



On isotropic and numerical equivalence of cycles

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Abstract

We study the conjecture claiming that, over a flexible field, *isotropic Chow groups* coincide with *numerical Chow groups* (with \mathbb{F}_p -coefficients). This conjecture is essential for understanding the structure of the isotropic motivic category and that of the tensor triangulated spectrum of Voevodsky category of motives. We prove the conjecture for the new range of cases. In particular, we show that, for a given variety X , it holds for sufficiently large primes p . We also prove the p -adic analogue. This permits to interpret integral numerically trivial classes in $\mathrm{CH}(X)$ as p^∞ -anisotropic ones.

Keywords Motive · Motivic cohomology · Chow groups · Numerical equivalence of cycles

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1 Introduction

The difference in complexity between algebraic geometry and topology can be visualised by the fact that in topology there is only one kind of a point, while in algebraic geometry there are many different types of points, namely, the spectra of all possible finitely generated extensions of the base field. It is a natural idea to supply algebro-geometric objects with their “local” versions parametrised by points of various kinds, versions whose complexity would be comparable to that of topological objects. With this in mind, the local versions of the Voevodsky motivic category $\mathrm{DM}(k; \mathbb{F}_p)$ were introduced in the article [11]. These, so-called, *isotropic motivic categories* provide realisation functors for “global” motives and are parametrized by finitely generated extensions of k . More precisely, the isotropic motivic category $\mathrm{DM}(F/F; \mathbb{F}_p)$ is obtained from the global category $\mathrm{DM}(F; \mathbb{F}_p)$ by killing the motives of all p -

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anisotropic varieties (the idea to use such a localisation goes back to T. Bachmann - [2]). Following [11], for any field E , we denote as $\tilde{E} = E(t_1, t_2, \dots) = E(\mathbb{P}^\infty)$ the *flexible closure* of it. Given a base field k , we may introduce the natural partial ordering (depending on the prime p) on the set of finitely generated extensions E/k . Namely $E/k \geq F/k$, if there is a correspondence $Q \rightsquigarrow P$ of degree prime to p , where $E = k(Q)$ and $F = k(P)$. Then we obtain the family of *isotropic realization functors*:

$$\psi_{E,p} : \mathrm{DM}(k) \rightarrow \mathrm{DM}(\tilde{E}/\tilde{E}; \mathbb{F}_p),$$

where E/k runs over all equivalence classes of finitely generated extensions of k under the equivalence defined by the ordering above. The passage to the flexible closure is needed here to make the target category “small”. Indeed, in the case of a flexible field, it is expected that the isotropic motivic category should be, in various respects, reminiscent of the topological motivic category $D(\mathbb{F}_p)$. In particular, homs between compact objects should be finite groups and Balmer’s tensor triangulated spectrum of this category should consist of a single point. But aside from similarities, there are visible differences. For example, it was computed for $p = 2$ —[11, Theorem 3.7] that the *isotropic motivic cohomology* of a point form an external algebra with generators - duals of Milnor’s operations. This, in turn, leads to the computation of the *stable isotropic homotopy groups of spheres* by F. Tanania, who identified these groups with the E_2 -term of the classical Adams spectral sequence—[9].

Another direction from which one may try to approach isotropic motives is the case of *pure motives*. The homs between the respective *isotropic Chow motives* are described by *isotropic Chow groups* $\mathrm{Ch}_{k/k}^*$. These groups are obtained from the usual Chow groups by moding out cycles coming from p -anisotropic varieties—see [11, 2.1]. The central here is [11, Conjecture 4.7] claiming that, in the case of a flexible field, such groups coincide with numerical Chow groups (with finite coefficients):

Conjecture 1.1 *Let k be a flexible field. Then $\mathrm{Ch}_{k/k}^* = \mathrm{Ch}_{\mathrm{Num}}^*$.*

It implies that (in the case of a flexible field) the category of isotropic Chow motives is equivalent to the category of *numerical Chow motives* (with \mathbb{F}_p -coefficients). This, in turn, shows that *isotropic realization functors* $\psi_{E,p}$ should provide “points” for the triangulated spectrum $\mathrm{Spc}(\mathrm{DM}_{\mathrm{gm}}(k))$ in the sense of Balmer ([3]) of (the compact part of) the Voevodsky motivic category. Thus, we get many new points of the spectrum.

In [11] Conjecture 1.1 was proven for varieties of dimension ≤ 5 , for divisors and for cycles of dimension ≤ 2 . The principal aim of the current article is to expand the range of this result. Namely, in Theorem 2.3 we will show that it still holds for varieties of dimension $\leq 2p$ as well as for cycles of dimension $< p$. In particular, for any given variety X , the conjecture is true for sufficiently large p .

In comparison to [11, Theorem 4.11], our method is much more transparent. It is based on the use of K_0 , as well as Adams and Steenrod operations. The Conjecture is equivalent to the fact that any p -numerically trivial class is p -anisotropic. For divisors, this is clear, since the Chow groups of the variety surject to the Chow groups of the generic representative of a very ample linear system. So, if the system is numerically trivial, then this generic divisor is anisotropic. At the same time, for a flexible field,

there is no need to distinguish between k and any finitely generated purely transcendental extension of it, so we have an “analogue” of our generic representative already over k , which will be anisotropic too. This immediately extends to numerically trivial complete intersections. For an arbitrary numerically trivial cycle $u \in \text{Ch}^r$, our strategy is to find $V \in K_0(X)$, such that $c_r(V) = u$ and all smaller classes are trivial. This can be done as long as $r \leq p$. Then using Adams operations we annihilate numerically all higher Chern classes of V , which is possible, if $\dim(X) < rp$. Our element u now is a part of a numerically trivial total Chern class of some $V \in K_0(X)$. It remains to prove that such a total Chern class is always anisotropic. This is done in Proposition 6.1 with the help of a nice geometric construction of Sect. 5 by induction on the dimension of a vector bundle V , using the case of a complete intersection.

The result is easily extended to the p -primary and p -adic case. The only tool we loose in such passage are Steenrod operations, which affects the range. In Theorems 7.2 and 7.5 we prove p -primary and p -adic analogues of Conjecture 1.1 for $\dim(X) \leq p$ and for cycles of dimension $< (p-1)$. This permits to interpret the integral numerically trivial classes in $\text{CH}(X)$ over any field k (of characteristic zero) as p^∞ -anisotropic ones over it’s flexible closure, where p is any prime $\geq \dim(X)$ - see Corollary 7.7. Thus, isotropic Chow groups can be used to approach (the classical) numerical equivalence of cycles with rational coefficients.

