Extracting Second Order System Matrices From State Space System

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I. Introduction

A second order system can be represented in the form

$$M\,\ddot{q}(t)+D\,\dot{q}(t)+K\,q(t)=F\,u(t) \tag{1}$$

$$y(t)=P_1\,q(t)+P_2\,\dot{q}(t) \tag{2}$$

where $M, D, K \in \Re^{n\times n}$ are the system mass, damping and stiffness matrices, respectively, $q(t)\in\Re^{n}$ represents the vector of generalised coordinates of the system, $u(t)\in\Re^{r}$ is the vector of applied forcing and the $F\in\Re^{n\times r}$ and $P_1, P_2 \in \Re^{p\times n}$ matrices represent locations of applied forces and observations respectively. For the purpose of this paper we allow the concept of the output *y(t)* to be a collection of displacements and velocities. The justification for this statement is that we are concerned only with redeeming the structure of the second order system matrices *M*, *D* and *K*, within this paper.

It is well known that the structure of the second order linear system permits representation in first order state space form

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}D\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ M^{-1}F\end{bmatrix}u(t) \tag{3}$$

$$y(t)=\begin{bmatrix}P_1 & P_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{4}$$

with $x_1(t)=q(t)$ and $x_2(t)=\dot{x}_1(t)$. The first order equations of motion illustrated above may be simplified where the definitions in the equation are apparent

$$\dot{x}(t)=A\,x(t)+B\,u(t) \quad , \quad y(t)=C\,x(t) \tag{5}$$

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Many applications exist which destroy the second order properties contained in the state space representation. One such example is the model reduction technique *balanced truncation* [1]. The purpose of this note is to find a similarity transformation, *T*, such that the zeros and identity can be re-established into the appropriate locations of the $\widetilde{A}$ and $\widetilde{B}$ matrices.

$$\widetilde{A} = T^{-1} A T \quad , \quad \widetilde{B} = T^{-1} B \quad , \quad \widetilde{C} = C T \tag{6a, b, c}$$

As an illustrative example to justify the ambitions of this note the author wishes to draw attention to the control problem. Currently the majority of linear control methods deal with the physical system in first order form however there is a growing subset of control techniques dealing with the control problem in second order form. Many obvious advantages of second order control over first order control exist: 1.) Physical insight of the system is preserved. 2.) Computational efficiency, since the dimension of the second order system is smaller than that of the state space form. 3.) Symmetry and structure of the systems can be preserved where desired. Thus the technique outlined in this note help regain some of these advantages.

The author wishes to draw attention here to previous work on the same ambition by Friswell *et al* [2] but the method proposed in this work is far simpler in application. The ambition of this note is to yield a simpler method rather than produce a new result. This fact the author feels is a worthwhile criterion. Friswell *et al* showed that a necessary condition for a system to be truly second order is that the equality $CB = 0$ holds true. Indeed this equality holds in the event of transformation to the *C* and *B* matrices. We assume here without loss of generality that the new mass matrix M_{new} is equal to the identity matrix of appropriate dimension.

A coordinate transformation known as a *Structural Preserving Equivalence* (SPE) [3] is known to exist which yields the appropriate form of the state space companion matrix, *A*. However the transformation pays no attention to the structure of the state space input matrix, *B*. We show here that the SPE has the form

$$T = \begin{bmatrix} X \\ X A \end{bmatrix}^{-1} \tag{7}$$

with $X \in \Re^{n \times 2n}$ being an arbitrary full rank matrix.

To see that *T* does reproduce the correct form for $\widetilde{A}$, perform the substitutions implied by Eqs. (6a) and (7) and observe that

$$X\,A\begin{bmatrix} X \\ X\,A \end{bmatrix}^{-1} \equiv \begin{bmatrix} 0 & I \end{bmatrix} \tag{8}$$

Noting that to yield the new forcing matrix $\widetilde{B}$ we pre-multiply the original forcing matrix B by T^{-1} hence we can gain a direct relationship to the arbitrary matrix *X*. We may split matrices *A*, *B* and *X* into *n* dimensional blocks

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \quad , \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad , \quad B = \begin{bmatrix} B_1{}^T & B_2{}^T \end{bmatrix}^T \tag{9}$$

to yield the result

$$\widetilde{B} = T^{-1}B = \begin{bmatrix} \left(X_1 B_1 + X_2 B_2\right) \\ \left(X_1 A_{11} + X_2 A_{21}\right)B_1 + \left(X_1 A_{12} + X_2 A_{22}\right)B_2 \end{bmatrix} \tag{10}$$

