Modal Control of Vibration in Rotating Machines and Other Generally Damped Systems

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Abstract

Second order matrix equations arise in the description of real dynamical systems. Traditional modal control approaches utilise the eigenvectors of the undamped system to diagonalise the system matrices. A regrettable consequence of this approach is the discarding of residual off-diagonal terms in the modal damping matrix. This has particular importance for systems containing skew-symmetry in the damping matrix which is entirely discarded in the modal damping matrix. In this paper a method to utilise modal control using the decoupled second order matrix equations involving non-classical damping is proposed. An example of modal control sucessfully applied to a rotating system is presented in which the system damping matrix contains skew-symmetric components.

Keywords: modal control, second order systems, general damping, non-proportional damping, rotordynamics

1 Introduction

Traditional control approaches, such as pole placement methods [1], deal with the physical system in first order state space form. The ambitions of this paper are to control the physical system in second order form. Very little literature is available in regards to direct second order control, see for example [2]. Many obvious advantages over first order control are available: 1.) Physical insight of the system is preserved. 2.) Computational efficiency, since the dimension of the second order system is smaller than that of the state space form. 3.) Symmetry and structure of the systems can be preserved where desired.

Many structural and dynamic systems are described by the second order equations of motion

$$\mathbf{M}_0\,\ddot{\mathbf{q}}(t) + \mathbf{D}_0\,\dot{\mathbf{q}}(t) + \mathbf{K}_0\,\mathbf{q}(t) = \mathbf{f}_{phy}(t) \quad \cdot \tag{1}$$

where $\mathbf{M}_0, \mathbf{D}_0, \mathbf{K}_0 \in \mathbb{R}^{n\times n}$ are the system mass, damping and stiffness matrices respectively, $\mathbf{q}(t) \in \mathbb{R}^n$ the vector of physical coordinates and $\mathbf{f}_{phy}(t) \in \mathbb{R}^r$ the vector of applied forces. For the sake of brevity this paper assumes that forces are available at all locations and as a consequence $r = n$.

Modal control is a particular control method in which the physical response of a system is divided into modes associated with their corresponding natural frequencies. A standard control approach is to move the natural frequencies into a stable region. The essence of modal control is that since the eigenvectors of a system do not contribute to the asympotic stability of a system then any effort expended on altering them represents wasted effort. This is the control approach utilised in this paper.

Traditional modal control for second order systems such as the *Independent Modal Space Control* (IMSC) method [3] proposed by Meirovitch and Baruh utilise the mass normalised left and right eigenvectors, $\mathbf{\Phi}_L$ and $\mathbf{\Phi}_R$, to diagonalise the system matrices. The coordinate transformation $\mathbf{q}(t) = \mathbf{\Phi}_R\,\mathbf{q}_m(t)$ is applied and the system matrices pre-multiplied by the transpose of the left eigenvectors, ${\mathbf{\Phi}_L}^T$

From

$$ {\mathbf{\Phi}_L}^T\,\mathbf{M}_0\,\mathbf{\Phi}_R\,\ddot{\mathbf{q}}_m + {\mathbf{\Phi}_L}^T\,\mathbf{D}_0\,\mathbf{\Phi}_R\,\dot{\mathbf{q}}_m + {\mathbf{\Phi}_L}^T\,\mathbf{K}_0\,\mathbf{\Phi}_R\,\mathbf{q}_m = {\mathbf{\Phi}_L}^T\,\mathbf{f}_{phy} \quad \cdot \tag{2} $$

one has

$$ \mathbf{I}\,\ddot{\mathbf{q}}_m + \mathbf{\Gamma}\,\dot{\mathbf{q}}_m + \mathbf{\Lambda}^2\,\mathbf{q}_m = {\mathbf{\Phi}_L}^T\,\mathbf{f}_{phy} \quad \cdot \tag{3} $$

where $\mathbf{q}_m(t)$ represents the modal coordinates of the system. For ease of reading the time script has been removed.

