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Exact smooth and sharp-fronted travelling waves of reaction–diffusion equations with Weak Allee effects☆

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ABSTRACT

We provide new exact forms of smooth and sharp-fronted travelling wave solutions of the reaction–diffusion equation, $\partial_t u = R(u) + \partial_x \left[D(u)\partial_x u\right]$, where the reaction term, $R(u)$, employs a Weak Allee effect. The resulting ordinary differential equation system is solved by means of constructing a power series solution of the heteroclinic trajectory in phase plane space. For specific choices of wavespeeds and standard Weak Allee reaction terms, extending the celebrated exact travelling wave solution of the FKPP equation with wavespeed $5/\sqrt{6}$, we determine a family of exact travelling wave solutions that are smooth or sharp-fronted.

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1. Introduction

Travelling waves arising from a general reaction–diffusion (RD) model over $x \in \mathbb{R}$, as well as reaction–diffusion models on the time-evolving domain $x \in (-\infty, L(t)]$, which we refer to as the *moving-boundary* (MB) model, arise in a wide range of applications and have been of continued interest in applied mathematics for over a century [1–17]. These RD and MB travelling waves are often examined via phase plane analysis of a resulting system of nonlinear ordinary differential equations (ODEs) [1,2], with some choices of reaction–diffusion functions leading to celebrated exact travelling wave solutions (c.f. [3–5]). The most common choice of reaction term in RD and MB models is logistic growth, which assumes that the population $u(x,t)$ will always thrive and survive [3]. Another choice of reaction term is the *Weak Allee effect* [3], which reduces the population growth rate in low population densities. However, explicit travelling wave solutions have not been reported. Furthermore, with the inclusion of standard degenerate diffusivity choices (c.f. [1,2,8–11]), numerically-computed solutions of the resulting sharp-fronted travelling waves are unreliable or inaccurate [1]. Consequently, obtaining exact forms of sharp-front travelling waves can aid the resulting analysis for travelling waves obeying similar reaction–diffusion mechanisms.

In this work, we examine the resulting nonlinear ODE system that arises from travelling wave analysis in RD and MB models [2]. Using a power series solution ansatz for the heteroclinic trajectory, we are able

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to obtain implicit analytic expressions for the travelling wave profile. For specific choices of wavespeeds and standard Weak Allee reaction terms, extending the celebrated exact travelling wave solution of the FKPP equation with wavespeed $5/\sqrt{6}$ [4,5], we determine a family of exact travelling wave solutions that are either smooth or sharp-fronted.

2. Travelling wave analysis in Reaction Diffusion and Moving-Boundary models

With reference to [2], we consider two non-dimensional reaction–diffusion models, describing the population $u \in [0,1]$. In the first, which we call the Reaction–Diffusion (RD) model, u evolves of the entire real line and is described via the following partial differential equation:

$$(\mathbf{RD}) : \begin{cases} \partial_t u(x,t) = R(u) + \partial_x \left[D(u)\partial_x u(x,t)\right], \\ \lim\limits_{x\to-\infty} u(x,t) = 1, \qquad \lim\limits_{x\to+\infty} u(x,t) = 0. \end{cases} \tag{1}$$

In the second, which we call the Moving-Boundary (MB) model [2], u evolves over the spatial domain $x \in (-\infty, L(t)]$ with a moving boundary condition at $x = L(t)$:

$$(\mathbf{MB}) : \begin{cases} \partial_t u(x,t) = R(u) + \partial_x \left[D(u)\partial_x u(x,t)\right], \\ \lim\limits_{x\to-\infty} u(x,t) = 1, \qquad u(L(t),t) = 0, \qquad \dfrac{\mathrm{d}L}{\mathrm{d}t} = -\kappa D(u)\partial_x u(x,t)|_{x\to L(t)^-}, \quad L(0) = L_0. \end{cases} \tag{2}$$

The moving boundary parameter, κ, relates the speed of the moving front, $\mathrm{d}L/\mathrm{d}t$, to the population flux, $-D(u)\partial_x u(x,t)$. For both models, we impose that both $R(u)$ and $D(u)$ have convergent power series on $[0,1]$ and that

$$0 \le D(u), R(u) < \infty, \quad u \in [0,1], \qquad R(u) = \mathcal{O}_s(u-1), \quad u \to 1^-. \tag{3}$$

