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Second-Order Elliptic PDE with Discontinuous Boundary Data

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We shall consider the weak formulation of a linear elliptic model problem with discontinuous Dirichlet boundary conditions. Since such problems are typically not well-defined in the standard $H^1 - H^1$ setting, we will introduce a suitable saddle point formulation in terms of weighted Sobolev spaces. Furthermore, we will discuss the numerical solution of such problems. Specifically, we employ an hp–discontinuous Galerkin method and derive an L^2-norm a posteriori error estimate. Numerical experiments demonstrate the effectiveness of the proposed error indicator in both the h– and hp–version setting. Indeed, in the latter case exponential convergence of the error is attained as the mesh is adaptively refined.

Keywords: Second-order elliptic PDE, discontinuous Dirichlet boundary conditions, inf-sup condition, hp-discontinuous Galerkin FEM, L^2-norm a posteriori error analysis, exponential convergence.

1. Introduction

On a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with straight edges and $N \geqslant 1$ corners $C_1, C_2, \ldots, C_N$, we consider the linear diffusion-reaction problem

$$-\Delta u + cu = f \qquad \text{in } \Omega \tag{1.1}$$

$$u = g \qquad \text{on } \Gamma, \tag{1.2}$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω, $c \in L^\infty(\Omega)$ is a nonnegative function, $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$ is a possibly discontinuous function on Γ whose precise regularity will be specified later. Throughout the paper we shall use the following notation. For a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$) we denote by $L^2(D)$ the space of all square-integrable functions on D, with norm $\|\cdot\|_{0,D}$. Furthermore, for an integer $k \in \mathbb{N}_0$, we let $H^k(D)$ be the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,D}$. The space $\mathring{H}^1(\Omega)$ is defined as the subspace of $H^1(\Omega)$ consisting of functions with zero trace on $\partial\Omega$.

Several variational formulations for elliptic problems with discontinuous Dirichlet boundary conditions exist. We mention the *very weak formulation* which is to find a solution $u \in L^2(\Omega)$ such that

$$-\int_\Omega u\Delta v \,\mathrm{d}\boldsymbol{x} + \int_\Omega cuv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} - \int_\Gamma g \nabla v \cdot \mathbf{n} \,\mathrm{d}s$$

for any $v \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$, where $\mathbf{n}$ denotes the unit outward normal vector to the boundary Γ. It is based on twofold integration by parts of (1.1) and incorporates the Dirichlet boundary data in a natural

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way. On the other hand, however, the numerical solution by means of a conforming finite element discretisation would require continuously differentiable test functions. In order to avoid this problem, the following saddle point formulation can be used (see Nečas (1962)): provided that $g \in H^{1/2-\varepsilon}(\partial\Omega)$, for some $\varepsilon \in [0, 1/2)$, find $u \in H^{1-\varepsilon}(\Omega)$ with $u|_{\Gamma} = g$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_{\Omega} cuv \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} fv \,\mathrm{d}\boldsymbol{x} \tag{1.3}$$

for all $v \in H^{1+\varepsilon}(\Omega) \cap \mathring{H}^1(\Omega)$. We note that the bilinear form on the left hand side is formally symmetric and corresponds to the standard form for the Poisson equation. For results dealing with related finite element approximations, we refer to Babuška (1971).

In the present paper, a new variational formulation for (1.1)–(1.2) is presented and analysed. Here, the emphasis shall be on Dirichlet boundary conditions which may exhibit (isolated) *discontinuities* and are essentially continuous otherwise. The formulation in this article is closely related to the saddle point formulation (1.3), however, it features Sobolev spaces which describe the local singularities in the analytical solution resulting from the discontinuities in the boundary data in a more specific way. More precisely, *weighted* Sobolev spaces which have been used in the context of regularity statements for second-order elliptic boundary value problems, see, e.g., Babuška & Guo (1988); Babuška & Guo (1989); Guo & Schwab (2006), will be used. We will establish well-posedness of the weak formulation in terms of an appropriate *inf-sup condition*.

In order to discretise the underlying PDE problem, we exploit the *hp*–version of the symmetric interior penalty discontinuous Galerkin (dG) finite element method, cf. Arnold *et al.* (2001), and the references cited therein. DG methods are ideally suited for realising *hp*–adaptivity for second-order boundary-value problems, an advantage that has been noted early on in the recent development of these methods; see, for example, Baumann & Oden (1999); Cockburn *et al.* (2000); Houston *et al.* (2002, 2007, 2008); Perugia & Schötzau (2002); Rivière *et al.* (1999); Stamm & Wihler (2010); Wihler *et al.* (2003) and the references therein. Indeed, working with discontinuous finite element spaces easily facilitates the use of variable polynomial degrees and local mesh refinement techniques on possibly irregularly refined meshes—the two key ingredients for *hp*–adaptive algorithms. A further advantage of interior penalty dG formulations is that they incorporate Dirichlet boundary conditions in a natural way irrespective of their smoothness (in fact, L^1-regularity is sufficient for well-posedness). With this in mind, we shall derive a computable a posteriori bound for the error measured in terms of the L^2–norm on Ω. On the basis of the resulting computable error indicators, adaptive *h*– and *hp*–mesh adaptation strategies will be investigated for a model second–order elliptic PDE with discontinuous boundary conditions. In particular, we shall show numerically that exploiting *hp*–mesh refinement leads to exponential convergence of the L^2–norm of the error as the finite element space is enriched.

The article is organised as follows: In Section 2 the new variational formulation of (1.1)–(1.2) will be presented. In addition, its well-posedness will be proved. Then, in Section 3, we will briefly review *hp*–version discontinuous Galerkin discretisations for the Laplace operator and derive an L^2-norm a posteriori error estimate. Additionally, the performance of the corresponding local error indicators is shown with a number of numerical experiments within an *h*– and *hp*–version adaptive framework. Finally, a few concluding remarks are made in Section 4.

2. Variational Formulation

2.1 *Weighted Sobolev Spaces*

Let $\mathscr{A} = \{A_i\}_{i=1}^M \subset \partial\Omega$, $A_i \neq A_j$ for $i \neq j$, be a finite set of points on the boundary of the polygonal domain Ω which are numbered in counter-clockwise direction along $\partial\Omega$; the points in $\mathscr{A}$ will signify the locations of the discontinuities in the Dirichlet boundary condition g in (1.2). Furthermore, we denote by $\Gamma_i \subset \Gamma$, $i = 1,2,\dots,M$, the (open) subset of Γ which connects the two points A_i and A_{i+1}; here, we set $A_{M+1} = A_1$. Moreover, let $\omega_i \in (0,2\pi]$ signify the interior angle of the polygon Ω at A_i. To each $A_i \in \mathscr{A}$, $i = 1,2,\dots,M$, we associate a weight $\alpha_i \in [0,1)$. These numbers are stored in a weight vector

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_M) \in [0,1)^M. \tag{2.1}$$

Moreover, for any number $k \in \mathbb{R}$, we use the notation $k\boldsymbol{\alpha} = (k\alpha_1, k\alpha_2, \dots, k\alpha_M)$ and $\boldsymbol{\alpha} + k = (\alpha_1 + k, \alpha_2 + k, \dots, \alpha_M + k)$. Furthermore, for a fixed number

$$\eta > 0, \tag{2.2}$$

we introduce the following weight function on Ω:

$$\Phi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \prod_{i=1}^M r_i(\boldsymbol{x})^{\alpha_i}, \qquad r_i(\boldsymbol{x}) = \min\{\eta^{-1}|\boldsymbol{x} - A_i|, 1\}.$$

Here, we assume that η is small enough, so that the open sectors

$$S_i = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x} - A_i| < \eta\}, \qquad i = 1,2,\dots,M, \tag{2.3}$$

do not intersect, i.e., $\overline{S}_i \cap \overline{S}_j = \emptyset$ if $i \neq j$. There holds, for $\boldsymbol{x} \in \Omega$, that

$$r_i(\boldsymbol{x}) = \begin{cases} \eta^{-1}|\boldsymbol{x} - A_i| & \text{if } \boldsymbol{x} \in S_i, \\ 1 & \text{if } \boldsymbol{x} \in \Omega \setminus S_i, \end{cases}$$

and $r_i \in C^0(\Omega)$, $i = 1,2,\dots,M$. Furthermore, setting

$$\mathscr{S} = \bigcup_{i=1}^M S_i, \qquad \Omega_0 = \Omega \setminus \overline{\mathscr{S}},$$

we have

$$\Phi_{\boldsymbol{\alpha}} = \begin{cases} r_i^{\alpha_i} & \text{if } \boldsymbol{x} \in S_i \text{ for some } i = 1,2,\dots M, \\ 1 & \text{if } \boldsymbol{x} \in \Omega_0. \end{cases} \tag{2.4}$$

Note that $\Phi_{\boldsymbol{\alpha}}$ is continuous on Ω. Furthermore, for $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^M$, we have

$$\Phi_{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2} = \Phi_{\boldsymbol{\alpha}_1}\Phi_{\boldsymbol{\alpha}_2}, \qquad \Phi_{\boldsymbol{\alpha}}^{-1} = \Phi_{-\boldsymbol{\alpha}}.$$

Then, for any integers $m \geqslant l \geqslant 0$, we define the weighted Sobolev spaces $H_{\boldsymbol{\alpha}}^{m,l}(\Omega)$ as the completion of the space $C^\infty(\overline{\Omega})$ with respect to the weighted Sobolev norms

$$\|u\|^2_{H_{\boldsymbol{\alpha}}^{m,l}(\Omega)} = \|u\|^2_{l-1,\Omega} + \sum_{k=l}^m |u|^2_{H_{\boldsymbol{\alpha}}^{k,l}(\Omega)}, \qquad l \geqslant 1,$$

$$\|u\|^2_{H_{\boldsymbol{\alpha}}^{m,0}(\Omega)} = \sum_{k=0}^m |u|^2_{H_{\boldsymbol{\alpha}}^{k,0}(\Omega)}.$$

Here,

$$|u|^2_{H^{k,l}_{\boldsymbol{\alpha}}(\Omega)} = \sum_{|\boldsymbol{\lambda}|=k} \left\| \Phi_{\boldsymbol{\alpha}+k-l} |\mathsf{D}^{\boldsymbol{\lambda}} u| \right\|^2_{0,\Omega}$$

is the $H^{k,l}_{\boldsymbol{\alpha}}$-seminorm in Ω, where

$$\mathsf{D}^{\boldsymbol{\lambda}} u = \frac{\partial^{|\boldsymbol{\lambda}|} u}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2}},$$

with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{N}_0^2$ and $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2$.

