \sup_{s\in[b^-,b)} \frac{c_1}{\sigma_s}(1 + |y|/\sigma_s)^{(2p-3)/4} \le c_2(1 + |y|)^{(2p-3)/4};$$

cf. the arguments used to derive the bounds in the proofs of corollaries 2-4.

But $(2p-3)/4 < p-1$ only when $p > 1/2$, so that condition (4.2) with $j = 1$ fails when $p \in (0, 1/2]$. However, with (4.2) and $j = 2$ we obtain, using a similar argument, the requirement that $(2p-5)/4 < p-1$, which holds for all $p > 0$. So (4.3) holds for all $p = \alpha > 0$.

(Case $p = 0$). Use Abramowitz and Stegun (1972, 9.6.13) so that in a neighborhood of 0,

$$K_0(x) = \lambda_1(x) I_0(x) + h(x),$$

where $\lambda_1(x) = -\{\ln(x/2) + \gamma\}$, $I_0(x)$ is a modified Bessel function, $h(x) = O(x^2)$ as $x \to 0$, and I_0 and h are entire functions. The MGF takes the form

$$\mathcal{M}(s) \sim c_1 \ln(1 - 2s/\beta)\} g(s) \qquad g(s) = 1 + O(\beta - 2s) \qquad s \to \beta/2.$$

Hence the conclusion of Example 6 of §7.7 applies and relative error is not preserved.

(Left tail). We now consider the left tail of the GIG distribution. Set scale parameter $\gamma = 1$ without loss in generality. In the case $p \geq 0$, use the notation $\hat{y} = \beta - 2\hat{s} \to \infty$ as $t \uparrow 0$ along with the expansions

$$K_p(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left\{ 1 + \frac{4p^2 - 1}{8x} + O(x^{-2}) \right\} \qquad x \to \infty \tag{7.93}$$

$$K_p'(x) = -\sqrt{\frac{\pi}{2x}} e^{-x} \left\{ 1 + \frac{4p^2 + 3}{8x} + O(x^{-2}) \right\} \qquad x \to \infty$$

from Abramowitz and Stegun (1972, 9.7.2 and 9.7.4). The saddlepoint equation is

$$t = \mathcal{K}'(\hat{s}) \sim \frac{-1}{\sqrt{\hat{y}}} \frac{K_p'(\sqrt{\hat{y}})}{K_p(\sqrt{\hat{y}})} + \frac{p}{\hat{y}} = \frac{1}{\sqrt{\hat{y}}} + \frac{p - 1/2}{\hat{y}} + O(\hat{y}^{-3/2}) \qquad \hat{y} \to \infty$$

when using the second-order expansions in (7.93). Since the expansions for $K_p(z)$ and $K_p'(z)$, viewed as functions of a complex variable z, are uniform over $|\arg(z)| < 3\pi/2$, they may be differentiated term by term to give expansions for higher-order derivatives (Copson, 1965, (vii), p. 10-11). This implies $\mathcal{K}''(\hat{s}) = \hat{y}^{-3/2} + O(\hat{y}^{-2})$.

The values for $\hat{u}$ and $\hat{w}$ are

$$\begin{aligned}
\hat{u} &\sim \frac{\hat{y}}{2} \sqrt{\hat{y}^{-3/2}} = \frac{1}{2} \hat{y}^{1/4} \to \infty \\
\frac{1}{2} \hat{w}^2 = \hat{s}t - \mathcal{K}(\hat{s}) &\sim \frac{\hat{y}}{2} \left(\frac{1}{\sqrt{\hat{y}}} \right) - \ln K_p(\sqrt{\hat{y}}) - \frac{p}{2} \ln \hat{y} \\
&\sim \frac{\sqrt{\hat{y}}}{2} + \frac{1}{4} \ln \hat{y} + \sqrt{\hat{y}} - \frac{p}{2} \ln \hat{y} \to \infty, \qquad \hat{y} \to \infty.
\end{aligned}$$