2 Isotropic Chow groups

Everywhere below k will be a field of characteristic zero.

Let $n \in \mathbb{N}$ and $\text{Ch}^* = \text{CH}^* / n$ be Chow groups modulo n .

Definition 2.1 Let $n \in \mathbb{N}$ and X be a scheme of finite type over k . We say that X is n -anisotropic, if degrees of all closed points on X are divisible by n .

Definition 2.2 Let $n \in \mathbb{N}$, X be a scheme over k and $x \in \text{CH}_r(X)$. Then x is n -anisotropic, if there exists a proper map $f : Y \rightarrow X$ from an n -anisotropic scheme Y and $y \in \text{CH}_r(Y)$ such that $f_*(y) = x$.

Now we can introduce *isotropic Chow groups*:

$$\text{Ch}_{k/k} := \text{Ch} / (n - \text{anisotropic classes}). \tag{1}$$

This defines an oriented cohomology theory with localisation in the sense of [10, Definition 2.1]. These groups describe homs in the category of *isotropic Chow motives* - see [11, Proposition 5.4].

From now on until Sect. 5, we will assume that $n = p$ is prime. Recall from [11, Definition 2.17] that the category of *isotropic Chow motives* is the full subcategory of $\text{DM}(k/k; \mathbb{F}_p)$ given by the image of $\text{Chow}(k, \mathbb{F}_p)$ under the isotropic realisation functor $\text{DM}(k; \mathbb{F}_p) \rightarrow \text{DM}(k/k; \mathbb{F}_p)$.

For an arbitrary field k , the isotropic motivic category may be pretty large. In particular, for algebraically closed field, it coincides with the original “global” motivic category (as there are no anisotropic varieties, in this case). But there is a large class of,

so-called, *flexible* fields for which isotropic realisation really simplifies things. A field k is called *flexible*, if it is a purely transcendental extension of infinite transcendence degree $k = k_0(t_1, t_2, \dots)$ of some other field. Any field can be embedded into its *flexible closure* by adjoining free parameters. Such an embedding is conservative on motives.

For a smooth projective variety $X \xrightarrow{\pi} \text{Spec}(k)$, there is a natural degree pairing:

$$\text{Ch}^*(X) \times \text{Ch}^*(X) \rightarrow \mathbb{F}_p,$$

given by $\langle a, b \rangle = \pi_*(a \cdot b)$. Since degrees of all zero-cycles on anisotropic varieties are divisible by p , it descends to $\text{Ch}_{k/k}^*$. Thus, we obtain a surjective map $\text{Ch}_{k/k}^* \rightarrow \text{Ch}_{Num}^*$ to Chow groups modulo numerical equivalence (with \mathbb{F}_p -coefficients). According to Conjecture 1.1, this map should be an isomorphism, if k is flexible.

This Conjecture implies that, for a flexible k , the category of isotropic Chow motives $\text{Chow}(k/k; \mathbb{F}_p)$ should be equivalent to the category of numerical Chow motives $\text{Chow}_{Num}(k, \mathbb{F}_p)$. The latter additive category is semisimple. Moreover, since the degree pairing is defined on the level of topological realization, the group of homs between two objects of the numerical category is a subquotient of homs of their topological realisations. In particular, such groups are finite. The above Conjecture implies that the same properties should hold for isotropic Chow motives.

In [11, Theorem 4.11] the Conjecture was proven in the following cases: for varieties of dimension ≤ 5 , for divisors and for cycles of dimension ≤ 2 . Our aim is to prove an extension of this result:

Theorem 2.3 *The Conjecture 1.1 holds for:*

- (1) $\dim(X) \leq 2p$;
- (2) Ch_r , with $r < p$;
- (3) $\text{Ch}^r(X)$, for $r \leq p$ and $\dim(X) < p^2 - p + r$.

In particular, for a given variety X , the Conjecture is valid for sufficiently large p .

3 An outline of the proof

The Conjecture 1.1 is equivalent to the fact that every numerically trivial class u belongs to the subgroup generated by push-forwards from anisotropic varieties.

We start with the main case: $u \in \text{Ch}^r(X)$, where $r \leq p$ and $\dim(X) < rp$. If $Z \subset X$ is a closed subvariety of pure co-dimension r , then the r -th Chern class of O_Z is equal to $(-1)^{r-1}(r-1)![Z]$, while smaller Chern classes are zero (as O_Z belongs to the r -th term of the topological filtration on $K_0(X)$). Thus, if $r \leq p$, then $(r-1)!$ is invertible and there is $V \in K_0(X)$ such that $c_r(V) = u$ and $c_i(V) = 0$, for $i < r$.

The next step is to make all Chern classes of V numerically trivial. This is achieved by applying an appropriate linear combination of Adams operations to V . Here it is essential that $\dim(X) < rp$. As a result, our class u is one of the Chern classes of some $V \in K_0(X)$ whose total Chern class is numerically trivial. In such a situation, the total Chern class of V will always be anisotropic.

Since we work modulo p , we may assume that V is a vector bundle. Passing to the variety of flags of V , we may assume that V is equal to a direct sum of very ample line bundles. In other words, we may assume that there is a set of very ample divisor classes $a_i, i \in \bar{N} = \{1, \dots, N\}$ such that $V = \bigoplus_{i=1}^N O(a_i)$. We will choose concrete cycles representing the above divisor classes. Since the ground field is flexible, we can always replace it by any finitely generated purely transcendental extension. So, we may think of our divisors as of generic representatives A_i of the respective linear systems. In particular, $\cup_i A_i$ is a divisor with strict normal crossings.