In addition, for $m \geqslant l \geqslant 1$, let us define the space $H^{m-\frac{1}{2},l-\frac{1}{2}}_{\boldsymbol{\alpha}}(\partial\Omega)$ as the trace space of $H^{m,l}(\Omega)$, equipped with the norm

$$\|u\|_{H^{m-\frac{1}{2},l-\frac{1}{2}}_{\boldsymbol{\alpha}}(\partial\Omega)} = \inf_{\substack{v \in H^{m,l}_{\boldsymbol{\alpha}}(\Omega) \\ v|_{\partial\Omega}=u}} \|v\|_{H^{m,l}_{\boldsymbol{\alpha}}(\Omega)}.$$

Finally, we denote by $\mathring{H}^{m,l}_{\boldsymbol{\alpha}}(\Omega)$ the subspace of $H^{m,l}_{\boldsymbol{\alpha}}(\Omega)$ consisting of functions with zero trace on $\partial\Omega$.

2.2 *Inequalities in $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$*

In order to describe the well-posedness of (1.1)–(1.2), the weighted Sobolev space $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ will play an important role. In the sequel, we shall collect a few inequalities which will be used for the analysis in this paper.

LEMMA 2.1 Let $I = (a,b) \subset \mathbb{R}$, $a < b$, be an open interval. Then, there holds the Poincaré-Friedrichs inequality

$$\int_a^b \phi(x)^2 \,\mathrm{d}x \leqslant \frac{(b-a)^2}{\pi^2} \int_a^b (\phi'(x))^2 \,\mathrm{d}x$$

for all $\phi \in H^1(a,b)$ with $\phi(a) = \phi(b) = 0$.

Proof. The bound follows from (Hardy *et al.*, 1952, Theorem 257) and a scaling argument. □

Applying the previous lemma, we shall prove the following result.

LEMMA 2.2 Consider a sector $S = \{(r,\theta) : 0 < r < R, \theta_0 < \theta < \theta_1\} \subset \mathbb{R}^2$, where (r,θ) denote polar coordinates in $\mathbb{R}^2$, and $R > 0$, $0 \leqslant \theta_0 < \theta_1 \leqslant 2\pi$ are constants. Furthermore, let $u \in L^2(S)$ with $\|r^\alpha \nabla u\|_{0,S} < \infty$ for some $\alpha \in [0,1)$, and $u|_{\partial S_<} = 0$, where $\partial S_< = \{(r,\theta) : 0 < r < R, \theta \in \{\theta_0, \theta_1\}\}$. Then, there holds

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_S r^{2\alpha} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}.$$

Proof. Using integration in polar coordinates, we get

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} = \int_0^R r^{2\alpha-1} \int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \,\mathrm{d}r. \tag{2.5}$$

Then, since for any $r \in (0,R)$ there holds $u(r,\theta_0) = u(r,\theta_1) = 0$, we can apply Lemma 2.1. This implies

$$\int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_{\theta_0}^{\theta_1} |\partial_\theta u|^2 \,\mathrm{d}\theta, \qquad 0 < r < R.$$

Furthermore, noticing that $|\partial_\theta u| \leqslant r\,|\nabla_{\boldsymbol{x}} u|$, we obtain

$$\int_{\theta_0}^{\theta_1} u^2 \,\mathrm{d}\theta \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} r^2 \int_{\theta_0}^{\theta_1} |\nabla_{\boldsymbol{x}} u|^2 \,\mathrm{d}\theta, \qquad 0<r<R.$$

Inserting this estimate into (2.5), leads to

$$\int_S r^{2\alpha-2} u(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{(\theta_1-\theta_0)^2}{\pi^2} \int_0^R r^{2\alpha+1} \int_{\theta_0}^{\theta_1} |\nabla_{\boldsymbol{x}} u|^2 \,\mathrm{d}\theta \,\mathrm{d}r.$$

Changing back to Cartesian coordinates $\boldsymbol{x}$, completes the proof. □

LEMMA 2.3 Given a weight vector $\boldsymbol{\alpha} \in [0,1)^M$. Then, there holds

$$\|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\Omega} \leqslant C\,\|u\|_{1,\Omega}$$

for any $u \in H^1(\Omega)$, where the constant $C>0$ only depends on $\boldsymbol{\alpha}$ and Ω.

Proof. Let S_i, $i=1,2,\ldots,M$, be the (sufficiently small) sectors from (2.3). Then, we recall the property (2.4) to write

$$\|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\Omega}^2 = \|u\|_{0,\Omega_0}^2 + \|\Phi_{-\boldsymbol{\alpha}} u\|_{0,\mathscr{S}}^2 = \|u\|_{0,\Omega_0}^2 + \sum_{i=1}^M \left\| r_i^{-\alpha_i} u \right\|_{0,S_i}^2. \tag{2.6}$$

If, for some $1 \leqslant i \leqslant M$, we have that $\alpha_i > 0$, then

$$\left\| r_i^{-\alpha_i} u \right\|_{0,S_i}^2 \leqslant C \left(\|u\|_{0,S_i}^2 + \left\| r_i^{1-\alpha_i} \nabla u \right\|_{0,S_i}^2 \right) \leqslant C\,\|u\|_{1,S_i}^2;$$

this follows from expressing the norms in terms of polar coordinates and from applying (Hardy *et al.*, 1952, Theorem 330). Inserting this into (2.6), gives the desired inequality. □

LEMMA 2.4 Consider a function $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, where $\alpha_i \in [0,1)$, $i=1,2,\ldots,M$. Then, there holds

$$\|\,|\nabla(\Phi_{\boldsymbol{\alpha}})|\,u\|_{0,\Omega} \leqslant \frac{1}{\pi} \max_{1\leqslant i\leqslant M} \alpha_i \omega_i\, |u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}.$$

Proof. Let S_i, $i=1,2,\ldots,M$, be the (sufficiently small) sectors from (2.3). Then, due to (2.4), we have

$$|\nabla(\Phi_{\boldsymbol{\alpha}})| = \begin{cases} |\nabla(r^{\alpha_i})| = \alpha_i r_i^{\alpha_i-1} & \text{if } \boldsymbol{x} \in S_i \text{ for some } i=1,2,\ldots,M, \\ 0 & \text{if } \boldsymbol{x} \in \Omega_0. \end{cases} \tag{2.7}$$

Hence,

$$\int_\Omega |\nabla(\Phi_{\boldsymbol{\alpha}})|^2 u^2 \,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^M \alpha_i^2 \int_{S_i} r_i^{2\alpha_i-2} u^2 \,\mathrm{d}\boldsymbol{x}. \tag{2.8}$$

Then, applying Lemma 2.2, we have

$$\int_{S_i} r_i^{2\alpha_i-2} u^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{\omega_i^2}{\pi^2} \int_{S_i} r^{2\alpha_i} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}.$$

Thus,

$$\int_\Omega |\nabla(\Phi_{\boldsymbol{\alpha}})|^2 u^2 \,\mathrm{d}\boldsymbol{x} \leqslant \sum_{i=1}^{M} \frac{\alpha_i^2\omega_i^2}{\pi^2} \int_{S_i} r^{2\alpha_i} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} \leqslant \frac{\max\limits_{1\leqslant i\leqslant M} \alpha_i^2\omega_i^2}{\pi^2} \int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \,\mathrm{d}\boldsymbol{x},$$

as required. $\square$

Furthermore, there holds the following Poincaré-Friedrichs inequality.

LEMMA 2.5 Consider a weight vector $\boldsymbol{\alpha} \in [0,1)^M$ and $\gamma \subseteq \partial\Omega$ with $\int_\gamma \mathrm{d}s > 0$. Then, there exists a constant $C > 0$ depending only on γ, Ω, and $\boldsymbol{\alpha}$ such that

$$\|u\|_{0,\Omega} \leqslant C\,|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}$$

for all functions $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ with $u|_\gamma = 0$ (in the trace sense). In particular, we have that $|\cdot|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}$ is a norm on $\mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$.