Also

$$\hat{u}/\hat{w}^3 \sim c_1 \hat{y}^{1/4} / \sqrt{\hat{y}}^{3/2} \to 0.$$

We now determine the dominating function needed for the condition (3.2). Using the asymptotic expansion for K_p, it follows that

$$\mathcal{M}(s) \sim (1-2s/\beta)^{-p/2-1/4} \exp\left(-\sqrt{\beta-2s}+\sqrt{\beta}\right) \qquad s \to -\infty.$$

Consequently, for fixed A_0 large and positive, if $\hat{s} < -A_0$ then

$$\begin{aligned}\left|\frac{\mathcal{M}(\hat{s}+iy/\hat{\sigma})}{\mathcal{M}(\hat{s})}\right| &\leq c_2 \frac{\left|\exp\left\{\sqrt{\beta-2\hat{s}}-\sqrt{\beta-2\hat{s}-2iy/\hat{\sigma}}\right\}\right|}{\left|1-\frac{2iy}{\hat{\sigma}(\beta-2\hat{s})}\right|^{p/2+1/4}} \\ &\leq c_2 \left|\exp\left\{\sqrt{\hat{y}}\left(1-\sqrt{1-\frac{2iy}{\hat{\sigma}\hat{y}}}\right)\right\}\right|, \end{aligned} \tag{7.94}$$

where $\hat{y} = \beta - 2\hat{s}$. It is sufficient to focus on the real part of the exponent of the exponential in the numerator of (7.94), because the modulus of the exponential of the imaginary part of the exponent is bounded above by 1. The real part of the exponent is given by

$$\sqrt{\hat{y}}\left[1-\mathrm{Re}(\sqrt{Re^{i\theta}})\right],$$

where $Re^{i\theta} = 1 - 2iy/(\hat{\sigma}\hat{y})$ is the polar representation of the quantity indicated. It is appropriate to take the positive square root here; using the identity $\cos\theta = 2\cos^2(\theta/2) - 1$, we obtain

$$\begin{aligned}\mathrm{Re}(R^{1/2}e^{i\theta/2}) &= 2^{-1/2}\left(1+\frac{4y^2}{\hat{\sigma}^2\hat{y}^2}\right)^{1/4}\left\{1+\left(1+\frac{4y^2}{\hat{\sigma}^2\hat{y}^2}\right)^{-1/2}\right\}^{1/2} \\ &= (1+x)^{1/4}\left\{\frac{1+(1+x)^{-1/2}}{2}\right\}^{1/2} \equiv f_1(x),\end{aligned}$$

where $x = 4y^2/(\hat{\sigma}\hat{y})^2$. Differentiating the logarithm of $\{f_1(x)\}$ with respect to x we find that (i) $f_1(x)$ is strictly increasing on $x \geq 0$ and (ii) the minimum value of $f_1(x)$ on $[0, \infty)$ is equal to 1. Now define

$$f_2(x) = \frac{1}{x}\{1 - f_1(x)\},$$

Clearly for $x \in (0, \infty)$, $f_2(x) < 0$; as $x \to \infty$, $-f_2(x) \sim 2^{-1/2}x^{-3/4} \to 0$; and as $x \to 0$, $f_2(x) \to -1/8$. Moreover, on any compact interval $[0, A]$, f_2 is bounded away from 0. It follows from the above facts that for A sufficiently large,

$$\sup_{x\in[0,A]} f_2(x) \sim -2^{-1/2}A^{-3/4}. \tag{7.95}$$

As noted above, $\hat{\sigma} \sim \hat{y}^{-3/4}$ so $\hat{\sigma}(\beta - 2\hat{s}) = \hat{\sigma}\hat{y} \sim \hat{y}^{1/4} \to \infty$, from which it follows that

$$\sup_{\hat{s}\in(-\infty,-A_0)} \frac{\sqrt{\hat{y}}}{\hat{\sigma}^2\hat{y}^2} \geq c_3 > 0,$$

for A_0 sufficiently large. Hence, for $\hat{s} \in (-\infty, -A_0)$, the RHS of (7.94) is bounded above by

$$c_2 \exp\left\{4y^2\left(\frac{\sqrt{\hat{y}}}{\hat{\sigma}^2\hat{y}^2}\right) f_2(x)\right\} \leq c_2 \exp\left\{c_3 y^2 f_2(x)\right\}, \tag{7.96}$$