Now, we will replace our data $(X; a_i, i \in \bar{N})$ by $(\tilde{X}; b_j, j \in \bar{M})$, where $\pi : \tilde{X} \rightarrow X$ is a blow up and $\pi^*c_\bullet(V) = c_\bullet(\tilde{V})$, where $\tilde{V} = \bigoplus_{j=1}^M O(b_j)$. We will call it the *Main construction*. Namely, π is the blow up in all the intersections $A_I = \cap_{i \in I} A_i, I \subset \bar{N}$. If $\tilde{\rho}_I$ is the class of the proper pre-image of A_I under π , then one can choose (the very ample classes equal modulo p to) $b_j := \sum_{|I| \geq j} \tilde{\rho}_I$ and check that the elementary symmetric functions of b_j s are π^* of those of a_i s. This time, though, the number of b_j s is no more than $\dim(X)$. Moreover, the top Chern class of \tilde{V} is a complete intersection which is still numerically trivial. By the standard arguments, the generic representative of this complete intersection is anisotropic. Applying the *Main construction* again we see that, modulo anisotropic classes, we may replace V by a vector bundle of dimension one less (as the top divisor class of the new arrangement will be exactly the class of the pre-image of the mentioned complete intersection $B_{\bar{M}} = \cap_j B_j$, and so, an anisotropic class). It remains to apply the induction on $\dim(V)$.

This proves the result for $\text{Ch}^r(X)$, where $r \leq p$ and $\dim(X) < rp$. Multiplying X by \mathbb{P}^{p-r} and passing to cycles of co-dimension exactly p , we obtain the whole case (3) of the Theorem. What remains is the case of $\text{Ch}_r(X)$, where $r < p$ and $\dim(X) > 2r + 1$. Here we show using the moving techniques of [11, Section 6] that, after an appropriate blow up, our numerically trivial class u is supported on some smooth divisor D and is numerically trivial already on D . The result follows by the induction on $\dim(X)$ from the main case above. This proves (2). Finally, (1) follows from (2) and (3), aside from the case $p = 2, \dim(X) = 4$, which is straightforward.

4 Adams and Steenrod operations

The purpose of this section is to annihilate numerically the Chern classes of our $V \in K_0(X)$ while keeping the r -th one: $c_r(V) = u$.

On K_0 we have the action of the Adams operations $\Psi_m, m \in \mathbb{Z}$, originally introduced in [1]. Here Ψ_m is a multiplicative operation characterized by the action $\Psi_m(L) = L^{\otimes m}$ on a line bundle L .

Let us describe, how these operations affect characteristic classes of vector bundles.

Proposition 4.1 *Let $\Psi = \sum_i \Psi_{m_i}$ and $V' = \Psi(V)$. Then*

$$c_d(V') = \left(\sum_i m_i^d \right) \cdot c_d(V) + \text{polynomial in } c_l(V), l < d.$$

Proof Since Ψ_m multiplies the Chern roots by m , we have $c_k(\Psi_m(V)) = m^k \cdot c_k(V)$. The rest follows from the Cartan’s formula. \square

Lemma 4.2 *Suppose $(p - 1) \nmid (d - r)$. Then there is a (finite) collection of integers $\{m_i | i \in I\}$ such that $\sum_i m_i^r \equiv 1 \pmod p$ and $\sum_i m_i^d \equiv 0 \pmod p$.*

Proof Since $(d - r)$ is not divisible by $(p - 1)$, we can find m such that $m^d \not\equiv m^r \pmod p$. Combining x copies of m with y copies of 1 , where x and y are any natural numbers such that $x \pmod p$ is the inverse of $(m^r - m^d)$ and $y \pmod p$ is $-x \cdot m^d$, we get the needed collection. \square

Let us now investigate the action of Steenrod operations on characteristic classes. Such characteristic classes are parametrized by the ordered collections $\vec{r} = (r_1, r_2, \dots)$ of non-negative integers, almost all zeroes. To such a collection we can assign two numbers: the degree $\|\vec{r}\| = \sum_i i \cdot r_i$ and the inner degree $|\vec{r}| = \sum_i r_i$. The characteristic class $c_{\vec{r}}(V)$ is the coefficient at $\vec{d}^{\vec{r}} = \prod_{i \geq 1} d_i^{r_i}$ of $\prod_l (\sum_{i \geq 0} d_i \cdot \lambda_i^l)$, where λ_l are Chern roots of V , where we assume $d_0 = 1$ —cf. [8, (2.3)]. Recall that Chern roots of V are “formal” elements λ_l such that the Chern classes $c_j(V)$ are the elementary symmetric functions $\text{sym}_j(\{\lambda_l\})$ of these (λ_l s can be realised as divisor classes on the variety $\text{Flag}(V)$ of complete flags of V —see [10, 2.3]). The mentioned coefficient $c_{\vec{r}}(V)$ is a symmetric function on roots, and so, by the Theorem on Symmetric Functions, is expressible as a polynomial in Chern classes c_j s. It belongs to $\text{CH}^{\|\vec{r}\|}$. In particular, the usual Chern class c_r corresponds to the collection $(r, 0, 0, \dots)$.

Express $c_{\vec{r}}$ as a polynomial in c_j s. Let us denote as $\kappa(\vec{r})$ the coefficient at c_n (where $n = \|\vec{r}\|$) in this polynomial. In other words,

$$c_{\vec{r}} = \kappa(\vec{r}) \cdot c_n + \text{decomposable terms.}$$

Let $\binom{|\vec{r}|}{r_1, r_2, \dots, r_k} = \frac{|\vec{r}|!}{r_1! r_2! \cdot \dots \cdot r_k!}$ be the multinomial coefficient.

Proposition 4.3

$$\kappa(\vec{r}) = (-1)^{\|\vec{r}\| - |\vec{r}|} \binom{|\vec{r}|}{r_1, r_2, \dots, r_k} \cdot \frac{\|\vec{r}\|}{|\vec{r}|}.$$

Proof Let μ_m be the “roots” of d_i s. That is, $d_i = \text{sym}_i(\{\mu_m\})$ is the i -th elementary symmetric function. Then

$$\prod_l (\sum_i d_i \lambda_i^l) = \prod_{l,m} (1 + \lambda_l \mu_m) = \prod_m (\sum_j c_j \mu_m^j) =: \sum_{\vec{r}, \vec{s}} \vec{d}^{\vec{r}} \vec{c}^{\vec{s}} \cdot \kappa(\vec{r}, \vec{s}).$$