The new damping matrix $\mathbf{\Gamma}$ is assumed diagonal with any remaining off-diagonal terms in the modal damping matrix traditionally discarded [4]. However, for rotating systems involving substantial gyroscopic terms ignoring these terms is in effect ignoring the gyroscopic terms themselves. Thus, it is proposed here to use the *Structure Preserving Transformations* (SPTs) developed by Garvey *et al* [5, 6] to diagonalise the second order system matrices and decouple the system equations of motion without need to discard any terms involved in the description of the system.

2 Structural Preserving Transformations

The notion of the *Lancaster Augmented Matrices* (LAMs) are introduced here such that the system may be represented in state space form. For a second order system there exists three LAMs which can be produced by inspection to be,

$$ \underline{\mathbf{A}_0} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{D}_0 \end{bmatrix} \quad , \ \underline{\mathbf{A}_1} = \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_0 \end{bmatrix} \quad , \ \underline{\mathbf{A}_2} = \begin{bmatrix} -\mathbf{D}_0 & -\mathbf{M}_0 \\ -\mathbf{M}_0 & \mathbf{0} \end{bmatrix} \quad \cdot \tag{4} $$

The LAMs allow the second order system to be represented in a reduced form

$$ \underline{\mathbf{A}_{k-1}}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_k}\,\underline{\dot{\mathbf{q}_A}} = \underline{\mathbf{f}_{A(3-k)}} \qquad k = 1, 2 \quad \cdot \tag{5} $$

The vectors $\underline{\mathbf{q}_A}$ and $\underline{\mathbf{f}_{A(3-k)}}$ may be defined

$$ \underline{\mathbf{q}_A} := \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \quad \underline{\mathbf{f}_{A1}} := \begin{bmatrix} \mathbf{f}_{phy} \\ \mathbf{0} \end{bmatrix} \quad \underline{\mathbf{f}_{A2}} := \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{phy} \end{bmatrix} \quad \cdot \tag{6} $$

A *Structural Preserving Transformation* (SPT) is a coordinate transformation applied to the LAMs representing a bijective mapping between linear systems. The specific nature of the transformation allows the preservation of the appropriate structure within the LAMs. The SPTs are defined simply by left and right $2n \times 2n$ transformation matrices, $\underline{\mathbf{T}_L}$ and $\underline{\mathbf{T}_R}$ respectively, allowing the definition

$$\underline{\mathbf{T}_L^T}\,\underline{\mathbf{A}_k}\,\underline{\mathbf{T}_R} = \underline{\mathbf{B}_k} \quad \forall \quad k = 0, 1, 2 \;\cdot \tag{7}$$

Thus the new LAMs are represented by $\underline{\mathbf{B}_k}$ containing the new second order system matrices $\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1$. The structure of the SPTs can be shown to have the following form

$$\underline{\mathbf{T}_L} = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_0^T & -\mathbf{G}_L\,\mathbf{M}_0^T \\ \mathbf{G}_L\,\mathbf{K}_0^T & \mathbf{F}_L + \frac{1}{2}\,\mathbf{G}_L\,\mathbf{D}_0^T \end{bmatrix}^{-1} \quad \underline{\mathbf{T}_R} = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_0 & -\mathbf{G}_R\,\mathbf{M}_0 \\ \mathbf{G}_R\,\mathbf{K}_0 & \mathbf{F}_R + \frac{1}{2}\,\mathbf{G}_R\,\mathbf{D}_0 \end{bmatrix}^{-1} \cdot \tag{8}$$

where $\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R \in \mathbb{R}^{n\times n}$ are arbitrary pre-defined matrices subject to the necessary constraint

$$\mathbf{F}_R\,\mathbf{G}_L^T + \mathbf{G}_R\,\mathbf{F}_L^T = 0 \;\cdot \tag{9}$$

