# Optimal Controller Designs for Rotating Machines - Penalising the Rate of Change of Control Forcing

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**ABSTRACT**

In the context of active control of rotating machines, standard optimal controller methods enable a trade-off to be made between (weighted) mean-square vibrations and (weighted) mean-square currents injected into magnetic bearings. One shortcoming of such controllers is that no concern is devoted to the voltages required. In practice, the voltage available imposes a strict limitation on the maximum possible rate of change of control force (force slew rate). This paper removes the aforementioned existing shortcomings of traditional optimal control.

**KEY WORDS**

Optimal control, Magnetic bearings, Force slew rate, Active control

## 1 INTRODUCTION

Consider the negative feedback control of a plant whose equations of motion are represented by

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{1}$$

where *x(t)* is the *2n x 1* vector state of the system, *u(t)* is the *r x 1* control vector applied to the system and *A* and *B* are the companion and control matrices respectively. The dot above the *x* denotes derivative with respect to time.

A feedback gain, *G*, can be calculated such that more desirable system properties can be found. This can be represented pictorially in figure 1.

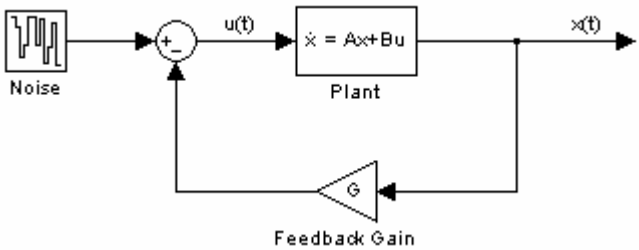


**Figure 1:** Pictorial representation of control system

One such approach to determine the feedback gains matrix is to utilise optimal control. Optimal control is best summarised by calculating an optimal feedback force which minimises a quadratic equation defined by

$$J = \frac{1}{2}\int_0^{\infty} x^T(t)Q\,x(t) + u^T(t)R\,u(t)\;\;dt \tag{2}$$

where *Q* represents a symmetric semi-definite weighting matrix governing the relative importance of the system state at time, *t*, and similarly *R* represents a symmetric positive definite matrix to weight the control effort.

The quadratic expression illustrated by equation (2) has threefold implication

- The quadratic nature of the expression ensures that
  - the positive and negative errors are weighted equally.
  - the larger errors are penalised more harshly than smaller errors.
- The integral penalises the more persistent error more harshly.

The standard approach to solve the optimal control problem is well understood and a great deal of literature is available for the problem. The standard procedure is to solve the Riccati equation [1] but methods exist to calculate the optimal control for second order systems [2] without the need to deal with the state space approach. The optimal control method is frequently referred to as the *Linear Quadratic Regulator* (LQR) control.

A major drawback of the traditional approach to optimal control is that no emphasis is placed on the rate at which the control effort can be applied. Control forces cannot be instantaneously applied and indeed several applications exist where the rate at which control forces can be applied is sufficiently important to warrant this work. One such area is in the field of electromagnetic bearings.

Consider magnetic bearings as a representative contemporary example of a control actuator for a dynamic system [3]. It is usual to operate these bearings with a bias current such that the net force produced by the bearing in a direction is linearly proportional to the control currents injected into the bearing. The maximum force achievable by the bearing is dependent on the maximum control currents which can be injected and could be identified as control input, *u(t)*. The role of conventional optimal control in trying to keep *u(t)* small is obvious here. Large currents would require thick conductors in the bearing and a higher current-rating in the power-amplifiers.

However, the maximum rate of change of force in a magnetic bearing is dependent on the rate of change of current. All magnetic bearings have some inductance thus a finite rate of change of current requires a finite voltage on top of the voltage required to drive a steady current. In many practical applications, the voltages associated with the rates of change of current are many times greater than the steady "IR" voltages. If the controller requires the magnetic bearing to produce very high rates of change of force then the power-amplifiers will require large internal voltages and the insulation between coils in the magnetic bearing will have to be large. Hence, for magnetic bearings, it is actually highly desirable to be able to develop optimal controllers which minimize some cost function that is determined by both control input and rate of control input.