From the $\vec{r} - \vec{s}$ symmetry, the coefficient $\kappa(\vec{r})$ at the monomial $\vec{d}^{\vec{r}} \cdot c_{\|\vec{r}\|}$ is the same as that at the monomial $d_{\|\vec{r}\|} \cdot \vec{c}^{\vec{r}}$. In other words, it is the coefficient at $\vec{c}^{\vec{r}}$ in the additive

characteristic class $\sum_l \lambda_l^n$, where $n = \|\vec{r}\|$. It is given by the Girard's formula [7, Problem 16-A]. We have:

$$\sum_{j \geq 1} \left(\sum_l \lambda_l^j \right) \cdot t^j = \sum_l \frac{\lambda_l t}{1 - \lambda_l t} = \frac{-t \left(\prod_l (1 - \lambda_l t) \right)'}{\prod_l (1 - \lambda_l t)} = \frac{-\sum_j (-1)^j j c_j t^j}{\sum_j (-1)^j c_j t^j}.$$

This fraction is equal to

$$\sum_{\vec{r} \neq \vec{0}} (-1)^{\|\vec{r}\| - |\vec{r}|} \binom{|\vec{r}|}{r_1, r_2, \dots, r_k} \cdot \frac{\|\vec{r}\|}{|\vec{r}|} \cdot \vec{c}^{\vec{r}} \cdot t^{\|\vec{r}\|},$$

as can be seen by observing that if we denote as $c_\bullet(t) = \sum_{r \geq 1} c_r t^r$ the t -normalized total Chern class, then

$$\begin{aligned} \log(1 + c_\bullet(-t)) &= - \sum_{n \geq 1} \frac{(-1)^n c_\bullet(-t)^n}{n} \\ &= - \sum_{n \geq 1} \sum_{\vec{r}; |\vec{r}|=n} (-1)^{\|\vec{r}\| - |\vec{r}|} \binom{|\vec{r}|}{r_1, \dots, r_k} \cdot \frac{\vec{c}^{\vec{r}}}{|\vec{r}|} t^{\|\vec{r}\|}, \end{aligned}$$

after which it remains only to differentiate and multiply by $-t$. □

It will be convenient to use an alternative slightly different notation for collections. Namely, we will denote the collection $\vec{r} = (r_1, r_2, \dots)$ also as $(1^{r_1}, 2^{r_2}, \dots)$, where we will list only terms with non-zero r_i s. To avoid confusion, we will denote the respective Chern classes as $s_{1^{r_1}, 2^{r_2}, \dots} = c_{\vec{r}}$.

We have the action of Steenrod operations $P^i : \text{Ch}^* \rightarrow \text{Ch}^{*+i(p-1)}$ on (mod p) Chow groups - [4, 12].

Proposition 4.4 *The Chern class $c_d(V)$ belongs to the subgroup generated by products of smaller Chern classes and images of Steenrod operations (on smaller Chern classes), for all $d \neq l \cdot p^l$, for $1 \leq l < p$.*

Proof Recall that $c_d = s_{1^d}$. Since the Total Steenrod operation $P^{Tot} = \sum_{k \geq 0} P^k$ acts on Chern roots by $P^{Tot}(\lambda) = (\lambda + \lambda^p)$ ([4, Remark 8.5]), we see that $P^k(s_{1^{d-k(p-1)}}) = s_{1^{d-kp}, p^k}$. The latter collection \vec{r} has $\|\vec{r}\| = d$ and $|\vec{r}| = d - k(p - 1)$ with only two non-zero terms $r_1 = d - kp$ and $r_p = k$. Thus, by Proposition 4.3, the respective Chern class is equal to

$$\begin{aligned} \frac{d}{d - k(p - 1)} \binom{d - k(p - 1)}{k} \cdot s_{1^d} + \text{decomp. terms} &= \frac{d}{k} \binom{d - kp + (k - 1)}{k - 1} \cdot s_{1^d} \\ &+ \text{decomp. terms.} \end{aligned}$$

Thus, if there exists, at least, one k such that the coefficient $\frac{d}{k} \binom{d - kp + (k - 1)}{k - 1}$ is not divisible by p , then s_{1^d} belongs to the subgroup generated by products of smaller

Chern classes and by the images of Steenrod operations applied to smaller Chern classes.

Suppose d is such that, for any natural k , $p \mid \frac{d}{k} \binom{d - kp + (k - 1)}{k - 1}$. Looking at $k = 1$ we see that either $d < p$, or $p \mid d$. In the latter case, let $d = pd_1$. Then the above divisibility holds for all k if and only if it holds for $p \mid k$. For $k = pk_1$, the respective coefficient is $\frac{d_1}{k_1} \binom{(d_1 - k_1 p)p + (k_1 - 1)p + (p - 1)}{(k_1 - 1)p + (p - 1)}$, p -adic valuation of which is equal to that of $\frac{d_1}{k_1} \binom{d_1 - pk_1 + (k_1 - 1)}{k_1 - 1}$. Thus, d_1 has the same property as d . By induction, we obtain that then $d = l \cdot p^t$, for some $l < p$ and $t \geq 0$. \square

Suppose, $r \leq p$ and $V \in K_0(X)$ is such that $c_r(V) = u$ is our numerically trivial class, while $c_i(V) = 0$, for $i < r$.

Proposition 4.5 *We can replace V (by another element of $K_0(X)$) to make $c_d(V)$ numerically trivial, for all $d < rp$, while keeping $c_r(V) = u$.*

Proof Let us prove by induction on $d < rp$ that we can make $c_i(V)$ numerically trivial, for $i \leq d$. By assumption, we have it for $d = r$. Suppose, $c_i(V) \stackrel{Num(p)}{\sim} 0$, for $i < d$. By [11, Proposition 4.6], Steenrod operations respect numerical equivalence. Hence, by Proposition 4.4, if $d \neq l \cdot p^t$, for $1 \leq l < p$, then $c_d(V)$ is numerically trivial as well. If $d = l \cdot p^t$, with $1 \leq l < p$, but $l \neq r$ (or $l \neq 1$, if $r = p$), then $(d - r)$ is not divisible by $(p - 1)$. Hence, by Proposition 4.1 and Lemma 4.2, we can find an appropriate linear combination Ψ of Adams operations which makes c_d numerically trivial, keeps c_r still equal to u and keeps all c_i numerically trivial, for $i < d$, and zero, for $i < r$. Hence, we can make the induction step as long as $\frac{d}{r}$ is not a power of p . This is always so for $d < rp$. \square

5 The main construction

Let X be a smooth projective variety, $a_i, i \in \overline{N}$ be very ample divisor classes on it and A_i be smooth cycles representing them, such that $\cup_{i \in \overline{N}} A_i$ is a divisor with strict normal crossings. Denote as A_I the faces $\cap_{i \in I} A_i$ of our divisor, where $I \subset \overline{N}$.

Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of all A_I s, for $I \subset \overline{N}$. Let $\tilde{\rho}_I$ be the class of the proper pre-image of A_I and ρ_I be (the pull-back of) $c_1(O(-1))$ of the I -th blow up. Then

$$\rho_I = \sum_{I \subset J} \tilde{\rho}_J \quad \text{and} \quad \pi^*(a_i) = \sum_{i \in I} \tilde{\rho}_I.$$

Let $2^{\overline{N}}$ be the set of subsets of \overline{N} . For any $I \subset \overline{N}$, consider $U_I \subset 2^{\overline{N}}$, given by $U_I = \{J \mid I \subset J\}$. Let's close the set of these subsets of $2^{\overline{N}}$ under \cap and \cup , and call the subsets obtained this way *good*. Clearly, these are exactly those subsets which have

the property that if I is an element, then so is J , for all $J \supset I$. For any good $U \subset 2^{\bar{N}}$, consider $\rho_U := \sum_{J \in U} \tilde{\rho}_J$. In particular, $\rho_{U_I} = \rho_I$.

Lemma 5.1 *For any good $U, W \subset 2^{\bar{N}}$ we have:*

$$(t - \rho_U)(t - \rho_W) = (t - \rho_{U \cap W})(t - \rho_{U \cup W}).$$

Proof A good subset containing I will necessarily contain any J with $I \subset J$. Let \tilde{E}_I be the proper pre-image of A_I under π . Then \tilde{E}_K and \tilde{E}_L don't intersect unless $K \subset L$, or $L \subset K$. Since for any $\tilde{\rho}_K$ from $(\rho_U - \rho_{U \cap W})$ and any $\tilde{\rho}_L$ from $(\rho_W - \rho_{U \cap W})$, neither $K \subset L$, nor $L \subset K$, we obtain that $(\rho_U - \rho_{U \cap W})(\rho_W - \rho_{U \cap W}) = 0$. Since, by the very definition, $\rho_{U \cup W} = \rho_U + \rho_W - \rho_{U \cap W}$ (recall, that $\rho_U = \sum_{J \in U} \tilde{\rho}_J$), we get the formula. \square

Remark 5.2 In other words, the 2-dimensional vector bundles $O(\rho_U) \oplus O(\rho_W)$ and $O(\rho_{U \cap W}) \oplus O(\rho_{U \cup W})$ have the same Chern classes in CH^* . Moreover, if A^* is an oriented cohomology theory and we denote as $\rho_U^A = c_1^A(O(\rho_U))$ and $\tilde{\rho}_J^A = c_1^A(O(\tilde{\rho}_J))$ the first A -Chern classes of the respective line bundles, then ρ_U^A is the A -formal sum $\sum_{J \in U} \tilde{\rho}_J^A$ and the identities $\rho_{U \cup W}^A = \rho_U^A + \rho_W^A - \rho_{U \cap W}^A$ and $(\rho_U^A - \rho_{U \cap W}^A)(\rho_W^A - \rho_{U \cap W}^A) = 0$ still hold. To see it, it is sufficient to observe that $\rho_{U \setminus U \cap W}^A$ and $\rho_{W \setminus U \cap W}^A$ have disjoint support. Thus, our 2-dimensional vector bundles have the same Chern classes in any oriented cohomology theory. In particular, looking at $c_1^{K_0}$, we get that our vector bundles coincide as elements of K_0 . \square

Let $U_k \subset 2^{\bar{N}}, k \in \bar{m}$ be a collection of good subsets. For any $P \subset \bar{m}$ denote as U_P the intersection $\cap_{k \in P} U_k$, and for $l \in \bar{m}$, denote as W_l the subset $\cup_{P \subset \bar{m}; |P|=l} U_P$. Let $V' = \oplus_{k \in \bar{m}} O(\rho_{U_k})$ and $V'' = \oplus_{l \in \bar{m}} O(\rho_{W_l})$. Then we have:

Proposition 5.3 *In the above situation, $[V'] = [V'']$ in $K_0(\tilde{X})$ and*

$$\prod_{k \in \bar{m}} (t - \rho_{U_k}) = \prod_{l \in \bar{m}} (t - \rho_{W_l}).$$

Proof Induction on m . For $m = 1$, it is a tautology. The induction step follows from Lemma 5.1 and Remark 5.2. \square

Let $\text{Ch}^* = \text{CH}^*/n$, for some $n \in \mathbb{N}$. Consider $U_i = U_{\{i\}}, i \in \bar{N}$. Then $\rho_{U_i} = \pi^*(a_i)$, while $\rho_{W_j} = \sum_{J \subset \bar{N}; |J| \geq j} \tilde{\rho}_J$. Let us choose some very ample representative b_j for $\rho_{W_j} \in \text{Ch}^1(\tilde{X})$. Then it follows from Proposition 5.3 that for $V = (\oplus_{i \in \bar{N}} O(a_i))$ and $\tilde{V} = (\oplus_{j \in \bar{N}} O(b_j))$, the bundles $\pi^*(V)$ and \tilde{V} have the same Chern classes in Ch^* .

Now we replace the data $(X; A_i, i \in \bar{N})$ by $(\tilde{X}; B_j, j \in \bar{N})$, where we choose some smooth representatives of our classes transversal to each other. Note that, in reality, some of these classes b_j s may be zero (modulo n) or anisotropic. In particular, by the very construction, $b_j = 0$, for $j > \dim(X)$, and so, we can replace \tilde{V} by a vector bundle of dimension $\leq \dim(X)$. Moreover, the top Chern class of this vector bundle is represented by a complete intersection (of very ample divisors).

6 Proof of the main theorem

Now we are ready to prove Theorem 2.3. We will first prove part (3), which is the main case.