where $x = 4y^2/(\hat{\sigma}\hat{y})^2$, and note also that for $\hat{s} \in (-\infty, -A_0)$, $(\hat{\sigma}\hat{y})^2$ is bounded below by a positive constant, c_4 say. Hence using (7.95) for y sufficiently large, the real part of the exponent is bounded above over $[0, 4y^2/c_4]$ by

$$\begin{aligned}\sup_{x\in[0,4y^2/c_4]} \left\{c_3 y^2 f_2(x)\right\} &\sim \left\{-c_3 y^2 2^{-1/2}\left(\frac{c_4}{4y^2}\right)^{3/4}\right\}\\ &\leq c_6 - c_5 y^{1/2},\end{aligned}$$

for all $\hat{s} \in (-\infty, -A_0)$, where c_5 and c_6 are positive constants which are sufficiently small and sufficiently large, respectively. Thus there exists a dominating function D of the form

$$D(y) = c_7 \exp(-c_5 y^{1/2}),$$

with $c_7 = \exp(c_6)$. The derivation when $p < 0$ is exactly the same and uses the identity $K_{-p}(x) = K_p(x)$. □

Example 6. (Logarithmic singularities: continuous case). If $\mathcal{M}$ has a logarithmic singularity at $b > 0$, there is no weak convergence as $s \to b$ by Proposition 3. In such settings and subject to additional assumptions, saddlepoint relative errors are not uniformly bounded.

Suppose $X \geq 0$ is absolutely continuous with MGF $\mathcal{M}(s) = \{-\ln(b-s)\}^m g(s) + h(s)$ for integer $m > 0$, where $g(s)$ and $h(s)$ are analytic on $\{s \in \mathbb{C} : \mathrm{Re}(s) \leq b\}$ with $g(b) \neq 0$. Examples of such distributions are given in Jensen (1995, §6.1) and Butler (2017, §8) and include densities of the form

$$f(t) \propto (\ln t)^{m-1}\frac{1}{t}e^{-t}, \qquad t > 1.$$

Additionally, assume there exists an $\varepsilon > 0$ such that the improper tilted density $e^{(b+\varepsilon)t} f(t)$ is ultimately non-decreasing. Then

$$\lim_{t\to\infty} \frac{f(t)}{\hat{f}(t)} = 0.$$

To show this, we consider the case in which $h(s) \equiv 0$ as this keeps expressions simple. The case with $h(s) \neq 0$ is the same argument but with more complicated expressions. The CGF is $\ln\{\mathcal{M}(s)\} = \mathcal{K}(s) = m \ln\{-\ln(b-s)\} + \ln g(s)$. As $\hat{s} \uparrow b$, the saddlepoint equation is

$$t = \mathcal{K}'(\hat{s}) \sim \frac{-m}{\hat{u}\ln\hat{u}} \qquad \hat{u} = b - \hat{s}$$

with $\mathcal{K}''(\hat{s}) \sim -m/\{\hat{u}^2 \ln\hat{u}\}$. The saddlepoint density is therefore

$$\hat{f}(t) \sim \frac{1}{\sqrt{2\pi}} \frac{\hat{u}\sqrt{-\ln\hat{u}}}{\sqrt{m}} (-\ln\hat{u})^m g(b) e^{-\hat{s}t}, \qquad \hat{s} \uparrow b. \tag{7.97}$$

The asymptotic form for $f(t)$ is given by a Tauberian theorem in Butler (2017, §8, Proposition 4) as

$$f(t) \sim m(\ln t)^{m-1} t^{-1} g(b) e^{-bt}, \qquad t \to \infty.$$

Extracting the essential parts of the ratio, and using $\ln t \sim -\ln\hat{u}$ and $\hat{u}t \sim -m/\ln\hat{u}$, then as $t \uparrow t_U$,

$$\frac{f(t)}{\hat{f}(t)} \sim c_1 \frac{(-\ln\hat{u})^{-3/2}}{\hat{u}t} e^{-\hat{u}t} \sim c_2(-\ln\hat{u})^{-1/2} \exp\left\{\frac{m}{\ln\hat{u}}\right\} \sim c_2(-\ln\hat{u})^{-1/2} \to 0. \qquad \square$$

Example 7. (Logarithmic singularities: discrete case). When the boundary singularity r for PGF $\mathcal{P}(z)$ is logarithmic, then saddlepoint approximations for mass functions lack uniformity in the right tail. The logarithmic series mass function and its independent convolutions in Butler (2017, §8) are examples. The logarithmic series mass function is the tilted mass function for a base mass function which is a Zipf's law with integer power.