Thus conventional optimal control may not provide adequate controller design. In section 2 an augmented system is introduced which allows the introduction of the rate of control input into the quadratic cost function and the optimal control policy is developed using the Euler-Lagrange equations in sections 3 and 4. Sections 5 and 6 present numerical examples to support this work.

## 2 A VARIANT TO TRADITIONAL OPTIMAL CONTROL

The purpose of this paper is take account of the rate of control forces applied to a system by acknowledging that forces cannot be instantaneously applied. The aim is to design practical controllers to overcome this consequence by determining the optimal feedback gain. Thus equation (2) can be altered to take account of this new condition.

$$J_{aug} = \frac{1}{2}\int_0^{\infty} x^T(t)Q\,x(t) + u^T(t)R\,u(t) + v^T(t)S\,v(t)\;dt \tag{3}$$

Here *v(t)* represents the first derivative of the control force with respect to time and *S* represents a symmetric positive definite weighting matrix of appropriate dimension to penalize the control rate.

The approach pursued in this paper is to rethink the pictorial representation of the system illustrated by figure 1. Suppose that an augmented plant can be constructed such that the input to this augmented plant is the rate of control rather than the normal control vector. A feedback gains matrix could be calculated to find the optimal rate of control force applied to the system. This concept is represented pictorially in figure 2.

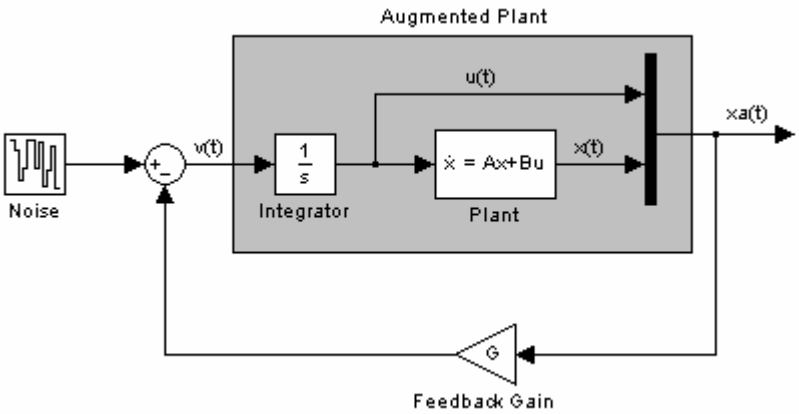


**Figure 2:** Augmented plant

As illustrated above an augmented state $x_a(t)$ can be formed which includes the control vector. This can be defined as,

$$x_a(t) = \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \tag{4}$$

Thus the augmented plant can be shown to have the following equations of motion

$$\begin{bmatrix} \dot{u}(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0^{r\times r} & 0^{r\times 2n} \\ B & A \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} I^{r\times r} \\ 0^{2n\times r} \end{bmatrix} v(t) \tag{5}$$

$$\dot{x}_a(t) = \tilde{A}\, x_a(t) + \tilde{B}\, v(t) \tag{6}$$

The quadratic cost illustrated by equation (2) can be rewritten in terms of the new augmented state equations such that it is equivalent to the cost function illustrated by equation (3)

$$J_{aug} = \frac{1}{2}\int_0^\infty x_a^{\,T}(t) Q_a\, x_a(t) + v^T(t) S\, v(t)\ dt \tag{7}$$

Here the new definition of $Q_a$ can be seen clearly to be,

$$Q_a = \begin{bmatrix} R & 0^{r\times 2n} \\ 0^{2n\times r} & Q \end{bmatrix} \tag{8}$$

## 3 CALCULATING THE OPTIMAL CONTROL GAIN

Equation (6) represents the dynamics of a first order system subjected to the constraint that the cost function given by equation (7) must be minimal over an infinite time horizon. This situation represents a constrained variational problem. In order to solve the optimal control problem the system must first be converted into an unconstrained control problem using the introduction of the co-state vector, $\lambda(t) \in \Re^n$. For the sake of brevity the time notation has been removed from here onwards. The function $H$ may be defined,

$$H(x_a, u, \lambda) = \frac{1}{2} x_a^{\,T} Q_a x_a + \frac{1}{2} v^T S v + \lambda^T \left( \tilde{A} x_a + \tilde{B} v - \dot{x}_a \right) \tag{9}$$

and

$$J_1 = \int_0^\infty H(x_a, u, \lambda)\ dt \tag{10}$$

In order for $J_1$ to be stationary the necessary conditions are given by the Euler-Lagrange equations [4].