Suppose $r \leq p$, $\dim(X) < rp$ and $u \in \text{Ch}^r(X)$ is a numerically trivial class. For any subvariety $Z \subset X$ of co-dimension r , O_Z belongs to the r -th component of the topological filtration on K_0 , it follows that $c_i(O_Z) = 0$, for $i < r$. At the same time, $c_r(O_Z) = (-1)^{r-1}(r-1)![Z]$ —[5, Example 15.3.1]. If $r \leq p$, the latter coefficient is invertible. Thus, we can find $V \in K_0(X)$, such that $c_i(V) = 0$, for $i < r$, and $c_r(V) = u$.

It follows from Proposition 4.5 that we can additionally make $c_d(V)$ numerically trivial, for all $d < rp$, that is, for all d (since $\dim(X) < rp$). Thus, the total Chern class $c_\bullet(V) = c_1 + c_2 + \dots$ is numerically trivial.

Proposition 6.1 *Let k be flexible, $n \in \mathbb{N}$ and $V \in K_0(X)$ be such that the total Chern class $c_\bullet(V)$ is n -numerically trivial. Then $c_\bullet(V)$ is n -anisotropic.*

Proof Since $1 + c_\bullet(n^m W) = (1 + c_\bullet(W))^{n^m}$ and we work with cycles modulo n , we may assume that V is a vector bundle. Indeed, the Chern classes of an arbitrary element $V_1 - V_2$ of K_0 will be the same as that of $V_1 + (n^m - 1)V_2$, for sufficiently large m (note that c_i s are zero, for $i > \dim(X)$ and so, $(1 + c_\bullet(V_2))$ raised to the power n sufficiently many times will be equal to 1 modulo n). Passing to the flag variety of V (and using the fact that both $\text{Ch}_{k/k}$ and Ch_{Num} are oriented cohomology theories and, in particular, satisfy the projective bundle axiom), we may assume that $V = \bigoplus_{i=1}^N O(a_i)$, where a_i are very ample divisor classes (recall, again, that we work modulo n and so, any class can be made very ample by adding an appropriate n -multiple of another class).

Note that it is sufficient to prove that $c_\bullet(V)$ becomes anisotropic after some blow up. Indeed, if $\pi : \tilde{X} \rightarrow X$ is such a blow up then $\pi^*(c_\bullet(V))$ is still numerically trivial, and if it is anisotropic, then so is $c_\bullet(V) = \pi_*\pi^*(c_\bullet(V))$.

We choose some smooth mutually transversal representatives A_i of our classes a_i and apply the *Main construction*, i.e. replace the data $(X; A_i, i \in \bar{N})$ by $(\tilde{X}; B_j, j \in \bar{M})$, where $\pi : \tilde{X} \rightarrow X$ is a blow up and $c_\bullet(\tilde{V}) = \pi^*(c_\bullet(V))$. In particular, $c_r(\tilde{V}) = \pi^*(u)$ and all Chern classes of \tilde{V} are still numerically trivial. This time, $M \leq \dim(X)$ and the top Chern class $c_M(\tilde{V}) = \prod_j b_j$ is a complete intersection. Now we proceed by induction on M (= dimension of our vector bundle, which we rename V). Indeed, $c_M(V)$ is a numerically trivial complete intersection. It follows from [11, Proposition 4.15] (using the flexibility of k) that by choosing appropriate representatives B_j of our divisorial classes, we can make $\bigcap_j B_j$ anisotropic. If we now apply the *Main construction* again, then the M -th divisor class in it will be given by the class of the proper pre-image of $\bigcap_j B_j$, and so, will be anisotropic too. So, we can safely remove this last line bundle, without changing the image of $u = c_r(V)$ in the isotropic Chow group $\text{Ch}_{k/k}^r$. Thus, we managed to decrease the dimension of V . The induction step is proven. \square

The above Proposition proves the case Ch^r , $r \leq p$, $\dim(X) < rp$. Let now $r \leq p$ and $\dim(X) < p^2 - p + r$. Consider $Y = X \times \mathbb{P}^{p-r}$ and $v = u \cdot h^{p-r} \in \text{Ch}^p(Y)$, where $h = c_1(O(1))$. Then v is still numerically trivial and, since $u = \pi_*(v)$ for the

natural projection $Y \xrightarrow{\pi} X$, it is sufficient to show that v is anisotropic. But, this time, the co-dimension is p and $\dim(Y) < p \cdot p$, so it is covered by the case proven above. The proof of part (3) is complete.

Next we will prove part (2). More precisely, we will show that the Conjecture holds for $\text{Ch}_r(X)$, where $r < p$ and $\dim(X) > 2r + 1$. We will do it by induction on the $\dim(X)$. Since r is less than half the dimension of X , by [11, Corollary 6.2], after an appropriate blow up, our numerically trivial class u may be represented by the class $[S']$ of a smooth subvariety. Blowing up S' , we may assume that u is supported on a smooth (but, possibly, disconnected) divisor E . Though, the cycle $[S]$ on it representing (the pull-back of) u may be non-smooth. We can employ the moving techniques from [11]. As a particular case of [11, Proposition 6.5] we have:

Proposition 6.2 *Let $n \in \mathbb{N}$ and X be a smooth projective connected variety, E be a divisor on it smooth outside closed n -anisotropic subscheme. Let $S \subset E$ be a closed subscheme on it of positive co-dimension. Then, over some finitely generated purely transcendental extension of k , there is an irreducible divisor Z on X such that $[Z] = [E] \in \text{Ch}^1(X)$, $S \subset Z$, Z is smooth outside an n -anisotropic subscheme on S , and the restriction $\text{CH}^*(X) \rightarrow \text{CH}^*(Z \setminus S)$ is surjective.*

Since our field is flexible, the same holds already over k . Thus, we may assume that u is represented by the class $[S]$ of a closed subvariety, which is contained in an irreducible divisor Z , which is smooth outside an anisotropic subscheme and such that the restriction $\text{Ch}^r(X) \rightarrow \text{Ch}^r(Z \setminus S)$ is surjective. Since $\dim(Z) > 2r$ and $\dim(S) = r$, it follows that $\text{Ch}^r(Z) = \text{Ch}^r(Z \setminus S)$. Thus, we obtain the surjection $\text{Ch}^r(X) \rightarrow \text{Ch}^r(Z)$.