The SPTs can be shown to yield the relationship between the old and new coordinate sets through the use of filters. The modal system $(\mathbf{q}_m, \mathbf{f}_{mod})$ is related to the original system through the relationship

$$\mathbf{q}_m = \mathbf{U}_0\,\mathbf{q} + \mathbf{U}_1\,\dot{\mathbf{q}} \tag{10}$$

$$\dot{\mathbf{q}}_m = \mathbf{U}_0\,\dot{\mathbf{q}} + \mathbf{U}_1\,\ddot{\mathbf{q}} \tag{11}$$

$$\mathbf{f}_{mod} = \mathbf{V}_0\,\mathbf{f}_{phy} + \mathbf{V}_1\,\dot{\mathbf{f}}_{phy} \tag{12}$$

where

$$\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_R}^{-1} \tag{13}$$

$$\begin{bmatrix} \mathbf{V}_0^T & \mathbf{V}_1^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \underline{\mathbf{T}_L} \tag{14}$$

Evidently knowledge of the physical accelerations is required.

3 Diagonalising Structural Preserving Transformations

We wish to decouple the original equations of motion such that the new system matrices $\mathbf{K}_1, \mathbf{D}_1$ and $\mathbf{M}_1$ are diagonal. It is possible to choose a non-unique SPT such that the entries in the new LAMs become diagonal. Such an SPT is referred to as a *diagonalising SPT* (DSPT) and a 4 step process of calculating the DSPT is presented here.

1. Calculate the left $(\underline{\mathbf{\Psi}_L})$ and right $(\underline{\mathbf{\Psi}_R})$ eigenvectors of reduced system

$$\underline{\mathbf{A}_0}\,\underline{\mathbf{q}_A} - \underline{\mathbf{A}_1}\,\underline{\dot{\mathbf{q}_A}} \;\cdot$$

2. Calculate the n *single degree of freedom* (SDOF) systems corresponding to conjugate eigenvalue pairs, $\lambda_{j(1,2)} = \alpha \pm i\beta$, found in part 1.

$$d_j = \lambda_{j1} + \lambda_{j2} \quad , \quad k_j = \frac{(\lambda_{j2} + \lambda_{j1})^2 - (\lambda_{j2} - \lambda_{j1})^2}{4} \quad , \quad m_j = 1 \quad \cdot \tag{15}$$

$j = 1, \cdots, n.$

3. Knowing the new diagonal system matrices form the new LAMs $\underline{\mathbf{B}_0}$ and $\underline{\mathbf{B}_1}$ representing the new diagonal system and calculate their corresponding left ($\underline{\mathbf{\Theta}_L}$) and right ($\underline{\mathbf{\Theta}_R}$) eigenvectors.

4. Since the two reduced systems have identical Jordan form appropriate scaling of the eigenvectors yields the following equality

$$\underline{\mathbf{\Psi}_L^T}\,\underline{\mathbf{A}_0}\,\underline{\mathbf{\Psi}_R} = \underline{\mathbf{\Lambda}} = \underline{\mathbf{\Theta}_L^T}\,\underline{\mathbf{B}_0}\,\underline{\mathbf{\Theta}_R} \qquad \underline{\mathbf{\Psi}_L^T}\,\underline{\mathbf{A}_1}\,\underline{\mathbf{\Psi}_R} = \underline{\mathbf{I}} = \underline{\mathbf{\Theta}_L^T}\,\underline{\mathbf{B}_1}\,\underline{\mathbf{\Theta}_R} \quad \cdot \tag{16}$$

where $\underline{\mathbf{\Lambda}}$ is the diagonal matrix of corresponding eigenvalues and $\underline{\mathbf{I}}$ is the identity matrix. Thus we may recognise that to get from the original LAM to the new LAM the following condition must be satisfied

$$\left(\underline{\mathbf{\Theta}_L}^{-T}\,\underline{\mathbf{\Psi}_L}^T\right)\,\underline{\mathbf{A}_0}\,\left(\underline{\mathbf{\Psi}_R}\,\underline{\mathbf{\Theta}_R}^{-1}\right) = \underline{\mathbf{B}_0} \quad \cdot \tag{17}$$

thus $\underline{\mathbf{T}_R} = \left(\underline{\mathbf{\Psi}_R}\,\underline{\mathbf{\Theta}_R}^{-1}\right)$ and $\underline{\mathbf{T}_L} = \left(\underline{\mathbf{\Psi}_L}\,\underline{\mathbf{\Theta}_L}^{-1}\right)$.