$$\frac{dH}{dx_a} - \frac{d}{dt}\left( \frac{dH}{d\dot{x}_a} \right) = 0 \tag{11}$$

$$\frac{dH}{dv} = 0 \tag{12}$$

Substituting the definition of $H$ into equations (11) and (12) yield the results

$$\dot{\lambda} = -Q_a x_a - \tilde{A}^T \lambda \tag{13}$$

$$v = -S^{-1} \tilde{B}^T \lambda \tag{14}$$

It is worth reminding the reader here that the matrices $Q_a$ and $S$ are symmetric hence equations (13) and (14) are simplified further by recognizing that $Q_a$ and $S$ equal their own transposes.

Equation (14) is substituted into the equations of motion represented by equation (6) and then combined with equation (13) to form the Hamiltonian system.

$$\begin{bmatrix} \dot{x}_a \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{A} & -\tilde{B}S^{-1}\tilde{B}^T \\ -Q_a & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} x_a \\ \lambda \end{bmatrix} = L \begin{bmatrix} x_a \\ \lambda \end{bmatrix} \tag{15}$$

The matrix $L$ is referred to as the Hamiltonian in this case and contains the necessary conditions for the control to minimize the quadratic cost function [5].

The co-state vector $\lambda$ is related to the augmented state $x_a$ of the system through [5]

$$\lambda = P\, x_a \tag{16}$$

This paper deals with the specific infinite horizon problem where the end condition is assumed to be equal to zero so that the final conditions may be $\lambda(\infty) = \underline{0}$. It is relatively simple to subject the system to a final settling state but this is not addressed in this paper. Therefore knowing the initial and final states of the system it is possible to solve the Hamiltonian by utilizing equation (16).

## 4 THE RICCATI EQUATION

A more robust method of finding the optimal control is to form the Riccati equation which may be solved backwards through time to give the matrix *P(t)*. The relationship between the co-state vector and control vector can be combined to yield the feedback control,

$$v = -S^{-1}\tilde{B}^T \lambda = -S^{-1}\tilde{B}^T P\, x_a = -G\, x_a \tag{17}$$

The Riccati equation may be formed by differentiating equation (16)

$$\dot{\lambda} = \dot{P}\, x_a + P\, \dot{x}_a \tag{18}$$

Substituting in equation (13) for $\dot{\lambda}$ and the equations of motion for $\dot{x}_a$ yields

$$\left(\dot{P} + P\tilde{A} + \tilde{A}^T P + Q - P\tilde{B}R^{-1}\tilde{B}^T P\right)x_a = 0 \tag{19}$$

Equation (19) holds for any arbitrary state $x_a$ starting with known initial conditions. Therefore the dependence of the state $x_a$ can be removed which implies $P$ must satisfy

$$\dot{P} = -P\tilde{A} - \tilde{A}^T P - Q + P\tilde{B}R^{-1}\tilde{B}^T P \tag{20}$$

This result is known as the Riccati equation and may be solved backwards through time [1] knowing the initial and final conditions for $P$.

## 5 NUMERICAL EXAMPLE 1

A spring-mass system with no damping is constructed such that equations of motion are governed by (*q(t)* represents a vector of displacements)

$$M\ddot{q}(t) + Kq(t) = Fu(t)$$

with matrices

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

subjected to initial conditions $\quad q(0) = [1 \quad -1 \quad 0]^T, \; \dot{q}(0) = [0 \quad 0 \quad 0]^T$

The second order system can be converted into first order state space form such that

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}$$

The optimal state weighting matrices are chosen to minimize the kinetic and potential energies of the system. The control and rate of control vectors are chosen arbitrarily

$$Q = \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}, \; R = 10^{-1} I, \; S = 10^{2} I$$

where $I$ is the identity matrix of appropriate dimension.