Suppose $X \geq 0$ and has PGF $\mathcal{P}(z) = \{-\ln(r-z)\}^m G(z)$ for $r > 1$ and integer $m > 0$, where $G(z)$ is analytic on $\{z \in \mathbb{C} : |z| \leq r\}$ with $G(r) \neq 0$. Then

$$\lim_{n\to\infty} \frac{p(n)}{\hat{p}(n)} = 0,$$

so relative error is not uniform as $n \to \infty$. To prove this limiting result, consider the setting with $\mathcal{P}(z) = \{-\ln(r-z)\}^m G(z)$. Take $\hat{y} = e^{\hat{s}}$ and let $\hat{u} = r - \hat{y}$. Solving the saddlepoint equation gives $n\hat{u} \sim -m\hat{y}/\ln\hat{u}$ so that $\ln n \sim -\ln\hat{u}$. Taking another derivative, $\mathcal{K}''(\hat{s}) \sim m\hat{y}^2/\{-\hat{u}^2 \ln\hat{u}\}$ so that

$$\hat{p}(n) \sim \frac{1}{\sqrt{2\pi}} \frac{\hat{u}\sqrt{-\ln\hat{u}}}{\sqrt{m}} \frac{1}{\hat{y}} (-\ln\hat{u})^m G(\hat{y}) \hat{y}^{-n}, \qquad n \to \infty. \tag{7.98}$$

The asymptotic behavior of $p(n)$ is given by

$$p(n) \sim \frac{m}{n} (\ln n)^{m-1} G(r) r^{-n}, \qquad n \to \infty, \tag{7.99}$$

and results from a Tauberian theorem in Butler (2017, §8, Proposition 5) which was derived using results from Flajolet and Odlyzko (1990). Combining (7.98) and (7.99) gives

$$\frac{p(n)}{\hat{p}(n)} \sim \sqrt{2\pi m}\left(\frac{\hat{y}}{r}\right)^n \frac{1}{\sqrt{-\ln(r-\hat{y})}} \to 0, \qquad n \to \infty, \tag{7.100}$$

since $(\hat{y}/r)^n < 1$ while $\hat{y} \to r$. □

Relative errors of saddlepoint approximations for finite mixtures behave in the expected way, as indicated in the next example.

Example 8. (Finite mixtures). Such mixture distributions are the base distributions which are tilted to form a regular exponential family. **(a)** Suppose X is Gamma $(1,1)$ with probability (w.p.) 1/2 and Gamma $(1/2, 2)$ w.p. 1/2. Then $X \in \mathfrak{D}(\mathcal{G}_1)$ as $s \to 1$ and $-X \in \mathfrak{D}(-\mathcal{G}_{1/2})$ as $s \to -\infty$. The limiting saddlepoint ratios are $\hat{\Gamma}(1)$ and $\hat{\Gamma}(1/2)/\Gamma(1/2)$ in the respective tails.

(b) Suppose X is Gamma $(1,1)$ w.p. 1/2 and Normal $(0,1)$ w.p. 1/2. Then $X \in \mathfrak{D}(\mathcal{G}_1)$ as $s \to 1$. The normal component dominates in the left tail and $X \in \mathfrak{D}(\mathcal{N})$ as $s \to -\infty$. The limiting saddlepoint ratios are $\hat{\Gamma}(1)$ and 1 in the respective tails. □

In this final example, we consider a distribution due to McCullagh (1994) in which weak convergence of the standardized tilted distribution Z_s does not occur as $s \uparrow b$, yet the relative error remains bounded as $t \uparrow t_U$.