It may be noted that the above process for finding the diagonalising SPT only requires one eigenvalue solution problem. The eigenvectors of the diagonal LAMs, $\underline{\mathbf{\Theta}_L}$ and $\underline{\mathbf{\Theta}_R}$, have a sparse form such that their calculation is trivial.

4 Independent Modal Control

To facilitate true independent modal control the modal equations of motion must be decoupled both externally and internally [7]. We have so far shown how to decouple the unforced equations of motion but the diagonalised system matrices remain coupled by the control forces unless the controller is designed independently such that the controller matrix remains decoupled. In practice this means that the force controller must be designed in the modal space. We can thus define the modal equations of motion as

$$\mathbf{M}_1\,\ddot{\mathbf{q}}_m + \mathbf{D}_1\,\dot{\mathbf{q}}_m + \mathbf{K}_1\,\mathbf{q}_m = \mathbf{f}_{mod} \quad \cdot \tag{18}$$

with $\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1 \in \mathbb{R}^{n\times n}$ the diagonal modal system matrices and $\mathbf{q}_m \in \mathbb{R}^n$ the modal coordinates.

Equation (18) represents n single degree of freedom (SDOF) systems corresponding to each mode of vibration. It is possible to use proportional-derivative control to directly affect the modal stiffness and damping properties of these modes. A controller of this form is introduced

$$\mathbf{f}_{mod} = \mathbf{G}_k\, \mathbf{q}_m + \mathbf{G}_d\, \dot{\mathbf{q}}_m \quad . \tag{19}$$

$\mathbf{G}_k$ and $\mathbf{G}_d$ represent the diagonal modal stiffness and damping gains matrices. Direct additon to the modal damping and stiffness matrices represents direct pole placement and has the advantage of being able to directly affect the poles of the system.

In general as many modes can be controlled as actuators available. As previously stated for the purpose of this paper the number of actuators is set to the number of modelled modes without loss of generality. For conventional second order control the modal force can be typically converted back into the physical domain fairly easily as illustrated by Baz and Poh [8]. For the SPT approach we have already defined the left filter and can see that the physical and modal forces are related by the relationship

$$\mathbf{f}_{mod} = \mathbf{V}_0\, \mathbf{f}_{phy} + \mathbf{V}_1\, \dot{\mathbf{f}}_{phy} \quad . \tag{20}$$

We can rearrange equation (20) to give the physical force in regards to the modal force

$$\dot{\mathbf{f}}_{phy} = \mathbf{V}_1^{-1}\, \left(\mathbf{f}_{mod} - \mathbf{V}_0\, \mathbf{f}_{phy}\right) \quad . \tag{21}$$

Since the modal filter illustrated by equation (21) represents a first order filter a necessary requirement is for the real eigenvalue components of $\mathbf{V}_1^{-1}\, \mathbf{V}_0 > 0$ for the filter to be stable.

5 Numerical Example

As a numerical example a finite element model of a rotor-disc system is considered with four degrees of freedom at each node (2 translational, 2 torsional). The rotor-disc system is illustrated in figure 1.