QR factorization may be used to solve the problem of finding a matrix *X* which can provide the necessary structure in the forcing matrix $\widetilde{B}$. Using QR factorization we define the orthogonal matrix $Q \in \Re^{2n \times 2n}$ and upper triangular matrix $R \in \Re^{2n \times r}$ such that

$$B = Q\,R \tag{11}$$

The matrix *Q* may be separated into

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \tag{12}$$

where Q_1 has dimension $2n$ x r and Q_2 has dimension $2n$ x $(2n - r)$. Therefore the matrix *X* can be solved such that

$$X = Y\,Q_2{}^T \tag{13}$$

Any matrix *X* satisfying equation (13) where $Y \in \Re^{n \times (2n-r)}$ is an arbitrary full rank matrix will satisfy $X\,B = \underline{0}$. Thus, the formulation yields the second order equations of motion in the form illustrated by equation (1) from first order equations of no specific structure. The inputs and the outputs of the system have physical realization but the generalized coordinates do not. Due to the non-uniqueness of the matrix *Y* we may observe that the set of all possible second order systems may be reached provided the initial system is non-defective. Indeed the new system matrices extracted from $\widetilde{A}$ are non-unique and the possibility to obtain more desirable system matrices is apparent.

It is worth noting at this point for systems where the dimensions $n = r = p$, if equality $C\,B = \underline{0}$ holds X can be arbitrarily set equal to C which will ensure that $P_{2\ new}$ will be equal to zero thus ensuring a full second order structure. This can be seen from the equality $X\,B = \underline{0}$.

II. Numerical Example

The initial A, B and C matrices are arbitrarily generated to be

$$A = \begin{bmatrix} 73 & 42 & 45 & 69 & 63 & 79 \\ 69 & 86 & 41 & 65 & 73 & 92 \\ 35 & 49 & 90 & 98 & 38 & 84 \\ 17 & 82 & 1 & 55 & 1 & 37 \\ 16 & 46 & 30 & 40 & 42 & 62 \\ 19 & 46 & 5 & 20 & 75 & 73 \end{bmatrix}, \quad B = \begin{bmatrix} 19 & 93 \\ 90 & 34 \\ 57 & 66 \\ 63 & 39 \\ 23 & 63 \\ 55 & 70 \end{bmatrix}, \quad C = \begin{bmatrix} 40 & 59 \\ 41 & 57 \\ 66 & 72 \\ 84 & 51 \\ 37 & 78 \\ 43 & 49 \end{bmatrix}^T$$

Through QR factorization Q_2 correct to 4 decimal places is found to be

$$Q_2 = \begin{bmatrix} -0.3359 & -0.1218 & -0.4028 & -0.3705 \\ -0.2907 & -0.5077 & 0.0824 & -0.2458 \\ 0.8732 & -0.0676 & -0.1288 & -0.1358 \\ -0.1578 & 0.8468 & -0.0864 & -0.1562 \\ -0.0322 & 0.0568 & 0.8881 & -0.0482 \\ -0.1190 & -0.0507 & -0.1347 & 0.8701 \end{bmatrix}$$

Setting Y to be equal to an n x $(2n\text{-}r)$ identity matrix, the new second order matrices reported to 4 decimal places can be extracted from the new transformed state space matrices assuming the new mass, M_{new}, is equal to the identity.

$$K_{new} = \begin{bmatrix} 10364.8151 & 6472.4562 & -4122.6942 \\ -27244.9504 & -14969.2395 & 9799.2364 \\ -1306.7350 & -738.7757 & 384.9582 \end{bmatrix}, \quad D_{new} = \begin{bmatrix} -230.2870 & 1.2515 & 73.3022 \\ 490.4981 & -152.1700 & -65.8903 \\ 28.0318 & 0.0948 & -36.5430 \end{bmatrix}$$

$$F_{new} = \begin{bmatrix} 2718.0236 & 1153.6079 \\ -3547.7159 & -9542.6998 \\ 670.6649 & -922.2557 \end{bmatrix}, \quad P_{1new} = \begin{bmatrix} -295.6068 & -312.8549 \\ -198.5391 & -251.8346 \\ 149.9955 & 187.1044 \end{bmatrix}^T, \quad P_{2new} = \begin{bmatrix} 6.1189 & 6.3411 \\ -0.7732 & -1.1320 \\ -3.9578 & -4.7629 \end{bmatrix}^T$$

III. Conclusion

This note has shown a method to recreate the second order equations of motion of the form illustrated by equation (1) from a first order system which has no specific form. A numerical method has demonstrated the simplicity and effectiveness of the algorithm.

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References

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