The augmented plant is constructed utilizing the form shown by equation (5) and the new weighting matrices satisfy equation (7). By standard optimal control approach the optimal controller gain for the augmented system is found to be,

$$G_{aug} = \begin{bmatrix} 0.2396 & 0.1132 & -0.2171 & 0.0512 & 0.0220 & 0.0346 & 0.0270 & 0.0142 \\ 0.1132 & 0.3335 & 0.0543 & -0.1938 & 0.0692 & 0.0378 & 0.0615 & 0.0291 \end{bmatrix}$$

The response to the initial conditions is illustrated in figure 3. This gives the quadratic cost, $J_{auq} = 161$.

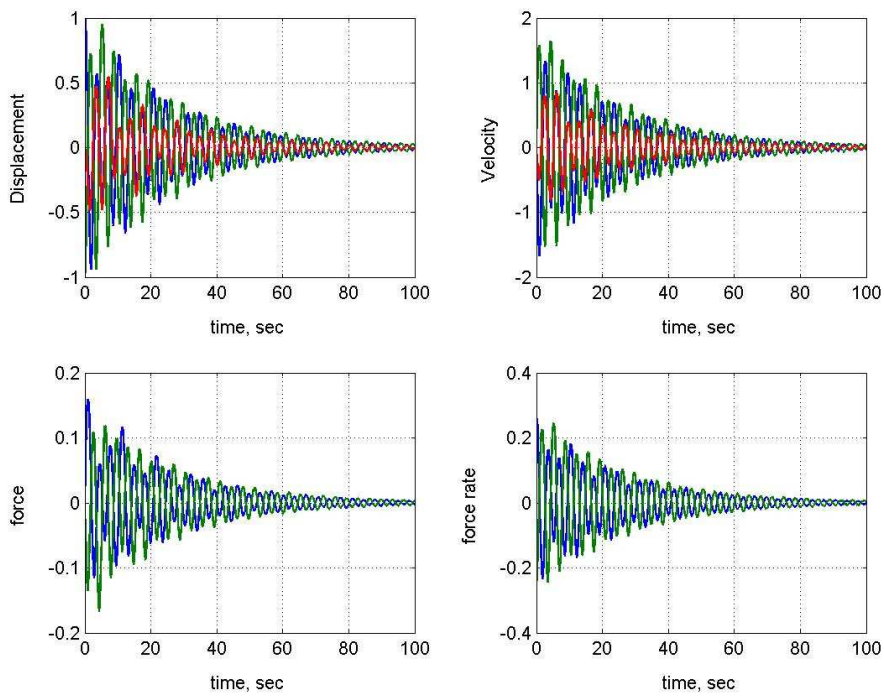


**Figure 3:** Optimal augmented system response to initial conditions

Traditional optimal control yields the controller applied to the original system illustrated by equation (1)

$$G_{lqr} = \begin{bmatrix} 2.8722 & -0.2698 & -0.7495 & 3.9676 & -0.0467 & -0.1413 \\ -0.1232 & 4.8708 & -3.0562 & -0.0467 & 4.4429 & 1.9914 \end{bmatrix}$$

Subjecting the traditional optimal controller to the same quadratic cost function illustrated by equation (7) yields the cost $J_{lqr} = 24{,}835$. The response is illustrated in figure 4. It is worth noting the necessary difference in scale for the forces and force rates in figures 3 and 4.

As apparent from figures 3 and 4, the LQR approach provides no weighting to the rate at which the force is applied so the force is applied more quickly bringing the state of the system under control much quicker than that of the augmented plant method. But examination of the costs defined from equation (7) of the two systems alone illustrates the expense of doing so. The LQR cost is approximately 154 times larger than that of the augmented system.