Proof. We first note that the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous for Lipschitz polygons in $\mathbb{R}^2$ (cf., e.g., (Adams & Fournier, 2003, Theorem 4.12)). Hence, there exists a constant $C > 0$ depending on Ω such that

$$\|u\|_{0,\Omega} \leqslant C\,\|u\|_{W^{1,1}(\Omega)}.$$

Moreover, applying the Poincaré-Friedrichs inequality in $W^{1,1}(\Omega)$, it follows that

$$\|u\|_{0,\Omega} \leqslant C\,\|u\|_{W^{1,1}(\Omega)} \leqslant C'\,\|\nabla u\|_{L^1(\Omega)},$$

for a constant $C' > 0$ depending on γ and Ω. Therefore, using Hölder's inequality, we obtain

$$\|u\|_{0,\Omega} \leqslant C' \int_\Omega |\nabla u| \,\mathrm{d}\boldsymbol{x} \leqslant C' \left(\int_\Omega \Phi_{\boldsymbol{\alpha}}^{-2} \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}.$$

Then, employing (2.4) yields

$$\int_\Omega \Phi_{\boldsymbol{\alpha}}^{-2} \,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{M} \int_{S_i} r_i^{-2\alpha_i} \,\mathrm{d}\boldsymbol{x} + \int_{\Omega_0} 1 \,\mathrm{d}\boldsymbol{x},$$

and using integration in polar coordinates, it follows that the above integrals are all bounded for $\alpha_i < 1$, $i = 1, 2, \ldots, M$. This completes the proof. $\square$

To close this section, we shall prove the following Green's type formula:

LEMMA 2.6 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector, and consider two functions $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ and $\phi \in H^2(\Omega)$. In addition, suppose that the trace of $u|_{\partial\Omega} \in L^2(\partial\Omega)$. Then,

$$\int_\Omega \Delta\phi\, u \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\nabla\phi \cdot \boldsymbol{n}) u \,\mathrm{d}s - \int_\Omega \nabla\phi \cdot \nabla u \,\mathrm{d}\boldsymbol{x} \tag{2.9}$$

holds true, where $\boldsymbol{n}$ denotes the outward unit vector to $\partial\Omega$.

Proof. Due to the density of $C^\infty(\overline{\Omega})$ in $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ we can choose a sequence $\{u_n\}_{n\geqslant 0} \subset C^\infty(\overline{\Omega})$ such that $\lim_{n\to\infty} \|u - u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} = 0$. Then, using Green's formula for smooth functions, we have

$$\int_\Omega \Delta\phi\, u_n \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\nabla\phi \cdot \boldsymbol{n}) u_n \,\mathrm{d}s - \int_\Omega \nabla\phi \cdot \nabla u_n \,\mathrm{d}\boldsymbol{x}$$

for any function $\phi \in C^\infty(\overline{\Omega})$. Furthermore, there holds

$$\left|\int_\Omega \Delta\phi\,(u_n - u)\,\mathrm{d}\boldsymbol{x}\right| \leqslant \|\phi\|_{2,\Omega}\,\|u-u_n\|_{0,\Omega} \xrightarrow{n\to\infty} 0,$$

and, using Lemma 2.3,

$$\begin{aligned}\left|\int_\Omega \nabla\phi\cdot\nabla(u_n - u)\,\mathrm{d}\boldsymbol{x}\right| &\leqslant \|\boldsymbol{\Phi}_{-\boldsymbol{\alpha}}\nabla\phi\|_{0,\Omega}\,\|\boldsymbol{\Phi}_{\boldsymbol{\alpha}}\nabla(u-u_n)\|_{0,\Omega}\\ &\leqslant C\,\|\phi\|_{2,\Omega}\,\|u-u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Furthermore, applying the trace theorem in $W^{1,1}(\Omega)$, yields

$$\begin{aligned}\left|\int_{\partial\Omega}(\nabla\phi\cdot\boldsymbol{n})(u_n-u)\,\mathrm{d}s\right| &\leqslant \sup_{\overline{\Omega}}|\nabla\phi|\,\|u-u_n\|_{L^1(\partial\Omega)}\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|\left(\|u-u_n\|_{L^1(\Omega)} + \|\nabla(u-u_n)\|_{L^1(\Omega)}\right)\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|\left(\|u-u_n\|_{0,\Omega} + \|\boldsymbol{\Phi}_{-\boldsymbol{\alpha}}\|_{0,\Omega}\,\|\boldsymbol{\Phi}_{\boldsymbol{\alpha}}\nabla(u-u_n)\|_{0,\Omega}\right)\\ &\leqslant C\sup_{\overline{\Omega}}|\nabla\phi|\,\|u-u_n\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

This implies the identity (2.9) for $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ and $\phi \in C^\infty(\overline{\Omega})$.

For $\phi \in H^2(\Omega)$, the density of $C^\infty(\overline{\Omega})$ in $H^2(\Omega)$ guarantees the existence of a sequence $\{\phi_n\}_{n\geqslant 0} \subset C^\infty(\overline{\Omega})$ with $\lim_{n\to\infty}\|\phi_n - \phi\|_{2,\Omega} = 0$. Then,

$$\int_\Omega \Delta\phi_n\,u\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega}(\nabla\phi_n\cdot\boldsymbol{n})u\,\mathrm{d}s - \int_\Omega \nabla\phi_n\cdot\nabla u\,\mathrm{d}\boldsymbol{x}$$

for all $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$. Similarly, as before, we have

$$\left|\int_\Omega \Delta(\phi_n - \phi)\,u\,\mathrm{d}\boldsymbol{x}\right| \leqslant \|\phi_n-\phi\|_{2,\Omega}\,\|u\|_{0,\Omega} \xrightarrow{n\to\infty} 0,$$

and, with Lemma 2.3,

$$\begin{aligned}\left|\int_\Omega \nabla(\phi_n-\phi)\cdot\nabla u\,\mathrm{d}\boldsymbol{x}\right| &\leqslant \|\boldsymbol{\Phi}_{-\boldsymbol{\alpha}}\nabla(\phi_n-\phi)\|_{0,\Omega}\,\|\boldsymbol{\Phi}_{\boldsymbol{\alpha}}\nabla u\|_{0,\Omega}\\ &\leqslant \|\phi_n-\phi\|_{2,\Omega}\,\|u\|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Moreover, using the trace theorem again, we obtain

$$\begin{aligned}\left|\int_{\partial\Omega}(\nabla(\phi_n-\phi)\cdot\boldsymbol{n})u\,\mathrm{d}s\right| &\leqslant \|\nabla(\phi_n-\phi)\|_{L^2(\partial\Omega)}\,\|u\|_{L^2(\partial\Omega)}\\ &\leqslant C\,\|\phi_n-\phi\|_{2,\Omega}\,\|u\|_{L^2(\partial\Omega)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

This completes the proof. □

2.3 *Weak Formulation*

The aim of this section is to introduce a weak formulation for the boundary value problem (1.1)–(1.2) and to discuss its well-posedness.

Let $g \in H_{\boldsymbol{\alpha}}^{\frac{1}{2},\frac{1}{2}}(\partial\Omega)$ in (1.2), where $\boldsymbol{\alpha}$ is the weight vector from (2.1) with $\alpha_i \in [0,1)$, $i = 1,2,\ldots,M$. Then, we call $u \in H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ with $u|_{\partial\Omega} = g$ a weak solution of (1.1)–(1.2) if

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega cuv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} \qquad \forall v \in \mathring{H}_{-\boldsymbol{\alpha}}^{1,1}(\Omega). \tag{2.10}$$

Writing the solution in the form $u = u_0 + G$, where $u_0 \in \mathring{H}_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ and $G \in H_{\boldsymbol{\alpha}}^{1,1}(\Omega)$ is a lifting of the boundary data g, i.e., $G|_\Gamma = g$, there holds

$$\int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega cu_0 v \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} - \int_\Omega \nabla G \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega cGv \,\mathrm{d}\boldsymbol{x} \quad \forall v \in \mathring{H}_{-\boldsymbol{\alpha}}^{1,1}(\Omega).$$

We note that this is a saddle point formulation on $\mathring{H}_{\boldsymbol{\alpha}}^{1,1}(\Omega) \times \mathring{H}_{-\boldsymbol{\alpha}}^{1,1}(\Omega)$. Its well-posedness will be discussed in the following.

We first show that the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega cuv \,\mathrm{d}\boldsymbol{x}$$

and the linear functional

$$\ell(v) = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} - \int_\Omega \nabla G \cdot \nabla v \,\mathrm{d}\boldsymbol{x} - \int_\Omega cGv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} - a(G,v)$$

are continuous. Here, we suppose that the lifting G is chosen such that

$$\|G\|_{H_{\boldsymbol{\alpha}}^{1,1}(\Omega)} \leqslant C\|g\|_{H_{\boldsymbol{\alpha}}^{\frac{1}{2},\frac{1}{2}}(\Gamma)} \tag{2.11}$$

for some fixed constant $C > 1$ independent of g.

PROPOSITION 2.1 There is a constant $C > 0$ (depending on Ω and $\boldsymbol{\alpha}$) such that

$$|a(u,v)| \leqslant C\,|u|_{H_{\boldsymbol{\alpha}}^{1,1}(\Omega)}\,|v|_{H_{-\boldsymbol{\alpha}}^{1,1}(\Omega)}$$

for all $u \in \mathring{H}_{\boldsymbol{\alpha}}^{1,1}(\Omega)$, $v \in \mathring{H}_{-\boldsymbol{\alpha}}^{1,1}(\Omega)$. Furthermore, for $f \in L^2(\Omega)$ and $g \in H_{\boldsymbol{\alpha}}^{\frac{1}{2},\frac{1}{2}}(\Gamma)$ we have

$$|\ell(v)| \leqslant C\left(\|f\|_{0,\Omega} + \|g\|_{H_{\boldsymbol{\alpha}}^{\frac{1}{2},\frac{1}{2}}(\Gamma)}\right)|v|_{H_{-\boldsymbol{\alpha}}^{1,1}(\Omega)}$$

for any $v \in \mathring{H}_{-\boldsymbol{\alpha}}^{1,1}(\Omega)$.