These bounds indicate that the diffusivity $D(u)$ is finite and non-negative and that the reaction $R(u)$ can only be a source term with a simple root at $u = 1$. Next, we transform the PDE system into travelling wave coordinates via $z = x - L_0 - ct$,. In the RD model, $z \in \mathbb{R}$, whereas in the MB model, $z \in (-\infty, 0]$ and $\mathrm{d}L/\mathrm{d}t = c$. By denoting

$$Q(z) = D(u)\frac{\mathrm{d}u}{\mathrm{d}z}, \tag{4}$$

we have, as shown in [2], that the heteroclinic trajectory $Q(u)$ satisfies the following nonlinear first-order ODE:

$$-\frac{\mathrm{d}Q(u)}{\mathrm{d}u} = c + \frac{R(u)D(u)}{Q(u)}, \qquad Q(1) = 0, \qquad \lim_{u\to 0^+} Q(u) = \begin{cases} 0, & (\mathbf{RD}) \\ -\dfrac{c}{\kappa}. & (\mathbf{MB}) \end{cases} \tag{5}$$

3. Power series solution of the heteroclinic trajectory $Q(u)$

In general, explicit solutions of Eq. (5), which is an Abel equation of the second kind [18], are not readily obtainable for all $R(u)$ and $D(u)$ choices. However, we can construct a power series solution for $Q(u)$ and, in a similar fashion, construct an implicit power series solution for $u(z)$. To do this, we first make the following power series ansatzes:

$$Q(u) = \sum_{n=1}^{\infty} \alpha_n (u-1)^n, \qquad R(u)D(u) = \sum_{n=1}^{\infty} \beta_n (u-1)^n, \qquad D(u) = \sum_{n=0}^{\infty} \delta_n (u-1)^n. \tag{6}$$

We note that the series in $Q(u)$ and $R(u)D(u)$ begin at $n = 1$, due to the boundary condition $Q(1) = 0$, as well as using the conditions stated in (3). By multiplying Eq. (5) by $Q(u)$, we obtain from a Cauchy product of power series that

$$-\sum_{n=1}^{\infty} \gamma_n (u-1)^n = c\sum_{n=1}^{\infty} \alpha_n (u-1)^n + \sum_{n=1}^{\infty} \beta_n (u-1)^n, \tag{7}$$

where γ_n is the discrete convolution of α_n with $(n+1)\alpha_{n+1}$:

$$\gamma_n = \sum_{k=1}^{n}(n-k+1)\alpha_k\alpha_{n-k+1} = \binom{n+1}{2}\sum_{k=1}^{n}\alpha_k\alpha_{n-k+1}. \tag{8}$$

Therefore, by examining each power of $(u-1)^n$, beginning with $n=1$, we have that

$$-\alpha_1^2 = c\alpha_1 + \beta_1 \quad \Longrightarrow \quad \alpha_1 = \frac{-c+\sqrt{c^2-4\beta_1}}{2} > 0, \tag{9}$$

since we know that $\beta_1 < 0$ and require $\alpha_1 > 0$. Next, for $n=2$, we have that

$$-3\alpha_1\alpha_2 = c\alpha_2 + \beta_2 \implies \alpha_2 = -\frac{\beta_2}{c+3\alpha_1}. \tag{10}$$

Finally, for $n \geq 3$, we have that

$$\alpha_n = -\frac{1}{c+(n+1)\alpha_1}\left[\beta_n + \binom{n+1}{2}\sum_{k=2}^{n-1}\alpha_k\alpha_{n-k+1}\right], \qquad n \geq 3. \tag{11}$$

Through this recursion formula, we are able to obtain the power series of $Q(u)$, and hence, a series solution for κ in the MB model:

$$(\mathbf{MB}): \quad \kappa = -\frac{c}{Q(0)} = \frac{c}{\sum_{n=1}^{\infty}(-1)^{n+1}\alpha_n}. \tag{12}$$

To determine $u(z)$, we use the same power series representations for $Q(u)$ and $D(u)$ and substitute these into Eq. (4):

$$\sum_{n=0}^{\infty}\delta_n(u-1)^n\frac{\mathrm{d}u}{\mathrm{d}z} = \sum_{n=1}^{\infty}\alpha_n(u-1)^n = (u-1)\sum_{n=0}^{\infty}\alpha_{n+1}(u-1)^n \tag{13}$$

To determine the power series representation of the quotient $D(u)/Q(u)$, we define

$$\frac{(u-1)D(u)}{Q(u)} = \sum_{n=0}^{\infty}\xi_n(u-1)^n \implies \delta_n = \sum_{k=0}^{n}\xi_k\alpha_{n-k+1}. \tag{14}$$