Using the results of H.Hironaka, we can resolve the singularities of Z (inside X) by blowing up anisotropic centers: $\pi : \tilde{X} \rightarrow X$. Here the proper pre-image \tilde{Z} of Z is smooth connected and the isotropic class $\pi^*(u) \in \text{Ch}_{k/k;r}^*(\tilde{X})$ is supported on \tilde{Z} and represented by the class $[\tilde{S}]$ of the proper pre-image of S (since the special divisor of π is anisotropic). Moreover, the maps $\pi^* : \text{Ch}_{k/k}^*(X) \xrightarrow{\cong} \text{Ch}_{k/k}^*(\tilde{X})$ and $\pi_Z^* : \text{Ch}_{k/k}^*(Z) \xrightarrow{\cong} \text{Ch}_{k/k}^*(\tilde{Z})$ are isomorphisms. Thus, the restriction $\text{Ch}_{k/k}^r(\tilde{X}) \rightarrow \text{Ch}_{k/k}^r(\tilde{Z})$ is surjective. Since $\pi^*(u) = [\tilde{S}]$ is numerically trivial on \tilde{X} , this cycle is numerically trivial already on \tilde{Z} (by the projection formula and surjectivity above). Thus, we managed to decrease the dimension of our variety (while keeping r fixed). There are two cases: 1) $\dim(Z) > 2r + 1$; 2) $\dim(Z) = 2r + 1$. In the first case, we conclude by the inductive assumption. In the second, since $r < p$, we see that $\dim(Z) < 2p$ and $u \in \text{Ch}^s(Z)$, where $s \leq p$, so the result follows from part (3) proven above.

Finally, part (1) follows from parts (2) and (3), aside from the case $p = 2$, $\dim(X) = 4$, where only the case $r = 2$ is non-trivial. This is a particular case of [11, Theorem 4.11]. Alternatively, by [11, Theorem 6.1], after an appropriate blow-up, u is a polynomial in divisorial classes, that is, a sum of classes of complete intersections. Then, as above, we can find a direct sum of very ample line bundles $V = \bigoplus_{i=1}^N \mathcal{O}(a_i)$ with $c_2(V) = u$ and $c_1(V) = 0$ (sufficient to see it for a complete intersection $[Z] = a \cdot b$, where $[O_Z] = ([O] - [O(-a)])([O] - [O(-b)])$). Using Proposition 4.5, we may assume that $c_3(V)$ is numerically trivial as well. Applying the Main con-

struction, we reduce to the case where $\dim(V) \leq 4$, keeping numerical triviality of $c_i(V)$, $i \leq 3$. But since $c_1(V) = 0$, we may assume that $V = U \oplus \Lambda^3 U$, for some 3-dimensional U , and $c_4(V) = c_3(V) \cdot d + c_2(V) \cdot d^2 + d^4$, where $d = c_1(\Lambda^3 U)$. Adding 4 copies of $O(d) = \Lambda^3 U$ to V , we obtain a vector bundle with c_2 equal u and c_\bullet numerically trivial (note that $\dim(X) = 4 < 8$). It remains to apply Proposition 6.1. Theorem 2.3 is proven.

7 Isotropic Chow groups with p -primary and p -adic coefficients

In this Section, Ch will denote CH/p^m —Chow groups with \mathbb{Z}/p^m -coefficients.

In complete analogy with the case of \mathbb{F}_p -coefficients one may define the *isotropic Chow motivic category with \mathbb{Z}/p^m -coefficients*—see [11, Section 5]. The Homs in this category are given by *isotropic Chow groups with \mathbb{Z}/p^m -coefficients* $\text{Ch}_{k/k} = \text{CH}_{k/k}(-, \mathbb{Z}/p^m)$ - see (1).

For a smooth projective variety $X \xrightarrow{\pi} \text{Spec}(k)$, we still have a natural degree pairing:

$$\text{Ch}^*(X) \times \text{Ch}^*(X) \rightarrow \mathbb{Z}/p^m,$$

giving the surjection $\text{Ch}_{k/k} \rightarrow \text{Ch}_{\text{Num}(p^m)}$ to numerical Chow groups with \mathbb{Z}/p^m -coefficients.

Conjecture 7.1 *If k is flexible, then for all m , $\text{CH}_{k/k}(X, \mathbb{Z}/p^m)$ coincides with $\text{CH}_{\text{Num}}(X, \mathbb{Z}/p^m)$.*

Another advantage of our Theorem 2.3 in comparison to [11, Theorem 4.11] is that it is easily extendable to the case of p -primary coefficients, though with some loss of range.

Theorem 7.2 *The Conjecture 7.1 holds for:*

- (1) $\dim(X) \leq p$;
- (2) Ch_r , for $r < p - 1$.

In particular, for any given X , the Conjecture is true for sufficiently large p .

Proof The main point is that all the tools used in the proof of Theorem 2.3, aside from Steenrod operations, will work also in the p -primary situation.

The Main case is: $u \in \text{Ch}^r(X)$, where $r \leq p$ and $\dim(X) < r + p - 1$. As $(r - 1)!$ is invertible in \mathbb{Z}/p^m , as above, we can find $V \in K_0(X)$, such that $c_r(V) = u$ and $c_i(V) = 0$, for $i < r$. For $r < d < r + p - 1$, the difference $(d - r)$ is not divisible by $(p - 1)$. So, we can annihilate numerically $c_d(V)$ using Proposition 4.1 and increasing induction on d by applying p^{m-1} times the linear combination of Adams operations, suggested by Lemma 4.2. Since $\dim(X) < r + p - 1$, this will make $c_\bullet(V)$ numerically-trivial. It remains to apply Proposition 6.1.

To prove (2), we need to show that the result holds for Ch_r , where $r < p - 1$ and $\dim(X) > 2r + 1$. This is done with the help of Proposition 6.2 exactly as in

the proof of Theorem 2.3 by induction on the $\dim(X)$. Namely, after an appropriate blow-up, u will be numerically trivial already on some smooth connected divisor Z , where either $\dim(Z) > 2r + 1$, in which case we are done by inductive assumption, or $\dim(Z) = 2r + 1$, which implies that $\dim(X) < \text{codim}(u) + (p - 1)$ and we are done by the Main case above.