Proof. There holds

$$\begin{aligned}
|a(u,v)| &\leqslant \|\Phi_{\boldsymbol{\alpha}}\nabla u\|_{0,\Omega}\,\|\Phi_{-\boldsymbol{\alpha}}\nabla v\|_{0,\Omega} + \|c\|_{L^\infty(\Omega)}\,\|u\|_{0,\Omega}\,\|v\|_{0,\Omega} \\
&\leqslant C\left(|u|_{H_{\boldsymbol{\alpha}}^{1,1}(\Omega)}\,|v|_{H_{-\boldsymbol{\alpha}}^{1,1}(\Omega)} + \|u\|_{0,\Omega}\,\|v\|_{0,\Omega}\right).
\end{aligned}$$

Furthermore, using the Poincaré-Friedrichs inequality and Lemma 2.5, we get

$$\|u\|_{0,\Omega}\,\|v\|_{0,\Omega} \leqslant C\,|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,|v|_{1,\Omega} \leqslant C\,|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}\,.$$

Hence,

$$|a(u,v)| \leqslant C\,|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}\,.$$

Moreover, employing the previous estimate and proceeding as before to estimate the L^2-norm, we obtain

$$|\ell(v)| \leqslant \|f\|_{0,\Omega}\|v\|_{0,\Omega} + |a(G,v)| \leqslant \|f\|_{0,\Omega}|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} + C\,|G|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}\,.$$

Then, applying (2.11), yields the stability bound for ℓ. □

Furthermore, the following inf-sup stability holds.

PROPOSITION 2.2 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector. Suppose that the weights α_i, $i = 1,2,\ldots M$, are sufficiently small so that

$$\mu := \frac{1}{\pi}\max_{1\leqslant i\leqslant M} \alpha_i\omega_i < \frac{1}{2}.$$

Then, there holds

$$\inf_{0\not\equiv u\in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\ \sup_{0\not\equiv v\in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \frac{a(u,v)}{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,|v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \geqslant \delta, \tag{2.12}$$

where

$$\delta = \frac{1-2\mu}{\sqrt{2(4\mu^2+1)}}.$$

Furthermore, we have that

$$\sup_{u\in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)} a(u,v) > 0 \qquad \forall v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega), v\not\equiv 0. \tag{2.13}$$

Proof. For $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, we define $\widetilde{v} = \Phi^2_{\boldsymbol{\alpha}}u$. Then, there holds

$$\begin{aligned}|\widetilde{v}|^2_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} &= \int_\Omega \Phi^2_{-\boldsymbol{\alpha}}|\nabla\widetilde{v}|^2\,\mathrm{d}\boldsymbol{x} \leqslant 2\int_\Omega \Phi^{-2}_{\boldsymbol{\alpha}}\left(\left|\nabla(\Phi^2_{\boldsymbol{\alpha}})\right|^2 u^2 + \Phi^4_{\boldsymbol{\alpha}}\,|\nabla u|^2\right)\mathrm{d}\boldsymbol{x}\\ &\leqslant 2\left(4\int_\Omega |\nabla\Phi_{\boldsymbol{\alpha}}|^2\,u^2\,\mathrm{d}\boldsymbol{x} + |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\right).\end{aligned}$$

Hence, applying Lemma 2.4, results in

$$|\widetilde{v}|^2_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \leqslant 2(4\mu^2+1)\,|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}\,. \tag{2.14}$$

In particular, it follows that $\widetilde{v} \in H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$.

Moreover, we observe that

$$a(u,\widetilde{v}) = \int_\Omega \nabla u\cdot\nabla\widetilde{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega cu\widetilde{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla u\cdot\nabla\left(\Phi^2_{\boldsymbol{\alpha}}u\right)\mathrm{d}\boldsymbol{x} + \int_\Omega c\,\Phi^2_{\boldsymbol{\alpha}}u^2\,\mathrm{d}\boldsymbol{x}.$$

Thus, since $c \geqslant 0$, we get

$$\begin{aligned} a(u,\widetilde{v}) &\geqslant \int_\Omega \left(\nabla u \cdot \nabla\left(\Phi_{\boldsymbol{\alpha}}^2\right) u + \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2\right) \mathrm{d}\boldsymbol{x} \\ &= 2\int_\Omega \Phi_{\boldsymbol{\alpha}} \nabla u \cdot \nabla\left(\Phi_{\boldsymbol{\alpha}}\right) u \,\mathrm{d}\boldsymbol{x} + \int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} \\ &\geqslant -\frac{1}{\mu}\int_\Omega \left|\nabla\left(\Phi_{\boldsymbol{\alpha}}\right)\right|^2 u^2 \,\mathrm{d}\boldsymbol{x} + (1-\mu)\int_\Omega \Phi_{\boldsymbol{\alpha}}^2 |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Recalling Lemma 2.4, leads to

$$a(u,\widetilde{v}) \geqslant -\mu\, |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} + (1-\mu)\, |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} \geqslant (1-2\mu)\, |u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}. \tag{2.15}$$

Now, combining (2.14) and (2.15), it follows that

$$\sup_{v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)} \frac{a(u,v)}{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)} |v|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \geqslant \frac{|u|_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}}{|\widetilde{v}|_{H^{1,1}_{-\boldsymbol{\alpha}}(\Omega)}} \frac{a(u,\widetilde{v})}{|u|^2_{H^{1,1}_{\boldsymbol{\alpha}}(\Omega)}} \geqslant \delta$$

for any $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$, $u \not\equiv 0$. Taking the infimum over all $u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ results in (2.12).

In addition, let $v \in \mathring{H}^{1,1}_{-\boldsymbol{\alpha}}(\Omega)$, $v \not\equiv 0$. Then,

$$\sup_{u \in \mathring{H}^{1,1}_{\boldsymbol{\alpha}}(\Omega)} a(u,v) \geqslant a(v,v) \geqslant \int_\Omega |\nabla v|^2 \,\mathrm{d}\boldsymbol{x}.$$

Due to $v|_\Gamma = 0$ and $v \not\equiv 0$, there holds $\|\nabla v\|_{0,\Omega} > 0$, and hence (2.13) holds. □

The above results, Propositions 2.1 and 2.2, imply the well-posedness of the variational formulation (2.10); cf., e.g., (Schwab, 1998, Theorem 1.15).

THEOREM 2.3 Let $\boldsymbol{\alpha} \in [0,1)^M$ be a weight vector, with α_i, $i = 1,2,\ldots,M$ sufficiently small such that

$$\max_{1 \leqslant i \leqslant M} \alpha_i \omega_i < \frac{\pi}{2}$$

is satisfied. Furthermore, suppose that $g \in H^{\frac{1}{2},\frac{1}{2}}_{\boldsymbol{\alpha}}(\partial\Omega)$ and $f \in L^2(\Omega)$ in (1.1)–(1.2). Then, there exists exactly one solution of the weak formulation (2.10) in $H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$.

3. Numerical Approximation

We shall now discuss the numerical approximation of the problem (1.1)–(1.2). To this end, we will consider hp-version interior penalty discontinuous Galerkin finite element methods. Particularly, we will derive an L^2-norm a posteriori error estimate which can be applied for adaptive purposes.

3.1 *Meshes, Spaces, and Element Edge Operators*

We consider shape-regular meshes $\mathscr{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint triangles and/or parallelograms $\{K\}_{K \in \mathscr{T}_h}$, i.e., $\overline{\Omega} = \bigcup_{K \in \mathscr{T}_h} \overline{K}$. Each element $K \in \mathscr{T}_h$ can then be affinely mapped onto the reference triangle $\widehat{T} = \{(\widehat{x},\widehat{y}) : -1 < \widehat{x} < 1, -1 < \widehat{y} < -\widehat{x}\}$ or the reference square $\widehat{S} = (-1,1)^2$, respectively. We allow the meshes to be 1-irregular, i.e., elements may contain hanging nodes. By h_K, we denote the diameter of an element $K \in \mathscr{T}_h$. We assume that these quantities are of bounded variation, i.e., there is a constant $\rho_1 \geqslant 1$ such that

$$\rho_1^{-1} \leqslant {}^{h_{K_\sharp}}\!/\!{}_{h_{K_\flat}} \leqslant \rho_1, \tag{3.1}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge. We store the elemental diameters in a vector $\boldsymbol{h}$ given by $\boldsymbol{h} = \{h_K : K \in \mathscr{T}_h\}$. Similarly, to each element $K \in \mathscr{T}_h$ we assign a polynomial degree $p_K \geqslant 1$ and define the degree vector $\boldsymbol{p} = \{p_K : K \in \mathscr{T}_h\}$. We suppose that $\boldsymbol{p}$ is also of bounded variation, i.e., there is a constant $\rho_2 \geqslant 1$ such that

$$\rho_2^{-1} \leqslant {}^{p_{K_\sharp}}\!/\!{}_{p_{K_\flat}} \leqslant \rho_2, \tag{3.2}$$

whenever $K_\sharp$ and $K_\flat$ share a common edge.