The coefficients of ξ_n can then be related to α_n and δ_n via the following difference equation:

$$\alpha_1\xi_n = \sum_{k=0}^{n}\delta_{n-k}A_k, \quad \text{where} \quad A_0 = 1 \quad \text{and} \quad A_{n+1} = -\frac{1}{\alpha_1}\sum_{k=0}^{n}A_k\alpha_{n-k+2}, \quad n \geq 0. \tag{15}$$

Therefore, we can use this expression for ξ_n in (4) and obtain

$$\int^u \frac{\alpha_1}{v-1}\left[\sum_{n=0}^{\infty}\xi_n(v-1)^n\right]\mathrm{d}v = \alpha_1(z-z_0). \tag{16}$$

Thus, we can determine an implicit solution $z(u)$ that solves the RD and MB models (1) and (2):

$$(\mathbf{RD}): \qquad \alpha_1(z-z_0) = \delta_0\ln(1-u) + \sum_{n=1}^{\infty}(u-1)^n\sum_{k=0}^{n}\left[\frac{\delta_{n-k}A_k}{n}\right], \tag{17}$$

$$(\mathbf{MB}): \qquad \alpha_1 z = \delta_0\ln(1-u) + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\delta_{n-k}A_k\left[\frac{(u-1)^n-(-1)^n}{n}\right]. \tag{18}$$

In these formulae, z_0 is an arbitrary constant of integration, while α_n and A_n are determined recursively via (11) and (15), respectively.

4. Exact travelling wave solutions in Weak Allee effect models

In general, the choices of $D(u)$ and $R(u)$, and hence, the choices of β_n and δ_n, give rise to a multitude of possibilities for the form of the implicit solution $z(u)$ described in (18). Nevertheless, there are specific function choices which give rise to key features of RD and MB travelling waves. In particular, we will focus on a family of Weak Allee effect reaction terms, due to their frequent appearance in reaction–diffusion models.

4.1. Quadratic $Q(u)$

From (11), it is worth noting that if $\beta_3 = -2\alpha_2^2$ and $\beta_n = 0$ for $n \geq 4$, then $\alpha_n = 0$ for $n \geq 3$ and $Q(u)$ reduces to a quadratic function. Furthermore, it can be shown from (15) that $A_n = (-\alpha_2/\alpha_1)^n$ and hence,

$$\alpha_1 z = \delta_0 \ln(1-u) + \sum_{n=1}^{\infty} \left(-\frac{\alpha_2}{\alpha_1}\right)^n \left[\frac{(u-1)^n - (-1)^n}{n}\right] \left[\sum_{k=0}^{n} \delta_k \left(-\frac{\alpha_1}{\alpha_2}\right)^k\right], \quad \kappa = \frac{c}{\alpha_1 - \alpha_2}. \tag{19}$$

As an example of a new explicit solution of MB travelling waves, we consider the case where diffusion is constant and a Weak Allee effect is employed in the reaction term [2,19,20]:

$$(\mathbf{MB}): \quad D(u) = 1, \qquad R(u) = u(1-u)(\rho u + 1 - \rho), \qquad \rho \in \left[\frac{1}{2}, 1\right). \tag{20}$$

By identifying $\delta_0 = 1$, $\beta_{1,2,3} = \{-1, -1-\rho, -\rho\}$, and all other coefficients being zero, we have that a particular wavespeed $\hat{c}$ gives rise to a quadratic form of $Q(u)$:

$$\begin{aligned} \hat{c} = \frac{2\rho - 1}{\sqrt{2\rho}} &\implies \alpha_1 = \frac{1}{\sqrt{2\rho}}, \quad \alpha_2 = \sqrt{\frac{\rho}{2}}, \quad \alpha_{n\geq 3} = 0 \\ &\implies Q(u) = \frac{(u-1)[1 + \rho(u-1)]}{\sqrt{2\rho}}, \quad \kappa = \frac{2\rho - 1}{1 - \rho}. \end{aligned} \tag{21}$$

By noting that $\alpha_2 = \rho\alpha_1$, we have from (19) that

$$\frac{z}{\sqrt{2\rho}} = \ln\left[\frac{(1-\rho)(1-u)}{1 - \rho(1-u)}\right] \iff u(z) = \frac{(1-\rho)\left[1 - \exp\left(\frac{z}{\sqrt{2\rho}}\right)\right]}{1 - \rho\left[1 - \exp\left(\frac{z}{\sqrt{2\rho}}\right)\right]}. \tag{22}$$