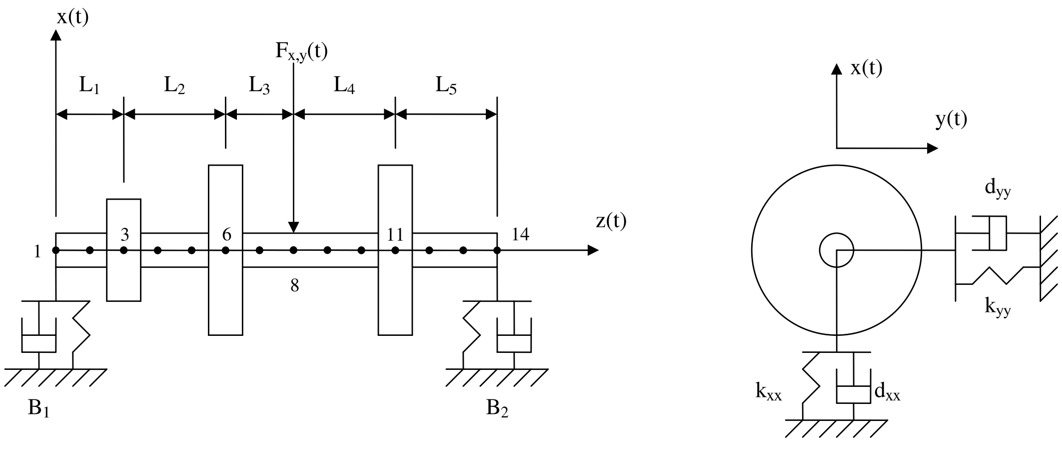

Figure 1: Example 1, Rotor-Disc system

The system is constructed from steel with Young's modulus, $E = 200$ GPa and density $\rho = 7800$ kg/m^3. The model is split into 13 equal-length elements of 0.1m and the discs have dimensions

Disc	Disc 1	Disc 2	Disc 3
Node	3	6	11
Thickness (m)	0.05	0.05	0.06
Inner diameter (m)	0.10	0.10	0.10
Outer diameter (m)	0.24	0.40	0.40

The bearings at each end of the rotor system are deliberately non-symmetric in the x-y directions with stiffness and damping properties

Bearing	Bearing 1	Bearing 2
Stiffness K_{xx} (MN/m)	50	50
Stiffness K_{yy} (MN/m)	70	70
Stiffness D_{xx} (N/m/s)	500	500
Stiffness D_{yy} (N/m/s)	700	700

Control forces can be applied at node 8 in the x and y-directions and similarly the displacements in the x-direction at this node are observed. For computational ease guyan reduction [9] is used to reduce the model to 6 degrees of freedom. The system is operated at 2,500 rpm.

As many modes can be controlled as actuators are available, thus the model allows for 2 modes to be controlled. It is decided to control the first two modes of vibration since these dominate the system response.

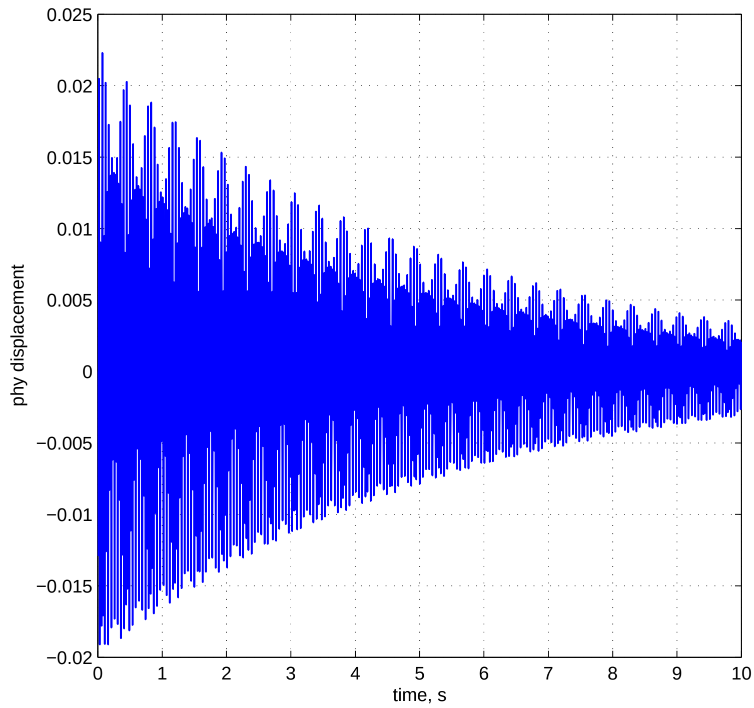

Figure 2: Example 1 SPT response to initial conditions: control off

The single degree of freedom systems corresponding to the first two modes in modal space are

$$\ddot{\mathbf{q}}_{m1} + 0.37850\,\dot{\mathbf{q}}_{m1} + 1.4467 \times 10^5\,\mathbf{q}_{m1} = \mathbf{f}_{m1} \tag{22}$$

$$\ddot{\mathbf{q}}_{m2} + 0.32708\,\dot{\mathbf{q}}_{m2} + 1.5772 \times 10^5\,\mathbf{q}_{m2} = \mathbf{f}_{m2} \tag{23}$$