Moreover, we shall define some suitable element edge operators that are required for the dG method. To this end, we denote by $\mathscr{E}_\mathscr{I}$ the set of all interior edges of the partition $\mathscr{T}_h$ of Ω, and by $\mathscr{E}_\mathscr{B}$ the set of all boundary edges of $\mathscr{T}_h$. In addition, let $\mathscr{E} = \mathscr{E}_\mathscr{I} \cup \mathscr{E}_\mathscr{B}$. The boundary ∂K of an element K and the sets $\partial K \setminus \partial\Omega$ and $\partial K \cap \partial\Omega$ will be identified in a natural way with the corresponding subsets of $\mathscr{E}$.

Let $K_\sharp$ and $K_\flat$ be two adjacent elements of $\mathscr{T}_h$, and $\boldsymbol{x}$ an arbitrary point on the interior edge $e \in \mathscr{E}_\mathscr{I}$ given by $e = \partial K_\sharp \cap \partial K_\flat$. Furthermore, let v and $\boldsymbol{q}$ be scalar- and vector-valued functions, respectively, that are sufficiently smooth inside each element $K_{\sharp/\flat}$. By $(v_{\sharp/\flat}, \boldsymbol{q}_{\sharp/\flat})$, we denote the traces of $(v, \boldsymbol{q})$ on e taken from within the interior of $K_{\sharp/\flat}$, respectively. Then, the averages of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$\langle\!\langle v \rangle\!\rangle = \frac{1}{2}(v_\sharp + v_\flat), \qquad \langle\!\langle \boldsymbol{q} \rangle\!\rangle = \frac{1}{2}(\boldsymbol{q}_\sharp + \boldsymbol{q}_\flat),$$

respectively. Similarly, the jumps of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$[\![v]\!] = v_\sharp \boldsymbol{n}_{K_\sharp} + v_\flat \boldsymbol{n}_{K_\flat}, \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_\sharp \cdot \boldsymbol{n}_{K_\sharp} + \boldsymbol{q}_\flat \cdot \boldsymbol{n}_{K_\flat},$$

respectively, where we denote by $\boldsymbol{n}_{K_{\sharp/\flat}}$ the unit outward normal vector on $\partial K_{\sharp/\flat}$, respectively. On a boundary edge $e \in \mathscr{E}_\mathscr{B}$, we set $\langle\!\langle v \rangle\!\rangle = v$, $\langle\!\langle \boldsymbol{q} \rangle\!\rangle = \boldsymbol{q}$, and $[\![v]\!] = v\boldsymbol{n}$, $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ denoting the unit outward normal vector on the boundary $\partial\Omega$.

Given a finite element mesh $\mathscr{T}_h$ and an associated polynomial degree vector $\boldsymbol{p} = (p_K)_{K \in \mathscr{T}_h}$, with $p_K \geqslant 1$ for all $K \in \mathscr{T}_h$, consider the hp-discretisation space

$$V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_K \in \mathbb{S}_{p_K}(K), K \in \mathscr{T}_h\}, \tag{3.3}$$

for the dG method. Here, for $K \in \mathscr{T}_h$, $\mathbb{S}_{p_K}(K)$ is either the space $\mathbb{P}_{p_K}(K)$ of all polynomials of total degree at most p_K on K or the space $\mathbb{Q}_{p_K}(K)$ of all polynomials of degree at most p_K in each coordinate direction on K.

3.2 *hp-dG Discretisation*

We will now consider the following hp-dG formulation for the numerical approximation of (1.1)–(1.2): find $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ such that

$$a_{\mathrm{DG}}(u_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \qquad \forall v \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p}). \tag{3.4}$$

Here,

$$a_{\mathrm{DG}}(w,v) = \int_\Omega \nabla_h w \cdot \nabla_h v \,\mathrm{d}\boldsymbol{x} - \int_\mathscr{E} \langle\!\langle \nabla_h w \rangle\!\rangle \cdot [\![v]\!] \,\mathrm{d}s - \int_\mathscr{E} [\![w]\!] \cdot \langle\!\langle \nabla_h v \rangle\!\rangle \,\mathrm{d}s + \gamma \int_\mathscr{E} \sigma [\![w]\!] \cdot [\![v]\!] \,\mathrm{d}s, \tag{3.5}$$

and

$$\ell_{\mathrm{DG}}(v) = \int_{\Omega} f v \,\mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla_h v \cdot \boldsymbol{n}) g \,\mathrm{d}s + \gamma \int_{\mathscr{E}_{\mathscr{B}}} \sigma g v \,\mathrm{d}s \tag{3.6}$$

are hp-version symmetric interior penalty dG forms. In these forms, ∇_h denotes the elementwise gradient operator, $\gamma > 0$ is a stability constant, and the function

$$\sigma = \frac{\mathtt{p}^2}{\mathtt{h}} \tag{3.7}$$

is defined by means of the two functions $\mathtt{h} \in L^\infty(\mathscr{E})$ and $\mathtt{p} \in L^\infty(\mathscr{E})$ given by

$$\mathtt{h}(\boldsymbol{x}) = \begin{cases} \min(h_{K_\sharp}, h_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathscr{E}_{\mathscr{I}}, \\ h_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial \Omega \in \mathscr{E}_{\mathscr{B}}, \end{cases}$$

$$\mathtt{p}(\boldsymbol{x}) = \begin{cases} \max(p_{K_\sharp}, p_{K_\flat}) & \text{for } \boldsymbol{x} \in \partial K_\sharp \cap \partial K_\flat \in \mathscr{E}_{\mathscr{I}}, \\ p_K & \text{for } \boldsymbol{x} \in \partial K \cap \partial \Omega \in \mathscr{E}_{\mathscr{B}}. \end{cases}$$

REMARK 3.1 Provided that $\gamma > 0$ is chosen sufficiently large (independently of the local element sizes and polynomial degrees), it is well-known that the dG form a_{DG} is coercive. More precisely, there is a constant $C > 0$ independent of $\mathscr{T}_h$ and $\boldsymbol{p}$ such that

$$a_{\mathrm{DG}}(v,v) \geqslant C \left(\|\nabla_h v\|_{0,\Omega}^2 + \gamma \int_{\mathscr{E}} \sigma |[\![v]\!]|^2 \,\mathrm{d}s \right)$$

for any $v \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$. In particular, the dG method (3.4) admits a unique solution $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$; see, e.g., Stamm & Wihler (2010) and the references therein.

3.3 *A Posteriori Error Estimation in the L^2-Norm*

We shall now derive a residual-based hp–a posteriori error estimate in the L^2-norm for the dG formulation (3.4). In this section we suppose that the dual problem

$$-\Delta\phi + c\phi = e_{\mathrm{DG}} \qquad \text{in } \Omega, \tag{3.8}$$

$$\phi = 0 \qquad \text{on } \Gamma, \tag{3.9}$$

has a solution $\phi \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ with continuous dependence on the data, i.e., there exists a constant $C > 0$ such that

$$\|\phi\|_{H^2(\Omega)} \leqslant C \|e_{\mathrm{DG}}\|_{0,\Omega}. \tag{3.10}$$

This is the case, for example, if Ω is a convex polygon since then $\Delta : H^2(\Omega) \cap \mathring{H}^1(\Omega) \to L^2(\Omega)$ is an isomorphism; cf. Babuška & Guo (1988); Dauge (1988); Grisvard (1985). Here, $e_{\mathrm{DG}} = u - u_{\mathrm{DG}}$ denotes the error, where $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ is the solution of (1.1)–(1.2) and $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ is the dG solution defined in (3.4).

Furthermore, we assume that the Dirichlet boundary data satisfies

$$g = u|_\Gamma \in L^2(\Gamma).$$

We start the development of the L^2-norm a posteriori error estimate by writing

$$\|e_{\mathrm{DG}}\|_{0,\Omega}^2 = \int_\Omega (-\Delta\phi + c\phi) e_{\mathrm{DG}} \,\mathrm{d}\boldsymbol{x} = \int_\Omega (-\Delta\phi + c\phi) u \,\mathrm{d}\boldsymbol{x} - \int_\Omega (-\Delta\phi + c\phi) u_{\mathrm{DG}} \,\mathrm{d}\boldsymbol{x}.$$

Applying Lemma 2.6 in the first integral and integrating by parts elementwise in the second integral, and noticing that $[\![\nabla\phi]\!] = 0$ on $\mathscr{E}_{\mathscr{I}}$ results in

$$\begin{aligned}\|e_{\mathrm{DG}}\|_{0,\Omega}^2 &= \int_\Omega (\nabla u\cdot\nabla\phi + cu\phi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}}\cdot\nabla\phi + cu_{\mathrm{DG}}\phi)\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} \nabla\phi\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi\cdot\boldsymbol{n})(u-u_{\mathrm{DG}})\,\mathrm{d}s\\ &= \int_\Omega f\phi\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}}\cdot\nabla\phi + cu_{\mathrm{DG}}\phi)\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla\phi\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s.\end{aligned}$$