This Weak Allee effect class can also be extended to RD travelling waves when $\rho = 1$, which allows $Q(0) = 0$:

$$(\mathbf{RD}): \quad \hat{c} = \frac{1}{\sqrt{2}} \implies Q(u) = \frac{u(u-1)}{\sqrt{2}}, \quad u(z) = \left[1 + \exp\left(\frac{z - z_0}{\sqrt{2}}\right)\right]^{-1}. \tag{23}$$

4.2. General polynomial $Q(u)$

In a similar fashion to quadratic solutions of $Q(u)$, we can also choose β_n via (11) to force a polynomial solution for $Q(u)$ of degree N, i.e. having α_N non-zero and $\alpha_n = 0$ for $n > N$. Due to the discrete convolution terms appearing in (11), we require that β_{2N-1} is non-zero, while $\beta_n = 0$ for $n \geq 2N$. We can the relate $\alpha_1, \ldots \alpha_N$ and c to $\beta_1, \ldots \beta_{2N-1}$ by means of a system $2N-1$ quadratic equations obtained from (11).

As a means of illustrating the more general forms of polynomial $Q(u)$, we consider a MB model whereby degenerate diffusion is employed alongside a family of Weak Allee models akin to those presented in [9]:

$$(\mathbf{MB}): \quad D(u) = u^n, \qquad R(u) = u[\sigma u(1 - u^{n+1}) + (1-u)(1 - \sigma - n\sigma)], \qquad \sigma \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right). \tag{24}$$

We note that this general reaction–diffusion model reduces to the quadratic $Q(u)$ explored in the previous subsection when $n = 0$. Furthermore, the degenerate diffusion $D(u)$ creates travelling waves with infinite slope at $x = L(t)$, while the population flux, $-D(u)\partial_x u$, remains finite [1,2]. For the wavespeed $\tilde{c} = \dfrac{(n+2)\sigma - 1}{\sqrt{(n+2)\sigma}}$, we determine from (11) that $Q(u)$ is a polynomial of degree $n+2$:

$$Q(u) = \frac{\sigma u(u^{n+1}-1) + (u-1)(1-\sigma-n\sigma)}{\sqrt{(n+2)\sigma}}, \quad \kappa = \frac{(n+2)\sigma - 1}{1-(n+1)\sigma}. \tag{25}$$

While the implicit travelling wave $z(u)$ can be obtained via integration or (15), for any permissible value of σ, it has been omitted here for brevity. For the special case when $c = 0$, the closed-form expression for $z(u)$ can be determined as

$$(\mathbf{MB}): \ \sigma = \frac{1}{n+2} \implies Q(u) = \frac{u^{n+3}-1}{n+2}, \quad z(u) = -\mathrm{B}_{u^{n+2}}\left(\frac{n+1}{n+2}, 0\right), \tag{26}$$

where $\mathrm{B}_x(p,q)$ is the incomplete Beta function [21]. Similar to the previous subsection, this class of reaction–diffusion functions can also be extended to RD travelling waves when $\sigma = (n+1)^{-1}$, which allows $Q(0) = 0$:

$$(\mathbf{RD}): \ \tilde{c} = \frac{1}{\sqrt{(n+1)(n+2)}} \implies Q(u) = \frac{u(u^{n+1}-1)}{\sqrt{(n+1)(n+2)}},$$
$$z(u) = z_0 - \sqrt{\frac{n+2}{n+1}}\mathrm{B}_{u^{n+1}}\left(\frac{n}{n+1}, 0\right). \tag{27}$$

We note that while this travelling wave solution is only permitted in the RD model, due to $Q(0) = 0$, the resulting travelling wave is still sharp-fronted and is identically zero for $z > z_0$ (c.f. [9]). Furthermore, due to the change in argument in the incomplete Beta function, we note that the leading edge of the RD travelling wave is proportional to $(z_0 - z)^{\frac{1}{n}}$, while the leading edge of the MB travelling waves are proportional to $(-z)^{\frac{1}{n+1}}$ [2]. This power law change demonstrates the qualitative differences between sharp-fronted RD and MB travelling waves.