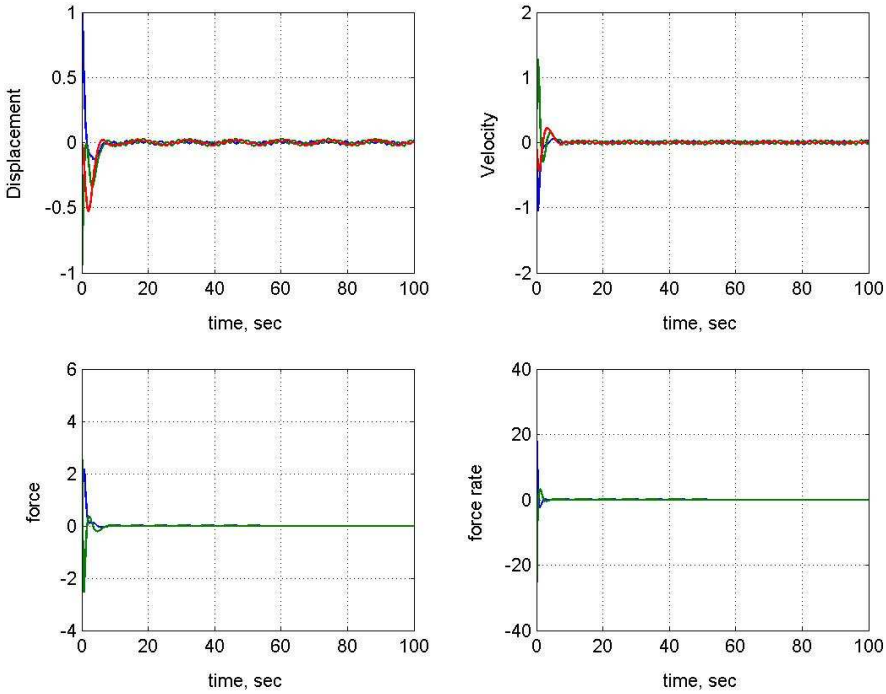


**Figure 4:** LQR system response to initial conditions

## 6 NUMERICAL EXAMPLE 2

A rotor dynamic system illustrated by figure 5 is modelled using finite elements with 4 degrees of freedom at each nodal point representing twist and displacement coordinates. The system model is reduced in size using Guyan reduction [6] to a smaller dimension and the optimal control system as outlined in this document applied. All dimensions marked on the figure are in millimeters (mm) and each element is 10mm in length. The steel shaft is of diameter 30mm and the diameters of the aluminimum discs illustrated at points Out 1 and Out 2 are of diameter 150mm. Bearings 1 and 2 constrain the system such that displacements at these points is zero.

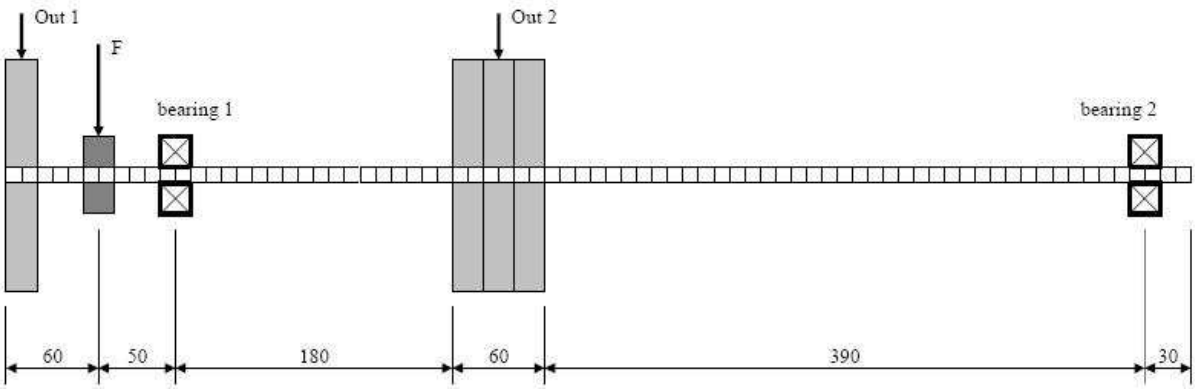

**Figure 5:** Rotor Dynamic Model

The initial conditions are such that the rotor system has been dropped yielding an initial velocity for the entire system being equal to 10 m/s. The control forces are applied at the location of the arrow as indicated. The output displacements are calculated at the centre of the two discs.