Moreover, for an arbitrary function $\phi_h \in V_{\mathrm{DG}}(\mathscr{T}_h,\boldsymbol{p})$, exploiting (3.4) with $v=\phi_h$, gives

$$\begin{aligned}\|e_{\mathrm{DG}}\|_{0,\Omega}^2 &= \int_\Omega f(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla_h u_{\mathrm{DG}}\cdot\nabla(\phi-\phi_h) + cu_{\mathrm{DG}}(\phi-\phi_h))\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla\phi\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s\\ &\quad + \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi_h\cdot\boldsymbol{n})g\,\mathrm{d}s - \gamma\int_{\mathscr{E}_{\mathscr{B}}} \sigma g\phi_h\,\mathrm{d}s - \int_{\mathscr{E}} \langle\!\langle\nabla_h u_{\mathrm{DG}}\rangle\!\rangle\cdot[\![\phi_h]\!]\,\mathrm{d}s\\ &\quad - \int_{\mathscr{E}} \langle\!\langle\nabla_h\phi_h\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s + \gamma\int_{\mathscr{E}} \sigma[\![u_{\mathrm{DG}}]\!]\cdot[\![\phi_h]\!]\,\mathrm{d}s.\end{aligned}$$

Using Green's formula in the second integral, leads to

$$\begin{aligned}\int_\Omega \nabla_h u_{\mathrm{DG}}\cdot\nabla(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \Delta_h u_{\mathrm{DG}}(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} + \sum_{K\in\mathscr{T}_h}\int_{\partial K} (\nabla u_{\mathrm{DG}}\cdot\boldsymbol{n}_K)(\phi-\phi_h)\,\mathrm{d}s\\ &= -\int_\Omega \Delta_h u_{\mathrm{DG}}(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} + \int_{\mathscr{E}} \langle\!\langle\nabla_h u_{\mathrm{DG}}\rangle\!\rangle\cdot[\![\phi-\phi_h]\!]\,\mathrm{d}s\\ &\quad + \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\,\mathrm{d}s,\end{aligned}$$

where Δ_h is the elementwise Laplace operator. Hence, using that $[\![\phi]\!] = \mathbf{0}$ on $\mathscr{E}$, yields

$$\begin{aligned}&\|e_{\mathrm{DG}}\|_{0,\Omega}^2\\ &\quad= \int_\Omega (f+\Delta_h u_{\mathrm{DG}} - cu_{\mathrm{DG}})(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\,\mathrm{d}s\\ &\qquad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla\phi\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla\phi\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s + \int_{\mathscr{E}_{\mathscr{B}}} (\nabla_h\phi_h\cdot\boldsymbol{n})g\,\mathrm{d}s\\ &\qquad - \gamma\int_{\mathscr{E}_{\mathscr{B}}} \sigma g\phi_h\,\mathrm{d}s - \int_{\mathscr{E}} \langle\!\langle\nabla_h\phi_h\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s + \gamma\int_{\mathscr{E}} \sigma[\![u_{\mathrm{DG}}]\!]\cdot[\![\phi_h]\!]\,\mathrm{d}s\\ &\quad= \int_\Omega (f+\Delta_h u_{\mathrm{DG}} - cu_{\mathrm{DG}})(\phi-\phi_h)\,\mathrm{d}\boldsymbol{x} - \int_{\mathscr{E}_{\mathscr{I}}} [\![\nabla_h u_{\mathrm{DG}}]\!]\langle\!\langle\phi-\phi_h\rangle\!\rangle\,\mathrm{d}s\\ &\qquad + \int_{\mathscr{E}_{\mathscr{I}}} \langle\!\langle\nabla_h(\phi-\phi_h)\rangle\!\rangle\cdot[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s - \int_{\mathscr{E}_{\mathscr{B}}} (\nabla_h(\phi-\phi_h)\cdot\boldsymbol{n})(g-u_{\mathrm{DG}})\,\mathrm{d}s\\ &\qquad - \gamma\int_{\mathscr{E}_{\mathscr{B}}} \sigma(g-u_{\mathrm{DG}})(\phi_h-\phi)\,\mathrm{d}s + \gamma\int_{\mathscr{E}_{\mathscr{I}}} \sigma[\![u_{\mathrm{DG}}]\!]\cdot[\![\phi_h-\phi]\!]\,\mathrm{d}s.\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality and noting that $p_K \geqslant 1$, $K \in \mathscr{T}_h$, gives

$$
\begin{aligned}
\|e_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant \Bigg(& \sum_{K\in\mathscr{T}_h} h_K^4 p_K^{-4} \|f + \Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}}\|_{0,\Omega}^2 + \sum_{K\in\mathscr{T}_h} h_K^3 p_K^{-3} \|[\![\nabla_h u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\partial\Omega}^2 \\
& + (\gamma^2+1) \sum_{K\in\mathscr{T}_h} h_K p_K \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\partial\Omega}^2 \\
& + (\gamma^2+1) \sum_{K\in\mathscr{T}_h} h_K p_K \|g - u_{\mathrm{DG}}\|_{0,\partial K\cap\partial\Omega}^2 \Bigg)^{\frac{1}{2}} \\
& \times \Bigg(\sum_{K\in\mathscr{T}_h} h_K^{-4} p_K^4 \|\phi - \phi_h\|_{0,K}^2 + \sum_{K\in\mathscr{T}_h} h_K^{-3} p_K^3 \|\phi - \phi_h\|_{0,\partial K}^2 \\
& \quad + \sum_{K\in\mathscr{T}_h} h_K^{-1} p_K \|\nabla_h(\phi - \phi_h)\|_{0,\partial K}^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Then, choosing $\phi_h \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ to be an elementwise optimal hp-interpolant (see, e.g., Babuška & Suri (1987a,b)), i.e., for any $K \in \mathscr{T}_h$,

$$
h_K^{-4} p_K^4 \|\phi - \phi_h\|_{0,K}^2 + h_K^{-3} p_K^3 \|\phi - \phi_h\|_{0,\partial K}^2 + h_K^{-1} p_K \|\nabla_h(\phi - \phi_h)\|_{0,\partial K}^2 \leqslant C \|\phi\|_{H^2(K)}^2,
$$

and recalling the regularity estimate (3.10), gives

$$
\|e_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant C \|e_{\mathrm{DG}}\|_{0,\Omega} \left(\sum_{K\in\mathscr{T}_h} \eta_K^2 \right)^{\frac{1}{2}},
$$

with

$$
\begin{aligned}
\eta_K^2 = & \sum_{K\in\mathscr{T}_h} h_K^4 p_K^{-4} \|f + \Delta_h u_{\mathrm{DG}} - c u_{\mathrm{DG}}\|_{0,\Omega}^2 + \sum_{K\in\mathscr{T}_h} h_K^3 p_K^{-3} \|[\![\nabla_h u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\partial\Omega}^2 \\
& + \sum_{K\in\mathscr{T}_h} h_K p_K \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial K\setminus\partial\Omega}^2 + \sum_{K\in\mathscr{T}_h} h_K p_K \|g - u_{\mathrm{DG}}\|_{0,\partial K\cap\partial\Omega}^2 .
\end{aligned} \tag{3.11}
$$

Hence, dividing both sides of the above inequality by $\|e_{\mathrm{DG}}\|_{0,\Omega}$ leads to the following result.

THEOREM 3.1 Suppose that the dual problem (3.8)–(3.9) fulfils (3.10), and that the Dirichlet boundary data $g \in L^2(\Gamma)$. Furthermore, let $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ denote the hp-dG solution from (3.4), and $u \in H^{1,1}_{\boldsymbol{\alpha}}(\Omega)$ the analytical solution of (1.1)–(1.2) for some weight vector $\boldsymbol{\alpha} \in [0,1)^M$. Then, the following a posteriori error estimate holds

$$
\|u - u_{\mathrm{DG}}\|_{0,\Omega}^2 \leqslant C \sum_{K\in\mathscr{T}_h} \eta_K^2,
$$

where $C > 0$ is a constant independent of the local element sizes $\boldsymbol{h}$ and polynomial degrees $\boldsymbol{p}$, and the local error indicators η_K, $K \in \mathscr{T}_h$, are defined in (3.11).

REMARK 3.2 We observe a slight suboptimality with respect to the polynomial degree in the last two terms of the local error indicators η_K defined in (3.11). This results from the fact that due to the possible presence of hanging nodes in $\mathscr{T}_h$, a nonconforming interpolant is used in the proof of Theorem 3.1. Indeed, in the absence of hanging nodes, the factor of $h_K p_K$ can be improved to $h_K p_K^{-1}$. We point out that the energy norm a posteriori error indicators derived in Houston *et al.* (2007, 2008), for example, suffer from a similar suboptimality with respect to the spectral order.

3.4 *Numerical Example*

On the rectangle $\Omega = (-1,1)\times(0,1)$, we consider the PDE problem: find u such that

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \Gamma. \end{aligned}$$

We choose the Dirichlet boundary data g in such a way that the analytical solution is given by

$$u(r,\theta) = \frac{1}{\pi}\theta,$$

where (r,θ) denote polar coordinates in $\mathbb{R}^2$. Note that g is smooth on $\partial\Omega$, except at the point $(0,0)$. Indeed, in Cartesian coordinates we have that

$$g(x,y=0) = \begin{cases} 1 & \text{for } x<0 \\ 0 & \text{for } x>0 \end{cases}, \qquad (x,y)\in\partial\Omega.$$

In addition, we remark that $u \notin H^1(\Omega)$. However, there holds $u \in H^{1,1}_{\alpha}(\Omega)$ for any $\alpha \in (0,1)$, where the weight function for this problem is given by $\Phi_\alpha(\boldsymbol{x}) = |\boldsymbol{x}|^\alpha$. Furthermore, u is analytic away from $(0,0)$ and belongs to the Babuška-Guo space (see, e.g., Babuška & Guo (1988))

$$B^1_\alpha(\Omega) = \left\{ v \in L^2\Omega) : |v|_{H^{k,1}_{\boldsymbol{\alpha}}(\Omega)} \leqslant C d^k k! \ \ \forall k \geqslant 1, \text{ and constants } C,d \in \mathbb{R} \right\}.$$

With this in mind, we might therefore be able to achieve exponential convergence when hp–mesh refinement is employed; cf. Schötzau & Schwab (2001).