Finally, part (1) follows from (2) and the case of divisors - [11, Statement 6.3, Proposition 4.15]. □

Remark 7.3 Over flexible fields, $\text{CH}_{k/k}(X, \mathbb{Z}/p^m)$ can be defined as $\text{CH}(X)/(p^m - \text{anisotropic classes})$. This is so, because, over such a field, classes divisible by p^m are automatically p^m -anisotropic, since it is so for divisors by [11, Statement 6.3], while up to blow up, any class is a polynomial in divisorial ones. □

Denote as $\text{CH}(X; \mathbb{Z}_p)$ the limit $\lim_m \text{CH}(X; \mathbb{Z}/p^m)$, that is, the completion of $\text{CH}(X)$ in the p -adic topology. For a smooth projective X , we have a natural degree pairing:

$$\text{CH}(X; \mathbb{Z}_p) \times \text{CH}(X; \mathbb{Z}_p) \rightarrow \mathbb{Z}_p,$$

and the numerical version $\text{CH}_{\text{Num}}(X; \mathbb{Z}_p)$ obtained by moding out the kernel of this pairing. This can be extended to an oriented cohomology theory using [11, Definition 4.3]. Since degree pairing is defined on the level of the topological realization in $H_{\text{Top}}(X; \mathbb{Z}_p) = H_{\text{Top}}(X; \mathbb{Z}) \otimes \mathbb{Z}_p$ which is a finitely generated \mathbb{Z}_p -module, it follows that $\text{CH}_{\text{Num}}(X; \mathbb{Z}_p)$ is a subquotient of $H_{\text{Top}}(X; \mathbb{Z}_p)$, and so, a finitely generated \mathbb{Z}_p -module too. As the target of the pairing has no \mathbb{Z}_p -torsion, we get that $\text{CH}_{\text{Num}}(X; \mathbb{Z}_p)$ is a free \mathbb{Z}_p -module of finite rank. By a similar reason, $\text{CH}_{\text{Num}}(X)$ is a free \mathbb{Z} -module of finite rank and the natural map $\text{CH}_{\text{Num}}(X) \otimes \mathbb{Z}_p \rightarrow \text{CH}_{\text{Num}}(X; \mathbb{Z}_p)$ is an embedding, which must be an isomorphism, since it is surjective modulo p^m , for any m . Thus, $\text{CH}_{\text{Num}}(X; \mathbb{Z}_p) = \text{CH}_{\text{Num}}(X) \otimes \mathbb{Z}_p$ - the numerical version $A_{(\infty)}^*$ for the free theory $A^* = \text{CH}(X) \otimes \mathbb{Z}_p$ - see [11, Section 4.1] (observe, that the theory $\text{CH}(X; \mathbb{Z}_p)$ itself is not even *constant*). It also shows that $\text{CH}_{\text{Num}}(X; \mathbb{Z}_p) = \lim_m \text{CH}_{\text{Num}}(X; \mathbb{Z}/p^m)$.

Similarly, we may define the *isotropic Chow groups with p -adic coefficients* as the limit

$$\text{CH}_{k/k}(X; \mathbb{Z}_p) = \lim_m \text{CH}_{k/k}(X; \mathbb{Z}/p^m).$$

For flexible fields, it is the same as the limit $\lim_m \text{CH}(X)/(p^m - \text{anisotropic classes})$. Thus, in this case, the image of $\text{CH}(X)$ in $\text{CH}_{k/k}(X; \mathbb{Z}_p)$ will be equal to $\text{CH}(X)/(p^\infty - \text{anisotropic classes})$, where a class is p^∞ -anisotropic, if it is p^m -anisotropic, for any m . There is a natural map

$$\text{CH}_{k/k}(X; \mathbb{Z}_p) \rightarrow \text{CH}_{\text{Num}}(X; \mathbb{Z}_p).$$

The following Conjecture is a consequence of Conjecture 7.1.

Conjecture 7.4 *Let k be flexible. Then $\text{CH}_{k/k}(X; \mathbb{Z}_p) = \text{CH}_{\text{Num}}(X; \mathbb{Z}_p)$.*

In particular, from Theorem 7.2 we obtain:

Theorem 7.5 *In the situation of Theorem 7.2, the Conjecture 7.4 holds true. In particular, for k -flexible and $p \geq \dim(X)$,*

$$\mathrm{CH}_{\mathrm{Num}}(X) = \mathrm{CH}(X)/(p^\infty - \text{anisotropic classes}).$$

Proof Since both $\mathrm{CH}_{k/k}(X; \mathbb{Z}_p)$ and $\mathrm{CH}_{\mathrm{Num}}(X; \mathbb{Z}_p)$ are limits of $\mathrm{CH}_{k/k}(X; \mathbb{Z}/p^m)$ and $\mathrm{CH}_{\mathrm{Num}}(X; \mathbb{Z}/p^m)$, respectively, it follows from Theorem 7.2 that Conjecture 7.4 holds true for $\dim(X) \leq p$, as well as for cycles of dimension $< p - 1$. In particular, for $p \geq \dim(X)$ and flexible k ,

$$\mathrm{CH}_{\mathrm{Num}}(X) \otimes \mathbb{Z}_p = \lim_m \mathrm{CH}(X)/(p^m - \text{anisotropic classes}).$$

The image of $\mathrm{CH}(X)$ in the left group is equal to $\mathrm{CH}_{\mathrm{Num}}(X)$, while on the right it is equal to $\mathrm{CH}(X)/(p^\infty - \text{anisotropic classes})$. Hence, the equality. \square

Remark 7.6 Since, for a variety X/k and a flexible closure $k_{flex} = k(\mathbb{P}^\infty)$, we have $\mathrm{CH}(X) = \mathrm{CH}(X_{k_{flex}})$, Theorem 7.5 gives a description of the numerical Chow groups over an arbitrary field. \square

Corollary 7.7 *Let k be flexible. Then:*

- (1) *Numerically trivial classes in $\mathrm{CH}(X)$ are exactly p^∞ -anisotropic classes, for $p \geq \dim(X)$.*
- (2) *The subgroup of p^∞ -anisotropic classes in $\mathrm{CH}(X)$ doesn't depend on the choice of a prime $p \geq \dim(X)$ and is a subgroup of a finite co-rank.*
- (3) *Torsion classes in $\mathrm{CH}(X)$ are p^∞ -anisotropic, for any $p \geq \dim(X)$.*

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