The weighting matrices are chosen to penalise the displacements of the system at the locations of the discs more harshly than other locations. Thus the Q-matrix is set equal to the identity matrix except for nodes 1, 2, 31, 32, 33, 34, 35 and 36, corresponding to the disc locations, and are weighted such that they are equal to 100. The weighting on the force and force rate are given values $10^2$I and $10^{-2}$I, respectively, where *I* is the identity matrix of appropriate dimension.

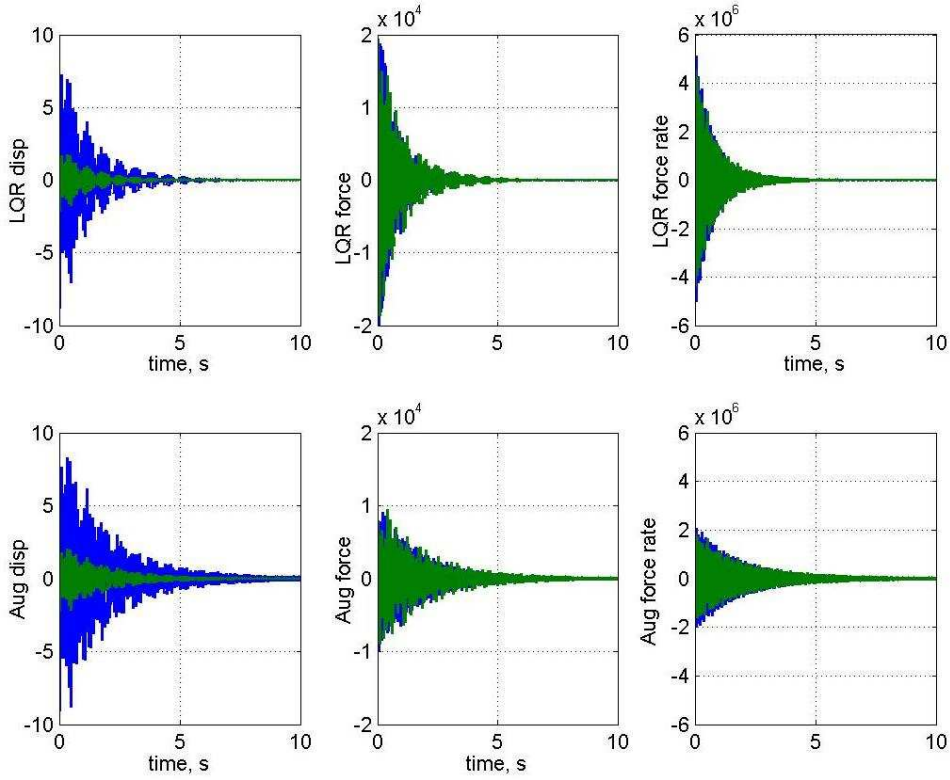


**Figure 6:** Rotor Dynamic Model Response to Control

The quadratic costs for the augmented and LQR control approaches are $4.4580x10^{10}$ and $6.4463x10^{10}$ respectively. This means that the augmented approach represents a sizeable reduction, approximately 30%, in cost compared with the LQR problem. Much of this reduction in cost is due to the peak control rate for the augmented system being approximately 40% of the peak control rate for the LQR system as illustrated in figure 6. This has immediate relevence to the magnitude of the force applied to the system resulting in a sizeable reduction in the magnitude of peak force again contributing to the reduction in quadratic cost.

## 7 CONCLUSION

In this paper an extension to the optimal control problem has been presented in which the rate of control has been incorporated. Numerical examples have been presented and compared with the traditional LQR approach.

For the first numerical example presented it could be argued that the weighting placed on the force rate is substantially larger than the weighting placed on the force so the LQR approach is immediately disadvantaged due to no inclusion of the rate to the control problem. This is precisely the key message that the authors are trying to present in this paper because there exist applications where the weighting on the rate at which the force can be applied will be much higher than weighting placed on the force itself. The second numerical example illustrates that the method yields itself to practical situations and again appropriately penalises the force rate as required.

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