Firstly, however, we investigate the practical performance of the a posteriori error estimate derived in Theorem 3.1 within an automatic h–version adaptive refinement procedure which is based on 1-irregular quadrilateral elements. The h–adaptive meshes are constructed by marking the elements for refinement/derefinement according to the size of the local error indicators η_K; this is done by employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively.

In Figure 1(a) we show the initial mesh and computed dG solution based on employing $p=2$, i.e., biquadratic polynomials. Furthermore, the computational mesh and dG solution are depicted in Figures 1(b) & (c) after 4 and 9 adaptive refinements have been undertaken, respectively. Here, we observe that the mesh has been significantly refined in the vicinity of the discontinuity present in g, as we would expect. Figure 2(a) shows the history of the actual and estimated $L^2(\Omega)$–norm of the error on each of the meshes generated based on employing h–adaptive mesh refinement. Here, we observe that the a posteriori bound over-estimates the true error by a consistent factor. Indeed, the effectivity index tends to a value of around 16 as the mesh is adaptively refined, cf. Figure 2(b).

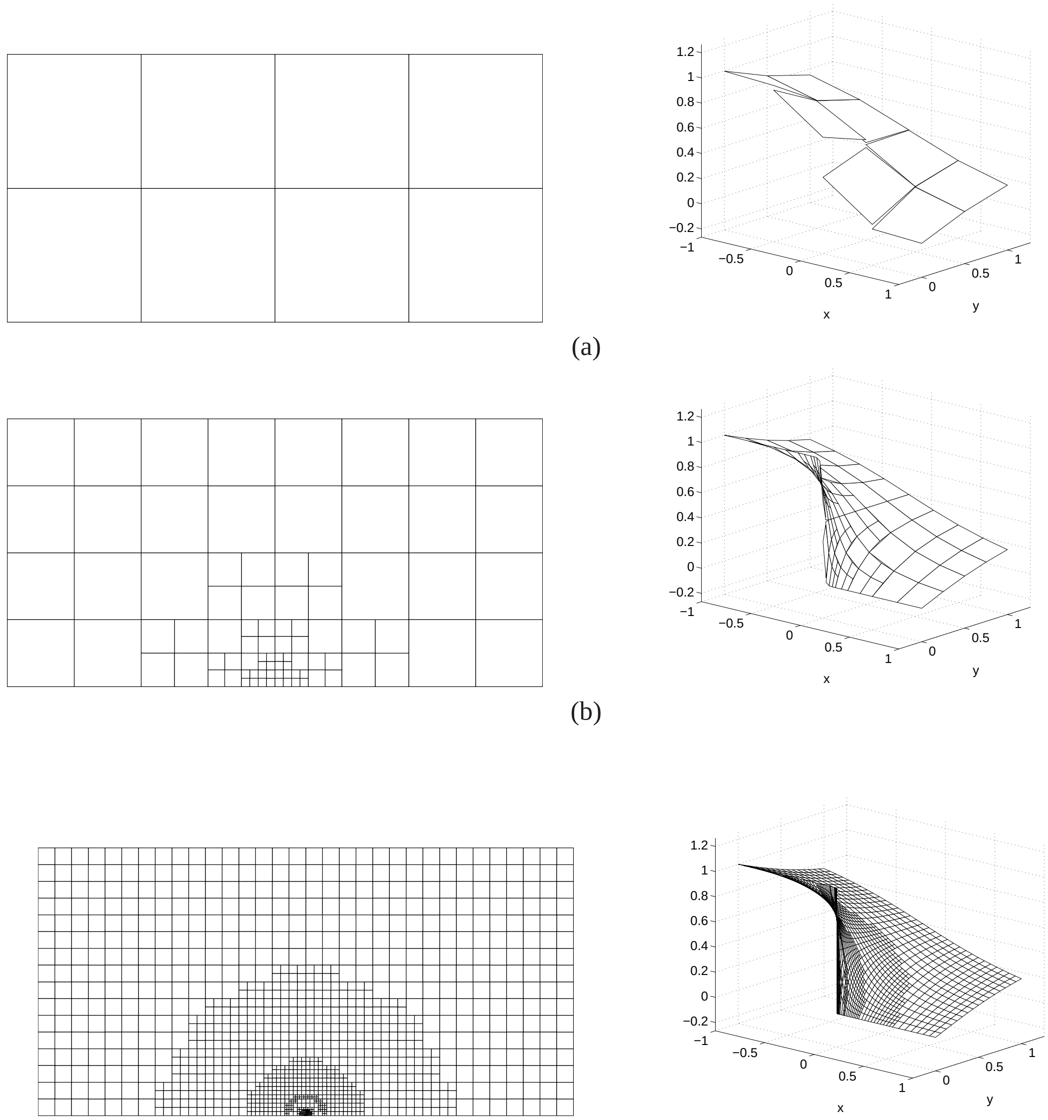


FIG. 1. *h*–Refinement. (a) Initial mesh and solution with 8 elements; Mesh and solution after: (b) 4 adaptive refinements, with 86 elements; (c) 9 adaptive refinements, with 1286 elements.

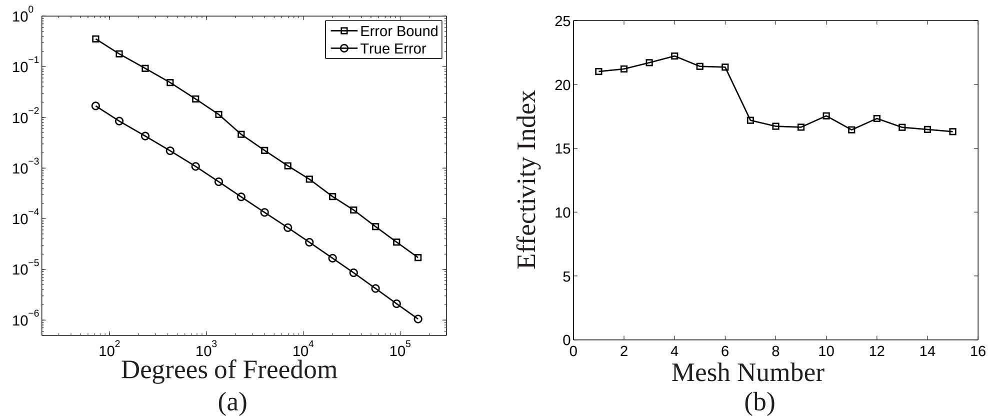


FIG. 2. h–Refinement. (a) Comparison of the actual and estimated $L^2(\Omega)$–norm of the error with respect to the number of degrees of freedom; (b) Effectivity indices.

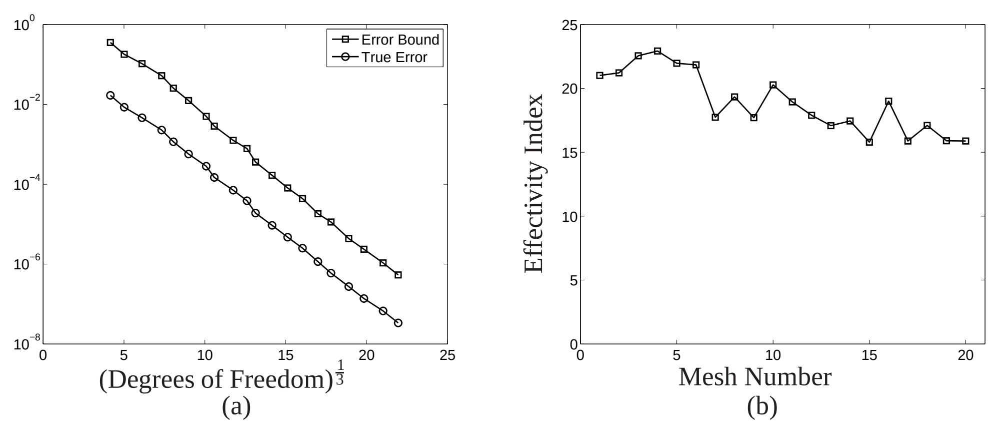


FIG. 3. hp–Refinement. (a) Comparison of the actual and estimated $L^2(\Omega)$–norm of the error with respect to the (third root of the) number of degrees of freedom; (b) Effectivity indices.

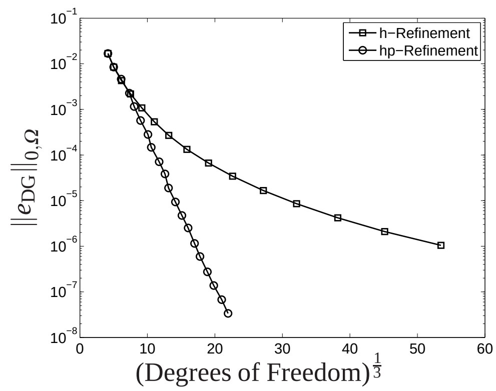


FIG. 4. Comparison between *h*– and *hp*–refinement.