4.3. Other Weak Allee effect models

We conclude by considering a final family of Weak Allee effect models that extend beyond polynomial $Q(u)$ solutions and is motivated by the celebrated exact travelling wave solution discussed in [4]:

$$(\mathbf{RD}): \quad D(u) = 1, \qquad R(u) = \frac{u}{r}(1-u^r), \qquad r > 0. \tag{28}$$

We note that in the case where $r = 1$, we retrieve the standard logistic growth reaction term and have an explicit solution for $c = 5/\sqrt{6}$ in [4]:

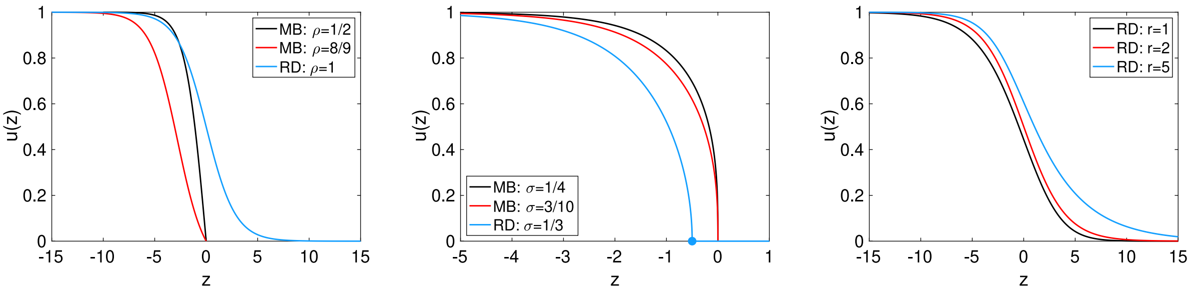
$$r = 1, \quad c = \frac{5}{\sqrt{6}} \implies u(z) = \left[1 + \exp\left(\frac{z - z_0}{\sqrt{6}}\right)\right]^{-2}, \quad Q(u) = \sqrt{\frac{2}{3}}u\left(\sqrt{u} - 1\right). \tag{29}$$

Motivated from this form of $Q(u)$, which incorporates rational powers of u, we adopt the following ansatz to solve (28):

$$Q(u) = \alpha u(u^\beta - 1) \implies \alpha = \sqrt{\frac{2}{r(r+2)}}, \quad \beta = \frac{r}{2}, \quad c = \frac{r+4}{\sqrt{2r(r+2)}}. \tag{30}$$

Therefore, for the special wavespeed $c = \dfrac{r+4}{\sqrt{2r(r+2)}}$, we can determine both the heteroclinic trajectory as well as the travelling wave solution to (28):

$$c = \frac{r+4}{\sqrt{2r(r+2)}} \implies Q(u) = \sqrt{\frac{2}{r(r+2)}}u\left(u^{r/2} - 1\right), \quad u(z) = \left[1 + \exp\left((z - z_0)\sqrt{\frac{r}{2(r+2)}}\right)\right]^{-\frac{2}{r}}. \tag{31}$$



(a) (b) (c)

Fig. 1. Explicit travelling wave solutions $u(z)$ for Weak Allee effect models described in (a) Section 4.1, (b) Section 4.2 with $n = 2$, and (c) Section 4.3. Note that all RD-labelled travelling waves are defined for $z \in \mathbb{R}$, while MB travelling waves are defined for $z \leq 0$. All smooth RD travelling waves are translated in z so that $z = 0$ corresponds to the inflection point of the travelling wave. The sharp-fronted RD travelling wave in (b) is identically zero for $z > z_0 = -1/2$. The travelling wave in (c) with $r = 1$ corresponds to the FKPP explicit solution described in [4].

5. Conclusion

In this work, we report new exact forms of smooth and sharp-fronted travelling wave solutions of reaction–diffusion equations that employ a Weak Allee effect. Motivated by polynomial solutions of the heteroclinic trajectory in phase plane space, we examine power series solutions of the travelling wave differential equation system. For a variety of Weak Allee effect models, we determine the necessary wavespeed conditions for an explicit heteroclinic trajectory solution, resulting in exact sharp-fronted and smooth travelling waves for reaction–diffusion equations on infinite domains and moving-boundary domains (Fig. 1).

This focus on explicit travelling wave solutions by means of an explicit heteroclinic trajectory provides exciting future avenues of travelling wave analysis. As shown in one particular family of Weak Allee effect models, the explicit heteroclinic trajectory need not be a polynomial to obtain an explicit travelling wave solution. Many reaction–diffusion model choices can result in other explicit solutions to the associated heteroclinic trajectory, which is a solution to an Abel equation of the second kind (c.f. [18]). Therefore, we anticipate that an even larger family of reaction–diffusion model choices can present explicit travelling wave solutions for specific wavespeed choices.

Data availability

No data was used for the research described in the article.

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