Optimal control is used to minimise the modal kinetic and potential energies such that controller gains are

$$\mathbf{G}_k = \begin{bmatrix} 4.9999 & 0 & 0 & \cdots & 0 \\ 0 & 4.9999 & 0 & \cdots & 0 \end{bmatrix} \quad , \quad \mathbf{G}_d = \begin{bmatrix} 4.1096 & 0 & 0 & \cdots & 0 \\ 0 & 4.1570 & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

The response of the system with the controller on is illustrated in figure 3.

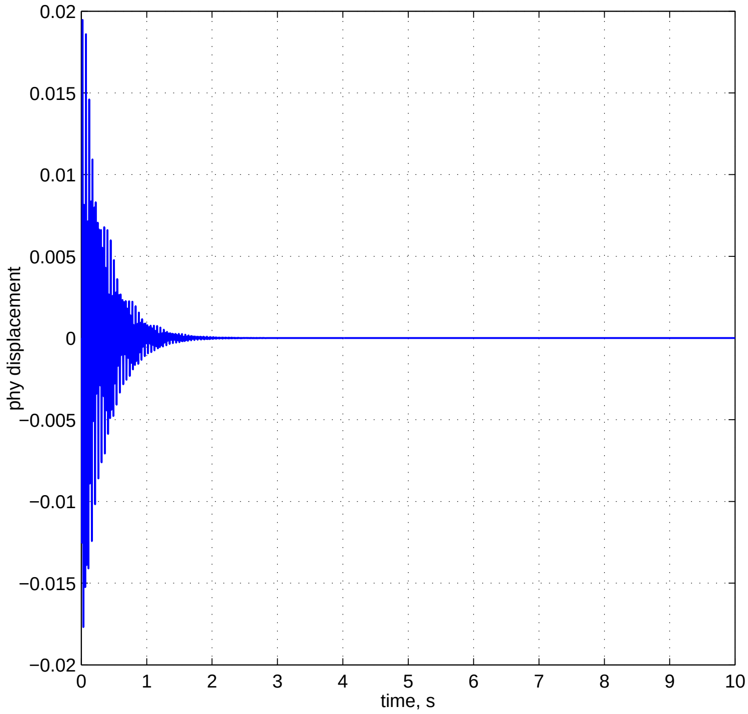

Figure 3: Example 1 SPT response to initial conditions: control on

As expected the response of the system decays much faster than that for the uncontrolled system with the displacement converging to zero much more rapidly. This is due to tageting the first two modes of vibration of the system which dominate the system response. The modal control technique is indeed sucessfully applied to bring the system under control.

6 Conclusions

It has been shown in this paper how to apply modal control to non-classically damped systems without throwing away system information. It has been demonstrated through examples that individual modes can be controlled.

The premise of this paper is to introduce possible new methods into the area of rotating machinery where skew-symmetry and gyroscopic coupling can be found in the system damping matrices. Conventional techniques maintain that skew-symmetry be ignored for the techniques to be usable.

Usually, systems require reduction in size due to numerical considerations. Traditional Guyan reduction models do not take into account damping properties. Alternative methods such as balanced truncation [4] traditionally place the system into state space form before reduction, thus destroying the second order properties of the system. Few methods have been developed to reduce the models in size for second order systems. Currently the methods for reducing second order systems appear to involve the use of state space form to balance the grammians [10]. It would thus be beneficial to develop second order model reduction methods that take into account damping whilst preserving second order form.

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