We now turn our attention to *hp*–mesh adaptation. Here, we again mark elements for refinement/derefinement according to the size of the local error indicators η_K based on employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively. Once an element $K \in \mathscr{T}_h$ has been flagged for refinement or derefinement, a decision must be made whether the local mesh size h_K or the local degree p_K of the approximating polynomial should be adjusted accordingly. The choice to perform either *h*–refinement/derefinement or *p*–refinement/derefinement is based on estimating the local smoothness of the (unknown) analytical solution. To this end, we employ the *hp*–adaptive strategy developed in Houston & Süli (2005), where the local regularity of the analytical solution is estimated from truncated local Legendre expansions of the computed numerical solution; see, also, Houston *et al.* (2003).

In Figure 3(a) we present a comparison of the actual and estimated $L^2(\Omega)$–norm of the error versus the third root of the number of degrees of freedom in the finite element space $V_{\mathrm{DG}}(\mathscr{T}_h, \boldsymbol{p})$ on a linear-log scale, for the sequence of meshes generated by our *hp*–adaptive algorithm. We remark that the third root of the number of degrees of freedom is chosen on the basis of the a priori error analysis carried out in Wihler *et al.* (2003); cf., also, Schötzau & Wihler (2003). Here, we observe that the error bound over-estimates the true error by a (reasonably) consistent factor; indeed, from Figure 3(b), we see that the computed effectivity indices are in the range 15–19 as the mesh is refined. Moreover, from Figure 3(a) we observe that the convergence lines using *hp*–refinement are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this problem. We point out that the slight suboptimality with respect to the polynomial degree in the last two terms of the local error indicator η_K defined in (3.11) does not adversely affect the quality of the local indicators, cf. Remark 3.2. Indeed, computations based on employing a modified local indicator $\hat{\eta}_K$, where $\hat{\eta}_K$ is defined in an analogous fashion to η_K with the factor of $h_K p_K$ in the last two terms in (3.11) replaced by $h_K p_K^{-1}$, leads to quantitatively similar behaviour of the $L^2(\Omega)$–norm of the error as the finite element space is enriched, cf. Houston *et al.* (2008). Indeed, for this particular example, the sequence of *hp*–refined meshes generated by the proposed adaptive algorithm is identical when either local indicator, i.e., η_K or $\hat{\eta}_K$, is employed. However, the effectivity indices are slightly improved to between 13–19

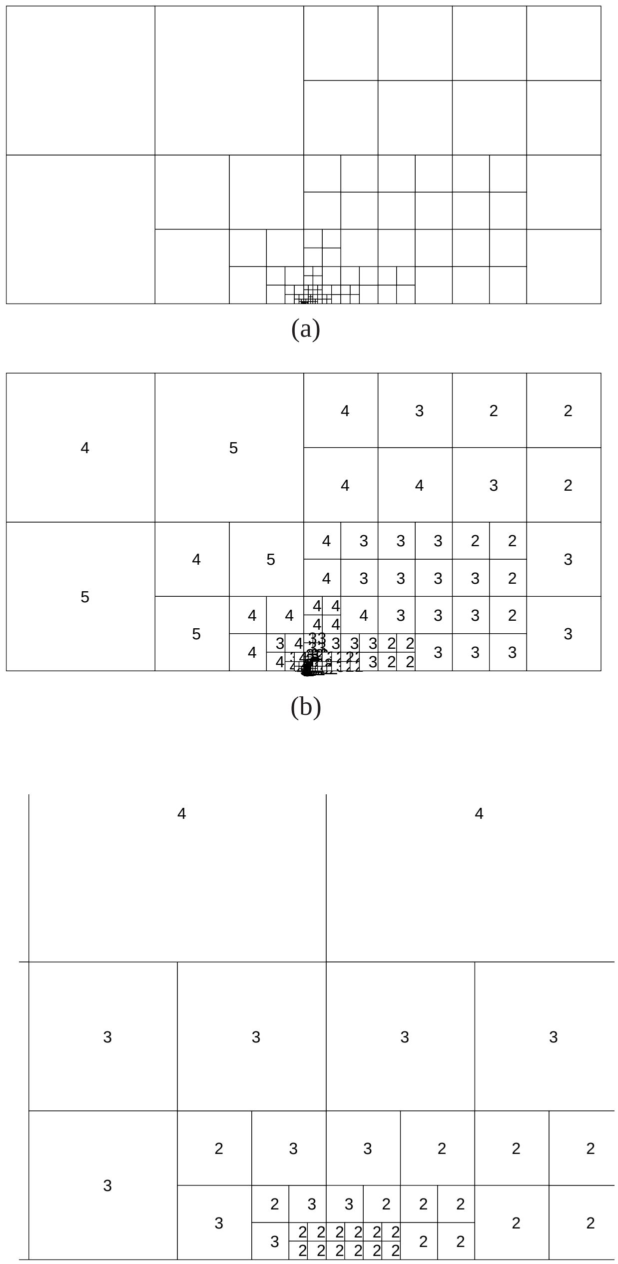
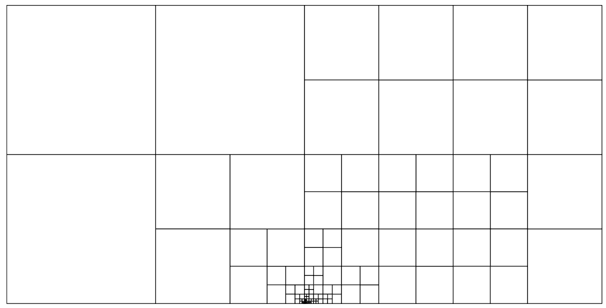
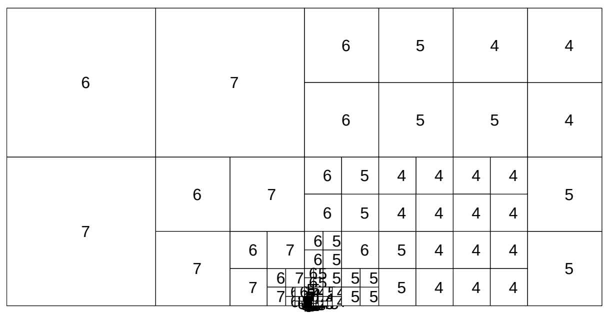


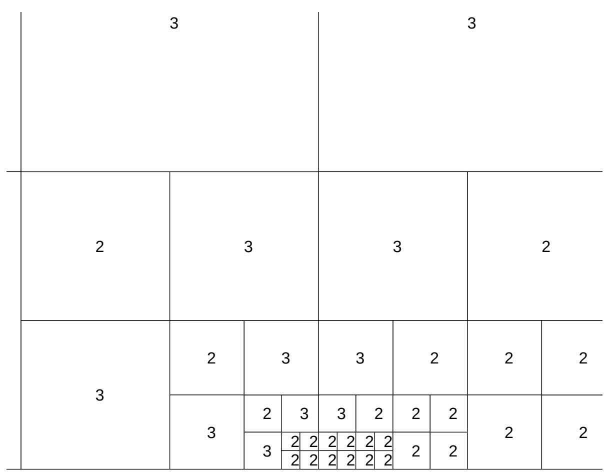
FIG. 5. *hp*–Mesh distribution after 9 adaptive refinements, with 134 elements and 2002 degrees of freedom: (a) *h*–mesh alone; (b) *hp*–mesh; (c) Zoom of (b).



(a)



(b)



(c)

FIG. 6. *hp*–Mesh distribution after 14 adaptive refinements, with 206 elements 4904 degrees of freedom: (a) *h*–mesh alone; (b) *hp*–mesh; (c) Zoom of (b).

when $\hat{\eta}_K$ is employed, in contrast to those computed using η_K. For brevity, these results have been omitted.

In Figure 4, we present a comparison between the actual $L^2(\Omega)$–norm of the error employing both h– and hp–mesh refinement. Here, we clearly observe the superiority of employing a grid adaptation strategy based on exploiting hp–adaptive refinement: on the final mesh, the $L^2(\Omega)$–norm of the error using hp–refinement is around three orders of magnitude smaller than the corresponding quantity computed when h-refinement is employed alone.

Finally, in Figures 5 & 6 we show the mesh generated using the proposed hp-version a posteriori error indicator stated in Theorem 3.1 after 9 and 14 hp-adaptive refinement steps, respectively. For clarity, we also show the h–mesh alone, as well as a zoom of the mesh in the vicinity of the origin. Here, we observe that h–refinement of the mesh has been performed in the vicinity of the discontinuity present in g, cf. above. Within this region, the polynomial degree has been kept at 2. Away from this region, the hp–adaptive algorithm increases the degree of the approximating piecewise polynomials where the analytical solution is smooth.

4. Conclusions

In this work, we have introduced a new variational framework for linear second-order elliptic PDE with discontinuous Dirichlet boundary conditions based on locally weighted Sobolev spaces. In particular, we have proved the well-posedness of the new setting by means of an inf-sup condition. In addition, we have proposed the use of symmetric hp–version interior penalty discontinuous Galerkin methods for the numerical approximation of such problems. For this discretisation scheme, we have derived an L^2–norm a posteriori error estimate whose performance within h– and hp–adaptive refinement procedures has been displayed with a model numerical experiment. Future work will deal with some extensions of the present setting to systems such as, e.g., the Stokes equations for cavity flow problems.

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