Example 9. In this example the standardized tilted distribution does not exhibit weak convergence. However, the relative saddlepoint error remains bounded though it does not converge to a constant. The density $f(t) = \phi(t)\{1 + \sin(2\pi t)/2\}$ has MGF

$$\mathcal{M}(s) = e^{s^2/2}\{1 + c\sin(2\pi s)\}, \qquad c = e^{-2\pi^2}/2, \qquad s \in \mathbb{R}. \tag{7.101}$$

The second $\mathcal{K}''(s)$ and third cumulants $\mathcal{K}'''(s)$ computed from (7.101) have period 1 and so the third standardized cumulant is also periodic without a limit. Here, the saddlepoint $\hat{s} = t + O(1)$ as $t \to \infty$ and the saddlepoint density ratio is

$$\frac{f(t)}{\hat{f}(t)} \sim \sqrt{\mathcal{K}''(t)}\frac{1+\sin(2\pi t)/2}{1+c\sin(2\pi t)} \exp\left[-2\pi^2\left\{\frac{c\cos(2\pi t)}{1+c\sin(2\pi t)}\right\}^2\right]$$

where the right side is purely 1-periodic. It attains maxima of 1.5 at $\{k + 1/4 : k \in \mathbb{Z}\}$ when sine terms are 1, and there are minima of 0.5 at $\{k + 3/4 : k \in \mathbb{Z}\}$ where sine terms are -1. □

Example 10. (Pareto and $\exp(-$**Pareto)).** Suppose P has a Pareto (p) distribution with density p/t^{p+1} for $t > 1$ and $p > 0$. The MGF is $\mathcal{M}_P(z) = p(-z)^p\Gamma(-p, -z)$ and convergent for $\{\mathrm{Re}(z) \leq 0\}$, where $\Gamma(-p, -z)$ denotes the incomplete gamma function, defined in e.g. Abramowitz and Stegun (1972). Since $b = 0 \in \mathcal{S}$, the right tail does not conform to the conditions of Proposition 1. There is no weak convergence of the standardized tilted variable Z_s as $s \to 0$ since the tilted cumulants of order p or greater diverge to ∞ as $s \to 0$. In the left tail, $\mathcal{M}_P(-z) \sim pz^{-1}e^{-z}\{1 + O(1/z)\}$ as $z \to \infty$ for $|\arg(z)| < 3\pi/2$ (Abramowitz and Stegun, 1972, 6.5.32) and so the MGF for $X = 1 - P$ (with $t_U = 0$) is $\mathcal{M}_X(z) = e^z\mathcal{M}_P(-z) \sim pz^{-1}\{1 + O(1/z)\}$. Thus, (5.1) holds and in the left tail $P \in \mathfrak{D}(-\mathcal{G}_1)$ as $t \downarrow 1$. The condition (5.4) also applies.

Take $Y = e^{-P}$ so Y has the density $f(t) = pt^{-1}(-\ln t)^{-p-1}$ for $t \in (0, e^{-1})$. The density differs from power law $1/t$ by the slowly varying factor $(-\ln t)^{-p-1}$ so $\beta = 1$ for an expansion such as (5.4). The slowly varying factor, however, makes the density integrable but it becomes divergent without this factor. The MGF $\mathcal{M}$ is an entire function. Using (5.4) in the right tail, $Y \in \mathfrak{D}(-\mathcal{G}_1)$ as $t \uparrow e^{-1}$ for all p. In the left tail as $t \downarrow 0$, the third standardized cumulant converges to ∞ as $s \to -\infty$ as shown in the next paragraph. Thus there can be no weak convergence as $s \to -\infty$.

For the $\exp\{-\text{Pareto}\}$ example, we consider the left tail of Y in which $t \downarrow 0$ and $s \to -\infty$ and show that the third standardized cumulant diverges to ∞ as $s \to -\infty$. First, an asymptotic expansion as $s \to -\infty$ for the MGF $\mathcal{M}$ of Y and its derivative $\mathcal{M}'$ are given by

$$\mathcal{M}(s) = p\int_0^{1/e} \frac{e^{st}}{t(-\ln t)^{p+1}}dt \qquad \text{and} \qquad \mathcal{M}'(s) = p\int_0^{1/e} \frac{e^{st}}{(-\ln t)^{p+1}}dt.$$

For $\mathcal{M}'$, make the substitution $v = -st$ so that

$$\mathcal{M}'(s) = \frac{-p}{s}\int_0^{-s/e} \frac{e^{-v}}{\{\ln(-s) - \ln v\}^{p+1}}dv = \frac{-p}{s\ln^{p+1}(-s)}\int_0^{-s/e} e^{-v}\left\{1 - \frac{\ln v}{\ln(-s)}\right\}^{-p-1}dv.$$

Expanding the integrand term in curly braces using the generalized binomial expansion and using Fubini's theorem to interchange integration and summation, then

$$\begin{aligned}\mathcal{M}'(s) &= \frac{-p}{s\ln^{p+1}(-s)}\int_0^{-s/e} e^{-v}\left\{1 + \frac{A_1\ln v}{\ln(-s)} + O\left(\frac{\ln^2 v}{\ln^2(-s)}\right)\right\}dv \\ &= \frac{-p}{s\ln^{p+1}(-s)}\left\{1 + O\left(\frac{1}{\ln(-s)}\right)\right\} \qquad (7.102)\end{aligned}$$

as $s \to -\infty$. Since the expansion in (7.102) is uniform in the sector $\{\mathrm{Re}(z) \leq 0\}$, it may be integrated to give $\mathcal{M}(z) = \ln^{-p}(-z)[1 + O\{1/\ln(-z)\}]$. Furthermore, second and third derivative computations from $\mathcal{K}(s) = -p\ln\{\ln(-s)\} + O\{1/\ln(-s)\}$, $s \in \mathbb{R}$, show that the third standardized cumulant has expansion $2\sqrt{\ln(-s)/p} \to \infty$ as $s \to -\infty$. □

Example 11. (Poisson (1)**).** The central limit theorem ensures that $X \in \mathfrak{D}(\mathcal{N})$. For $j = 1$ and 2 the limiting survival ratio is 1. Taking $n = 400$, then $S(400) = 5.759 \times 10^{-870}$ and computations give $p(400)/\hat{p}(400) = 0.9998$. For $j = 1, 2$ the survival ratios at 400 are 0.9999 and 1.0004 respectively. For $j = 3$ and 4 such limits should be 0 and ∞ and at 400 they attain values 0.3002 and 6.008 respectively. □

Example 12. (Discretized normal, Balkema et al. 1999a). Consider the mass function $p(n) = c\phi(n)$ with $c = \sqrt{2\pi}/\theta_3(0, e^{-1/2}) \simeq 1.0^8535$, where 0^8 indicates a string of 8 zeros, and θ_3 is a theta function (NIST DLMF, 20.2.3). The MGF can be derived as $\theta_3(is/2, e^{-1/2})/\theta_3(0, e^{-1/2})$. All cumulants, as functions of s, are purely oscillatory with period 1 hence weak convergence of the standardized tilted distribution Z_s does not occur. If, however, $s_\gamma \uparrow b = \infty$ over the subsequence $s_\gamma \in \{\gamma, \gamma+1, \ldots\}$ for $\gamma \in [0, 1)$, then all cumulants converge to finite limits which depend only on the value of γ. The third cumulant versus γ resembles $c \sin(2\pi\gamma)$ and accordingly the weak limit over subsequence $\{\gamma, \gamma + 1, \ldots\}$ is in $\mathfrak{D}(\mathcal{N})$ if $\gamma = 0$ or $1/2$, in $\mathfrak{D}(\mathcal{G})$ if $\gamma \in (0, 1/2)$, and in $\mathfrak{D}(-\mathcal{G})$ if $\gamma \in (1/2, 1)$. If $\gamma = 0$ or $1/2$, then (6.4) holds with $\kappa_\infty \simeq 0.9^6788$ or $\kappa_\infty \simeq 1.0^62112$ respectively, where 9^6 and 0^6 are strings of six 9's and 0's, respectively. In both cases, the limiting values for $S(n)/\hat{S}_2(n)$ are $\eta^{-1/4} = 0.8825$. □